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COMPACT OPERATORS AND INTEGRAL EQUATIONS IN THE BV_p SPACE

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Abstract. In this paper, compact operators and integral equations in terms of integral kernel in the BV_p space will be considered. The BV_p space is the space of all bounded p-variation on a compact interval. The integral used in this paper is of Stieltjes-type. The integral is an integration with respect to a function of bounded p-variation. A fractional Brownian motion in the stochastic integral is a processes of bounded p-variation.

In this paper, let BV_p be the space of all bounded p-variation defined on a compact interval $[a,b]$. We shall consider compact operators $K : BV_p \to BV_p$ where K are defined by

$$(K\varphi)(x) = \int_a^b k(x,t)\varphi(t)dg(t).$$

Integral equations considered are of the form

$$\varphi = \psi + \lambda K\varphi.$$

The integral $\int_a^b f(t)dg(t)$ used here is the Kurzweil-Henstock-Young integral, which is a Stieltjes-type integral. When $f \in BV_p$ and $g \in BV_q$, where $\frac{1}{p} + \frac{1}{q} > 1$ and $p,q \geq 1$, then the integral $\int_a^b f(t)dg(t)$ exists, see [3, 7, 12]. The kurzweil-Henstock approach is used to handle the integral, for the Kurzweil-Henstock approach, see [8, 11].

1. Preliminaries

In this section, we shall present some results proved by L.C.Young in 1936, see [7, 12].

Definition 1. *Let $f : [a,b] \to \mathbb{R}$ and let $0 < p < \infty$. Given a partition $P = \{[u_i, v_i]\}_{i=1}^n$ of $[a,b]$, let*

$$V_p(f,P;[a,b]) = \left(\sum_{i=1}^n |f(v_i) - f(u_i)|^p\right)^{1/p}.$$

The p-variation of f is defined by

$$V_p(f;[a,b]) = \sup_P V_p(f,P;[a,b]).$$

We say that f is of bounded p-variation on $[a,b]$ or $f \in BV_p[a,b]$, if $V_p(f;[a,b]) < \infty$. In this paper, we always denote $BV_p[a,b]$ and $V_p(f;[a,b])$ by BV_p and $V_p(f)$, respectively.

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From Jensen's inequality, we have

$$\left(\sum |f(v_i)-f(u_i)|^q\right)^{1/q} \le \left(\sum |f(v_i)-f(u_i)|^p\right)^{1/p}$$

whenever $0<p<q$. If $f\in BV_p$ and $0<p<q$, then $V_q(f;[a,b]) \le V_p(f;[a,b]) < \infty$, i.e., $f\in BV_q$. Hence, we have $BV_p \subseteq BV_q$.

If $f\in BV_p$, then f is bounded. Thus $\|f\|_\infty$ exists. Let $\|f\|_{V_p} = V_p(f)+\|f\|_\infty$. It is well-known that $(BV_p, \|\cdot\|_{V_p})$ is a Banach space, if $p\ge 1$.

Theorem 1. *[3, Theorem 2.2; 12, p. 256, (6.2)]Let $f\in BV_p$ and $g\in BV_q$, with $p,q>0$ and $\frac{1}{p}+\frac{1}{q}>1$. Then, for any partition $P=\{[t_i,t_{i+1}]\}_{i=1}^n$ of $[a,b]$ and $\xi=t_i$ for some $i=1,2,\dots,n+1$,*

$$\left|\sum_{i=1}^n f(t_{i+1})\left(g(t_{i+1})-g(t_i)\right) - f(\xi)\left(g(b)-g(a)\right)\right| \le \left\{1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\} V_p(f)V_q(g),$$

where $\zeta\left(\frac{1}{p}+\frac{1}{q}\right)=\sum_{n=1}^\infty n^{-(\frac{1}{p}+\frac{1}{q})}$.

Corollary 1. *[3, Corollary 2.3; 12, p. 257, (6.4)]Let $f\in BV_p$ and $g\in BV_q$, with $p,q>0$ and $\frac{1}{p}+\frac{1}{q}>1$. Then, for any two partitions, $D=\{[t_i,t_{i+1}]\}_{i=1}^n$ and $D'=\{[s_j,s_{j+1}]\}_{j=1}^m$ of $[a,b]$, with any $\xi_i\in[t_i,t_{i+1}], \eta_j\in[s_j,s_{j+1}]$, we have*

$$\left|(D)\sum f(\xi_i)\left(g(t_{i+1})-g(t_i)\right) - (D')\sum f(\eta_j)\left(g(s_{j+1})-g(s_j)\right)\right| \le 2\left\{1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\} V_p(f)V_q(g),$$

where $\zeta(\frac{1}{p}+\frac{1}{q}) = \sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$.

A finite collection $P=\{I\}$ of nonoverlapping closed subintervals of $[a,b]$ is said to be a partition of $[a,b]$ if $\cup_{I\in P} I = [a,b]$. Let δ be a positive function on $[a,b]$ and I be a closed subinterval of $[a,b]$. An interval point-pair (I,ξ) is said to be δ-fine if $\xi\in I\subset(\xi-\delta(\xi),\xi+\delta(\xi))$. A finite collection of interval-point pairs, $D=\{(I,\xi)\}$, is called a δ-fine division of $[a,b]$ if each (I,ξ) is δ-fine and $\{I\}$ is a partition of $[a,b]$.

Now we shall introduce Kurzweil-Henstock-Young integrals.

Definition 2. *[3, Definition 1.2] Let $f,g:[a,b]\to\mathbb{R}$. Then f is said to be* Kurzweil-Henstock-Young integrable *(or $\mathcal{KHY}$-integrable) to real number A on $[a,b]$ with respect to g if for every $\epsilon>0$, there exists a positive function δ defined on $[a,b]$ such that for every δ-fine division $D=\{(I,\xi)\}$ of $[a,b]$, we have*

$$|S(f,\delta,D)-A|\le\epsilon,$$

where $S(f,\delta,D)=(D)\sum f(\xi)g(I)$, $g(I)=g(v)-g(u)$ and $I=[u,v]$. In this paper, the value A is denoted by $\int_a^b f dg$.

Theorem 2 (Existence Theorem)**.** *[3, Theorem 3.5] Let $f\in BV_p$ and $g\in BV_q$, with $p,q\ge 1$ and $\frac{1}{p}+\frac{1}{q}>1$, then f is $\mathcal{KHY}$-integrable with respect to g.*

The following Theorem has been proved for the Young integral in [12, p. 266], we shall prove it for the $\mathcal{KHY}$-integral. This theorem is crucial in this paper.

