Solvability and Approximation of Nonlinear Functional Mixed Volterra-Fredholm Equation in Banach Space

Chinedu Nwaigwe

Department of Mathematics, Rivers State University, Nigeria
nwaigwe.chinedu@ust.edu.ng

Keywords: Mixed Volterra-Fredholm equations; Functional Integral equations; Banach contraction principle; Trapezoidal rule; Numerical Quadrature; Collocation method, Experimental order of convergence.

Abstract

This study probes into the existence of a unique solution and the numerical approximation of a nonlinear functional Volterra-Fredholm integral equations of the mixed type and second kind. Based on the Lipschitz constants of the functional and kernel, a Bielecki's norm is defined and used to modify a distance inequality on a constructed self-map. The map is shown to be contractive, thereby establishing solvability. The problem is then approximated by collocating at discrete points and use of a composite multi-dimensional numerical quadrature approximation. A new Grownwall-type inequality is proposed, and used, to prove the second order of convergence of the numerical scheme. Numerical experiments are provided to verify the theoretical results.

1 Introduction

Integral equations occur in many applications of practical interest. They occur as mathematical models of problems in epidemiology [1, 2], optimal control and economics [3, 4, 5, 6], physical sciences [7, 8, 9], chemical sciences [10], and even as more tractable reformulation of some more difficult mathematical problems [11, 12].

The present research investigates the following nonlinear functional mixed Volterra-Fredholm second kind integral equation problem: find $u(x)$ such

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

that

$$u(x) = g(x) + f\left(x, \int_{t=a}^{x}\int_{y=a}^{b} k(t,y,u(y))\,dy\,dt\right), \qquad x \in [a,b] \subset \mathbb{R}, \quad (1)$$

where $g : \mathbb{R} \to \mathbb{R}, f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. More assumptions on g, k and f will be given below. Given a problem such as (1), some fundamental questions naturally arise, namely (i) does the problem have a solution? (ii) if it does have a solution, is the solution unique? and (iii) how can we find this unique solution if it actually exists? This paper is devoted to answering these three questions. To answer the first two, we only impose Lipschitz continuity on k and f but not contractivity. Then the Generalized Banach fixed point theorem allows to answer the two questions. To answer the last questions, we adopt numerical quadrature rules.

The problems of solvability and approximation of some functional integral equations have attracted much research attention due to the above listed applicability of these equations. For instance, [13] proved the existence of the unique solution of

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y k(x,y,s,t,u(s,t))\,ds\,dt, \qquad x,y \in \mathbb{R}_+.$$

In 2018, Kazemi and Ezzati [14] used the Petryshyn's fixed point theorem to prove an existence theorem for the nonlinear functional Volterra equation:

$$u(x) = q\left(t, u(\alpha(t)), u(\beta(t)), \int_0^{\phi(t)} h(t,s,u(\gamma(s)))\,ds\right), \qquad t \in [0,a], a \in \mathbb{R}.$$

Darbo's fixed-point theorem is employed in [15] to demonstrate the existence of solution of nonlinear functional stochastic integral equations of the form:

$$\begin{aligned} u(x) =& f\left(t, \int_0^t v(t,s,u(s))\,dB(s), u(\alpha(t))\right) \cdot g\left(t, \int_0^t w(t,s,u(s))\,dB(s), u(\beta(t))\right), \\ & t \in [0,1]. \end{aligned}$$

The modified homotopy perturbation method and Petryshyn's fixed point theorem are adopted in [16] to study some other forms of nonlinear functional Volterra integral equations:

$$\begin{aligned} u(s) =& q\left(s, \mu(\xi(s)), \int_0^{\alpha(s)} h(s,r,u(\phi(r)))\,dr \int_0^{\beta(s)} f(s,r,u(\psi(r)))\,dr\right), \\ & s \in [0,c],\ c \in \mathbb{R}_+. \end{aligned}$$

Other nonlinear functional Volterra integral equations have been investigated, using the technique of measures of noncompactness [17, 18] and Krasnoselskii Fixed Point Theorem [19]. In [20], nonlinear functional convolution

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

type Volterra equations are also investigated while nonlinear fractional integral equations are investigated in [21]. Other studies include [22, 23, 24, 25], see also [26, 27, 28].

In 2021, Bazm and coworkers [29] used a technique based on the Picard iterative method to prove the existence and uniqueness of the following integral equation:

$$u(x) = g(x) + f\left(x, \int_0^x k(x,t,u(t))\,dt\right), \qquad x \in [0,a],\ a \in \mathbb{R}_+. \tag{2}$$

They also approximated the solution of (2) using two numerical schemes. In the present study, we extend the problem in [29] by considering the mixed Volterra-Fredholm equation (1) instead of the Volterra form (2).

