# Volterra and Fredholm integral equations in locally convex spaces, and Leader-type contractions in gauge spaces

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**Abstract**

There are the several different methods concerning studies of quadratic, quadratic fractional, linear and nonlinear integral equations of Volterra and Fredholm types in various spaces. Method of studies of such equations in locally convex spaces with approximation procedure and presented here is new, general, optimal and precise, and also is different from those known in the literature. Examples provided in this paper illustrate chosen aspects of our method. The main tools used here are based on our special cases of very general and stronger periodic point, fixed point and approximation theorems concerning not necessarily continuous dynamic systems in not necessarily sequentially complete or separable gauge spaces.

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## 1 Introduction

The fundamental papers of Volterra concerning integral equations are initiated by paper [27] and are the milestone in the history of integral equations. These papers, papers of Fredholm initiated by paper [15] and papers of many researchers (we give some important references, but they are not exhaustive) in this field provide new perspectives on the investigations and new ideas and techniques together with the different areas in which the topic of solutions of integral equations has had its influence; they were motivated by questions arising in radiative transfer theory, neutron transport theory, kinetic theory of gases, plasma physics, electro-magnetic fluid dynamics, dynamic models of chemical

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reactors, traffic theory, queuing theory, mathematical physics, mechanics, engineering, biology, bioengineering and economics. In particular, the theory of Volterra and Fredholm integral equations in abstract settings (e.g., in Banach spaces, Fréchet spaces, and locally convex spaces) has received increasing attention (see e.g., [1, 2, 6, 7, 9, 10, 11, 12, 20, 31] and references therein).

Recall that beautiful results of Leader [21] generalize the results [2, 4, 5, 7, 8, 13, 14, 16, 20, 22, 23, 24, 25, 26], and many others. Leader defines new contractions $T : X \to X$ in not necessarily complete metric spaces $(X, d)$ and with assumption that $T$ have complete graphs (i.e. closed in $Y^2$ where $Y$ is the completion of $X$) and proved that such contractions $T$ (which are not necessarily continuous) have contractive fixed points; point $w \in X$ such that $Fix_X(T) = \{w\}$ and $\lim_{m\to\infty} d(T^{[m]}w^0, w) = 0$ for all $w^0 \in X$, is called a contractive fixed point of $T$. Presentation concerning generality and some structural properties of Leader contractions in metric spaces was fully exploited in Jachymski [17] and Jachymski and Jóźwik [18].

In this paper, we establish new and general convergence, existence and uniqueness theorems (see Theorems 4.1, 4.2, 5.1 and 5.2) concerning solutions of various types (quadratic, quadratic fractional, nonlinear, linear) of Volterra and Fredholm integral equations (see Section 2) in locally convex spaces. These theorems are applications of our convergence, existence and uniqueness results contained in [29, 30] and concerning periodic and fixed points of Leader type contractions in gauge spaces (see Theorems 3.1 and 3.2). Examples are provided (see Examples 7.1-7.17).

Motivations of studies, some questions and provided here efficient tools and techniques for new investigations of Volterra and Fredholm types operators in locally convex spaces are also described in this paper. Established here results are new even in $\mathbb{R}$ and in Banach spaces.

# 2 Definitions, notations and remarks

First, we recall the following two definitions.

**Definition 2.1 .** *Let* $E$ be a vector space over a field $\mathbb{R}$, and let $\mathcal{A}$ be an index set.

(A) The map $P : E \to [0; \infty)$ is called *seminorm* if: $(i)$ $\forall_{u\in E}\forall_{\lambda\in\mathbb{R}}\{P(\lambda u) = |\lambda| \cdot P(u)\}$ (*homogenity*; so, in particular, $P(0) = 0$). $(ii)$ $\forall_{u,v\in E}\{P(u + v) \leq P(u) + P(v)\}$ (*triangle inequality*).

(B) A topological vector space $(E, \mathcal{T})$, such that there is a family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of continuous seminorms $P_\alpha : E \to [0; \infty)$, $\alpha \in \mathcal{A}$, on $E$ and $\mathcal{T}$ is a locally convex topology on $E$ generated by the family $\mathcal{P}_{\mathcal{A}}$, is called *locally convex space* and is denoted by $(E, \mathcal{P}_{\mathcal{A}})$.

(C) The family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of seminorms $P_\alpha : E \to [0; \infty)$, $\alpha \in \mathcal{A}$, on $E$ is called *separating* if $\forall_{u\in E}\{u \neq 0 \Rightarrow \exists_{\alpha_0\in\mathcal{A}}\{P_{\alpha_0}(u) > 0\}\}$.