Theorem 3. *If $f \in BV_p$ and $g \in BV_q$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then*

$$\left|\int_a^b f dg\right| \leq \|f\|_\infty V_q(g) + 3\left\{1 + \zeta(\frac{1}{p} + \frac{1}{q})\right\} V_p(f) V_q(g).$$

Proof. Let $\epsilon > 0$ be given. By Theorem 2, there exists a positive function δ such that for every δ-fine division $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$,

$$\left|\int_a^b f dg - (D)\sum f(\xi_i)(g(v_i) - g(u_i))\right| < \epsilon. \tag{1}$$

By Theorem 1, Corollary 1 and inequality (1), we have

$$\begin{aligned}
\left|\int_a^b f dg\right| &\leq \left|f(a)(g(b)-g(a))\right| + \left|\int_a^b f dg - f(a)(g(b)-g(a))\right| \\
&\leq \|f\|_\infty V_q(g) + \left|\int_a^b f dg - (D)\sum f(\xi_i)(g(t_{i+1}) - g(t_i))\right| \\
&\quad + \left|(D)\sum f(\xi_i)(g(t_{i+1}) - g(t_i)) - \sum f(t_{i+1})(g(t_{i+1}) - g(t_i))\right| \\
&\quad + \left|\sum f(t_{i+1})(g(t_{i+1}) - g(t_i)) - f(a)(g(b)-g(a))\right| \\
&\leq \|f\|_\infty V_q(g) + \epsilon + 2\left\{1 + \zeta(\frac{1}{p} + \frac{1}{q})\right\} V_p(f)V_q(g) + \left\{1 + \zeta(\frac{1}{p} + \frac{1}{q})\right\} V_p(f)V_q(g)
\end{aligned}$$

Since ϵ is arbitrary,

$$\left|\int_a^b f dg\right| \leq \|f\|_\infty V_q(g) + 3\left\{1 + \zeta(\frac{1}{p} + \frac{1}{q})\right\} V_p(f) V_q(g).$$

□

In this note, from now onwards, we always assume that $p, q \geq 1$.

2. Continuous linear operators in BV_p spaces

An operator $T : (BV_p, \|\cdot\|_{V_p}) \longrightarrow (BV_q, \|\cdot\|_{V_q})$ is said to be continuous if $\|T\varphi_n - T\varphi\|_{V_q} \longrightarrow 0$ as $n \longrightarrow \infty$ whenever $\|\varphi_n - \varphi\|_{V_p} \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $T : (BV_p, \|\cdot\|_{V_p}) \longrightarrow (BV_q, \|\cdot\|_{V_q})$ be linear. Suppose T is continuous. Define

$$\|T\| = \sup\left\{\frac{\|T\varphi\|_{V_q}}{\|\varphi\|_{V_p}} : \varphi \in BV_p\right\}.$$

Thus $\|T\|$ is a norm of T. Note that $\|T\varphi\|_{V_q} \leq \|T\|\|\varphi\|_{V_p}$ for all φ, if T is continuous.

Let $B(BV_p)$ be the space of all continuous operators from $(BV_p, \|\cdot\|_{V_p})$ to $(BV_p, \|\cdot\|_{V_p})$. Let $(BV_p)^\star$ be the space of all continuous linear functionals defined on BV_p.

Theorem 4. *$B(BV_p)$ and $(BV_p)^\star$ are complete.*

Proof. The proof is standard, see [10, p.221, proposition 3]. □

The Banach-Steinhaus theorem, the open mapping theorem, the bounded inverse theorem and the closed graph theorem hold true for linear continuous operators from the BV_p space to any Banach spaces. We shall only state the Banach-Steinhaus theorem.

Theorem 5 (Banach-Steinhaus)**.** *[10, P 232, Proposition 13 and Problem 26] For each $n \in \mathbb{N}$, let $T_n : (BV_p, \|\cdot\|_{V_p}) \to (BV_q, \|\cdot\|_{V_q})$ be linear and continuous. If for each φ in BV_p, the sequence $\{T_n\varphi\}$ converges to a value $T\varphi$, i.e., $\|T_n\varphi - T\varphi\|_{V_q} \to 0$ as $n \to \infty$, then $T : BV_p \to BV_q$ is a continuous linear operator.*

We remark that the BV_q can be replaced by any Banach spaces.

Theorem 6. *Let $f \in BV_p$ and $g \in BV_q$. If $q \le p$, then $fg \in BV_p$ and*

$$V_p(fg) \le \|g\|_\infty V_p(f) + \|f\|_\infty V_p(g).$$

Proof. Let $P = \{[u,v]\}$ be a partition of $[a,b]$. Since $g \in BV_q$ and $q \le p$, $g \in BV_p$. By Minkowski inequality, we have

$$\begin{aligned}
\left(\sum_P |(fg)(v) - (fg)(u)|^p\right)^{1/p} &= \left(\sum_P |f(v)g(v) - f(u)g(u)|^p\right)^{1/p} \\
&= \left(\sum_P |f(v)g(v) - f(u)g(v) + f(u)g(v) - f(u)g(u)|^p\right)^{1/p} \\
&\le \left(\sum_P |f(v)g(v) - f(u)g(v)|^p\right)^{1/p} + \left(\sum_P |f(u)g(v) - f(u)g(u)|^p\right)^{1/p} \\
&\le \left(\sum_P \|g\|_\infty^p |f(v) - f(u)|^p\right)^{1/p} + \left(\sum_P \|f\|_\infty^p |g(v) - g(u)|^p\right)^{1/p} \\
&\le \|g\|_\infty V_p(f) + \|f\|_\infty V_p(g).
\end{aligned}$$

Since the partition P is arbitrary, $V_p(fg) \le \|g\|_\infty V_p(f) + \|f\|_\infty V_p(g)$ and $fg \in BV_p$. $\square$

Notice that

$$\|g\|_\infty V_p(f) + \|f\|_\infty V_p(g) \le \|g\|_{V_p} V_p(f) + \|f\|_\infty \|g\|_{V_p} = \|g\|_{V_p}(V_p(f) + \|f\|_\infty) = \|f\|_{V_p}\|g\|_{V_p}.$$

Hence we have

Corollary 2. *Let $f \in BV_p$ and $g \in BV_q$. If $q \le p$, then*

$$V_p(fg) \le \|f\|_{V_p}\|g\|_{V_p}.$$

Example 1. *Let $g \in BV_q$, $f_n \in BV_p$ and $\varphi \in BV_p$. By Theorem 6, $\varphi f_n \in BV_p$. Hence, by Theorem 2, $\int_a^b \varphi f_n dg$ exists if $\frac{1}{p} + \frac{1}{q} > 1$. Let $T_n : (BV_p, \|\cdot\|_{V_p}) \to \mathbb{R}$ be defined by $T_n\varphi = \int_a^b \varphi f_n dg$, where $f_n \in BV_p$ with $p, q \ge 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. Then T_n is continuous. Suppose $\lim_{n\to\infty} \int_a^b \varphi f_n dg$ exists. Then, by Theorem 5, $T : (BV_p, \|\cdot\|_{V_p}) \to \mathbb{R}$ defined by $T\varphi = \lim_{n\to\infty} \int_a^b \varphi f_n dg$ is continuous.*