In this current study, the questions of existence, uniqueness and approximation of (1) are tackled. The present work is novel in two parts. First, solvability is proved by using the Banach Contraction principle without any contraction assumption. Secondly, a second order accurate numerical scheme is proposed, with a detailed convergence proof based on a New Discrete Grownwall's inequality. Consequently, the paper is arranged as follows: We construct a norm and use the Banach contraction principle to prove the existence and uniqueness of solution of (1) in section 2, while a numerical approximation is formulated in section 3, with detailed convergence analysis in section 4. Some numerical examples are presented and discussed in section 5, and some concluding remarks are made in section 6.

2 Solvability

We begin this section with some well-known useful definitions and theorems.

Definition 2.1 (Fixed Point). *Let U be a non-empty set and $T : U \to U$. Then $u^* \in U$ is called a fixed point of T if:*

$$T(u^*) = u^*.$$

Definition 2.2 (Contraction). *An operator $T : U \to U$ is called a contraction if there is $\alpha_0 \in [0,1)$ such that*

$$|T(u) - T(v)| \le \alpha_0 |u - v| \quad \text{for all } u, v \in U.$$

T is called Lipschitz continuous if $\alpha_0 \ge 0$.

Remark 2.1. *Note that it is more restrictive to be a contraction than Lipschitz continuous. In this work, we only impose Lipschitz continuity, but not contractivity, in establishing existence and uniqueness.*

The following theorem is central in the proof of the solvability of (1).

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

Theorem 2.1 (Banach Contraction Principle). *Suppose U is a non-empty closed subset of a Banach space $\mathbb{B}$, and $T : U \to U$. If T^m is a contraction for some $m > 0$, then T has a unique fixed point in U.*

To investigate the existence and uniqueness of the solution of (1), we make the following assumptions:

(A) $g, f, \int_{t=a}^{x} \int_{y=a}^{b} k(t, y, u(y))\, dy\, dt \in C[a, b]$.

(B) The functional f, is Lipschitz continuous with respect to the second argument, and with a Lipschitz constant $\alpha_1 \geq 0$:

$$|f(x, z_1) - f(x, z_2)| \leq \alpha_1 |z_1 - z_2|, \qquad \text{for all } z_1, z_2 \in C[a, b]. \quad (3)$$

(C) The kernel, k is Lipschitz continuous with respect to the last argument, with Lipschitz constant $\alpha_2 \geq 0$:

$$|k(x, y, u_1) - k(x, y, u_2)| \leq \alpha_2 |u_1 - u_2| \quad \text{for all } u_1, u_2 \in C[a, b]. \quad (4)$$

With the Lipschitz constants, α_1, α_2 available, we are now in position to define a Bielecki's norm [30].

Definition 2.3 (Bielecki's Norm). *For $u \in C[a, b]$, we define the norm:*

$$|||u||| = \max_{a \leq x \leq b} \{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |u(x)|\}, \quad (5)$$

It is quite straightforward to show that (5) defines a norm, indeed:
(i)$|||u||| > 0$ and $u = 0 \Rightarrow |||u||| = \max_{a \leq x \leq b}\{0\} = 0$ and $|||u||| = 0 \Rightarrow \max_{a \leq x \leq b}\{e^{-(\alpha_1 \alpha_2 (b-a)+1)x}|u(x)|\} = 0$ which can only be true if $u = 0$. Hence, $|||u||| = 0 \iff u = 0$.
(ii) Let $\lambda \in \mathbb{R}$, then

$$\begin{aligned} |||\lambda u||| &= \max_{a \leq x \leq b} \{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |\lambda u(x)|\} \\ &= |\lambda| \max_{a \leq x \leq b} \{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |u(x)|\} \\ &= |\lambda| |||u|||. \end{aligned}$$

(iii) Let $u, v \in C[a, b]$, then

$$\begin{aligned} |||u + v||| &= \max_{a \leq x \leq b} \{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |u(x) + v(x)|\} \\ &\leq \max_{a \leq x \leq b} \left\{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |u(x)| + e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |v(x)|\right\} \\ &\leq \max_{a \leq x \leq b} \left\{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |u(x)|\right\} + \max_{a \leq x \leq b} \left\{e^{-(\alpha_1 \alpha_2 (b-a)+1)x} |v(x)|\right\} \\ &= |||u||| + |||v|||. \end{aligned}$$

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

Theorem 2.2 (Solvability). *Suppose assumptions A - C are true, then the nonlinear functional mixed integral equation* (1) *has a unique solution.*

Proof. Define the operator T by

$$(Tu)(x) := g(x) + f\left(x, \int_{t=a}^{x} \int_{y=a}^{b} k(t, y, u(y))\, dy\, dt\right).$$

Then assumption (A) above guarantees that $T : C[a, b] \to C[a, b]$.

