Fixed Point Theorems for Meir-Keeler Multivalued Maps and Application

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Abstract: This work is intended for a generalization of Darbo fixed point theorem by using multivalued condensing operators, and a measure of weak noncompactness which does not necessarily possesses the maximum property. Moreover, we prove some fixed point theorems in Banach algebras which satisfy an appointed weak sequential condition (P). As an application, we establish the existence of solutions for a functional integral inclusion.

Key words : Measure of weak noncompactness, Meir-Keeler multimap, Darbo fixed point theorem, integral equation.
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1 Introduction

Measure of noncompactness plays an important role in nonlinear analysis which has been introduced by Kuratowski [25]. Darbo [15], in 1955, used this measure to present a fixed point theorem for a k-set-contraction mapping in a nonempty bounded closed and convex subset of Banach spaces. Then, Sadovskii [29] established a more general result for condensing mapping. Recently, many studies have

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appeared as generalizations of Darbo's fixed point theorem (DFPT) [2–4, 9, 22]. In [2, 4], authors extended DFPT for some classes of condensing mappings and they proved a different existence of fixed point theorems. Also, in [3], Aghajani et al. presented a new generalization with the introduction of the concept of Meir-Keeler condensing mapping under an arbitrary measure of noncompactness which does not fulfill maximum property. In 2019, Belhadj et al. [9] investigated some generalizations of DFPT for Meir-Keeler condensing mappings involving a measure of weak noncompactness. Moreover, they gave a fixed point result of Krasnoselskii type and proved some coupled fixed point theorems for single-valued mappings. They applied these results to investigate the existence of solutions for a system of functional integral equations of Volterra type.

In this paper, we present a version of Meir-Keeler theorem for condensing multivalued mappings with respect to a measure of weak noncompactness and we present some generalizations of DFPT by considering a measure of weak noncompactness, which does not necessarily satisfy the maximum property. This motivates us to establish some Krasnoselskii's type fixed point results for the sum of two multivalued mappings on a closed convex subset of a Banach space. Moreover, we are interested in the theory of fixed point in Banach algebras. Inspired by Dhage's paper [18], we prove a Krasnoselskii type fixed point theorem in a Banach algebra which satisfies a certain weak sequential condition (P) where the measure of weak noncompactness satisfies the condition (m). Furthermore, we use the class of multivalued operator $\frac{I-\Gamma''}{\Gamma}$ (see [12]) to presnt new fixed point theorems for the sum $\Gamma\Gamma' + \Gamma''$ of certain operators and we study the existence of solutions for functional integral inclusion

$$\rho(t) \in f(\rho(t)) \int_0^t F(s, \rho(s))ds + \int_0^t g(s, \rho(s))ds \quad t \in I = [0, 1],$$

where $\mathcal{Q}$ is a Banach space, $f : \mathcal{Q} \to \mathcal{Q}$, $F : I \times \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ and $g : I \times \mathcal{Q} \to \mathcal{Q}$.

2 Preliminaries

Suppose that $\mathcal{Q}$ denote a Banach space equipped with the norm $\|\cdot\|$ and with the zero element θ. For the sake of brevity

1.
2. $\mathcal{P}(\mathcal{Q}) = \{\mathcal{G} \subset \mathcal{Q} : \mathcal{G}$ is nonempty $\}$,
3. $\mathcal{P}^b(\mathcal{Q}) = \{\mathcal{G} \subset \mathcal{Q} : \mathcal{G}$ is nonempty and bounded$\}$,
4. $\mathcal{P}^{bc}(\mathcal{Q}) = \{\mathcal{G} \subset \mathcal{Q} : \mathcal{G}$ is nonempty, bounded and closed $\}$,
5. $\mathcal{P}^{bcc}(\mathcal{Q}) = \{\mathcal{G} \subset \mathcal{Q} : \mathcal{G}$ is nonempty, bounded, closed and convex$\}$.

Let $B_\varepsilon(\theta)$ be the closed ball depending on the center θ and radius ε. If $\mathcal{G} \in \mathcal{P}(\mathcal{Q})$ then $\overline{\mathcal{G}}$, $conv\mathcal{G}$, $\overline{conv\mathcal{G}}$, and $\overline{\mathcal{G}}^\omega$, indicate the closure, the convex hull, the closed convex hull, and the weak closure of the subset $\mathcal{G}$, respectively. Furthermore, $\rho_n \rightharpoonup \rho$ and $\rho_n \longrightarrow \rho$ indicate the weak convergence and the strong convergence of a sequence $\{\rho_n\}$ to ρ, respectively.

A multivalued mapping $\Gamma : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ is said to have the weakly sequentially closed graph property if for each sequence $\{\rho_n\} \subset \mathcal{G}$ with $\rho_n \rightharpoonup \rho$ in $\mathcal{G}$ and for each sequence $\{\varrho_n\}$ with $\varrho_n \in \Gamma(\rho_n)$, for all

$n \in \mathbb{N}$, $\varrho_n \rightharpoonup \varrho$ in $\mathcal{Q}$ implies $\varrho \in \Gamma(\rho)$.

A multivalued mapping $\Gamma : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ is said to be weakly completely continuous property if Γ has weakly sequentially closed graph property and, for each $\mathcal{G} \in \mathcal{P}^b(\mathcal{Q})$, $\Gamma(\mathcal{G})$ is a relatively weakly compact subset of $\mathcal{Q}$.

A single-valued mapping $\Gamma : \mathcal{G} \to \mathcal{Q}$ is said to be weakly sequentially continuous property whenever for each sequence $\{\rho_n\} \subset \mathcal{G}$ with $\rho_n \rightharpoonup \rho \in \mathcal{G}$, we have $\Gamma(\rho_n) \rightharpoonup \Gamma(\rho)$.

A multivalued mapping $\Gamma : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ is said to be sequentially weakly upper semicompact property in $\mathcal{G}$ if for each sequence $\{\rho_n\}$ in $\mathcal{G}$ such that $\rho_n \rightharpoonup \rho$ and for each $\varrho_n \in \Gamma(\rho_n)$, the sequence $\{\varrho_n\}$ has a weakly convergent subsequence in $\Gamma(\rho)$.

For any $\Delta \in \mathcal{P}^b(\mathcal{Q})$, let

$$\|\Delta\| = \sup\{\|d\|,\, d \in \Delta\}.$$

For $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{P}^{bc}(\mathcal{Q})$ and $d_1 \in \mathcal{G}_1$, we define

$$\mathcal{D}(d_1, \mathcal{G}_2) = \inf\{\|d_1 - d_2\|,\, d_2 \in \mathcal{G}_2\},$$

and

$$\chi(\mathcal{G}_1, \mathcal{G}_2) = \sup_{d_1 \in \mathcal{G}_1} \mathcal{D}(d_1, \mathcal{G}_2).$$

The function $d_H : \mathcal{P}^{bc}(\mathcal{Q}) \times \mathcal{P}^{bc}(\mathcal{Q}) \to \mathbb{R}_+$ is defined by

$$d_H(\mathcal{G}_1, \mathcal{G}_2) = \max\{\chi(\mathcal{G}_1, \mathcal{G}_2), \chi(\mathcal{G}_2, \mathcal{G}_1)\}$$

is a metric on $\mathcal{P}^{bc}(\mathcal{Q})$ known as the Hausdorff metric on $\mathcal{Q}$ (see [21]). It can be seen that $d_H(\{\theta\}, \mathcal{G}) = \|\mathcal{G}\|$ for any $\mathcal{G} \in \mathcal{P}^{bc}(\mathcal{Q})$.

Proposition 2.1 *[21] If $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{P}^{bc}(\mathcal{Q})$, then $d_H(\mathcal{G}_1, \mathcal{G}_2) \le r$ if and only if*

$$\mathcal{G}_1 \subset \mathcal{G}_2 + B_\varepsilon(\theta) \text{ and } \mathcal{G}_2 \subset \mathcal{G}_1 + B_\varepsilon(\theta).$$

A multivalued mapping $\Gamma : \mathcal{G} \to \mathcal{P}^{bc}(\mathcal{Q})$ is said to be $\mathcal{D}$-Lipschitzian if there is a nondecreasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with the condition $\psi(0) = 0$ such that

$$d_H(\Gamma\rho, \Gamma\varrho) \le \psi(\|\rho - \varrho\|),$$

for each $\rho, \varrho \in \mathcal{G}$. The function ψ is said to be a $\mathcal{D}$-function of Γ on $\mathcal{G}$. For the function $\psi(t) = \alpha t$ with $t > 0$, then Γ is said to be a Lipschitzian multivalued mapping. Especially, for $\alpha < 1$, Γ is said to be a multivalued contraction. With the condition $\psi(t) < t$ for $t > 0$, Γ is said to be a nonlinear contraction multivalued mapping with contraction function ψ.

A mapping $\Gamma : \mathcal{Q} \longrightarrow \mathcal{Q}$ is called $\mathcal{D}$-Lipschitzian if, for each $\rho, \varrho \in \mathcal{Q}$,

$$\|\Gamma\rho - \Gamma\varrho\| \le \psi(\|\rho - \varrho\|).$$

Definition 2.1 *A function $\mu : \mathcal{P}^b(\mathcal{Q}) \to \mathbb{R}_+$ is called a measure of weak noncompactness (MWNC, for short) on $\mathcal{Q}$ if the following conditions are satisfied:*

1.

(1) $\mu(\overline{conv}(\Delta)) = \mu(\Delta)$ *for all* $\Delta \in \mathcal{P}^b(\mathcal{Q})$.