31 Jan 2024 18:25:26 PST
230601-Boonpogkrong Version 2 - Submitted to J. Integr. Eq. Appl.

Example 2. *Let $\{f_j\}$ and $\{h_i\}$ be sequences of functions in BV_p and $g \in BV_q$ with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. Let $T_n : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ be defined by $(T_n\varphi)(x) = \sum_{j=1}^n \int_a^b h_j(x)\varphi(t) f_j(t) dg(t) = \sum_{j=1}^n h_j(x) \int_a^b \varphi(t) f_j(t) dg(t)$. By Theorem 6, T_n is well-defined. Then each $T_n : BV_p \to BV_p$ is a linear operator of finite rank. Hence T_n is continuous. Suppose $\{T_n\varphi\}$ is $\|\cdot\|_{V_p}$-convergent to $T\varphi$ in BV_p for each $\varphi \in BV_p$. Then, by the Banach-Steinhaus Theorem (Theorem 5), $T : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ is continuous.*

We shall discuss the compactness of T in the next section.

Operators in the BV_p space have also been discussed in [4, 5, 8].

3. Compact operators in the BV_p space

A sequence $\{\varphi_n\}$ in $(BV_p, \|\cdot\|_{V_p})$ is said to be bounded if $\{\varphi_n\}$ is bounded under $\|\cdot\|_{V_p}$. Let $C \subseteq BV_p$. C is said to be compact if for any bounded sequence in C there exists a $\|\cdot\|_{V_p}$-convergent subsequence.

An operator $T : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ is said to be compact if for any bounded sequence $\{\varphi_n\}$ in BV_p, there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{T\varphi_{n_k}\}$ is convergent in BV_p.

Using subsequence argument, as in Banach spaces, if $T : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ is linear and compact, then T is continuous.

The rank of an operator is the dimension of its range. It is well-known that every finite rank continuous linear operator acting between Banach spaces is compact.

Example 3. *Let $\{f_j\}$ and $\{h_i\}$ be sequences of functions in BV_p and $g \in BV_q$ with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. Suppose $K : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ is defined by*

$$(K\varphi)(x) = \int_a^b \left(\sum_{j=1}^n h_j(x) f_j(t) \right) \varphi(t) dg(t).$$

Then

$$(K\varphi)(x) = \int_a^b \left(\sum_{j=1}^n h_j(x) f_j(t) \right) \varphi(t) dg(t) = \sum_{j=1}^n h_j(x) \int_a^b \varphi f_j dg = \sum_{j=1}^n h_j(x) \alpha_j,$$

where $\alpha_j = \int_a^b \varphi f_j dg \in \mathbb{R}$. Thus K is a linear operator of finite rank. Therefore K is compact.

Here for easy reference, we shall prove that K is compact. Let $\{\varphi_k\}_{k=1}^\infty$ be a bounded sequence in BV_p. Hence $\{\|\varphi_k\|_{V_p}\}$ is bounded in $\mathbb{R}$. Then, by Theorem 3 and Corollary 2, we have

$$\begin{aligned}
|\alpha_{j,k}| = \left| \int_a^b \varphi_k f_j dg \right| &\leq \|\varphi_k f_j\|_\infty V_q(g) + 3\left\{1 + \zeta(\frac{1}{p} + \frac{1}{q})\right\} V_p(\varphi_k f_j) V_q(g) \\
&\leq \|\varphi_k\|_\infty \|f_j\|_\infty V_q(g) + 3\left\{1 + \zeta(\frac{1}{p} + \frac{1}{q})\right\} \|f\|_{V_p} \|\varphi_k\|_{V_p} V_q(g),
\end{aligned}$$

i.e., $\{\alpha_{j,k}\}_{k=1}^\infty$ is bounded in $\mathbb{R}$, for $j = 1, 2, \ldots, n$. By the BolzanoWeierstrass theorem, there exists a subsequence $\{\alpha_{j,k_l}\}$ of $\{\alpha_{j,k}\}_{k=1}^\infty$, converging in $\mathbb{R}$, for $j = 1, 2, \ldots, n$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$

such that for any $s,t \geq N$, $|\alpha_{j,k_s} - \alpha_{j,k_t}| < \epsilon$. Thus

$$\begin{aligned}
\|K\varphi_{k_s} - K\varphi_{k_t}\|_{V_p} &= V_p(K\varphi_{k_s} - K\varphi_{k_t}) + \|K\varphi_{k_s} - K\varphi_{k_t}\|_\infty \\
&= V_p\left(\sum_{j=1}^n h_j\alpha_{j,k_s} - \sum_{j=1}^n h_j\alpha_{j,k_t}\right) + \left\|\sum_{j=1}^n h_j\alpha_{j,k_s} - \sum_{j=1}^n h_j\alpha_{j,k_t}\right\|_\infty \\
&\leq \sum_{j=1}^n |\alpha_{j,k_s} - \alpha_{j,k_t}|\, V_p(h_j) + \sum_{j=1}^n |\alpha_{j,k_u} - \alpha_{j,k_v}|\, \|h_j\|_\infty \\
&< \epsilon \sum_{j=1}^n V_p(h_j) + \epsilon \sum_{j=1}^n \|h_j\|_\infty.
\end{aligned}$$

Hence, $\{K\varphi_{k_l}\}$ is a Cauchy sequence in BV_p under $\|\cdot\|_{V_p}$. Therefore, there exists $\psi \in BV_p$ such that $\|K\varphi_{k_l} - \psi\|_{V_p} \to 0$ as $k_l \to \infty$. Hence K is compact. Therefore, K is $\|\cdot\|_{V_p}$-continuous. We remark that we can use the same idea to prove that K is $\|\cdot\|_{V_p}$-continuous without using the fact that compactness implies continuity.

Next we shall prove a result for a countably infinite dimensional rank.