Now, we have

$$\begin{aligned}
|(Tu)(x) - (Tv)(x)| &= \left| f\left(x, \int_{t=a}^{x} \int_{y=a}^{b} k(t, y, u(y))\, dy\, dt\right) \right. \\
&\quad \left. - f\left(x, \int_{t=a}^{x} \int_{y=a}^{b} k(t, y, v(y))\, dy\, dt\right) \right| \\
&\leq \alpha_1 \left| \int_{t=a}^{x} \int_{y=a}^{b} k(t, y, u(y))\, dy\, dt \right. \\
&\quad \left. - \int_{t=a}^{x} \int_{y=a}^{b} k(t, y, v(y))\, dy\, dt \right| \\
&\leq \alpha_1 \int_{t=a}^{x} \int_{y=a}^{b} \left| k(t, y, u(y)) - k(t, y, v(y)) \right| dy\, dt \\
&\leq \alpha_1 \alpha_2 \int_{t=a}^{x} \int_{y=a}^{b} \left| u(y) - v(y) \right| dy\, dt. \qquad (6)
\end{aligned}$$

Then multiplying both sides of (6) by $e^{-(\alpha_1\alpha_2(b-1)+1)x}$ gives:

$$
\begin{aligned}
&e^{-(\alpha_1\alpha_2(b-a)+1)x}|(Tu)(x)-(Tv)(x)| \\
&\le e^{-(\alpha_1\alpha_2(b-a)+1)x}\alpha_1\alpha_2\int_{t=a}^{x}\int_{y=a}^{b}\Big|u(y)-v(y)\Big|dy\,dt \\
&= e^{-(\alpha_1\alpha_2(b-a)+1)x}\alpha_1\alpha_2\int_{t=a}^{x}\int_{y=a}^{b}e^{(\alpha_1\alpha_2(b-1)+1)t}e^{-(\alpha_1\alpha_2(b-1)+1)t}\Big|u(y)-v(y)\Big|dy\,dt \\
&= e^{-\beta x}\alpha_1\alpha_2\int_{t=a}^{x}\int_{y=a}^{b}e^{\beta t}e^{-\beta t}\Big|u(y)-v(y)\Big|dy\,dt \quad (\text{with } \beta=\alpha_1\alpha_2(b-a)+1>0)\,. \\
&\le e^{-\beta x}\alpha_1\alpha_2\int_{t=a}^{x}\int_{y=a}^{b}e^{\beta t}\max_{a\le t\le x; a\le y\le b}\left(e^{-\beta t}\Big|u(x)-v(x)\Big|\right)dy\,dt \\
&\le e^{-\beta x}\alpha_1\alpha_2\int_{t=a}^{x}\int_{y=a}^{b}e^{\beta t}\max_{a\le y\le b}\left(e^{-\beta y}\Big|u(y)-v(y)\Big|\right)dy\,dt \quad \left(\text{since } \max_{a\le t\le x}(f(t))\le\max_{a\le x\le b}(f(x))\right). \\
&= e^{-\beta x}\alpha_1\alpha_2|||u-v|||\int_{t=a}^{x}\int_{y=a}^{b}e^{\beta t}dy\,dt = e^{-\beta x}\alpha_1\alpha_2(b-a)|||u-v|||\int_{t=a}^{x}e^{\beta t}\,dt \\
&= e^{-\beta x}\alpha_1\alpha_2(b-a)|||u-v|||\int_{t=a}^{x}e^{\beta t}\,dt \\
&= \alpha_1\alpha_2(b-a)|||u-v|||e^{-\beta x}\left(\frac{e^{\beta x}-e^{\beta a}}{\beta}\right) = \frac{\alpha_1\alpha_2(b-a)}{\beta}|||u-v|||\left(1-e^{\beta(a-x)}\right) \\
&\le \frac{\alpha_1\alpha_2(b-a)}{\beta}|||u-v||| \quad \left(\text{because } a-x\le 0 => e^{\beta(a-x)}\le 1\right) \\
&= \frac{\alpha_1\alpha_2(b-a)}{\alpha_1\alpha_2(b-a)+1}|||u-v|||. \qquad (7)
\end{aligned}
$$

The left hand side of (7) is equal to $|||Tu-Tv|||$, and since $\frac{\alpha_1\alpha_2(b-a)}{\alpha_1\alpha_2(b-a)+1}$ is non-negative and strictly less than one, we have derived the following inequality

$$|||Tu-Tv|||<|||u-v|||, \qquad (8)$$

which shows that the operator, T, is a contraction. Hence by the Banach contraction principle, T has a unique fixed point, meaning that problem (1) has a unique solution. □

3 Numerical Algorithm

The solution of problem (1) is approximated in this section. Let $N\in\mathbb{Z}^+$ with $N>1$ and

$$h:=\frac{b-a}{N},\quad x_n=a+nh \text{ for } n=0,1,\cdots,N;\qquad \text{and } \Omega_h=\{x_n:n=0,1,\cdots,N\}.$$

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

We also define the grid functions

$$u_n \approx u(x_n) \text{ for each } n, \text{ and } \xi_j^N := \begin{cases} \frac{1}{2}, \text{ if } j = 0, N, \\ 1, \text{ otherwise,} \end{cases} . \tag{9}$$