(2) *Monotonicity: For all* $\Delta_1, \Delta_2 \in \mathcal{P}^b(\mathcal{Q})$, *we have*

$$\Delta_1 \subset \Delta_2 \Longrightarrow \mu(\Delta_1) \leq \mu(\Delta_2).$$

(3) *The set* $Ker\mu = \{\Delta \in \mathcal{P}^b(\mathcal{Q}) : \mu(\Delta) = 0\}$ *(which is called the kernel of the MWNC* μ*) is nonempty and if* $\mu(\Delta) = 0$, *then* Δ *is relatively weakly compact in* $\mathcal{Q}$.

(4) *If* $\{\Delta_n\}$ *is a decreasing sequence of* $\overline{\mathcal{P}}^b(\mathcal{Q})$ *satisfying* $\lim\limits_{n\to\infty} \mu(\Delta_n) = 0$, *then the set* $\Delta_\infty = \bigcap\limits_{n=1}^{\infty} \Delta_n$ *is nonempty.*

A consequence of the definition

$$\mu(\overline{\Delta}^{\omega}) = \mu(\overline{\Delta}) = \mu(\Delta).$$

From (4), $\bigcap\limits_{n=1}^{\infty} \Delta_n = \Delta_\infty$ belongs to $Ker\mu$, since $\mu(\Delta_\infty) \leq \mu(\Delta_n)$ for every n and $\lim\limits_{n\to\infty} \mu(\Delta_n) = 0$.

Definition 2.2 *The MWNC μ is said to be regular if it satisfies the following conditions:*

1.

- *the kernel of μ consists of all possible relatively weakly compact subsets of $\mathcal{Q}$ (the complete property).*
- $\mu(c\Delta) = c\mu(\Delta)$ *for all* $c > 0$ *for all* $\Delta \in \mathcal{P}^b(\mathcal{Q})$.
- $\mu(\Delta_1 + \Delta_2) \leq \mu(\Delta_1) + \mu(\Delta_2)$ *for all* $\Delta_1, \Delta_2 \in \mathcal{P}^b(\mathcal{Q})$.
- $\mu(\Delta_1 \cup \Delta_2) = \max\{\mu(\Delta_1), \mu(\Delta_2)\}$ *for all* $\Delta_1, \Delta_2 \in \mathcal{P}^b(\mathcal{Q})$ *(the maximum property)* .

The above Definitions are a general format of the De Blasi MWNC β [16] defined by

$$\beta(\Delta) = \inf\Big\{\varepsilon > 0, \text{ there is a weakly compact set } \mathcal{K} \text{ such that } \Delta \subseteq \mathcal{K} + B_\varepsilon(\theta)\Big\},$$

where $\Delta \in \mathcal{P}^b(\mathcal{Q})$.
For more details on the properties of MWNC we refer the reader to [1, 5, 6, 23, 24].
In [22], the following MWNC was introduced, which does not satisfy the maximum property. Let $\mathcal{Q}$

be a nonreflexive Banach space and let $(\mathcal{Q} \times \mathcal{Q}, \|.\|_1)$ be the product Banach space with the standard norm $\|(\rho, \varrho)\|_1 = \max\{\|\rho\|, \|\varrho\|\}$. For all $\Delta \subseteq \mathcal{P}^b(\mathcal{Q}) \times \mathcal{P}^b(\mathcal{Q})$, we pose

$$\mu(\Delta) = \max\{\beta(\Delta_1), diam(\Delta_2)\},$$

where $\Delta_i : \Delta \to \mathcal{Q}$ are the natural projections for $i = 1, 2$.

A multivalued $\Gamma : \mathcal{G} \longrightarrow \mathcal{P}(\mathcal{Q})$ is said to be $\mathcal{D}$-set-Lipschitzian (with respect to μ) if there is a nondecreasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with the condition $\psi(0) = 0$ such that

$$\mu(\Gamma\Delta) \leq \psi(\mu(\Delta)),$$

for each $\Delta \in \mathcal{P}^b(\mathcal{Q})$ such that $\Gamma(\Delta)$ is bounded. For the function $\psi(t) = \alpha t$, Γ is said to be a α-set-Lipschitzian and for $\alpha < 1$, Γ is said to be a α-set-contraction. Having condition $\psi(t) < t$, for all $t > 0$, Γ is said to be a nonlinear $\mathcal{G}$-set-contraction. Especially, for $\mu = diam(.)$, Γ is said to be a diametrically $\mathcal{D}$-Lipschitzian, diametrically $\alpha-$Lipschitzian, diametrically $\alpha-$contraction and diametrically nonlinear contraction, respectively.

We recall the following result.

Theorem 2.1 *([28]) Suppose that $\mathcal{Q}$ is a metrizable locally convex linear topological space and $\mathcal{G}$ be a weakly compact and convex subset of $\mathcal{Q}$. If $\Gamma : \mathcal{G} \to \mathcal{P}^{bcc}(\mathcal{G})$ has weakly sequentially closed graph property, then Γ possesses a fixed point.*

3 Fixed point results in Banach space

The notion of Meir-Keeler contraction has been introduced by Meir and Keeler [27] and an interesting fixed point theorem has been proved. In this section, the notion of Meir-Keeler condensing multivalued mappings via an arbitrary MWNC on a Banach space $\mathcal{Q}$ is presented and some fixed point results are proved.

Definition 3.1 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \mathcal{P}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. The mapping $\Gamma : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ is a Meir-Keeler condensing mapping (MKCMM) (with respect to μ) if for each $\varepsilon > 0$, there is $\delta > 0$ such that for each $\Delta \in \mathcal{P}^b(\mathcal{G})$*

$$\epsilon \leq \mu(\Delta) < \epsilon + \delta \Rightarrow \mu(T\Delta \cap \mathcal{G}) < \epsilon. \tag{3.1}$$

Theorem 3.1 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Let $\Gamma : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ be a multivalued mapping such that*

1. *Γ has weakly sequentially closed graph,*
2. *Γ is a MKCMM,*
3. *for each $\rho \in \mathcal{G}$, $\Gamma\rho \cap \mathcal{G} \neq \varnothing$,*
4. *$\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{G})$.*

Then Γ admits a fixed point.

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Proof Consider

$$\mathcal{G}_0 = \overline{conv}^{\omega}(\Gamma\mathcal{G} \cap \mathcal{G}) \text{ and } \mathcal{G}_{n+1} = \overline{conv}^{\omega}(\Gamma\mathcal{G}_n \cap \mathcal{G}) \text{ for all } n \geq 0.$$

We see that $(\mathcal{G}_n)$ is a positive decreasing sequence in $\mathcal{Q}$. Then $(\varepsilon_n = \mu(\mathcal{G}_n))$ is a decreasing sequence of real numbers. We pose $r = \lim_{n\to\infty} \varepsilon_n$. If $r \neq 0$, there exists $n \in \mathbb{N}$ such that $r \leq \mu(\mathcal{G}_n) < r + \delta(r)$. By assumption 1, we have

$$\mu(\mathcal{G}_{n+1}) = \mu(\overline{conv}^{\omega}(\Gamma\mathcal{G}_n \cap \mathcal{G})) = \mu(\Gamma\mathcal{G}_n \cap \mathcal{G}) < r,$$

which is absurd. Then, we deduce that $r = 0$. Since $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$, for all $n \geq 0$, from the axiom (4) of Definition 2.1, we deduce that $\mathcal{G}_\infty = \bigcap_{n=0}^{\infty} \mathcal{G}_n$ is a weakly closed convex subset of $\mathcal{G}$ such that $\mu(\mathcal{G}_\infty) = 0$, which means that $\mathcal{G}_\infty$ is weakly compact. We pose

$$\mathcal{U} = \{\Delta \subset \mathcal{G}_\infty, \Delta \text{ is weakly compact, convex and } \Gamma\Delta \cap \mathcal{G} \subset \Delta\}.$$

Since

$$\Gamma\mathcal{G}_\infty \cap \mathcal{G} \subset \Gamma\mathcal{G}_n \cap \mathcal{G} \subset \overline{conv}^{\omega}(\Gamma\mathcal{G}_n \cap \mathcal{G}) = \mathcal{G}_{n+1} \subset \mathcal{G}_n$$

for all $n \geq 0$, we have

$$\Gamma\mathcal{G}_\infty \cap \mathcal{G} \subset \bigcap_{n=0}^{\infty} \mathcal{G}_n = \mathcal{G}_\infty.$$

We deduce that $\mathcal{G}_\infty \in \mathcal{U}$, so, $\mathcal{U} \neq \varnothing$. On the other hand, $\mathcal{U}$ is partially ordered by

$$\Delta_1 \preceq \Delta_2 \Leftrightarrow \Delta_1 \supset \Delta_2, \text{ for all } \Delta_1, \Delta_2 \in \mathcal{U}.$$

Every chain $\mathcal{J}$ in $\mathcal{U}$ has the finite intersection property, so, as $\mathcal{G}_\infty$ is weakly compact, the intersection of all members of any chain in $(U, \preceq)$ is nonempty, that is, $B = \bigcap_{\Delta\in\mathcal{J}} \Delta \neq \varnothing$. Since

$$\Gamma B \cap \mathcal{G} \subset \Gamma\Delta \cap \mathcal{G} \subset \Delta,$$

for all $\Delta \in \mathcal{J}$, we get

$$TB \cap \mathcal{G} \subset \bigcap_{\Delta\in\mathcal{J}} \Delta = B,$$

that is, $\Gamma\prime \in \mathcal{U}$, and it is an upper bound of $\mathcal{J}$. By Zorn's lemma, we deduce that $\mathcal{U}$ has a maximal element say Δ_0. Let $\rho \in \Delta_0$. We have

$$\Gamma(\rho) \cap \mathcal{G} \subset \Gamma\Delta_0 \cap \mathcal{G} \subset \Delta_0$$

and so, we deduce that $T\rho \cap \Delta_0 \neq \varnothing$, for all $\rho \in \Delta_0$. Put $\Delta_1 = \overline{conv}^{\omega}(\Gamma\Delta_0 \cap \Delta_0)$. Therefore,

$$\Gamma\Delta_0 \cap \Delta_0 \subset \Gamma\Delta_0 \cap \mathcal{G} \subset \Delta_0.$$

So, $\Delta_1 \subset \Delta_0$. By the maximality of Δ_0, we have $\Delta_1 = \Delta_0$. It is clear that Δ_1 is convex, weakly compact and $\Gamma\Delta_1 \subset \Gamma\Delta_0 \cap \Delta_0 \subset \Delta_1$. Hence, by Theorem 2.1, $\Gamma : \Delta_1 \to \mathcal{P}^{bcc}(\Delta_1)$ has a fixed point.