Lemma 1. *Let $\{f_j\}$ be sequence of functions in BV_p and $g \in BV_q$ with $p,q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. Suppose $\sum_{j=1}^\infty \|f_j\|_{V_p} < \infty$ and $\{h_j\}$ a sequence of functions such that, for each $x \in [a,b]$, $|h_j(x)| \leq A(x) < \infty$, for all j and $\sum_{j=1}^\infty h_j(x)f_j(t)$ exists for any $x,t \in [a,b]$. Then, for each $x \in [a,b]$, $\sum_{j=1}^\infty h_j(x)f_j(t) \in BV_p$ and for each $\varphi \in BV_p$,*

$$\int_a^b \left(\sum_{j=1}^\infty h_j(x)f_j(t)\right)\varphi(t)dg(t) = \sum_{j=1}^\infty h_j(x) \int_a^b f_j(t)\varphi(t)dg(t).$$

Proof. Let $m,n \in \mathbb{N}$ be fixed. For any fixed x, we have

$$V_p\left(\sum_{j=m}^n h_j(x)f_j(t)\right) \leq \sum_{j=m}^n |h_j(x)| V_p\left(f_j(t)\right) \leq A(x)\sum_{j=m}^n V_p(f_j).$$

Then

$$\begin{aligned}
V_p\left(\sum_{j=m}^\infty h_j(x)f_j(t)\right) &= V_p\left(\lim_{n\to\infty}\sum_{j=m}^n h_j(x)f_j(t)\right) \\
&\leq A(x)\lim_{n\to\infty}\sum_{j=m}^n V(f_j) = A(x)\sum_{j=m}^\infty V_p(f_j) \leq A(x)\sum_{j=m}^\infty \|f_j\|_{V_p}.
\end{aligned}$$

Thus $V_p\left(\sum_{j=m}^\infty h_j(x)f_j(t)\right) \to 0$ as $m \to \infty$. Hence, for each $x \in [a,b]$,

$$V_p\left(\sum_{j=1}^\infty h_j(x)f_j(t)\right) < \infty.$$

Therefore $\sum_{j=1}^\infty h_j(x)f_j(t) \in BV_p$, for each $x \in [a,b]$. We remark that this result, in fact, is a consequence of the completeness of BV_p.

Notice that, by Theorem 3 and Corollary 2,

$$\begin{aligned}
&\left|\int_a^b \left(\sum_{j=n+1}^{\infty} h_j(x)f_j(t)\right)\varphi(t)dg(t)\right| \\
&\quad \le \left\|\sum_{j=n+1}^{\infty} h_j(x)f_j(t)\varphi(t)\right\|_{\infty} V_q(g) + 3\left\{1+\zeta\left(\frac{1}{p_1}+\frac{1}{q}\right)\right\} V_p\left(\sum_{j=n+1}^{\infty} h_j(x)f_j(t)\varphi(t)\right) V_q(g) \\
&\quad \le \|\varphi\|_{\infty}\left\|\sum_{j=n+1}^{\infty} h_j(x)f_j(t)\right\|_{\infty} V_q(g) + 3\left\{1+\zeta\left(\frac{1}{p_1}+\frac{1}{q}\right)\right\}\|\varphi(t)\|_{V_p}\left\|\sum_{j=n+1}^{\infty} h_j(x)f_j(t)\right\|_{V_p} V_q(g).
\end{aligned}$$

Hence $\left|\int_a^b (\sum_{j=n+1}^{\infty} h_j(x)g_j(t))\varphi(t)\, dt\right| \to 0$ as $n\to\infty$. Observe that

$$\int_a^b \left(\sum_{j=1}^{\infty} h_j(x)f_j(t)\right)\varphi(t)dg(t) - \sum_{j=1}^{n}\int_a^b h_j(x)f_j(t)\varphi(t)dg(t) = \int_a^b \left(\sum_{j=n+1}^{\infty} h_j(x)f_j(t)\right)\varphi(t)dg(t).$$

So

$$\lim_{n\to\infty}\int_a^b \left(\sum_{j=1}^{n} h_j(x)f_j(t)\right)\varphi(t)dg(t) = \int_a^b \left(\sum_{j=1}^{\infty} h_j(x)f_j(t)\right)\varphi(t)dg(t).$$

□

The following lemma is proved by Abel's transformation, see [1, Ch.2, p.365], which is crucial in this paper.

Lemma 2. *Let $\{a_j\}$ be a sequence in a normed spaces S with norm $\|\cdot\|_S$ and $\{b_j\}$ a real-valued sequence such that $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty}|b_{j+1}-b_j|$ exist. Then $\lim_{j\to\infty} b_j$, $\sum_{j=1}^{\infty} a_j b_j$ exist and, for each $m = 1,2,\ldots,$*

$$\left\|\sum_{j=m}^{\infty} a_j b_j\right\|_S \le 2A\sum_{j=m}^{\infty}|b_{j+1}-b_j| + \left\|\sum_{k=m}^{\infty} a_k\right\|_S |b|, \tag{2}$$

where $A = \sup_n \|\sum_{j=1}^{n} a_j\|_S$ and $b = \lim_{j\to\infty} b_j$.

Lemma 3. *Let $\{f_j\}$ and $\{h_j\}$ be sequences of functions in BV_p and $g \in BV_q$ with $p,q \ge 1$ and $\frac{1}{p}+\frac{1}{q} > 1$. Suppose that $\sum_{j=1}^{\infty}\|f_j\|_{V_p} < \infty$ and for each $x \in [a,b]$, $\sum_{j=1}^{\infty} h_j(x)$ exists.*

(i) Then, for each x, there exists $0 < A(x) < \infty$ such that $|h_j(x)| \le A(x)$ for each j and $\sum_{j=1}^{\infty} h_j(x)f_j(t)$ exists for any $x,t \in [a,b]$.

(ii) Let $\varphi \in BV_p$ and $\alpha_j = \int_a^b f_j(t)\varphi(t)dg(t)$. Then $\sum_{j=1}^{\infty}|\alpha_{j+1}-\alpha_j| < \infty$ and $\sum_{j=1}^{\infty} h_j\alpha_j \in BV_p$.

Proof. (i) By given condition, for each x, $\sum_{j=1}^{\infty} h_j(x)$ exists. Hence there exists $0 < A(x) < \infty$ such that $|h_j(x)| \le A(x)$ for each j. Apply Lemma 2 to two real-valued sequences with $a_j = h_j(x)$ and $b_j = f_j(t)$ we have $\sum_{j=1}^{\infty} h_j(x)f_j(t)$ exists for any $x,t \in [a,b]$.

(ii) First, by Theorem 3 and Corollary 2, we have

$$|\alpha_{j+1}-\alpha_j| = \left|\int_a^b (f_{j+1}(t)-f_j(t))\varphi(t)dg(t)\right|$$
$$\leq \|\varphi\|_\infty \|f_{j+1}-f_j\|_\infty V_q(g) + 3\left\{1+\zeta(\frac{1}{p_1}+\frac{1}{q})\right\}\|f_{j+1}-f_j\|_{V_p}\|\varphi\|_{V_p}V_q(g).$$

Hence $\sum_{j=1}^\infty |\alpha_{j+1}-\alpha_j| < \infty$.