(D) If the family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ is separating on $E$, then $\mathcal{T}$ is a Hausdorf locally convex topology on $E$ and $(E, \mathcal{P}_{\mathcal{A}})$ is called *Hausdorff locally*



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*convex space.*

**Definition 2.2** (see [13]). *Let $X$ be a (nonempty) set, and let $\mathcal{A}$ be an index set.*

(A) The distance $D : X^2 \to [0;\infty)$ is called *pseudometric* (or the *gauge*) on $X$ if: $(i)$ $\forall_{u\in X}\{D(u,u)=0\}$ (it is not required that $D(u,v)=0$ implies $u=v$). $(ii)$ $\forall_{u,v\in X}\{D(u,v)=D(v,u)\}$ (*symmetry*). $(iii)$ $\forall_{u,v,w\in X}\{D(u,v)\leq D(u,w)+D(w,v)\}$ (*triangle inequality*).

(B) Each family $\mathcal{D}_{\mathcal{A}}=\{D_\alpha : \alpha\in\mathcal{A}\}$ of pseudometrics $D_\alpha : X^2\to[0;\infty)$, $\alpha\in\mathcal{A}$, on $X$ is called *gauge* on $X$.

(C) The gauge $\mathcal{D}_{\mathcal{A}}=\{D_\alpha : \alpha\in\mathcal{A}\}$ on $X$ is called *separating* if $\forall_{u,w\in E}\{u\neq w \Rightarrow \exists_{\alpha_0\in\mathcal{A}}\{D_{\alpha_0}(u,w)>0\}\}$.

(D) Let the family $\mathcal{D}_{\mathcal{A}}=\{D_\alpha : \alpha\in\mathcal{A}\}$ be gauge on $X$. The topology $\mathcal{T}(\mathcal{D}_{\mathcal{A}})$ having as a subbase the family $\mathcal{B}(\mathcal{D}_{\mathcal{A}})=\{B(u,D_\alpha,\varepsilon_\alpha) : u\in X, \varepsilon_\alpha>0, \alpha\in\mathcal{A}\}$ of all balls $B(u,D_\alpha,\varepsilon_\alpha)=\{v\in X : D_\alpha(u,v)<\varepsilon_\alpha\}$ with $u\in X$, $\varepsilon_\alpha>0$, and $\alpha\in\mathcal{A}$ is called topology *induced* by $\mathcal{D}_{\mathcal{A}}$ on $X$.

(E) A topological space $(X,\mathcal{T})$ such that there is a gauge $\mathcal{D}_{\mathcal{A}}$ on $X$ with $\mathcal{T}=\mathcal{T}(\mathcal{D}_{\mathcal{A}})$ is called a *gauge space* and is denoted by $(X,\mathcal{D}_{\mathcal{A}})$.

(F) If the family $\mathcal{D}_{\mathcal{A}}=\{D_\alpha : \alpha\in\mathcal{A}\}$ is separating on $X$, then the topology $\mathcal{T}(\mathcal{D}_{\mathcal{A}})$ is Hausdorff and $(X,\mathcal{D}_{\mathcal{A}})$ is called *Hausdorff gauge space.*

(G) We say that a sequence $(w_m : m\in\mathbb{N})\subset X$ is *$\mathcal{D}_{\mathcal{A}}$-convergent in $X$* if there exists $w\in X$ such that $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}D_\alpha(w_m,w)=0\}$. We will use $(w_m : m\in\mathbb{N})\subset X$ as a sequence and as a set as the situation demands.

Throughout this paper, for each $n\in\mathbb{N}$, let

$$\mathbb{A}_n=\{A=[0;a_1]\times[0;a_2]\times...\times[0;a_n]\subset\mathbb{R}^n : a_k>0,\ k=1,2,...,n\}$$

and, for each $A\in\mathbb{A}_n$ and $t=(t_1,...,t_n)\in A$, let $A(t)$ be defined by

$$A(t)=\{\tau=(\tau_1,...,\tau_n)\in\mathbb{R}^n : 0\leq\tau_k\leq t_k, k=1,...,n\}.$$

Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space and let $\mathcal{C}(A,E)$ denote a family of all continuous maps $y : A\to E$. Clearly, $\mathcal{C}(A,E)$ is a vector space over a field $\mathbb{R}$.

Let us make the following assumptions, notations and remarks (I) and (II):

(I)

(I.1) $n\in\mathbb{N}$ and $A\in\mathbb{A}_n$ is an arbitrary and fixed set. $(E,\mathcal{P}_{\mathcal{A}})$ is a locally convex space. $\mathcal{C}(A,E)$ denotes a family of all continuous maps $y : A\to E$.