The following approximation is considered:

$$\begin{aligned} u_n &= g_n + f(x_n, I_n), \quad n = 0, 1, 2, \cdots, N, \\ I_n &:= \begin{cases} h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \eta_i^n \xi_j^N k(x_i, x_j, u_j), \text{ if } n > 0, \\ 0, \text{ if } n = 0. \end{cases} \end{aligned} \tag{10}$$

This is an $(N+1) \times (N+1)$ nonlinear system.
Define the vector of unknowns

$$\vec{u} = (u_0, u_1, \cdots, u_N)^T,$$

and vector of nonlinear functions

$$\vec{Q} = \vec{Q}(\vec{u}) = (q_1(\vec{u}), q_2(\vec{u}), \cdots, q_N(\vec{u}))^T,$$

where

$$\begin{aligned} q_0(\vec{u}) &= u_0 - g_0 - f(a, 0), \\ q_n(\vec{u}) &= u_n - g_n - f(x_n, I_n) \quad 1, 2, \cdots, N. \end{aligned} \tag{11}$$

Note that for $n > 0$, we get

$$\begin{aligned} \frac{\partial I_n}{\partial u_m} &= h^2 \sum_{i=0}^{n} \left[\eta_i^n \frac{\partial}{\partial u_m} \left(\sum_{j=0}^{N} \xi_j^N k(x_i, x_j, u_j) \right) \right] \\ &= h^2 \sum_{i=0}^{n} \left[\eta_i^n \xi_m^N \frac{\partial k(x_i, x_m, z)}{\partial z} \bigg|_{z=u_m} \right], \quad \text{for } 0 \le m \le N, 0 < n \le N. \end{aligned} \tag{12}$$

Using chain rule, we obtain,

$$\frac{\partial f(x_n, I_n)}{\partial u_m} = \frac{\partial f(x_n, I_n)}{\partial I_n} \frac{\partial I_n}{\partial u_m} = \frac{\partial I_n}{\partial u_m} \left(\frac{\partial f(x_n, y)}{\partial y} \right) \bigg|_{y=I_n}.$$

So that

$$\frac{\partial q_n}{\partial u_m} = \begin{cases} 1 - \frac{\partial f(x_n, I_n)}{\partial u_m} & \text{if } m = n, \\ -\frac{\partial f(x_n, I_n)}{\partial u_m} & \text{if } m \neq n. \end{cases} \tag{13}$$

Hence, the Jacobian matrix $A \in \mathbb{R}^{(N+1)\times(N+1)}$, is

$$A_{0,m} = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

while for $n = 1, 2, \cdots, N$, we have

$$A_{n,m} = \frac{\partial q_n(\vec{u})}{\partial u_m}$$
$$= \begin{cases} 1 - h^2 \sum_{i=0}^{n} \left[\eta_i^n \xi_m^N \frac{\partial k(x_i, x_m, y)}{\partial y} \Big|_{y=u_m} \right] \frac{\partial f(x_n, z)}{\partial z} \Big|_{z=I_n}, & \text{if } m = n, \\ -h^2 \sum_{i=0}^{n} \left[\eta_i^n \xi_m^N \frac{\partial k(x_i, x_m, y)}{\partial y} \Big|_{y=u_m} \right] \frac{\partial f(x_n, z)}{\partial z} \Big|_{z=I_n} & \text{if } m \neq n. \end{cases} \tag{15}$$

for $m = 0, 1, 2, \cdots, N$.

The above system is computed using the following Newton algorithm:

$$\begin{cases} \text{Choose } \epsilon \in \mathbb{R}^+, \\ \text{Set } \vec{u}^0 = \vec{u}(0). \\ \text{for } s = 0, 1, \cdots \text{ do the following :} \\ \vec{u}^{s+1} = \vec{u}^s - A(\vec{u}^s)\vec{Q}(\vec{u}^s), \\ \qquad \text{if } \|\vec{u}^{s+1} - \vec{u}^s\| < \epsilon, \text{ terminate the loop,} \\ \qquad \text{otherwise, continue the loop.} \end{cases} \tag{16}$$

This completes the formulation of the numerical scheme. The proposed scheme is implemented in an in-house python program. The convergence of the scheme is discussed in the next section.

4 Convergence Analysis

The convergence of the proposed scheme is analyzed in this section. The following Lemmas are useful for the results we aim to obtain in this section.

Lemma 4.1 (See [12]). *Let $k \in C^4([a, b] \times [a, b] \times \mathbb{R})$ and the interval $[a, b]$ is subdivided by $x_i = a + ih$, $i = 0, 1, 2, \cdots, N$ then*

$$\int_{t=a}^{x_n} \int_{y=a}^{b} k(t, y, u(y)) = h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N k(x_i, x_j, u(x_j)) + \frac{(b-a)^2}{12} h^2 \Pi, \tag{17}$$

where $\Pi > 0$ is independent of n, N and h.