As a consequence of Theorem 3.1, we have:

Corollary 3.1 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Suppose that $\Gamma : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{G})$ is a multivalued mapping such that*

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1. Γ *has weakly sequentially closed graph,*

2. Γ *is a MKCMM,*

3. $\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{G})$.

Then Γ *possesses a fixed point.*

Corollary 3.2 *([9]) Suppose that* $\mathcal{Q}$ *is a Banach space,* $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q}) \cap \mathcal{P}^b(\mathcal{Q})$ *and* μ *be an MWNC on* $\mathcal{Q}$*. Assume that MKCMM* $\Gamma : \mathcal{G} \to \mathcal{G}$ *is weakly sequentially continuous. Then* Γ *admits a fixed point.*

The following result gives a partial answer to Problem 1.3 in [19] when we consider a MKCMM as a particular case of a diametrically multivalued contraction.

Theorem 3.2 *Suppose* $\mathcal{Q}$ *be a Banach space and* $\mathcal{G} \in \overline{\mathcal{P}}(\mathcal{Q})$*. Suppose that* $\Gamma : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ *is a multivalued mapping such that, for each* $\rho \in \mathcal{G}$*,* $Tx \cap \mathcal{G} \neq \varnothing$*. Assume that for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that, for each* $\Delta \in \mathcal{P}(\mathcal{G})$*,*

$$\epsilon \leq diam(\Delta) < \epsilon + \delta \Rightarrow diam(\Gamma\Delta \cap \mathcal{G}) < \epsilon.$$

If $\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$*, then* Γ *has a unique fixed point.*

Proof We pose

$$\mathcal{G}_0 = \overline{\Gamma\mathcal{G} \cap \mathcal{G}}^{\omega} \text{ and } \mathcal{G}_{n+1} = \overline{\Gamma\mathcal{G}_n \cap \mathcal{G}}^{\omega} \text{ for all } n \geq 0.$$

It is well known that $diam(\cdot)$ is an MWNC in the sense of Definition 2.1 which enjoys property (4). The proof of Theorem 3.1 guarantees the existence of a $\Gamma-$invariant nonempty weakly closed convex subset Δ_0 with $diam(\Delta_0) = 0$, which means that Δ_0 is a singleton. Hence, Γ has a strict fixed point. To show uniqueness, we assume that there exist two different fixed points $\rho_0, \rho_1 \in \mathcal{G}$ and we put $\Delta_0 = \{\rho_0,\ \rho_1\}$. We have

$$diam(\Delta_0) = diam(\Gamma\Delta_0 \cap \mathcal{G}) < \|\rho_0 - \rho_1\|,$$

which is absurd, so, Γ has a unique fixed point.

Lim [26] and [30], Suzuki presented the implication of L-functions and specified the Meir-Keeler contractions in metric spaces. In the same way, Aghajani [3] characterized the notion of Meir-Keeler condensing operator via L-functions.

Definition 3.2 *([26]) A function* $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *is called an* $L-$*function (resp. strictly L-function) if* $\sigma(0) = 0$*,* $\sigma(s) > 0$ *for all* $s \in (0, +\infty)$ *and for each* $s \in (0, +\infty)$ *there exists* $\delta > 0$ *such that* $\sigma(t) \leq s$ *(resp.* $\sigma(t) < s$*), for each* $t \in [s, s + \delta]$.

Example 3.1 *An upper semi-continuous and nondecreasing function* $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\sigma(0) = 0$ *is said to be a* $\mathcal{D}-$*function [17]. If* $\sigma(t) < t$ *for all* $t > 0$*, the function* σ *is said to be a non linear contraction. It is clear that a non linear contraction is an* $L-$*function. For example,* $\sigma(t) = \alpha t$*, where* $0 \leq \alpha < 1$ *is an* $L-$*function.*

Applying the same technique in [3], we can prove the following characterization of Meir-Keeler condensing multivalued mappings.

Proposition 3.1 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \mathcal{P}^b(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. $\Gamma : \mathcal{G} \to \mathcal{P}^b(\mathcal{Q})$ is a MKCMM if and only if there is an L−function (resp. strictly L-function) σ such that*

$$\mu(\Gamma\Delta \cap \mathcal{G}) < \sigma(\mu(\Delta)), \quad (resp. \;\; \mu(\Gamma\Delta \cap \mathcal{G}) \leq \sigma(\mu(\Delta))),$$

for each $\Delta \in \mathcal{P}^b(\mathcal{G})$ with $\mu(\Delta) > 0$.

As a consequence of Theorem 3.1 and Proposition 3.1, we obtain the following fixed point result.

Theorem 3.3 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Suppose that $\Gamma : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ is a multivalued mapping such that*

1. *Γ has weakly sequentially closed graph,*
2. *there is an L−function σ (resp. strictly L-function) such that*

$$\mu(\Gamma\Delta \cap \mathcal{G}) < \sigma(\mu(\Delta)), \quad (resp. \;\; \mu(\Gamma\Delta \cap \mathcal{G}) \leq \sigma(\mu(\Delta))),$$

   for each $\Delta \in \mathcal{P}^b(\mathcal{G})$ with $\mu(\Delta) > 0$,

3. *$\Gamma\rho \cap \mathcal{G} \neq \varnothing$, for each $\rho \in \mathcal{G}$,*
4. *$\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{G})$.*

Then Γ admits a fixed point.

In particular, if $\mathcal{G}$ is invariant under Γ, we deduce the following result.

Corollary 3.3 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Suppose that $\Gamma : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ is a multivalued mapping such that*

1. *Γ has weakly sequentially closed graph,*
2. *there is an L−function (resp. strictly L-function) σ such that for each $\Delta \in \mathcal{P}^b(S)$ with $\mu(\Delta) \neq \varnothing$,*

$$\mu(\Gamma\Delta) < \sigma(\mu(\Delta)), \quad (resp. \;\; \mu(\Gamma\Delta) \leq \sigma(\mu(\Delta))),$$

3. *$\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{G})$.*

Then Γ possesses a fixed point.

The following result is a generalization of Theorem 3.1 in [12].

Corollary 3.4 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Suppose that $\Gamma : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ is a multivalued mapping such that*

1. *T has weakly sequentially closed graph,*

2. there is a non linear contraction σ such that

$$\mu(\Gamma\Delta \cap \mathcal{G}) \leq \sigma(\mu(\Delta)),$$

for each $\Delta \in \mathcal{P}^b(\mathcal{G})$ with $\mu(\Delta) > 0$,

3. $\Gamma\rho \cap \mathcal{G} \neq \varnothing$, for each $\rho \in \mathcal{G}$,

4. $\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$.

Then Γ has a fixed point.

Corollary 3.5 *[9] Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q}) \cap \mathcal{P}^b(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Suppose that $\Gamma : \mathcal{G} \to \mathcal{G}$ is a weakly sequentially continuous and σ be an $L-$function such that*

$$\mu(\Gamma\Delta) < \sigma(\mu(\Delta)),$$

for each $\Delta \in \mathcal{P}^b(\mathcal{G})$ with $\mu(\Delta) \neq 0$. Then Γ has a fixed point.

Corollary 3.6 *[28] Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be an MWNC on $\mathcal{Q}$. Suppose that $\Gamma : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{G})$ is a multivalued mapping such that*

1. Γ has weakly sequentially closed graph,

2. Γ is an $\alpha-$set-contraction,

3. $\Gamma(\mathcal{G}) \in \mathcal{P}^b(\mathcal{G})$.

Then Γ has a fixed point.

In the following, we state some Krasnoselskii's fixed point results.

Theorem 3.4 *Suppose that $\mathcal{Q}$ is a Banach space, $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ and μ be a subadditive and complete MWNC on $\mathcal{Q}$. Suppose that $\Gamma, \Gamma'' : \mathcal{G} \to \overline{\mathcal{P}}^{bcc}(\mathcal{Q})$ are multivalued mappings such that*

1. Γ has weakly sequentially closed graph,

2. Γ is a MKCMM,

3. Γ'' is weakly completely continuous,

4. $\Gamma(\mathcal{G})$, $\Gamma''(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$,

5. $\Gamma(\rho) + \Gamma''(\rho) \in \mathcal{P}^{bcc}(\mathcal{G})$, for each $\rho \in \mathcal{G}$.