Denote by $(\mathcal{C}(A,E),\mathcal{D}_{\mathcal{A}})$ a gauge space with a gauge $\mathcal{D}_{\mathcal{A}}=\{D_\alpha : \alpha\in\mathcal{A}\}$ defined on $\mathcal{C}(A,E)$ as follows:

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in\mathcal{C}(A,E)}\{D_\alpha(x,y)=\sup_{t\in A}P_\alpha[x(t)-y(t)]\}.$$

If, in particular, $E=\mathbb{R}$, then $\mathcal{D}_{\mathcal{A}}=\{D\}$ and $(\mathcal{C}(A,\mathbb{R}),D)$ is a gauge space with gauge $D$ defined by

$$\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{D(x,y)=\sup_{t\in A}|x(t)-y(t)|\}.$$

(I.2) Define a Volterra quadratic integral equation

$$y(t)=f(t,y(t))+\int_{A(t)}K(t,\tau,y(h(\tau)))d\tau,\ t\in A, \tag{2.1}$$

and a Fredholm quadratic integral equation

$$y(t)=f(t,y(t))+\int_{A}K(t,\tau,y(h(\tau)))d\tau,\ t\in A, \tag{2.2}$$

in which $f\in\mathcal{C}(A\times E,E)$, $h\in\mathcal{C}(A,A)$ and $K\in\mathcal{BC}(A\times A\times E,E)$ (that is $K$ is bounded and continuous), the maps $f$, $h$ and $K$ are given maps, and $y\in\mathcal{C}(A,E)$ is an unknown map to be determined.

We say that $\mathcal{V}$ is a Volterra operator on $\mathcal{C}(A,E)$ if $(\mathcal{V}y)(t)=f(t,y(t))$ $+\int_{A(t)}K(t,\tau,y(h(\tau)))d\tau$, $y\in\mathcal{C}(A,E)$, $t\in A$. We say that $\mathcal{F}$ is a Fredholm operator on $\mathcal{C}(A,E)$ if $(\mathcal{F}y)(t)=f(t,y(t))+\int_{A}K(t,\tau,y(h(\tau)))d\tau$, $y\in\mathcal{C}(A,E)$, $t\in A$.

Clearly, $(\mathcal{C}(A,E),\mathcal{V})$ and $(\mathcal{C}(A,E),\mathcal{F})$ are (single-valued) dynamic systems on $\mathcal{C}(A,E)$. Indeed, let $y\in\mathcal{C}(A,E)$. Then, for any $t,t_0\in A$ we have

$$\begin{aligned}
&(\mathcal{V}y)(t)-(\mathcal{V}y)(t_0)\\
=\;&f(t,y(t))-f(t_0,y(t_0))+\int_{A(t)}[K(t,\tau,y(h(\tau)))-K(t_0,\tau,y(h(\tau)))]d\tau\\
&+\int_{A(t)}K(t_0,\tau,y(h(\tau)))d\tau-\int_{A(t_0)}K(t_0,\tau,y(h(\tau)))d\tau\\
=\;&f(t,y(t))-f(t_0,y(t_0))+\int_{A(t)}[K(t,\tau,y(h(\tau)))-K(t_0,\tau,y(h(\tau)))]d\tau\\
&+\int_{A(t_0)\backslash A(t)}K(t_0,\tau,y(h(\tau)))d\tau-\int_{A(t)\backslash A(t_0)}K(t_0,\tau,y(h(\tau)))d\tau.
\end{aligned}$$

Hence, since the maps $f$, $y$ and $h$ are continuous, $K$ is bounded and continuous, $\lim_{t\to t_0}\mu(A(t)\backslash A(t_0))=0$ and $\lim_{t\to t_0}\mu(A(t_0)\backslash A(t))=0$, follows $(\mathcal{V}y)(t)\to(\mathcal{V}y)(t_0)$ as $t\to t_0$. So $\mathcal{V}y$ is continuous in $t_0$. Therefore, $\mathcal{V}:\mathcal{C}(A,E)\to\mathcal{C}(A,E)$, as was to be shown. The proof that $(\mathcal{C}(A,E),\mathcal{F})$ is a dynamic system is similar but easier. Here $\mu$ denotes the Lebesgue measure in $\mathbb{R}^n$.

Define a dynamic system $(\mathcal{C}(A,E),\mathcal{W})$ where $\mathcal{W}=\mathcal{V}$ or $\mathcal{W}=\mathcal{F}$.