Lemma 4.2 (Grownwall-Type Lemma). *Let $\Delta > 0$ and $w(x)$ be a non-negative function satisfying*

$$w(x) \leq \Delta + \alpha_1 \alpha_2 \frac{(b-a)}{2} \int_a^b w(s)\, ds. \tag{18}$$

If

$$\frac{\alpha_1\alpha_2(b-a)^2}{2} < 1, \tag{19}$$

then

$$w(x) \le \frac{\Delta}{1-\frac{\alpha_1\alpha_2(b-a)^2}{2}}. \tag{20}$$

Proof. Set $C_0 = \frac{\alpha_1\alpha_2(b-a)}{2}$. Then (18) is:

$$w(x) \le \Delta + C_0\int_a^b w(s)\,ds = \Delta + C_0 H, \tag{21}$$

where

$$H := \int_a^b w(s)\,ds \le \int_a^b (\Delta + C_0 H)\,ds = (\Delta + C_0 H)(b-a).$$

That is,

$$H \le \frac{\Delta(b-a)}{1-C_0(b-a)}.$$

Putting this into (21) gives

$$w(x) \le \Delta + C_0 H \le \frac{\Delta}{1-C_0(b-a)}$$

which proves (20).

□

Lemma 4.3 (Discrete Grownwall-Type Lemma)**.** *Let* $\Delta > 0$ *and* $\{p_n\}_{n=0}^N$ *be nonnegative grid functions satisfying*

$$p_n \le \Delta + \frac{K_0}{2}h\sum_{i=1}^{N}(p_{i-1}+p_i)\,. \tag{22}$$

If $K_0(b-a) < 1$*, then*

$$p_n \le \frac{\Delta}{1-K_0(b-a)}. \tag{23}$$

Proof. Inequality (22) can be written as:

$$p_n \le \Delta + \frac{K_0}{2}\sum_{i=1}^{N}\left(\int_{x_{i-1}}^{x_i} p_{i-1}\,dx + \int_{x_i}^{x_i+1} p_i\,dx\right), \qquad n \ge 0. \tag{24}$$

Define the function [29]:

$$w(x) = \begin{cases} p_i, \text{ if } x \in (x_{i-1}, x_i], \\ 0, \qquad\qquad\qquad\qquad \text{else }. \end{cases}$$

Then, (24) gives

$$\begin{aligned} w(x_n) &\leq \Delta + \frac{K_0}{2} \sum_{i=1}^{N} \left(\int_{x_{i-1}}^{x_i} w(t)\, dt + \int_{x_i}^{x_i+1} w(t)\, dt \right) \\ &= \Delta + \frac{K_0}{2} \left(\int_{x_0}^{x_N} w(t)\, dx + \int_{x_1}^{x_N+1} w(t)\, dx \right) \\ &\leq \Delta + K_0 \int_{x_0}^{x_N} w(t). \end{aligned} \tag{25}$$

Since x_n is arbitrary in $[a, b]$, then the last inequality can be written as:

$$w(x) \leq \Delta + K_0 \int_a^b w(t). \tag{26}$$

Hence, by Lemma 4.2, inequality (26) gives:

$$w(x) \leq \frac{\Delta}{1 - K_0(b-a)},$$

in particular,

$$p_n = w(x_n) \leq \frac{\Delta}{1 - K_0(b-a)}.$$

□

We are ready to prove the convergence of the proposed scheme.

Theorem 4.1 (Convergence)**.** *Let conditions A - C be satisfied and* $k \in C^4([a, b] \times [a, b] \times \mathbb{R})$. *If*

$$\alpha_1 \alpha_2 (b-a)^2 < 1, \tag{27}$$

then the scheme (10) *converges to the unique solution of* (1) *with the second order of convergence.*

Proof. Collocating (1) at $x_n \in \Omega_h$ gives:

$$u(x_n) = g(x_n) + f\left(x, \int_{t=a}^{x_n} \int_{y=a}^{b} k(t, y, u(y))\, dy\, dt\right). \tag{28}$$

Lemma 4.1 gives

$$u(x_n) = g(x_n) + f\left(x, h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N k(x_i, x_j, u(x_j)) + \frac{(b-a)^2}{12} h^2 \Pi\right). \quad (29)$$

Subtracting (10) from (29), we have:

$$\begin{aligned}
e_n &:= |u(x_n) - u_n| \\
&= \left| f\left(x, h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N k(x_i, x_j, u(x_j)) + \frac{(b-a)^2}{12} h^2 \Pi\right) \right. \\
&\quad \left. - f\left(x, h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N k(x_i, x_j, u_j)\right) \right| \\
&\leq \alpha_1 \left| h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N k(x_i, x_j, u(x_j) - h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N k(x_i, x_j, u_j) \right| \quad (30) \\
&\quad + \alpha_1 \left| \frac{(b-a)^2}{12} h^2 \Pi \right| \\
&\leq \alpha_1 h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N \left| k(x_i, x_j, u(x_j) - k(x_i, x_j, u_j) \right| + \alpha_1 \left| \frac{(b-a)^2}{12} h^2 \Pi \right| \\
&\leq \alpha_1 \alpha_2 h^2 \sum_{i=0}^{n} \sum_{j=0}^{N} \xi_i^n \xi_j^N \left| u(x_j) - u_j \right| + \alpha_1 \left| \frac{(b-a)^2}{12} h^2 \Pi \right| \\
&= \alpha_1 \alpha_2 h^2 n \sum_{j=0}^{N} \xi_j^N e_j + \alpha_1 \left| \frac{(b-a)^2}{12} h^2 \Pi \right| \\
&\leq \alpha_1 \alpha_2 h (b-a) \sum_{j=0}^{N} \xi_j^N e_j + \alpha_1 \frac{(b-a)^2}{12} h^2 \Pi \\
&= \alpha_1 \alpha_2 (b-a) \frac{h}{2} \sum_{j=1}^{N} (e_{j-1} + e_j) + \alpha_1 \frac{(b-a)^2}{12} h^2 \Pi. \quad (31)
\end{aligned}$$

By Lemma 4.3, inequality (31) gives:

$$\begin{aligned}
e_n &\leq \left(\frac{1}{1 - \alpha_1 \alpha_2 (b-a)^2} \right) \frac{\alpha_1 (b-a)^2}{12} h^2 \Pi \\
&= O(h^2).
\end{aligned}$$

□

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

5 Numerical Examples and Results

Some numerical examples verify the convergence and access the performance of the proposed scheme. These examples are constructed via the method of manufactured solutions [31, 32, 33, 34, 35, 36, 37]. The error (in infinity norm) and experimental order of convergence (EOC) are computed by using the formulae:

$$Error_h = \max_{x_n \in \Omega_h} |u_n - u(x_n)|, \tag{32}$$

$$EOC = \frac{log\left(\frac{Error_h}{Error_{h/2}}\right)}{log(2)}, \tag{33}$$

see [38].

5.0.1 Example 1

Consider the following problem:

$$u(x) = -\frac{x^6(1+e^2)^2}{64} + e^x + x^2 \left(\int_{t=0}^{x} \int_{y=0}^{1} k(t,y,u(y))\, dy\, dt \right)^2, \quad x \in [0,1], \tag{34}$$

where,

$$k(t,y,u(y)) = tyu^2(y),$$

and whose exact solution is $u(x) = e^x$. This problem is solved with the proposed scheme on a sequence of grids with $N = 20 \times 2^p, p = 0, 1, \cdots, 5$ sub-intervals. The numerical results are tabulated in Table 1. The second order of convergence is obviously verified.

Table 1: Numerical Results for Example 1. N = number of sub-intervals. Errors are computed with infinity norm. EOC = Experimental Order of Convergence.

N	Error (in Infinity Norm)	EOC
20	0.014878449829527352	-
40	0.00341798195608245	2.1220075353601846
80	0.0008268315504746937	2.047479435185898
160	0.00020375084013002365	2.020787432533474
320	5.059657036543541e-05	2.009694506248465
640	1.2608206059194771e-05	2.004676578248638

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

5.0.2 Example 2

In this example, we consider the problem:

$$u(x) = g(x) + \frac{x^2 \int_0^x \int_0^1 k(t,y,u(y))\,dy\,dt}{\left(1 + \int_0^x \int_0^1 k(t,y,u(y))\,dy\,dt\right)^2}, \tag{35}$$

where,

$$k(t,y,u(y)) = \frac{t^3 y^5}{1+u^2(y)},$$

$$g(x) = -\frac{x^6\left(-\frac{\pi}{68} - \frac{19\tan^{-1}\left(\frac{3}{2}\right)}{544} - \frac{\log(5)}{68} + \frac{\log(2)}{136} + \frac{19\tan^{-1}\left(\frac{1}{2}\right)}{544} + \frac{\log(13)}{136} + \frac{\tan^{-1}(2)}{17}\right)}{x^8\left(-\frac{\pi}{68} - \frac{19\tan^{-1}\left(\frac{3}{2}\right)}{544} - \frac{\log(5)}{68} + \frac{\log(2)}{136} + \frac{19\tan^{-1}\left(\frac{1}{2}\right)}{544} + \frac{\log(13)}{136} + \frac{\tan^{-1}(2)}{17}\right)^2 + 1.0}$$
$$+ 2x^4 + 3x^2 + 3.$$

$u(x) = 2x^4 + 3x^2 + 3$ is the exact solution of this problem. This problem is also solved on a sequence of grids with $N = 20 \times 2^p, p = 0, 1, \cdots, 5$ subintervals. The numerical results which are displayed in Table 2 show that the scheme converges and with second order. This, again, verifies the theoretical results.