For each $x \in [a,b]$, apply Lemma 2 to two real-valued sequences, $a_j = h_j(x)$ and $b_j = \alpha_j$, we have $\sum_{j=1}^\infty h_j(x)\alpha_j$ exists.

Now apply Lemma 2 to a real-valued sequence $\{\alpha_j\}$ and a sequence $\{h_j\}$ in the normed space with norm $\|\cdot\|_{V_p}$, we have $\sum_{j=1}^\infty h_j\alpha_j$ exists under norm $\|\cdot\|_{V_p}$. Thus $\left\{\sum_{j=1}^n h_j\alpha_j\right\}$ is Cauchy under $\|\cdot\|_{V_p}$. Therefore, there exists $\psi \in BV_p$ such that $\left\|\sum_{j=1}^n h_j\alpha_j - \psi\right\|_{V_p} \to 0$ as $n\to\infty$. Therefore $\sum_{j=1}^\infty h_j\alpha_j \in BV_p$. □

Theorem 7. *Let $\{f_j\}$ and $\{h_j\}$ be sequences of functions in BV_p such that $\sum_{j=1}^\infty \|f_j\|_{V_p} < \infty$, for each $x\in[a,b]$, $\sum_{j=1}^\infty h_j(x)$ exists and $g \in BV_q$ with $p,q \geq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$. Let $K, K_n : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ be linear operators defined by*

$$(K_n\varphi)(x) = \int_a^b \left(\sum_{j=1}^n h_j(x)f_j(t)\right)\varphi(t)dg(t).$$

and

$$(K\varphi)(x) = \int_a^b \left(\sum_{j=1}^\infty h_j(x)f_j(t)\right)\varphi(t)dg(t).$$

Then $\|K_n - K\| \to 0$ as $n\to\infty$. Furthermore K is continuous and compact.

Proof. By Lemmas 1 and 3 (i), for each $x,t\in[a,b]$, $\sum_{j=1}^\infty h_j(x)g_j(t)$ exists and

$$(K\varphi)(x) = \int_a^b \left(\sum_{j=1}^\infty h_j(x)f_j(t)\right)\varphi(t)dg(t) = \sum_{j=1}^\infty h_j(x)\int_a^b \varphi f_j dg = \sum_{j=1}^\infty h_j(x)\alpha_j,$$

where $\alpha_j = \int_a^b \varphi(t)f_j(t)dg(t)$. Thus $(K-K_n)\varphi(x) = \sum_{j=n+1}^\infty h_j(x)\alpha_j$.

Hence, $\|(K-K_n)\varphi\|_{V_p} = \|\sum_{j=n+1}^\infty h_j\alpha_j\|_{V_p}$. By Lemma 3 (ii),

$$\|(K-K_n)(\varphi)\|_{V_p} \to 0 \text{ as } n\to\infty.$$

By the Banach-Steinhaus theorem (Theorem 5), K is continuous. Now, we shall prove that $\|K_n - K\| \to 0$ as $n\to\infty$. Recall that $\sum_{j=1}^\infty |\alpha_{j+1}-\alpha_j| < \infty$. Let $\alpha = \lim_{j\to\infty}\alpha_j$. Apply Lemma 2 to $\{a_j\}$, where $a_j = h_j$ with norm $\|\cdot\|_{V_p}$, and $b_j = \alpha_j$, use inequality (2), we have

$$\|(K-K_n)\varphi\|_{V_p} = \left\|\sum_{j=n+1}^\infty h_j\alpha_j\right\|_{V_p} \leq 2\sup_j\left\|\sum_{k=1}^j h_k\right\|_{V_p}\sum_{j=n+1}^\infty|\alpha_j-\alpha_{j+1}| + \left\|\sum_{k=n+1}^\infty h_k\right\|_{V_p}|\alpha|.$$

Now we shall estimate $\sum_{j=n+1}^{\infty}|\alpha_{j+1}-\alpha_j|$ and $|\alpha|$ in the above inequality. Since $\|\varphi\|_{V_p}=V_p(\varphi)+\|\varphi\|_\infty$, we have $\|\varphi\|_\infty\le\|\varphi\|_{V_p}$ and $V_p(\varphi)\le\|\varphi\|_{V_p}$. Then, by Theorem 3,

$$\begin{aligned}|\alpha_{j+1}-\alpha_j| &= \left|\int_a^b (f_{j+1}(t)-f_j(t))\varphi(t)dg(t)\right|\\ &\le \|\varphi\|_\infty\|f_{j+1}-f_j\|_\infty V_q(g)+3\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|f_{j+1}-f_j\|_{V_p}\|\varphi\|_{V_p}V_q(g)\\ &\le \|\varphi\|_{V_p}\|f_{j+1}-f_j\|_{V_p}V_q(g)+3\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|f_{j+1}-f_j\|_{V_p}\|\varphi\|_{V_p}V_q(g)\\ &\le 4\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|\varphi\|_{V_p}\|f_{j+1}-f_j\|_{V_p}V_q(g)\end{aligned}$$

Note that $\alpha=\lim_{j\to\infty}\alpha_j$ and $\alpha_j=\int_a^b\varphi(t)f_j(t)dg(t)$. By Theorem 3 and Corollary 2,

$$\begin{aligned}|\alpha| = \lim_{j\to\infty}|\alpha_j| &= \lim_{j\to\infty}\left|\int_a^b \varphi(t)f_j(t)dg(t)\right|\\ &\le \|\varphi\|_\infty\lim_{j\to\infty}\|f_j\|_\infty V_q(g)+3\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|\varphi\|_{V_p}\lim_{j\to\infty}\|f_j\|_{V_p}V_q(g)\\ &\le \|\varphi\|_\infty\sum_{j=1}^{\infty}\|f_j\|_\infty V_q(g)+3\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|\varphi\|_{V_p}\sum_{j=1}^{\infty}\|f_j\|_{V_p}V_q(g)\\ &\le \|\varphi\|_{V_p}\sum_{j=1}^{\infty}\|f_j\|_{V_p}V_q(g)+3\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|\varphi\|_{V_p}\sum_{j=1}^{\infty}\|f_j\|_{V_p}V_q(g)\\ &\le 4\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|\varphi\|_{V_p}\sum_{j=1}^{\infty}\|f_j\|_{V_p}V_q(g).\end{aligned}$$