Let $(\mathcal{C}(A,E),\mathcal{W}^{[m]})$, $m\in\mathbb{N}$, be a (single-valued) dynamic system, that is $\mathcal{W}^{[m+1]}:\mathcal{C}(A,E)\to\mathcal{C}(A,E)$, $m\in\{0\}\cup\mathbb{N}$. Clearly, a sequence $(\mathcal{W}^{[m+1]}:m\in\{0\}\cup\mathbb{N})$ satisfies $(\mathcal{V}^{[m+1]}y)(t)=f(t,(\mathcal{V}^{[m]}y)(t))+\int_{A(t)}K(t,\tau,(\mathcal{V}^{[m]}y)(h(\tau)))d\tau$ when $\mathcal{W}=\mathcal{V}$ and $(\mathcal{F}^{[m+1]}y)(t)=f(t,(\mathcal{F}^{[m]}y)(t))+\int_{A}K(t,\tau,(\mathcal{F}^{[m]}y)(h(\tau)))d\tau$ when $\mathcal{W}=\mathcal{F}$, $t\in A$. For each $y^0\in\mathcal{C}(A,E)$, a sequence $(\mathcal{W}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})\subset\mathcal{C}(A,E)$ of iterations starting at $y^0\in\mathcal{C}(A,E)$ is well defined. Here $\mathcal{W}^{[0]}=I_{\mathcal{C}(A,E)}$ - identity on $\mathcal{C}(A,E)$.

Let $\mathcal{Y}_{\mathcal{W}}$ be a set such that $\mathcal{Y}_{\mathcal{W}}=\mathcal{Y}_{\mathcal{V}}$ or $\mathcal{Y}_{\mathcal{W}}=\mathcal{Y}_{\mathcal{F}}$, where $\mathcal{Y}_{\mathcal{V}}$ and $\mathcal{Y}_{\mathcal{F}}$ are the sets of all solutions $y\in\mathcal{C}(A,E)$ of equations (2.1) and (2.2), respectively.

(I.3) Define a Volterra integral equation

$$y(t) = f(t) + g(t) \int_{A(t)} K(t, \tau, y(h(\tau)))d\tau,\ t \in A, \tag{2.3}$$

and a Fredholm integral equation

$$y(t) = f(t) + g(t) \int_{A} K(t, \tau, y(h(\tau)))d\tau,\ t \in A, \tag{2.4}$$

in which $f \in \mathcal{C}(A, E)$, $g \in \mathcal{C}(A, \mathbb{R})$ with $\max_{t\in A} |g(t)| = \lambda > 0$, $h \in \mathcal{C}(A, A)$ and $K \in \mathcal{BC}(A\times A\times E, E)$, the maps $f$, $g$, $h$ and $K$ are given maps, and $y \in \mathcal{C}(A, E)$ is an unknown map to be determined. We say that $\mathcal{V}$ is a Volterra operator on $\mathcal{C}(A, E)$ if $(\mathcal{V}y)(t) = f(t) + g(t) \int_{A(t)} K(t, \tau, y(h(\tau)))d\tau$, $t \in A$. We say that $\mathcal{F}$ is a Fredholm operator on $\mathcal{C}(A, E)$ if $(\mathcal{F}y)(t) = f(t) + g(t) \int_A K(t, \tau, y(h(\tau)))d\tau$, $t \in A$.

Clearly, $(\mathcal{C}(A, E), \mathcal{V})$ and $(\mathcal{C}(A, E), \mathcal{F})$ are dynamic systems on $\mathcal{C}(A, E)$. Indeed, let $y \in \mathcal{C}(A, E)$. Then, for any $t, t_0 \in A$ we have

$$\begin{aligned}
& (\mathcal{V}y)(t) - (\mathcal{V}y)(t_0) \\
= \ & f(t) - f(t_0) + g(t) \int_{A(t)} [K(t, \tau, y(h(\tau))) - K(t_0, \tau, y(h(\tau)))]d\tau \\
& +[g(t) - g(t_0)] \int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau \\
& +g(t_0)[\int_{A(t_0)\setminus A(t)} K(t_0, \tau, y(h(\tau)))d\tau - \int_{A(t)\setminus A(t_0)} K(t_0, \tau, y(h(\tau)))d\tau].
\end{aligned}$$

Hence, since $f$, $g$, $y$ and $h$ are continuous, $K$ is bounded and continuous, $\lim_{t\to t_0} \mu(A(t)\setminus A(t_0)) = 0$ and $\lim_{t\to t_0} \mu(A(t_0)\setminus A(t)) = 0$, follows $(\mathcal{V}y)(t) \to (\mathcal{V}y)(t_0)$ as $t \to t_0$. So $\mathcal{V}y$ is continuous in $t_0$. Therefore, $\mathcal{V} : \mathcal{C}(A, E) \to \mathcal{C}(A, E)$, as was to be shown. The proof that $(\mathcal{C}(A, E), \mathcal{F})$ is a dynamic system is similar.

Define a dynamic system $(\mathcal{C}(A, E), \mathcal{W})$ where $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{F}$.