Then $\Gamma + \Gamma''$ has a fixed point.

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Proof By assumption 5, $\Gamma+\Gamma'':\mathcal{G}\to\mathcal{P}^{bcc}(\mathcal{Q})$ is defined and by Lemma 2.2 (see [8]), $\Gamma+\Gamma''$ enjoy weakly sequentially closed graph

Let $\Delta\in\mathcal{P}^{b}(\mathcal{Q})$. Since μ is subadditive,

$$\mu((\Gamma+\Gamma'')(\Delta))\leq\mu(\Gamma(\Delta)+\Gamma''(\Delta))\leq\mu(\Gamma(\Delta))+\mu(\Gamma''(\Delta)).$$

As Γ'' is weakly compact and μ is complete, we have

$$\mu((\Gamma+\Gamma'')(\Delta))\leq\mu(\Gamma(\Delta)).$$

Hence, $\Gamma+\Gamma''$ is a MKCMM. By Corollary 3.1, we conclude the result.

Now, we present the next Theorem as an interesting result in the metric fixed point theory.

Theorem 3.5 *Suppose that* $(\mathcal{Q},\|\cdot\|)$ *is a Banach space and* $\mathcal{G}\in\overline{\mathcal{P}}^{bcc}(\mathcal{Q})$. *Suppose that* $\Gamma:\mathcal{Q}\to\mathcal{P}^{bc}(\mathcal{Q})$ *and* $\Gamma'':\mathcal{G}\to\mathcal{P}(\mathcal{Q})$ *are multivalued mappings such that*

1. Γ *is sequentially weakly upper semicompact,*
2. *there is a right continuous and nondecreasing strictly* $L-$*function* σ *such that for each* $\rho,\varrho\in S$,
$$d_H(\Gamma\rho,\Gamma\varrho)\leq\sigma(\|\rho-\varrho\|),\tag{3.2}$$
3. Γ'' *is weakly completely continuous,*
4. $\Gamma(\mathcal{Q})$, $\Gamma''(\mathcal{G})\in\mathcal{P}^{b}(\mathcal{Q})$,
5. $\Gamma(\rho)+\Gamma''(\rho)\in\mathcal{P}^{bcc}(\mathcal{G})$, *for each* $\rho\in\mathcal{G}$.

Then $\Gamma+\Gamma''$ *has a fixed point.*

Proof

Let $\Delta\in\mathcal{P}^{b}(\mathcal{Q})$ and $\rho>0$ be such that $\beta(\Delta)<\rho$. From the definition of the De Blasi MWNC, there is a weakly compact set $\mathcal{K}$ such that $\Delta\subseteq\mathcal{K}+B_{\rho}(\theta)$. Then, for each $\rho\in\Delta$, there is $\varrho\in\mathcal{K}$ such that $\|\rho-\varrho\|\leq\rho$. Hence,

$$d_H(\Gamma(\rho),\Gamma(\varrho))\leq\sigma(\|\rho-\varrho\|)\leq\sigma(\rho).$$

Proposition 2.1 implies that

$$\Gamma(\rho)\subseteq\Gamma(\varrho)+B_{\sigma(\rho)}(\theta).$$

Thus,

$$\Gamma\Delta\subseteq\Gamma\mathcal{K}+B_{\sigma(\rho)}(\theta)\subseteq\overline{\Gamma\mathcal{K}}^{\omega}+B_{\sigma(\rho)}(\theta).$$

Let $\{\rho_n\}\subseteq\mathcal{K}$ and $\varrho_n\in\Gamma(\rho_n)$. According to the Eberlein-Šmulian theorem (see [20, Theorem 8.12.4, p. 549]), there is a subsequence $\{\rho_{n_k}\}$ which is weakly converges to $\rho\in\mathcal{K}$. By assumption 1, $\{\varrho_{n_k}\}$ has a subsequence which is weakly converges to $\varrho\in\Gamma(\rho)$. Hence, $\overline{\Gamma\mathcal{K}}^{\omega}$ is weakly compact and we obtain

$$\beta(\Gamma\Delta)\leq\sigma(\rho).$$

Letting $\rho \to \beta(\Delta)$ and by right continuity assumption of σ, we get

$$\beta(\Gamma\Delta) \leq \sigma(\beta(\Delta)).$$

Hence, Γ is a MKCMM. Since Γ is sequentially weakly upper semicompact, we have Γ has a weakly sequentially closed graph and the proof is concluded by Theorem 3.4.

In the particular case when Γ and Γ'' are single-valued mappings, we obtain the following result.

Theorem 3.6 *[9] Suppose that $(\mathcal{Q}, \|\cdot\|)$ is a Banach space and $\mathcal{G} \in \overline{\mathcal{P}}^{bcc}(\mathcal{Q}) \cap \mathcal{P}^b(\mathcal{Q})$. Suppose that $\Gamma : \mathcal{Q} \to \mathcal{Q}$ and $\Gamma'' : \mathcal{G} \to \mathcal{Q}$ are w.s.c. mappings such that*

1. *there is a right continuous and nondecreasing strictly $L-$function σ such that for each $\rho, \varrho \in \mathcal{G}$*

$$\|\Gamma(\rho) - \Gamma(\varrho)\| \leq \sigma(\|\rho - \varrho\|),$$

2. *Γ'' is weakly compact,*
3. *$\Gamma(\rho) + \Gamma''(\rho) \in \mathcal{G}$, for each $\rho \in \mathcal{G}$.*

Then $\Gamma + \Gamma''$ has a fixed point.

4 Fixed point theorems in Banach algebras

Consider a Banach algebra $\mathcal{Q}$ endowed with a norm such that

$$\|\rho \cdot \varrho\| \leq \|\rho\|\|\varrho\| \text{ for all } \rho, \varrho \in \mathcal{Q}.$$

Generally, the multiplication of two weakly sequentially convergent sequences is not weakly sequentially convergent. In [12, 13] the authors introduced a class of Banach algebras which satisfy a certain weak sequential condition (P):

$$(P) \text{ If } \{\rho_n\}, \{\varrho_n\} \subset \mathcal{Q} \text{ such that } \rho_n \rightharpoonup \rho \text{ and } \varrho_n \rightharpoonup \varrho, \text{ then } \rho_n \cdot \varrho_n \rightharpoonup \rho \cdot \varrho.$$

Note that if $\mathcal{Q}$ enjoys condition (P), then $C(I, \mathcal{Q})$ is also a Banach algebra with condition (P) where I is a Hausdorff compact space.

Definition 4.1 *[7] We say that an MWNC μ on a Banach algebra $\mathcal{Q}$ enjoys the condition (m) if for any bounded sets Δ_1, Δ_2 of $\mathcal{Q}$,*

$$\mu(\Delta_1\Delta_2) \leq \|\Delta_1\|\mu(\Delta_2) + \|\Delta_2\|\mu(\Delta_1).$$

Note that the MWNC $diam(.)$ satisfies condition (m).

Now, we state some fixed point theorems.

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Theorem 4.1 *Suppose that $\mathcal{Q}$ is a Banach algebra with condition (P), $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$ and μ be a subadditive MWNC on $\mathcal{Q}$ with condition (m). Suppose that $\Gamma, \Gamma', \Gamma'' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ are multivalued mappings such that*

1. *Γ, Γ' and Γ'' are sequentially weakly upper semicompact,*
2. *Γ, Γ' and Γ'' are $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-functions σ_Γ, $\sigma_{\Gamma'}$ and $\sigma_{\Gamma''}$, respectively,*
3. *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $r \in [\varepsilon, \varepsilon + \delta[$*

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \|\Gamma(\mathcal{G})\|\sigma_{\Gamma'}(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

4. *$\Gamma(\mathcal{G})$, $\Gamma'(\mathcal{G})$, $\Gamma''(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$.*
5. *$\Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho) \in \mathcal{P}^{bcc}(\mathcal{G})$, for each $\rho \in \mathcal{G}$,*

Then $\Gamma\Gamma' + \Gamma''$ possesses a fixed point.

Proof By assumption (4), the mapping $\Gamma\Gamma' + \Gamma'' : \mathcal{G} \to \mathcal{P}^{bcc}(\mathcal{G})$ is well defined. Let $\{\rho_n\} \subset \mathcal{G}$, $\rho_n \rightharpoonup \rho \in \mathcal{G}$ and $\varrho_n \in \Gamma(\rho_n)\Gamma'(\rho_n) + \Gamma''(\rho_n) \rightharpoonup \varrho \in \mathcal{G}$, $\varrho_n = a_n b_n + c_n$ such that $a_n \in \Gamma(\rho_n)$, $b_n \in \Gamma'(\rho_n)$ and $c_n \in \Gamma''(\rho_n)$. By assumption 1, there exist subsequences $\{a_{n_k}\}$ of $\{a_n\}$, $\{b_{n_k}\}$ of $\{b_n\}$ and $\{c_{n_k}\}$ of $\{c_n\}$ such that $a_{n_k} \rightharpoonup a \in \Gamma(\rho)$, $b_{n_k} \rightharpoonup b \in \Gamma'(\rho)$ and $c_{n_k} \rightharpoonup c \in \Gamma''(\rho)$.
The (P) condition guarantees that

$$\varrho_{n_k} = a_{n_k} b_{n_k} + c_{n_k} \rightharpoonup \varrho = ab + c \in \Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho).$$

Hence, $\Gamma\Gamma' + \Gamma''$ has weakly sequentially closed graph.
Let $\Delta \in \mathcal{P}^b(\mathcal{G})$, we obtain

$$\mu(\Gamma(\Delta)\Gamma'(\Delta) + \Gamma''(\Delta)) \leq \|\Gamma(\mathcal{G})\|\mu(\Gamma'(\Delta)) + \|\Gamma'(\mathcal{G})\|\mu(\Gamma(\Delta)) + \mu(\Gamma''(\Delta)).$$

By assumption 2, we have

$$\mu(\Gamma(\Delta)\Gamma'(\Delta) + \Gamma''(\Delta)) \leq \|\Gamma'(\mathcal{G})\|\sigma_\Gamma(\mu(\Delta)) + \|\Gamma(\mathcal{G})\|\sigma_{\Gamma'}(\mu(\Delta)) + \sigma_{\Gamma''}(\mu(\Delta)) = \sigma(\mu(\Delta)),$$

where $\sigma(t) = \|\Gamma'(\mathcal{G})\|\sigma_\Gamma(t) + \|\Gamma(\mathcal{G})\|\sigma_{\Gamma'}(t) + \sigma_{\Gamma''}(t)$, $t \geq 0$, is a srtictly L-function. By Corollary 3.3, $\Gamma\Gamma' + \Gamma''$ has a fixed point.
The following result is a direct consequence of Theorem 4.1 and Theorem 3.2.