Thus

$$\begin{aligned}\|(K-K_n)\| &= \sup_{\varphi}\frac{\|(K-K_n)\varphi\|_{V_p}}{\|\varphi\|_{V_p}}\\ &\le 2\sup_j\left\|\sum_{k=1}^{j}h_k\right\|_{V_p}\sum_{j=n+1}^{\infty}4\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\|f_{j+1}-f_j\|_{V_p}V_q(g)\\ &\quad+\left\|\sum_{k=n+1}^{\infty}h_k\right\|_{V_p}4\left\{1+\zeta(\frac{1}{p}+\frac{1}{q})\right\}\sum_{j=1}^{\infty}\|f_j\|_{V_p}V_q(g).\end{aligned}$$

Hence $\|K_n-K\|\to 0$ as $n\to\infty$. Note that each K_n is a linear operator of finite rank. Hence each K_n is compact. Thus K is compact, i.e., for any bounded sequence $\{\varphi_n\}$ in BV_p, there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{K\varphi_{n_k}\}$ is convergent in BV_p under $\|\cdot\|_{V_p}$. □

Now we shall consider nonlinear operators. Let $h(t,s)$ be a Carathéodory function from $[a,b]\times\mathbb{R}$ to $\mathbb{R}$, i.e., the function $h(t,\cdot)$ is continuous for all $t\in[a,b]$ and the function $h(\cdot,s)$ is measurable for every $s\in\mathbb{R}$, [6, p. 349, Chapter 5]

Let φ be a function defined on $[a,b]$ and $H\varphi$ a function defined on $[a,b]$ and $(H\varphi)(t)=h(t,\varphi(t))$. The operator H is called a Nemytskii operator. In the following, we assume that the following Lipschitz condition holds for the function $h(t,s)$, i.e., there exists a positive numbers κ such that

$$|h(t_1,s_1)-h(t_2,s_2)|\le\kappa(|t_1-t_2|+|s_1-s_2|) \tag{3}$$

for all $t_1,t_2\in[a,b]$ and all $s_1,s_2\in\mathbb{R}$.

Lemma 4. *Let H be a Nemytskii operator defined as above and $\varphi\in BV_p$. Then*

(i) $V_p(H\varphi)\le 2\kappa(|b-a|+V_p(\varphi))$*;*

(ii) $\|H\varphi\|_\infty\le\kappa(|b-a|+2\|\varphi\|_\infty+|h(t_0,M_\varphi)|)$*, where $t_0\in[a,b]$ is fixed and* $h(t_0,M_\varphi)=\sup\{h(t_0,s): -\|\varphi\|_\infty\le s\le\|\varphi\|_\infty\}$*;*

(iii) the operator $H:(BV_p,\|\cdot\|_{V_p})\to(BV_p,\|\cdot\|_{V_p})$ and H maps a bounded sequence in $(BV_p,\|\cdot\|_{V_p})$ to a bounded sequence in $(BV_p,\|\cdot\|_{V_p})$.

Proof. By inequality (3), we get

$$|h(t_1,\varphi(t_1))-h(t_2,\varphi(t_2))|^p\le 2^p\kappa^p\left(|t_1-t_2|^p+|\varphi(t_1)-\varphi(t_2)|^p\right).$$

Hence

$$\begin{aligned} V_p^p(h(t,\varphi(t)))&\le 2^p\kappa^p\left(V_p^p(t)+V_p^p(\varphi(t))\right)\\ &\le 2^p\kappa^p\left(V_1^p(t)+V_p^p(\varphi(t))\right)\\ &=2^p\kappa^p\left(|b-a|^p+V_p^p(\varphi(t))\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} V_p(h(t,\varphi(t)))&\le\left(2^p\kappa^p\left(|b-a|^p+V_p^p(\varphi(t))\right)\right)^{\frac{1}{p}}\\ &<2\kappa\left((|b-a|^p)^{\frac{1}{p}}+\left(V_p^p(\varphi(t))\right)^{\frac{1}{p}}\right)\\ &=2\kappa\left(|b-a|+V_p(\varphi(t))\right). \end{aligned}$$

Therefore (i) holds. Thus $(H\varphi)(t)=h(t,\varphi(t))$ is in BV_p if $\varphi\in BV_p$. Furthermore $\{V_p(H\varphi_n)\}$ is bounded if $\{V_p(\varphi_n)\}$ is bounded.

On the other hand,

$$|h(t,\varphi(t))|\le\kappa(|t-t_0|+|\varphi(t)-\varphi(t_0)|)+|h(t_0,\varphi(t_0))|$$

Suppose $\|\varphi\|_\infty\le\alpha$, i.e., $-\alpha\le\varphi(t)\le\alpha$ and all $t\in[a,b]$. Recall that for a fixed t_0, $h(t_0,\cdot)$ is continuous on $[-\alpha,\alpha]$. Thus, there exists $M_\varphi\in[-\alpha,\alpha]$ such that $|h(t_0,\varphi(t_0))|\le|h(t_0,M_\varphi)|$. Hence,

$$\|H\varphi\|_\infty\le\kappa(|b-a|+2\|\varphi\|_\infty)+|h(t_0,M_\varphi)| \tag{4}$$

Thus (ii) holds. Therefore, $\{\|H\varphi_n\|_\infty\}$ is bounded if $\{\|\varphi_n\|_\infty\}$ is bounded. Consequently, (iii) holds. ∎

Let $k(x,t)=\sum_{j=1}^{\infty} h_j(x)f_j(t)$ and $K:(BV_p,\|\cdot\|_{V_p})\to(BV_p,\|\cdot\|_{V_p})$ be defined as in the Theorem 7, i.e.,

$$(K\varphi)(x)=\int_a^b k(x,t)\varphi(t)dg(t)=\int_a^b\left(\sum_{j=1}^{\infty} h_j(x)f_j(t)\right)\varphi(t)dg(t).$$

Then the composite operator KH maps $(BV_p,\|\cdot\|_{V_p})$ space to $(BV_p,\|\cdot\|_{V_p})$ space. Let $\varphi\in BV_p$, Then $(KH)(\varphi)\in BV_p$ and for each $x\in[a,b]$

$$((KH)(\varphi))(x)=(K(H\varphi))(x)=\int_a^b k(x,t)h(t,\varphi(t))\,dt.$$

The composite operator KH is called a Hammerstein operator.