Let $(\mathcal{C}(A, E), \mathcal{W}^{[m]})$, $m \in \mathbb{N}$, be a single-valued dynamic system, that is $\mathcal{W}^{[m+1]} : \mathcal{C}(A, E) \to \mathcal{C}(A, E)$, $m \in \{0\} \cup \mathbb{N}$. Clearly, a sequence $(\mathcal{W}^{[m+1]} : m \in \{0\}\cup\mathbb{N})$ satisfies $(\mathcal{V}^{[m+1]}y)(t) = f(t)+g(t) \int_{A(t)} K(t, \tau, (\mathcal{V}^{[m]}y)(h(\tau)))d\tau$ for $t \in A$ when $\mathcal{W} = \mathcal{V}$ and $(\mathcal{F}^{[m+1]}y)(t) = f(t) + g(t) \int_A K(t, \tau, (\mathcal{F}^{[m]}y)(h(\tau)))d\tau$ for $t \in A$ when $\mathcal{W} = \mathcal{F}$. For each $y^0 \in \mathcal{C}(A, E)$, a sequence $(\mathcal{W}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N}) \subset\mathcal{C}(A, E)$ of iterations starting at $y^0 \in \mathcal{C}(A, E)$ is well defined. Here $\mathcal{W}^{[0]} = I_{\mathcal{C}(A,E)}$ - identity on $\mathcal{C}(A, E)$.

Let $\mathcal{Y}_{\mathcal{W}}$ be a set such that $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{V}}$ or $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{F}}$, where $\mathcal{Y}_{\mathcal{V}}$ and $\mathcal{Y}_{\mathcal{F}}$ are the sets of all solutions $y \in \mathcal{C}(A, E)$ of equations (2.3) and (2.4), respectively.

(I.4) Define a Volterra quadratic integral equation

$$y(t) = f(t) + g(t, y(t))\int_{A(t)}K(t, \tau, y(h(\tau)))d\tau,\ t \in A, \tag{2.5}$$

and a Fredholm quadratic integral equation

$$y(t) = f(t) + g(t, y(t))\int_{A}K(t, \tau, y(h(\tau)))d\tau,\ t \in A, \tag{2.6}$$



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in which $f \in \mathcal{C}(A, \mathbb{R})$, $g \in \mathcal{C}(A \times \mathbb{R}, \mathbb{R})$, $h \in \mathcal{C}(A, A)$ and $K \in \mathcal{BC}(A \times A \times \mathbb{R}, \mathbb{R})$, the maps $f, g, h$ and $K$ are given maps, and $y \in \mathcal{C}(A, \mathbb{R})$ is an unknown map to be determined. We say that $\mathcal{V}$ is a Volterra operator on $\mathcal{C}(A, E)$ if $(\mathcal{V}y)(t) = f(t) + g(t, y(t)) \int_{A(t)} K(t, \tau, y(h(\tau)))d\tau$, $y \in \mathcal{C}(A, \mathbb{R})$, $t \in A$. We say that $\mathcal{F}$ is a Fredholm operator on $\mathcal{C}(A, \mathbb{R})$ if $(\mathcal{F}y)(t) = f(t) + g(t, y(t)) \int_A K(t, \tau, y(h(\tau)))d\tau$, $y \in \mathcal{C}(A, \mathbb{R})$, $t \in A$.

Clearly, $(\mathcal{C}(A, \mathbb{R}), \mathcal{V})$ and $(\mathcal{C}(A, \mathbb{R}), \mathcal{F})$ are (single-valued) dynamic systems on $\mathcal{C}(A, \mathbb{R})$. Indeed, let $y \in \mathcal{C}(A, \mathbb{R})$. Then, for any $t, t_0 \in A$ we have

$$\begin{aligned}
& (\mathcal{V}y)(t) - (\mathcal{V}y)(t_0) \\
= \; & f(t) - f(t_0) + g(t, y(t)) \textstyle\int_{A(t)} [K(t, \tau, y(h(\tau))) \\
& -K(t_0, \tau, y(h(\tau)))]d\tau + [g(t, y(t)) - g(t_0, y(t_0))] \textstyle\int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau \\
& +g(t_0, y(t_0))[\textstyle\int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau - \int_{A(t_0)} K(t_0, \tau, y(h(\tau)))d\tau] \\
= \; & f(t) - f(t_0) + g(t, y(t)) \textstyle\int_{A(t)} [K(t, \tau, y(h(\tau))) \\
& -K(t_0, \tau, y(h(\tau)))]d\tau + [g(t, y(t)) - g(t_0, y(t_0))] \textstyle\int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau \\
& +g(t_0, y(t_0))[\textstyle\int_{A(t_0) \backslash A(t)} K(t_0, \tau, y(h(\tau)))d\tau - \int_{A(t) \backslash A(t_0)} K(t_0, \tau, y(h(\tau)))d\tau].
\end{aligned}$$

Hence, since $f$, $g$, $y$ and $h$ are continuous, $K$ is bounded and continuous, $\lim_{t \to t_0} \mu(A(t) \backslash A(t_0)) = 0$ and $\lim_{t \to t_0} \mu(A(t_0) \backslash A(t)) = 0$, follows $(\mathcal{V}y)(t) \to (\mathcal{V}y)(t_0)$ as $t \to t_0$. So $\mathcal{V}y$ is continuous in $t_0$. Therefore, $\mathcal{V} : \mathcal{C}(A, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$, as was to be shown. The proof that $(\mathcal{C}(A, \mathbb{R}), \mathcal{F})$ is a dynamic system is similar.