Corollary 4.1 *Suppose that $\mathcal{Q}$ is a Banach algebra with condition (P) and $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$. Suppose that $\Gamma, \Gamma', \Gamma'' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ are multivalued mappings such that*

1. *Γ, Γ' and Γ'' are diametrically $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions σ_Γ, $\sigma_{\Gamma'}$ and $\sigma_{\Gamma''}$, respectively,*
2. *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $r \in [\varepsilon,\ \varepsilon + \delta[$*

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \|\Gamma(\mathcal{G})\|\sigma_{\Gamma'}(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

3. $\Gamma(\mathcal{G}),\ \Gamma'(\mathcal{G}),\ \Gamma''(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$.

4. $\Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho) \in \mathcal{P}(\mathcal{G})$, *for each* $\rho \in \mathcal{G}$,

Then $\Gamma\Gamma' + \Gamma''$ *has a fixed point.*

Lemma 4.1 *[11] Suppose that* $\mathcal{Q}$ *is a Banach algebra which satisfies condition* (P)*. Then for each* $\Delta_1 \in \mathcal{P}^b(\mathcal{Q})$ *and weakly compact subset* Δ_2 *of* $\mathcal{Q}$*, we get*

$$\beta(\Delta_1\Delta_2) \leq \|\Delta_2\|\beta(\Delta_1).$$

We present the next theorem as a generalization of Theorem 4.3 in [12].

Theorem 4.2 *Suppose that* $\mathcal{Q}$ *is a Banach algebra with condition* (P) *and* $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$ *and* μ *be a subadditive and complete MWNC on* $\mathcal{Q}$ *with condition* (m)*. Suppose that* $\Gamma, \Gamma', \Gamma'' : \mathcal{G} \to \mathcal{P}^{bc}(\mathcal{Q})$ *are multivalued mappings such that*

1. Γ *and* Γ'' *are sequentially weakly upper semicompact,*

2. Γ' *is weakly completely continuous,*

3. Γ *and* Γ'' *are* $\mathcal{D}$*-set-Lipschitzian with* $\mathcal{D}$*-functions* σ_Γ *and* $\sigma_{\Gamma''}$*, respectively,*

4. *for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* $r \in [\varepsilon,\ \varepsilon + \delta[$

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

5. $\Gamma(\mathcal{G}),\ \Gamma'(\mathcal{G}),\ \Gamma''(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$,

6. $\Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho) \in \mathcal{P}^{bcc}(\mathcal{G})$, *for each* $\rho \in \mathcal{G}$,

Then $\Gamma\Gamma' + \Gamma''$ *has a fixed point.*

Proof Let $\{\rho_n\}$ be a sequence in $\mathcal{Q}$ such that $\rho_n \rightharpoonup \rho \in \mathcal{G}$ and $\varrho_n \in \Gamma'(\rho_n)$. As the set $\{\rho_n :\ n \in \mathbb{N}\}$ is bounded and Γ' is weakly completely continuous, we have $\Gamma'(\{\rho_n,\ n \in \mathbb{N}\})$ is relatively weakly compact. Then $\{\varrho_n\} \subset \Gamma'\{(\rho_n)\}$ has a weakly convergent subsequence. Hence, Γ' is sequentially weakly upper semicompact. . Similar to the proof of Theorem 4.1, $\Gamma\Gamma' + \Gamma'' : \mathcal{G} \to \mathcal{P}^{bcc}(\mathcal{G})$ is well defined and enjoys the weakly sequentially closed graph.

Let $\Delta \in \mathcal{P}^b(\mathcal{G})$. From lemma 4.1 and assumption 3, we have

$$\begin{aligned}
\mu(\Gamma(\Delta)\Gamma'(\Delta) + \Gamma''(\Delta)) &\leq \mu(\Gamma(\Delta)\Gamma'(\Delta)) + \mu(\Gamma''(\Delta)) \\
&\leq \|\Gamma'(\mathcal{G})\|\mu(\Gamma(\Delta)) + \mu(\Gamma''(\Delta)) \\
&\leq \|\Gamma'(\mathcal{G})\|\sigma_\Gamma(\mu(\Delta)) + \sigma_{\Gamma''}(\mu(\Delta)) \\
&= \sigma(\mu(\Delta)),
\end{aligned}$$

where $\sigma(t) = \|\Gamma'(\mathcal{G})\|\sigma_\Gamma(t) + \sigma_{\Gamma''}(t)$, for $t \geq 0$ is a strictly L-function. By Corollary 3.3, $\Gamma\Gamma' + \Gamma''$ has a fixed point.

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Theorem 4.3 *Suppose that $\mathcal{Q}$ is a Banach algebra with condition (P) and $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$. Suppose that $\Gamma, \Gamma'' : \mathcal{Q} \to \mathcal{P}^{bc}(\mathcal{Q})$ and $\Gamma' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ are multivalued mappings such that*

1. *Γ and Γ'' are sequentially weakly upper semicompact,*
2. *Γ' is weakly completely continuous,*
3. *Γ and Γ'' are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions σ_Γ and $\sigma_{\Gamma''}$, respectively,*
4. *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $r \in [\varepsilon, \ \varepsilon + \delta[$*

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

5. *$\Gamma(\mathcal{Q})$, $\Gamma'(\mathcal{G})$, $\Gamma''(\mathcal{Q}) \in \mathcal{P}^b(\mathcal{Q})$,*
6. *$\Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho) \in \mathcal{P}^{bcc}(\mathcal{G})$, for each $\rho \in \mathcal{G}$,*

Then $\Gamma\Gamma' + \Gamma''$ has a fixed point.

Proof Similar to the proof of Theorem 3.5, we can prove that the operator Γ and Γ'' are $\mathcal{D}$-set-Lipschitzian (with respect to β) with $\mathcal{D}$-functions σ_Γ and σ''_Γ, respectively. Thus, by Theorem 4.2, we conclude the result.

Corollary 4.2 *Suppose that $\mathcal{Q}$ is a Banach algebra with condition (P) and $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$. Suppose that $\Gamma, \Gamma'' : \mathcal{Q} \to \mathcal{Q}$ and $\Gamma' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ are multivalued mappings such that*

1. *Γ and Γ'' are w.s.c,*
2. *Γ' is weakly completely continuous,*
3. *Γ and Γ'' are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions σ_Γ and $\sigma_{\Gamma''}$, respectively,*
4. *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $r \in [\varepsilon, \ \varepsilon + \delta[$*

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

5. *$\Gamma(\mathcal{Q})$, $\Gamma'(\mathcal{G})$, $\Gamma''(\mathcal{Q}) in \mathcal{P}^b(\mathcal{Q})$,*
6. *$\Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho) \in \mathcal{P}(\mathcal{G})$, for each $\rho \in \mathcal{G}$,*

Then $\Gamma\Gamma' + \Gamma''$ has a fixed point.

Corollary 4.3 *(See [12, Theorem 4.6]) Suppose that $\mathcal{Q}$ is a Banach algebra with condition (P) and $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$. Suppose that $\Gamma : \mathcal{Q} \to \mathcal{P}^{bc}\mathcal{Q}$ and $\Gamma' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ are multivalued mappings such that*

1. *Γ is sequentially weakly upper semicompact,*

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2. Γ' *is weakly completely continuous,*

3. Γ *is Lipschitzian with Lipschitz constant* α *such that* $\alpha\|\Gamma'(\mathcal{G})\| < 1$,

4. $\Gamma(\mathcal{Q})$, $\Gamma'(\mathcal{G}) \in \mathcal{P}^b(\mathcal{Q})$,

5. $\Gamma(\rho)\Gamma'(\rho) \in \mathcal{P}^{bcc}(\mathcal{G})$, *for each* $\rho \in \mathcal{G}$,

Then $\Gamma\Gamma'$ *has a fixed point.*

In [12, 14], the authors established a new class of mappings of the form $\frac{I-\Gamma''}{\Gamma}$ and used it to prove some existence theorems on the equation $\rho \in \Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho)$.

Definition 4.2 *([12]) Suppose that* $\mathcal{Q}$ *is a Banach algebra,* $\Gamma, \Gamma'' : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ *be multivalued mappings. We say that the mapping* $\frac{I-C}{A}$ *is well defined on* $\rho \in \mathcal{Q}$ *and we write* $\varrho \in \left(\frac{I-\Gamma''}{\Gamma}\right)(\rho)$ *if* $\rho \in \Gamma(\rho)\varrho + \Gamma''(\rho)$.