Corollary 3. *The nonlinear Hammerstein operator $KH:(BV_p,\|\cdot\|_{V_p})\to(BV_p,\|\cdot\|_{V_p})$ given above is compact.*

Proof. By Lemma 4 (iii), the Nemytskii operator H maps every bounded sequence under $\|\cdot\|_{V_p}$ to a bounded sequence under $\|\cdot\|_{V_p}$. By Theorem 7, K is compact. Therefore the composite Hammerstein operator KH from $(BV_p,\|\cdot\|_{V_p})$ to $(BV_p,\|\cdot\|_{V_p})$ is compact. $\square$

4. Integral equations

The Fredholm-Stieltjes integral equation of the second kind is an equation of the form

$$\varphi(x)=\psi(x)+\lambda\int_a^b k(x,t)\varphi(t)dg(t), \tag{5}$$

where $\psi:[a,b]\to\mathbb{R}$ and $k:[a,b]\times[a,b]\to\mathbb{R}$. The function k is known as the integral kernel.

In this section, let $\psi,\varphi\in BV_p$, we first discuss the case when the integral kernel is separable, i.e., $k(x,t)=h(x)f(t)$. By the standard method we can show that the integral equation (5) has a unique solution. We shall write it down for easy reference. Suppose that $h,f\in BV_p$ and $g\in BV_q$.

The equation (5) becomes

$$\begin{aligned}\varphi(x)&=\psi(x)+\lambda\int_a^b h(x)f(t)\varphi(t)dg(t)\\&=\psi(x)+\lambda h(x)\alpha,\end{aligned} \tag{6}$$

where $\alpha=\int_a^b f(t)\varphi(t)dg(t)$. Multiply $f(x)$ the both side of the above equality, then integrate with respect to $g(x)$, we get

$$\int_a^b\varphi(x)f(x)dg(x)=\int_a^b\psi(x)f(x)dg(x)+\lambda\alpha\int_a^b h(x)f(x)dg(x),$$

i.e.,

$$\alpha=\int_a^b\psi(x)f(x)dg(x)+\lambda\alpha\int_a^b h(x)f(x)dg(x).$$

Note that by Theorems 3 and 6, $\int_a^b \psi(x)f(x)dg(x)$ and $\int_a^b h(x)f(x)dg(x)$ exist. If $\lambda\int_a^b h(x)f(x)dg(x) \neq 1$, then we get

$$\alpha = \frac{\int_a^b \psi(x)f(x)dg(x)}{1-\lambda\int_a^b h(x)f(x)dg(x)}.$$

Hence, from Equation (6),

$$\varphi(x) = \psi(x) + \lambda h(x)\alpha = \psi(x) + \lambda h(x)\frac{\int_a^b \psi(t)f(t)dg(t)}{1-\lambda\int_a^b h(t)f(t)dg(t)}.$$

is the unique solution of a Fredholm-Stieltjes integral equation with separable integral kernel whenever $\lambda\int_a^b h(t)f(t)dg(t) \neq 1$.

For the case when $\lambda\int_a^b h(t)f(t)dg(t) = 1$, the equation has no solution, if $\int_a^b \psi(t)f(t)dg(t) \neq 0$.

Let $h_j, f_j \in BV_p$, $g \in BV_q$, for $j = 1,2,\ldots,n$, and

$$k(x,t) = \sum_{j=1}^{n} h_j(x)f_j(t)$$

for $x,t \in [a,b]$. Then the corresponding Fredholm integral equation has properties analogous to the above case.

Linear Fredholm equations and the Kurzweil-Henstock integral have been addressed in [5].

Let us now return to the case in which the kernel $k(x,t) = \sum_{j=1}^{\infty} h_j(x)f_j(t)$ and the operator K are the same as in Theorem 7. Let $\psi \in BV_p$ and $T\varphi = \psi + \lambda K\varphi$ for $\varphi \in BV_p$. Let $\|K\| = \sup\frac{\|K\varphi\|_{V_p}}{\|\varphi\|_{V_p}}$, where supremum is over all $\varphi \in BV_p$.

Theorem 8. *Suppose there exists $\mu > 0$ such that $\|K\varphi\|_{V_p} \leq \mu\|\varphi\|_{V_p}$ for each φ, i.e., $\|K\| \leq \mu$. For any λ with $0 < \lambda < \frac{1}{\mu}$, there exists a unique fixed point $\phi \in BV_p$, i.e.,*

$$\phi(x) = \psi(x) + \lambda(K\phi)(x) = \psi(x) + \lambda\int_a^b k(x,t)\phi(t)dg(t).$$

This solution ϕ is given by a convergent Neumann series $\phi(x) = \sum_{i=1}^{\infty}\lambda^i K^i\psi$ and $\|\phi\|_{V_p} \leq \frac{1}{1-\lambda\mu}\|\psi\|_{V_p}$, where $K^1 = K$ and $K^i\psi = K\left(K^{i-1}\psi\right)$, $i = 2,3,4,\ldots$.

Proof. The proof is standard. Let $\phi_0(x) = \psi(x)$, $\phi_n(x) = \psi + \lambda K\phi_{n-1}(x)$, $n = 1,2,\ldots$. Then $\phi_{n+1}(x) = \sum_{i=0}^{n+1}\lambda^i K^i\phi(x)$, $n = 1,2,\ldots$. Since $\|K\| \leq \mu$, we have

$$\|K^i\psi\|_{V_p} = \|KK^{i-1}\psi\|_{V_p} \leq \mu\|K^{i-1}\psi\|_{V_p} \leq \mu^i\|\psi\|_{V_p}.$$

Thus, for any $m,n \in \mathbb{N}$, we have

$$\|\phi_n - \phi_m\|_{V_p} = \left\|\sum_{i=m+1}^{n}\lambda^i K^i\psi\right\|_{V_p} \leq \sum_{i=m+1}^{n}\lambda^i\left\|K^i\psi\right\|_{V_p} = \left(\sum_{i=m+1}^{n}(\lambda\mu)^i\right)\|\psi\|_{V_p}.$$

Since $0 < \lambda\mu < 1$ by our assumptions, the sequence $\{\phi_n\}$ is Cauchy under $\|\cdot\|_{V_p}$. Hence $\lim_{n\to\infty}\phi_n = \sum_{i=0}^{\infty}\lambda^i K^i\psi$ exists in BV_p, if $0 < \lambda < \frac{1}{\mu}$.