Table 2: Numerical Results for Example 2

N	Error (in Infinity Norm)	EOC
20	4.3830124472066245e-06	-
40	1.0395131173623895e-06	2.0760148126138316
80	2.5329666186735267e-07	2.037007993952602
160	6.252749962243342e-08	2.0182653318807704
320	1.5533862196548398e-08	2.0090742545260905
640	3.8713103833742935e-09	2.0045226868915984

5.0.3 Example 3

As a third and final example, we consider the problem:

$$u(x) = g(x) + \sqrt{6 + \left(\int_{t=0}^{x} \int_{y=0}^{1} k(t,y,u(y))\,dy\,dt\right)^4}, \quad x \in [0,1], \tag{36}$$

where,

$$k(t,y,u(y)) = \frac{t^4 y^2 y \cos(u(y))}{12+y^2},$$
$$g(x) = -0.00444444444444444(x^{20}(-72\sqrt{3}tan^{-1}(\sqrt{3}/6)+35)^4 + 303750)^{0.5} + \cos^{-1}(x^2).$$

The exact solution is $u(x) = \cos^{-1}(x^2)$. This example is solved with the proposed scheme on a sequence of grids with $N = 2 \times 2^p, p = 0, 1, \cdots, 6$ sub-intervals. The numerical results are shown in Table 3. One can see that the proposed scheme is extremely accurate for this problem. Indeed, even with just two sub-intervals an error on the order of 10^{-8} is achieved, and the error vanishes almost to the machine epsilon on just a few grid refinements. This also verifies the second order of convergence and the good performance of the proposed method.

Table 3: Numerical Results for Example 3. Results Show Second-Order Convergence.

N	Error (in Infinity Norm)	EOC
2	2.789777076017761e-08	-1
4	5.415667914120516e-11	9.008794852903595
8	5.927258683822764e-12	3.1917023486849017
16	1.1826095657460955e-12	2.3253911707783583
32	2.7355895316244123e-13	2.112050172967271
64	6.794564899308637e-14	2.009398699867462
128	1.8651746696275295e-14	1.8650704265767835

6 Conclusion

The existence and uniqueness of solution of nonlinear functional mixed Volterra-Fredholm integral equation has been established via Banach contraction principle. A numerical scheme is proposed to approximate the solution. The convergence of the method is rigorously proved via a new discrete Grownwall inequality, and numerical examples are constructed via manufactured solutions and used to verify the theoretical results. The results show that the scheme (i) is highly accurate (second order of convergence), and (ii) performs well even on relatively coarse grids.

References

[1] K Maleknejad and M Hadizadeh. A new computational method for volterra-fredholm integral equations. *Computers & Mathematics with Applications*, 37(9):1–8, 1999.

[2] Abdul-Majid Wazwaz. A reliable treatment for mixed volterra–fredholm integral equations. *Applied Mathematics and Computation*, 127(2-3):405–414, 2002.

[3] Abdul Jerri. *Introduction to integral equations with applications*. John Wiley & Sons, 1999.

[4] Shouchuan Hu, Mohammad Khavanin, and WAN Zhuang. Integral equations arising in the kinetic theory of gases. *Applicable Analysis*, 34(3-4):261–266, 1989.

[5] Jan Prüss. *Evolutionary integral equations and applications*, volume 87. Birkhäuser, 2013.

[6] Donal Oregan. Existence results for nonlinear integral equations. *Journal of Mathematical Analysis and Applications*, 192(3):705–726, 1995.

[7] MA Abdou. On a symptotic methods for fredholm–volterra integral equation of the second kind in contact problems. *Journal of Computational and Applied Mathematics*, 154(2):431–446, 2003.

[8] Aline C Rocha, Marcio A Murad, Christian Moyne, Saulo P Oliveira, and Tien D Le. A new methodology for computing ionic profiles and disjoining pressure in swelling porous media. *Computational Geosciences*, 20(5):975–996, 2016.

[9] Tien-Dung Le, Christian Moyne, MA Murad, and SA Lima. A two-scale non-local model of swelling porous media incorporating ion size correlation effects. *Journal of the Mechanics and Physics of Solids*, 61(12):2493–2521, 2013.

[10] Mohsen Rabbani. An iterative algorithm to find a closed form of solution for hammerstein nonlinear integral equation constructed by the concept of cosm-rs. *Mathematical Sciences*, 13(3):299–305, 2019.

[11] Abdul-Majid Wazwaz. *First Course In Integral Equations, A*. World Scientific Publishing Company, 2015.

[12] Sanda Micula. On some iterative numerical methods for mixed volterra–fredholm integral equations. *Symmetry*, 11(10):1200, 2019.

2 Aug 2022 08:43:54 PDT
220129-Nwaigwe Version 2 - Submitted to J. Integr. Eq. Appl.

[13] Nicolaie Lungu and Ioan A Rus. On a functional volterra-fredholm integral equation, via picard operators. *Journal of Mathematical Inequalities*, 3(4):519–527, 2009.

[14] Manouchehr Kazemi and Reza Ezzati. Existence of solutions for some nonlinear volterra integral equations via petryshyn's fixed point theorem. *International Journal of Nonlinear Analysis and Applications*, 9(1):1–12, 2018.