Define a dynamic system $(\mathcal{C}(A, \mathbb{R}), \mathcal{W})$ where $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{F}$.

Let $(\mathcal{C}(A, \mathbb{R}), \mathcal{W}^{[m]})$, $m \in \mathbb{N}$, be a (single-valued) dynamic system, that is $\mathcal{W}^{[m+1]} : \mathcal{C}(A, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$, $m \in \{0\} \cup \mathbb{N}$. Clearly, a sequence $(\mathcal{W}^{[m+1]} : m \in \{0\} \cup \mathbb{N})$ satisfies

$$(\mathcal{V}^{[m+1]}y)(t) = f(t) + g(t, (\mathcal{V}^{[m]}y)(t)) \textstyle\int_{A(t)} K(t, \tau, (\mathcal{V}^{[m]}y)(h(\tau)))d\tau$$

when $\mathcal{W} = \mathcal{V}$ and

$$(\mathcal{F}^{[m+1]}y)(t) = f(t) + g(t, (\mathcal{F}^{[m]}y)(t)) \textstyle\int_A K(t, \tau, (\mathcal{F}^{[m]}y)(h(\tau)))d\tau$$

when $\mathcal{W} = \mathcal{F}$, $t \in A$. For each $y^0 \in \mathcal{C}(A, \mathbb{R})$, a sequence $(\mathcal{W}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N}) \subset \mathcal{C}(A, \mathbb{R})$ of iterations starting at $y^0 \in \mathcal{C}(A, \mathbb{R})$ is well defined. Here $\mathcal{W}^{[0]} = I_{\mathcal{C}(A, \mathbb{R})}$ - identity on $\mathcal{C}(A, \mathbb{R})$.

Let $\mathcal{Y}_{\mathcal{W}}$ be a set such that $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{V}}$ or $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{F}}$, where $\mathcal{Y}_{\mathcal{V}}$ and $\mathcal{Y}_{\mathcal{F}}$ are the sets of all solutions $y \in \mathcal{C}(A, \mathbb{R})$ of equations (2.5) and (2.6), respectively.

(I.5) Define a Volterra quadratic integral equation

$$y(t) = f(t, y(t)) + g(t, y(t)) \textstyle\int_{A(t)} K(t, \tau, y(h(\tau)))d\tau, \; t \in A, \tag{2.7}$$

and a Fredholm quadratic integral equation

$$y(t) = f(t, y(t)) + g(t, y(t)) \textstyle\int_A K(t, \tau, y(h(\tau)))d\tau, \; t \in A, \tag{2.8}$$

in which $f \in \mathcal{C}(A \times \mathbb{R}, \mathbb{R})$, $g \in \mathcal{C}(A \times \mathbb{R}, \mathbb{R})$, $h \in \mathcal{C}(A, A)$ and $K \in \mathcal{BC}(A \times A \times \mathbb{R}, \mathbb{R})$, the maps $f$, $g$, $h$ and $K$ are given maps, and $y \in \mathcal{C}(A, \mathbb{R})$ is an unknown map to be determined.

We say that $\mathcal{V}$ is a Volterra operator on $\mathcal{C}(A, \mathbb{R})$ if $(\mathcal{V}y)(t) = f(t, y(t)) + g(t, y(t)) \int_{A(t)} K(t, \tau, y(h(\tau)))d\tau$, $y \in \mathcal{C}(A, \mathbb{R})$, $t \in A$. We say that $\mathcal{F}$ is a Fredholm operator on $\mathcal{C}(A, \mathbb{R})$ if $(\mathcal{F}y)(t) = f(t, y(t)) + g(t, y(t)) \int_A K(t, \tau, y(h(\tau)))d\tau$, $y \in \mathcal{C}(A, \mathbb{R})$, $t \in A$.