Let $\phi(x) = \sum_{i=0}^{\infty} \mu^i (K^i \psi)(x)$. Then $\phi(x) = \lim_{n\to\infty} \phi_n(x)$. By Theorem 7, K is $\|\cdot\|_{V_p}$-continuous. From the iteration equation, $\phi_n(x) = \psi + \lambda K \phi_{n-1}(x)$ and the continuity of K, we have

$$\begin{aligned}
\lim_{n\to\infty} \phi_n(x) &= \psi + \lambda \lim_{n\to\infty} K\phi_{n-1}(x) \\
&= \psi + \lambda \lim_{n\to\infty} \int_a^b k(x,t)\phi_{n-1}(t)dg(t) \\
&= \psi + \lambda \int_a^b k(x,t)\left(\lim_{n\to\infty} \phi_{n-1}(t)\right) dg(t) \\
&= \psi + \lambda \int_a^b k(x,t)\phi(t)dg(t).
\end{aligned}$$

Thus, if $0 < \lambda < \frac{1}{\mu}$, we can find $\phi \in BV_p$ such that $\phi(x) = \psi + \lambda \int_a^b k(x,t)\phi(t)dg(t)$.

Now we shall prove that the fixed point ϕ is unique. Suppose that there are two fixed points, namely ϕ and Φ. Then $\phi = \psi + \lambda K\phi$ and $\Phi = \psi + \lambda K\Phi$. Therefore $\phi - \Phi = \lambda K(\phi - \Phi)$ and

$$\|\phi - \Phi\|_{V_p} = \lambda \|K(\phi - \Phi)\|_{V_p} \le \lambda \|K\| \|(\phi - \Phi)\|_{V_p} \le \lambda\mu \|(\phi - \Phi)\|_{V_p}.$$

Hence $(1 - \lambda\mu)\|\phi - \Phi\|_{V_p} \le 0$. Recall that $1 - \lambda\mu > 0$. It imply that $\|\phi - \Phi\|_{V_p} = 0$. Thus $\|\phi - \Phi\|_{V_p} = 0$. Consequently $\phi = \Phi$. Therefore the fixed point ϕ is unique. □

Remark 1. *We note that although μ is difficult to locate. However, according to Theorem 8, if λ is a sufficiently small positive number, then a unique fixed point exists.*

Theorem 9 (Tychonoff's theorem)**.** *[9, Theorem A] Let C be a convex subset of a locally convex topological vector space. If T is a continuous operator which maps C into a compact subset of C, then T has a fixed point in C.*

Theorem 10. *Let $T\varphi = \psi + \lambda(KH)\varphi$, where KH is given before Corollary 3. Assume that $\psi \in BV_p$, $\frac{3\kappa|b-a| + |h(t_0, M_{2\psi})|}{\|\psi\|_{V_p}} \le \beta$ and $\|K\| \le \mu$, $0 < \lambda < \frac{1}{\mu(\beta + 4\kappa)}$. Then $T : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ has a fixed point.*

Proof. By Lemma 4 (i) and (ii), we have

$$\|H\varphi\|_{V_p} = V_p(H\varphi) + \|H\varphi\|_\infty \le 3\kappa|b-a| + 2\kappa\|\varphi\|_{V_p} + |h(t_0, M_\varphi)|.$$

Since K is continuous, we have

$$\|KH\varphi\|_{V_p} \le \|K\| \|H\varphi\|_{V_p} \le \|K\| \left(3\kappa|b-a| + 2\kappa\|\varphi\|_{V_p} + |h(t_0, M_\varphi)|\right).$$

Let $C = \{\varphi \in BV_p : \|\varphi\|_{V_p} \leq 2\|\psi\|_{V_p}\}$. Then C is convex and bounded. For every $\varphi \in C$, we have

$$\begin{aligned}
\|T\varphi\|_{V_p} &\leq \lambda\|(KH)\varphi\|_{V_p} + \|\psi\|_{V_p} \\
&\leq \lambda\|K\| \left(3\kappa|b-a| + 2\kappa\|\varphi\|_{V_p} + |h(t_0, M_\varphi)|\right) + \|\psi\|_{V_p} \\
&\leq \lambda\mu \left(3\kappa|b-a| + 4\kappa\|\psi\|_{V_p} + |h(t_0, M_{2\psi})|\right) + \|\psi\|_{V_p} \\
&= \lambda\mu\|\psi\|_{V_p} \left(\frac{3\kappa|b-a| + |h(t_0, M_{2\psi})|}{\|\psi\|_{V_p}} + 4\kappa\right) + \|\psi\|_{V_p} \\
&< \frac{1}{\mu(\beta+4\kappa)}\mu\|\psi\|_{V_p}\,(\beta+4\kappa) + \|\psi\|_{V_p} \\
&= 2\|\psi\|_{V_p}.
\end{aligned}$$

Then $TC \subseteq C$. T is compact since KH is compact. Thus TC is compact. Hence, by Tychonoff's theorem, T has a fixed point in C. $\square$

Remark 2. *We note that although* $\frac{1}{\mu(\beta+4\kappa)}$ *is difficult to locate. However, according to Theorem 10, if* λ *is a sufficiently small positive number,* $T : (BV_p, \|\cdot\|_{V_p}) \to (BV_p, \|\cdot\|_{V_p})$ *has a fixed point.*

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References

[1] *T. M. Apostol*: Mathematical Analysis, Addison Wesley, Reading, 1957.

[2] *K. K. Aye and P. Y. Lee*: The dual of the space of functions of bounded variation, Math. Bohem. **131(1)** (2006), 1–9.

[3] *V. Boonpogkrong and T. S. Chew*: On integral with integrators in BV_p, Real Anal. Exch. **30(1)** (2004), 193–200.

[4] *D. Bugajewski, J. Gulgowski and P. Kasprzak* : On continuity and compactness of some nonlinear operators in the spaces of functions of bounded variation, Ann. di Mat. Pura ed Appl. **195** (2016), 1513–1530.

[5] *M. Federson and R. Bianconi*: Linear Fredholm integral equations and the integral of Kurzweil, J. Appl. Anal. **8(1)** (2002), 83–110.

[6] *M. A. Krasnosel'skii*: Integral Operators in Spaces of Summable Functions. English Edition, Noordhoff International, 1976.

[7] *E. R. Love and L. C. Young*: On fractional integration by parts, Proc. London Math. Soc., Ser. 2, **44** (1938), 1–35.

[8] *G. A. Monteiro, A. Slavík and M. Tvrdý*: Kurzweil-Stieltjes Integral Theory and Applications. Ser. Real Anal. **Vol. 15**, World Scientific, 2018.

[9] *S. A. Morrisf and E. S. Noussair*: The Schauder-Tychonoff fixed point theorem and applications. Matematický časopis **25(2)** (1975), 165–172.

[10] *H. L. Royden*: Real Analysis, Third edition, Macmillan, 1989.

[11] *S. Schwabik and G. Ye* : Topics in Banach Space Integration. Ser. Real Anal. **Vol. 10**, World Scientific, 2005.

[12] *L. C. Young*: An inequality of the Hölder type, connected with Stieltjes integration, Acta Math. **67** (1936), 251–282.

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