Theorem 4.4 *Suppose that* $\mathcal{Q}$ *is a Banach algebra with condition* (P), $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$ *and* μ *be a subadditive complete MWNC on* $\mathcal{Q}$ *with condition* (m)*. Suppose that* $\Gamma : \mathcal{Q} \to \mathcal{P}^{bcc}(\mathcal{Q})$, $\Gamma'' : \mathcal{Q} \to \mathcal{P}^{cp,bcc}(\mathcal{Q})$ *and* $\Gamma' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ *are multivalued mappings such that*

1. Γ *and* Γ'' *are sequentially weakly upper semicompact,*

2. Γ' *is weakly completely continuous,*

3. Γ *and* Γ'' *are* $\mathcal{D}$*-set-Lipschitzian with* $\mathcal{D}$*-functions* σ_Γ *and* $\sigma_{\Gamma''}$*, respectively,*

4. for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* $r \in [\varepsilon,\ \varepsilon + \delta[$

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

5. $\left(\frac{I-\Gamma''}{\Gamma}\right)^{-1}\Gamma'(\varrho) \in \mathcal{P}^{bcc}(\mathcal{G})$, *for each* $\varrho \in \mathcal{G}$,

6. $\Gamma(\mathcal{Q})$, $\Gamma'(\mathcal{G})$, $\Gamma''(\mathcal{Q}) \in \mathcal{P}^b(\mathcal{Q})$.

Then $\Gamma\Gamma' + \Gamma''$ *has a fixed point.*

Proof Choose $\varrho \in \mathcal{G}$ and let $z \in \Gamma'(\varrho)$. We pose

$$\sigma_z : \mathcal{Q} \to \mathcal{P}^{bcc}(\mathcal{Q}),\ \rho \mapsto \Gamma\rho \cdot z + \Gamma''\rho.$$

Since $\Gamma\rho$ is closed and $\Gamma''\rho$ is compact, σ_z is well defined. We prove that σ_z has weakly sequentially closed graph. Let $\{\rho_n\} \subseteq \mathcal{Q}$, $\rho_n \rightharpoonup \rho \in \mathcal{Q}$ and $\varrho_n \in \Gamma\rho_n \cdot z + \Gamma''\rho_n, \varrho_n = \alpha_n z + \gamma_n \rightharpoonup \varrho$ such that $\alpha_n \in \Gamma(\rho_n)$ and $\gamma_n \in \Gamma''(\rho_n)$. By assumption 1, there exist subsequences, for simplicity, $\alpha_n \rightharpoonup \alpha \in \Gamma(\rho)$ and $\gamma_n \rightharpoonup \gamma \in \Gamma''(\rho)$. Since the right hand multiplication operator $R_z(\rho) = \rho.z$ is a continuous linear operator, then it is weakly continuous. It follows that

$$\varrho_n = \alpha_n z + \gamma_n \rightharpoonup \alpha z + \gamma \in \sigma_z(\rho).$$

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Hence, $\varrho \in \sigma_z(\rho)$. Thus, σ_z has weakly sequentially closed graph.

Next, We show that σ_z is Meir-Keeler condensing. Let $\Delta \in \mathcal{P}^b(\mathcal{G})$. We obtain that

$$\begin{aligned} \mu(\sigma_z(\Delta)) &\leq \mu(\Gamma(\Delta)z) + \mu(\Gamma''(\Delta)) \\ &\leq \|z\|\mu(\Gamma(\Delta)) + \mu(\Gamma''(\Delta)) \\ &\leq \|z\|\sigma_\Gamma(\mu(\Delta)) + \sigma_{\Gamma''}(\mu(\Delta)) \\ &\leq \sigma(\mu(\Delta)), \end{aligned}$$

where $\sigma(t) = \|z\|\sigma_\Gamma(t) + \sigma_{\Gamma''}(t)$. From assumption 3, σ is a strictly L-function, then σ_z is Meir-Keeler condensing. Applying Corollary 3.3 for the operator σ_z, there exists $\rho \in \mathcal{Q}$ such that $\rho \in \Gamma(\rho)\cdot z + \Gamma''(\rho)$. Therefore, $z \in \left(\frac{I-\Gamma''}{\Gamma}\right)(\rho)$. So, $\left(\frac{I-\Gamma''}{\Gamma}\right)(\rho) \cap \Gamma'(\varrho) \neq \varnothing$. Therefore, the multivalued mapping $\left(\frac{I-\Gamma''}{\Gamma}\right)^{-1}$ is well defined on $\Gamma'(\mathcal{G})$. By assumption (4), the mapping

$$\mathcal{T} = \left(\frac{I-\Gamma''}{\Gamma}\right)^{-1}\Gamma' : \mathcal{G} \to \mathcal{P}^{bcc}(\mathcal{G})$$

is well defined. We claim that $\mathcal{T}$ has weakly sequentially closed graph. Let $\{\varrho_n\} \subset \mathcal{G}$, $\varrho_n \rightharpoonup \varrho \in \mathcal{G}$ and $\rho_n \in \mathcal{T}(\varrho_n)$ with $\rho_n \rightharpoonup \rho$. We have $\rho_n \in \Gamma(\rho_n)\Gamma'(\varrho_n) + \Gamma''(\rho_n)$. Then $\rho_n = \alpha_n\mu_n + \gamma_n$, where $\alpha_n \in \Gamma(\rho_n)$, $\mu_n \in \Gamma'(\varrho_n)$ and $\gamma_n \in \Gamma''(\rho_n)$. Since Γ and Γ'' are sequentially weakly upper semicompact, we assume that $\alpha_n \rightharpoonup \alpha \in \Gamma(\rho)$ and $\gamma_n \rightharpoonup \gamma \in \Gamma''(\rho)$. Since Γ' is weakly completely continuous. and the subset $\{\Gamma'(\varrho_n), n \in \mathbb{N}\}$ is relatively weakly compact, then we can suppose that $\mu_n \rightharpoonup \mu \in \Gamma'(\varrho)$.

Condition (P) guarantees that

$$\alpha_n\mu_n + \gamma_n \rightharpoonup \alpha\mu + \gamma \in \Gamma(\rho)\Gamma'(\varrho) + \Gamma''(\rho).$$

Also, $\rho \in \Gamma(\rho)\Gamma'(\varrho) + \Gamma''(\rho) \Rightarrow \rho \in \left(\frac{I-\Gamma''}{\Gamma}\right)^{-1}\Gamma'(\varrho) = \mathcal{T}(\varrho)$. Also, we have

$$\mathcal{T}(\mathcal{G}) \subset \Gamma(\mathcal{T}(\mathcal{G}))\Gamma'(\mathcal{G}) + \Gamma''(\mathcal{T}(\mathcal{G})).$$

Taking into account assumption 6, we conclude that $\mathcal{T}(\mathcal{G})$ is bounded. If $\mu(\mathcal{T}(\mathcal{G})) > 0$, we get

$$\begin{aligned} \mu(\mathcal{T}(\mathcal{G})) &\leq \mu(\Gamma(\mathcal{T}(\mathcal{G})).\Gamma'(\mathcal{G})) + \mu(\Gamma''(\mathcal{T}(\mathcal{G}))) \\ &\leq \|\Gamma'(\mathcal{G})\|\mu(\Gamma(\mathcal{T}(\mathcal{G}))) + \mu(\Gamma''(\mathcal{T}(\mathcal{G}))) \\ &\leq \|\Gamma\prime(\mathcal{G})\|\sigma_\Gamma(\mu(\mathcal{T}(\mathcal{G}))) + \sigma_{\Gamma''}(\mu(\mathcal{T}(\mathcal{G}))) \\ &< \mu(\mathcal{T}(\mathcal{G})), \end{aligned}$$

which is absurd. So, $\mu(\mathcal{T}(\mathcal{G})) = 0$. Hence, $\mathcal{T}(\mathcal{G})$ is relatively compact. According to Theorem 2.1, $\mathcal{T}$ has a fixed point ρ which is a solution of the equation $\rho \in \Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho)$.

As a consequence of the above theorem, we present the following results.

Corollary 4.4 *Suppose that $\mathcal{Q}$ is a Banach algebra with condition (P) and $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$. Suppose that $\Gamma : \mathcal{Q} \to \mathcal{P}^{bcc}(\mathcal{Q})$, $\Gamma'' : \mathcal{Q} \to \mathcal{P}^{cp,bcc}(\mathcal{Q})$ and $\Gamma' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ are multivalued mappings such that*

1. *Γ and Γ'' are sequentially weakly upper semicompact,*
2. *Γ' is weakly completely continuous,*

3. Γ *and* Γ'' *are* $\mathcal{D}$*-Lipschitzian with* $\mathcal{D}$*-functions* σ_Γ *and* $\sigma_{\Gamma''}$*, respectively,*

4. *for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* $r \in [\varepsilon,\ \varepsilon + \delta[$

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

5. $\left(\frac{I-\Gamma''}{\Gamma}\right)^{-1}\Gamma'(\varrho) \in \mathcal{P}^{bcc}(\mathcal{G})$*, for each* $\varrho \in \mathcal{G}$*,*

6. $\Gamma(\mathcal{Q})$*,* $\Gamma'(\mathcal{G})$*,* $\Gamma''(\mathcal{Q}) \in \mathcal{P}^b(\mathcal{Q})$*.*

Then $\Gamma\Gamma' + \Gamma''$ *has a fixed point.*

Proof Similar to the proof of Theorem 3.5, we prove that Γ and Γ'' are $\mathcal{G}$-Lipschtzian (with respect to β) with $\mathcal{D}$-functions σ_Γ and σ_Γ'' respectively.