Clearly, $(\mathcal{C}(A, \mathbb{R}), \mathcal{V})$ and $(\mathcal{C}(A, \mathbb{R}), \mathcal{F})$ are (single-valued) dynamic systems on $\mathcal{C}(A, \mathbb{R})$. Indeed, let $y \in \mathcal{C}(A, \mathbb{R})$. Then, for any $t, t_0 \in A$ we have

$$\begin{aligned}
& (\mathcal{V}y)(t) - (\mathcal{V}y)(t_0) \\
= \; & f(t, y(t)) - f(t_0, y(t_0)) + g(t, y(t)) \textstyle\int_{A(t)} [K(t, \tau, y(h(\tau))) \\
& -K(t_0, \tau, y(h(\tau)))]d\tau + [g(t, y(t)) - g(t_0, y(t_0))] \textstyle\int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau \\
& +g(t_0, y(t_0))[\textstyle\int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau - \int_{A(t_0)} K(t_0, \tau, y(h(\tau)))d\tau] \\
= \; & f(t, y(t)) - f(t_0, y(t_0)) + g(t, y(t)) \textstyle\int_{A(t)} [K(t, \tau, y(h(\tau))) \\
& -K(t_0, \tau, y(h(\tau)))]d\tau + [g(t, y(t)) - g(t_0, y(t_0))] \textstyle\int_{A(t)} K(t_0, \tau, y(h(\tau)))d\tau \\
& +g(t_0, y(t_0))[\textstyle\int_{A(t_0) \setminus A(t)} K(t_0, \tau, y(h(\tau))d\tau - \int_{A(t) \setminus A(t_0)} K(t_0, \tau, y(h(\tau))d\tau].
\end{aligned}$$

Hence, since $f$, $g$, $y$ and $h$ are continuous, $K$ is bounded and continuous, $\lim_{t \to t_0} \mu(A(t) \setminus A(t_0)) = 0$ and $\lim_{t \to t_0} \mu(A(t_0) \setminus A(t)) = 0$, follows $(\mathcal{V}y)(t) \to (\mathcal{V}y)(t_0)$ as $t \to t_0$. So $\mathcal{V}y$ is continuous in $t_0$. Therefore, $\mathcal{V} : \mathcal{C}(A, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$, as was to be shown. The proof that $(\mathcal{C}(A, \mathbb{R}), \mathcal{F})$ is a dynamic system is similar.

Define a dynamic system $(\mathcal{C}(A, \mathbb{R}), \mathcal{W})$ where $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{F}$.

Let $(\mathcal{C}(A, \mathbb{R}), \mathcal{W}^{[m]})$, $m \in \mathbb{N}$, be a dynamic system, that is $\mathcal{W}^{[m+1]} : \mathcal{C}(A, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$, $m \in \{0\} \cup \mathbb{N}$. Here

$$(\mathcal{V}^{[m+1]}y)(t) = f(t, (\mathcal{V}^{[m]}y)(t)) + g(t, (\mathcal{V}^{[m]}y)(t)) \textstyle\int_{A(t)} K(t, \tau, (\mathcal{V}^{[m]}y)(h(\tau)))d\tau$$

when $\mathcal{W} = \mathcal{V}$ and

$$(\mathcal{F}^{[m+1]}y)(t) = f(t, (\mathcal{F}^{[m]}y)(t)) + g(t, (\mathcal{F}^{[m]}y)(t)) \textstyle\int_A K(t, \tau, (\mathcal{F}^{[m]}y)(h(\tau)))d\tau$$

when $\mathcal{W} = \mathcal{F}$, $t \in A$, $m \in \{0\} \cup \mathbb{N}$. For each $y^0 \in \mathcal{C}(A, \mathbb{R})$, a sequence $(\mathcal{W}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N}) \subset \mathcal{C}(A, \mathbb{R})$ of iterations starting at $y^0 \in \mathcal{C}(A, \mathbb{R})$ is well defined. Here $\mathcal{W}^{[0]} = I_{\mathcal{C}(A,\mathbb{R})}$ - identity on $\mathcal{C}(A, \mathbb{R})$.

Let $\mathcal{Y}_{\mathcal{W}}$ be a set such that $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{V}}$ or $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{F}}$, where $\mathcal{Y}_{\mathcal{V}}$ and $\mathcal{Y}_{\mathcal{F}}$ are the sets of all solutions $y \in \mathcal{C}(A, \mathbb{R})$ of equations (2.7) and (2.8), respectively.