[15] Farshid Mirzaee and Nasrin Samadyar. Extension of darbo fixed-point theorem to illustrate existence of the solutions of some nonlinear functional stochastic integral equations. *International Journal of Nonlinear Analysis and Applications*, 11(1):413–421, 2020.

[16] Amar Deep, Ashish Kumar, Syed Abbas, and Mohsen Rabbani. Solvability and numerical method for non-linear volterra integral equations by using petryshyns fixed point theorem. *International Journal of Nonlinear Analysis and Applications*, 13(1):1–28, 2022.

[17] İsmet Özdemir, Ümit Çakan, and Bekir İlhan. On the existence of the solutions for some nonlinear volterra integral equations. In *Abstract and Applied Analysis*, volume 2013. Hindawi, 2013.

[18] İsmet Özdemir and Ümit Çakan. On the solutions of a class of nonlinear functional integral quations in space c [0, a]. *Journal of Mathematics and Applications*, 38, 2015.

[19] Ümit ÇAKAN and İsmet ÖZDEMİR. An application of krasnoselskii fixed point theorem to some nonlinear functional integral equations. *Nevşehir Bilim ve Teknoloji Dergisi*, 3(2):66–73, 2014.

[20] Edgardo Alvarez and Carlos Lizama. Application of measure of noncompactness to volterra equations of convolution type. *Journal of Integral Equations and Applications*, 28(4):441–458, 2016.

[21] Ü Çakan. On monotonic solutions of some nonlinear fractional integral equations. *Nonlinear Funct. Anal. Appl.*, 22(2):259–273, 2017.

[22] K Maleknejad, R Mollapourasl, and K Nouri. Study on existence of solutions for some nonlinear functional–integral equations. *Nonlinear Analysis: Theory, Methods & Applications*, 69(8):2582–2588, 2008.

[23] Hemant Kumar Pathak et al. Study on existence of solutions for some nonlinear functional-integral equations with applications. *Mathematical Communications*, 18(1):97–107, 2013.

[24] PM Fitzpatrick. Klaus deimling, nonlinear functional analysis. *Bulletin (New Series) of the American Mathematical Society*, 20(2):277–280, 1989.

[25] Józef Banaś and Beata Rzepka. On existence and asymptotic stability of solutions of a nonlinear integral equation. *Journal of Mathematical Analysis and Applications*, 284(1):165–173, 2003.

[26] Manochehr Kazemi, Hamid Mottaghi Golshan, Reza Ezzati, and Mohsen Sadatrasoul. New approach to solve two-dimensional fredholm integral equations. *Journal of Computational and Applied Mathematics*, 354:66–79, 2019.

[27] Manochehr Kazemi. Triangular functions for numerical solution of the nonlinear volterra integral equations. *Journal of Applied Mathematics and Computing*, 68(3):1979–2002, 2022.

[28] Manochehr Kazemi and AR Yaghoobnia. Application of fixed point theorem to solvability of functional stochastic integral equations. *Applied Mathematics and Computation*, 417:126759, 2022.

[29] Sohrab Bazm, Pedro Lima, and Somayeh Nemati. Analysis of the euler and trapezoidal discretization methods for the numerical solution of nonlinear functional volterra integral equations of urysohn type. *Journal of Computational and Applied Mathematics*, page 113628, 2021.

[30] Adam Bielecki. Une remarque sur la méthode de banach-cacciopoli-tikhonov dans la théorie des équations différentielles ordinaires. *Bull. Acad. Polon. Sci*, 4(5):261–264, 1956.

[31] Chinedu Nwaigwe. Analysis and application of a convergent difference scheme to nonlinear transport in a brinkman flow. *International Journal of Numerical Methods for Heat & Fluid Flow*, 2020.

[32] Patrick J Roache. The method of manufactured solutions for code verification. In *Computer Simulation Validation*, pages 295–318. Springer, 2019.

[33] William L Oberkampf and Timothy G Trucano. Verification and validation benchmarks. *Nuclear engineering and Design*, 238(3):716–743, 2008.

[34] Benjamin Daniel Dudson, Jens Madsen, John Omotani, Peter Hill, Luke Easy, and Michael Løiten. Verification of bout++ by the method of manufactured solutions. *Physics of Plasmas*, 23(6):062303, 2016.

[35] Benjamin Grier, Richard Figliola, Edward Alyanak, and José Camberos. Discontinuous solutions using the method of manufactured solutions on finite volume solvers. *AIAA Journal*, 53(8):2369–2378, 2015.

[36] Li Wang, Wei-jiang Zhou, and Chu-qun Ji. Verification of a chemical nonequilibrium flows solver using the method of manufactured solutions. *Procedia Engineering*, 99:713–722, 2015.

[37] Jipu Wang, William Martin, and Benjamin Collins. The application of method of manufactured solutions to method of characteristics in planar geometry. *Annals of Nuclear Energy*, 121:295–304, 2018.

[38] Chinedu Nwaigwe. *Coupling methods for 2d/1d shallow water flow models for flood simulations.* PhD thesis, University of Warwick, 2016.