Corollary 4.5 *Suppose that* $\mathcal{Q}$ *is a Banach algebra with condition* (P) *and* $\mathcal{G} \in \mathcal{P}^{bcc}(\mathcal{Q})$*. Suppose that* $\Gamma, \Gamma'' : \mathcal{Q} \to \mathcal{Q}$ *and* $\Gamma' : \mathcal{G} \to \mathcal{P}(\mathcal{Q})$ *are multivalued mappings such that*

1. Γ *and* Γ'' *are weakly sequentially continuous,*

2. Γ' *is weakly completely continuous,*

3. Γ *and* Γ'' *are* $\mathcal{D}$*-Lipschitzian with* $\mathcal{D}$*-functions* σ_Γ *and* $\sigma_{\Gamma''}$*, respectively,*

4. *for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* $r \in [\varepsilon, \varepsilon + \delta[$

$$\|\Gamma'(\mathcal{G})\|\sigma_\Gamma(r) + \sigma_{\Gamma''}(r) < \varepsilon,$$

5. $\left(\frac{I-\Gamma''}{\Gamma}\right)^{-1}\Gamma'(\varrho) \in \mathcal{P}^{bcc}(\mathcal{G})$*, for each* $\varrho \in \mathcal{G}$*,*

6. $\Gamma(\mathcal{Q})$*,* $\Gamma'(\mathcal{G})$*,* $\Gamma''(\mathcal{Q})\mathcal{P}^b(\mathcal{Q})$*.*

Then $\Gamma\Gamma' + \Gamma''$ *has a fixed point.*

5 Application

Suppose that $\mathcal{Q}$ is a real Banach algebra which satisfies condition (P) and let $I = [0, 1]$. We discuss the existence of solution for the nonlinear integral inclusion

$$\rho(t) \in \Gamma(\rho(t)) \int_0^t \mathcal{F}(\kappa, \rho(\kappa))d\kappa + \int_0^t g(\kappa, \rho(\kappa))d\kappa \quad t \in I, \tag{5.1}$$

where $\Gamma : \mathcal{Q} \to \mathcal{Q}$, $\mathcal{F} : I \times \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ and $g : I \times \mathcal{Q} \to \mathcal{Q}$. The integral in (5.1) is the Pettis integral and by a solution of (5.1) we mean a function $\rho \in C(I, \mathcal{Q})$ such that

$$\rho(t) = \Gamma(\rho(t)) \int_0^t v(\kappa)d\kappa + \int_0^t g(\kappa, \rho(\kappa))d\kappa, \tag{5.2}$$

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for some $v \in L^1(I, \mathcal{Q})$ which is a scalarly measurable function satisfying $v(\kappa) \in \mathcal{F}(\kappa, \rho(\kappa))$ for a.e. $\kappa \in I$ such that v is Pettis integrable on I.
For each $t \in I$ and $\rho \in \mathcal{Q}$, we pose

$$\|\mathcal{F}(t,\rho)\| = \sup\{|v|, v \in \mathcal{F}(t,\rho)\}.$$

We now examine the existence of a solution to Equation (5.1) under the following conditions.

(H_1) The mapping Γ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function σ_Γ and $L = \sup\limits_{\rho \in \mathcal{Q}} \|\Gamma \rho\|$.

(H_2) For every continuous function $\rho : I \to \mathcal{Q}$ there is a scalarly measurable function $v : I \to \mathcal{Q}$ such that $v(t) \in \mathcal{F}(t, \rho(t))$ a.e. on I and v is Pettis integrable on I.

(H_3) There is a $\mathcal{D}$-function $\sigma_{\mathcal{F}}$ such that for each bounded subset Ω of $\mathcal{Q}$ and all $t \in I$,

$$diam(\mathcal{F}([0,t] \times \Omega)) \leq \sigma_{\mathcal{F}}(diam(\Omega)).$$

(H_4) For any $r > 0$, there exists $h_r \in L^1(I)$ such that $\|\mathcal{F}(t,\rho)\| \leq h_r(t)$ for a.e. $t \in I$ and all $\rho \in \mathcal{Q}$ with $|\rho| \leq \rho$. Let

$$M_r = \int_0^1 h_r(\kappa) d\kappa.$$

(H_5) For each continuous function $\rho : I \to \mathcal{Q}$, $g(\cdot, \rho)$ is Pettis integrable on I.

(H_6) There exists a $\mathcal{D}$-function σ_g such that for each $\rho, \varrho \in \mathcal{Q}$, we have

$$|g(t,\rho) - g(t,\varrho)| \leq \sigma_g(|\rho - \varrho|) \text{ for a.e. } t \in I.$$

(H_7) There are a function $\alpha \in L^1(I, \mathbb{R}_+)$ and a continuous nondecreasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,\rho)| \leq \alpha(\kappa)\theta(|\rho|) \quad \text{for a.e. } t \in I$$

and all $\rho \in \mathcal{Q}$.

(H_8) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $L\sigma_{\mathcal{F}}(r) + M_r\sigma_\Gamma(r) + \sigma_g(r) < \varepsilon$ for each $r \in [\varepsilon, \varepsilon + \delta[$.

Theorem 5.1 *Suppose that conditions* $(H_1) - (H_8)$ *are satisfied. If there is* $R > 0$ *such that*

$$LM_R + \theta(R)\|\alpha\|_{L^1} < R,$$

then the inclusion (5.1) *has a solution in* $C(I, \mathcal{Q})$.

Proof Let $\Delta = C(I, \mathcal{Q})$. Consider $\mathcal{G} = \{\rho \in \Delta, \|\rho\| \leq R\}$. We see that $\mathcal{G}$ is a convex and closed subset. Let

$$\Gamma'(\rho)(t) = \left\{ \int_0^t v(\kappa)\, d\kappa, v \text{ is Pettis integrable}, v(\kappa) \in \mathcal{F}(\kappa, \rho(\kappa)) \text{ for a.e. } \kappa \in I \right\},$$

$$\Gamma''(\rho)(t) = \int_0^t g(\kappa, \rho(\kappa)) d\kappa.$$

In the next few steps we will prove that the mappings Γ, Γ' and Γ'' verify all the assumptions of Corollary 4.1.

Step I. We claim that $\Gamma : \Delta \to \Delta$ is well defined and $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function σ_Γ.
Let $\rho \in \mathcal{Q}$ and $t_1, t_2 \in I$, we obtain

$$||\Gamma\rho(t_1) - \Gamma\rho(t_2)|| \leq \sigma_\Gamma(||\rho(t_1) - \rho(t_2)||).$$

As σ_Γ is continuous and $\sigma_\Gamma(0) = 0$, we conclude that $\Gamma\rho \in \Delta$.
For all $\rho_1, \rho_2 \in \Delta$, we have

$$\begin{aligned} ||\Gamma(\rho_1(t)) - \Gamma(\rho_2(t))|| &\leq \sup_{t \in I} \sigma_\Gamma(||\rho_1(t) - \rho_2(t)||) \\ &\leq \sigma_\Gamma(\sup_{t \in I} ||\rho_1(t) - \rho_2(t)||) \\ &= \sigma_\Gamma(||\rho_1 - \rho_2||). \end{aligned}$$

Then,

$$||\Gamma(\rho_1) - \Gamma(\rho_2)|| \leq \sigma_\Gamma(||\rho_1 - \rho_2||).$$

Hence, Γ is $\mathcal{D}$-Lipschitzian.

Step II. We prove that Γ'' is well defined and diametrically $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function σ_g. To see this, let $t_1, t_2 \in I$ with $t_1 < t_2$. Without loss of generality, assume that $\Gamma''\rho(t_1) - \Gamma''\rho(t_2) \neq 0$. By Hahn-Banach theorem, there exists $\phi \in \mathcal{Q}^*$ such that $|\phi| = 1$ and

$$\begin{aligned} |\Gamma''\rho(t_1) - \Gamma''\rho(t_2)| &= \phi(\Gamma''\rho(t_1) - \Gamma''\rho(t_2)) \\ &= \phi(\int_{t_1}^{t_2} g(\kappa, \rho(\kappa)) d\kappa) \\ &\leq \int_{t_1}^{t_2} |g(\kappa, \rho(\kappa))| d\kappa \\ &\leq \int_{t_1}^{t_2} \sup_{\kappa \in I} \theta(|\rho(\kappa)|)\, \alpha(\kappa) d\kappa \\ &\leq \theta(\|\rho\|) \int_{t_1}^{t_2} \alpha(\kappa) d\kappa. \end{aligned}$$

From assumption (H_7), Γ'' is well defined.
Let $\Omega \in \mathcal{P}^b(\mathcal{G})$ and let $t \in I$. Without loss of generality, assume that $\Gamma''\rho_1(t) - \Gamma''\rho_2(t) \neq 0$. By Hahn-Banach theorem, there exists $\phi \in \mathcal{Q}^*$ such that $|\phi| = 1$ and $|\Gamma''\rho_1(t) - \Gamma''\rho_2(t)| = \phi(\Gamma''\rho_1(t) - \Gamma''\rho_2(t))$.

Thus

$$\begin{aligned}
|\Gamma''\rho_1(t) - \Gamma''\rho_2(t)| &= \phi(\int_0^t g(\kappa,\rho_1(\kappa)) - g(\kappa,\rho_2(\kappa))\ d\kappa) \\
&\leq \int_0^t |g(\kappa,\rho_1(\kappa)) - g(\kappa,\rho_2(\kappa))|\ d\kappa \\
&\leq \int_0^t \sigma_g(|\rho_1(\kappa) - \rho_2(\kappa)|)\ d\kappa \\
&\leq \sigma_g(||\rho_1 - \rho_2||).
\end{aligned}$$

Then,

$$||\Gamma''\rho_1 - \Gamma''\rho_2|| \leq \sigma_g(||\rho_1 - \rho_2||).$$

Consequently, Γ'' is $\mathcal{D}$-Lipschitzian.

Step III. By assumption (H_2), $\Gamma' : \mathcal{G} \to \mathcal{P}(\Delta)$ is well defined on $C(I, \mathcal{Q})$. We claim that $\Gamma'\rho \in \mathcal{P}(\Delta)$ for each $\rho \in \mathcal{G}$.

Let $\rho \in \mathcal{G}$ and $w \in \Gamma'\rho$ such that $w(t) = \int_0^t v(\kappa)\, d\kappa$, where v is Pettis integrable and $v(\kappa) \in \mathcal{F}(\kappa, \rho(\kappa))$ for a.e. $\kappa \in I$. let $t_1, t_2 \in I$ with $t_1 < t_2$. Without loss of generality, assume that $w(t_1) - w(t_2) \neq 0$. By Hahn-Banach theorem, there exists $\phi \in \mathcal{Q}^*$ with $|\phi| = 1$ and $|w(t_1) - w(t_2)| = \phi(w(t_1) - w(t_1))$. Thus

$$\begin{aligned}
|w(t_1) - w(t_2)| &= \phi(\int_{t_1}^{t_2} v(\kappa)\, d\kappa) \\
&\leq \int_{t_1}^{t_2} |v(\kappa)|\, d\kappa \\
&\leq \int_{t_1}^{t_2} h_\rho(\kappa)\, d\kappa.
\end{aligned}$$

From assumption (H_4), we obtain $w \in \Delta$. Hence, $\Gamma'\rho \in \mathcal{P}(\Delta)$.

Now, we prove that Γ' is diametrically $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\sigma_{\mathcal{F}}$.

Let $\Omega \in \mathcal{P}^b(\mathcal{G})$ and $t \in I$, we get

$$\begin{aligned}
\Gamma'(\Omega)(t) &= \Big\{ \int_0^t v(\kappa)\ d\kappa, v(\kappa) \in \mathcal{F}(\kappa, \rho(\kappa)),\ \rho \in \Omega \Big\} \\
&\subseteq t\overline{conv}(\mathcal{F}([0,t] \times \Omega[0,t])).
\end{aligned}$$

Using the properties of $diam(\cdot)$ MWNC, we have

$$\begin{aligned}
diam(\Gamma'(\Omega)(t)) &\leq tdiam(\overline{conv}(\mathcal{F}([0,t] \times \Omega[0,t]))) \\
&\leq diam(\mathcal{F}([0,t] \times \Omega[0,t])) \\
&\leq \sigma_{\mathcal{F}}(diam(\Omega[0,t])) \\
&\leq \sigma_{\mathcal{F}}(diam(\Omega)).
\end{aligned}$$

Then

$$\sup_{t \in I} diam(\Gamma'(\Omega)(t)) \leq \sigma_{\mathcal{F}}(diam(\Omega)).$$

Hence,

$$diam(\Gamma'(\Omega)) = \sup_{t \in I} diam(\Gamma'(\Omega)(t)) \leq \sigma_{\mathcal{F}}(diam(\Omega)).$$

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Thus, Γ' is diametrically $\mathcal{D}$-Lipschitzian.

Step IV. We claim that $\Gamma(\rho)\Gamma'(\rho)+\Gamma''(\rho) \subseteq \mathcal{G}$ for all $\rho \in \mathcal{G}$. As $f, \mathcal{F}$ and g are bounded, we obtain that Γ, Γ' and Γ'' are also bounded.
Let $u \in \Gamma(\rho)\Gamma'(\rho) + \Gamma''(\rho)$ such that

$$u(t) = \Gamma(\rho(t)) \int_0^t v(\kappa)d\kappa + \int_0^t g(\kappa, \rho(\kappa))d\kappa,$$

where $v : I \to \mathcal{Q}$ is a scalarly measurable Pettis integrable function with $v(\kappa) \in \mathcal{F}(\kappa, \rho(\kappa))$ for a.e. $\kappa \in I$. Then

$$\begin{aligned} |u(t)| &\leq |\Gamma(\rho(t))| \int_0^t |v(\kappa)|d\kappa + \int_0^t |g(\kappa, \rho(\kappa))|d\kappa \\ &\leq L \int_0^t h_R(\kappa)d\kappa + \int_0^t \alpha(\kappa)\, \theta(|\rho(\kappa)|)d\kappa \\ &\leq LM_R + \theta(R)\|\alpha\|_{L^1} \\ &< R. \end{aligned}$$

Hence, by Corollary 4.1, $\Gamma\Gamma' + \Gamma''$ has a fixed point.

Example 5.1 *Let $\mathcal{Q} = I\!R$.*

- *We define*

$$\begin{aligned} \Gamma : \mathcal{Q} &\longrightarrow \mathcal{Q} \\ \rho &\longmapsto \begin{cases} \frac{e^{2t}-1}{e^{2t}+1} + ln\left(1+\frac{\rho}{8}\right), & -7 \leq \rho \leq 7 \\ ln\frac{15}{8} & else, \end{cases} \end{aligned}$$

Let $\rho_1, \rho_2 \in [-7, 7]$ and $t \in I$, we have

$$\begin{aligned} |\Gamma(\rho_1(t)) - \Gamma(\rho_2(t))| &= \left| ln\left(1+\frac{|\rho_1(t)|}{8}\right) - ln\left(1+\frac{|\rho_2(t)|}{8}\right)\right| \\ &\leq \left| ln\left(\frac{1+\frac{|\rho_1(t)|}{8}}{1+\frac{|\rho_2(t)|}{8}}\right)\right| \\ &\leq ln\left(1+\frac{|\rho_1(t)-\rho_2(t)|}{8}\right) \\ &= \sigma_\Gamma(|\rho_1(t)-\rho_2(t)|), \end{aligned}$$

where $\sigma_\Gamma(t) = ln(1+\frac{t}{8})$ is a $\mathcal{D}$-function.
Next, if $\rho_1, \rho_2 \in I\!R \setminus [-7, 7]$, then we have

$$|\Gamma(\rho_1(t)) - \Gamma(\rho_2(t))| = 0 \leq \sigma_\Gamma(|\rho_1(t) - \rho_2(t)|).$$

- *The mapping $\mathcal{F} : I \times \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ is defined by*

$$\mathcal{F}(t, \rho) = \left[0, \frac{t}{t^2+1} e^{-t} \cos(\sqrt{|\rho|})\right].$$

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For each bounded subset Ω of $\mathcal{Q}$ and $t \in I$

$$\begin{aligned} \mathcal{F}([0,t] \times \Omega[0,t]) &= \left[0, \frac{t}{t^2+1} e^{-t} \cos\left(\sqrt{|\ \Omega[0,t]\ |}\right)\right] \\ &\subset \left[0, \sqrt{|\ \Omega[0,t]\ |}\right] \\ &\subset \left[0, \sqrt{|\ diam\Omega[0,t]\ |}\right], \end{aligned}$$

so,

$$diam(\mathcal{F}([0,t] \times \Omega[0,t])) \leq \sqrt{|\ diam\Omega[0,t]\ |} = \sigma_{\mathcal{F}}(diam(\Omega[0,t])),$$

where $\sigma_{\mathcal{F}}(t) = \sqrt{t}$.
There exists an integrable function $h(t) = \frac{t}{t^2+1}$ such that $\|\mathcal{F}(t,\rho)\| \leq \frac{1}{2} ln2$.

- *The mapping $g : I \times \mathcal{Q} \to \mathcal{Q}$ is defined by*

$$g(t,\rho) = \frac{e^{-2t}}{2(1+\mid sin\rho \mid)}.$$

For each $\rho, \varrho \in \mathcal{Q}$ and for a.e. $t \in I$, we have

$$\begin{aligned} |g(t,\rho) - g(t,\varrho)| &= \left|\frac{e^{-2t}}{2(1+\mid sin\rho \mid)} - \frac{e^{-2t}}{2(1+\mid sin\varrho \mid)}\right| \\ &= \frac{e^{-2t}}{2}\left|\frac{\mid sin\rho \mid - \mid sin\varrho \mid}{(1+\mid sin\rho \mid)(1+\mid sin\varrho \mid)}\right| \\ &\leq \frac{1}{2} sin \mid \rho - \varrho \mid \\ &\leq \frac{1}{2} \mid \rho - \varrho \mid \\ &= \sigma_g(|\rho - \varrho|) \end{aligned}$$

where $\sigma_g(t) = \frac{t}{2}$.
Moreover, for all $\rho \in \mathcal{Q}$ and for a.e. $t \in I$, we have $\mid g(t,\rho) \mid = \alpha(t)\theta(\mid \rho \mid)$, where $\alpha(t) = \frac{e^{-2t}}{2}$ and $\theta(\mid \rho \mid) = \frac{1}{1+|sin\rho|}$.

So, the conditions of Theorem 5.1 are satisfied.

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