(I.6) It is an important fact that (I.5) contains, as a special case, the quadratic integral equations of fractional order of the forms

$$y(t) = f(t, y(t)) + g(t, y(t)) \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t, \tau, y(h(\tau)))d\tau, \; t \in [0; a_1], \quad (2.9)$$

and

$$y(t) = f(t, y(t)) + g(t, y(t)) \int_0^{a_1} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t, \tau, y(h(\tau)))d\tau, \; t \in [0; a_1], \quad (2.10)$$

in which $a_1 > 0$, $\alpha > 0$, $f \in \mathcal{C}([0;a_1] \times \mathbb{R}, \mathbb{R})$, $g \in \mathcal{C}([0;a_1] \times \mathbb{R}, \mathbb{R})$, $h \in \mathcal{C}([0;a_1], [0;a_1])$ and $K \in \mathcal{BC}([0;a_1] \times [0;a_1] \times \mathbb{R}, \mathbb{R})$. The maps $f$, $g$, $h$ and $K$ are given maps, and $y \in \mathcal{C}([0;a_1], \mathbb{R})$ is an unknown map to be determined.

Observe that $(\mathcal{C}([0;a_1], \mathbb{R}), \mathcal{W}^{[m]})$, $m \in \mathbb{N}$, is a dynamic system, that is $\mathcal{W}^{[m+1]} : \mathcal{C}([0;a_1], \mathbb{R}) \to \mathcal{C}([0;a_1], \mathbb{R})$, $m \in \{0\} \cup \mathbb{N}$, where

$$\begin{aligned}
&(\mathcal{V}^{[m+1]}y)(t) \\
= \quad & f(t, (\mathcal{V}^{[m]}y)(t)) + g(t, (\mathcal{V}^{[m]}y)(t)) \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t, \tau, (\mathcal{V}^{[m]}y)(h(\tau))) d\tau
\end{aligned}$$

when $\mathcal{W} = \mathcal{V}$ and

$$\begin{aligned}
&(\mathcal{F}^{[m+1]}y)(t) \\
= \quad & f(t, (\mathcal{F}^{[m]}y)(t)) + g(t, (\mathcal{F}^{[m]}y)(t)) \int_0^{a_1} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t, \tau, (\mathcal{F}^{[m]}y)(h(\tau))) d\tau
\end{aligned}$$

when $\mathcal{W} = \mathcal{F}$, $t \in [0.a_1]$, $m \in \{0\} \cup \mathbb{N}$.

To prove this fact, first we see that

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \leq \frac{a_1^\alpha}{\Gamma(\alpha+1)}. \tag{2.11}$$

Next, if $y \in \mathcal{C}(A, \mathbb{R})$, $t, t_0 \in A$ and $\mathcal{W} = \mathcal{V}$, then

$$\begin{aligned}
&(\mathcal{V}y)(t) - (\mathcal{V}y)(t_0) \\
= \quad & f(t, y(t)) - f(t_0, y(t_0)) + g(t, y(t)) \int_{A(t)} [\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t, \tau, y(h(\tau))) \\
& - \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau)))] d\tau \\
& + [g(t, y(t)) - g(t_0, y(t_0))] \int_{A(t)} \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau))) d\tau \\
& + g(t_0, y(t_0)) [\int_{A(t)} \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau))) d\tau \\
& - \int_{A(t_0)} \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau))) d\tau] \\
= \quad & f(t, y(t)) - f(t_0, y(t_0)) + g(t, y(t)) \int_{A(t)} [\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t, \tau, y(h(\tau))) \\
& - \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau)))] d\tau \\
& + [g(t, y(t)) - g(t_0, y(t_0))] \int_{A(t)} \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau))) d\tau \\
& + g(t_0, y(t_0)) [\int_{A(t_0) \setminus A(t)} \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau))) d\tau \\
& - \int_{A(t) \setminus A(t_0)} \frac{(t_0-\tau)^{\alpha-1}}{\Gamma(\alpha)} K(t_0, \tau, y(h(\tau))) d\tau].
\end{aligned}$$



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Hence, it follows that $(\mathcal{V}y)(t) \to (\mathcal{V}y)(t_0)$ as $t \to t_0$, that $\mathcal{V}y$ is continuous in $t_0$. Therefore, $\mathcal{V} : \mathcal{C}(A, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$, as was to be shown. The proof that $(\mathcal{C}(A, \mathbb{R}), \mathcal{F})$ is a dynamic system is similar.

For each $y^0 \in \mathcal{C}([0; a_1], \mathbb{R})$, a sequence $(\mathcal{W}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N}) \subset \mathcal{C}([0; a_1], \mathbb{R})$ of iterations starting at $y^0 \in \mathcal{C}([0; a_1], \mathbb{R})$ is well defined. Here $\mathcal{W}^{[0]} = I_{\mathcal{C}([0;a_1],\mathbb{R})}$ - identity on $\mathcal{C}([0; a_1], \mathbb{R})$.

Let $\mathcal{Y}_{\mathcal{W}}$ be a set such that $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{V}}$ or $\mathcal{Y}_{\mathcal{W}} = \mathcal{Y}_{\mathcal{F}}$, where $\mathcal{Y}_{\mathcal{V}}$ and $\mathcal{Y}_{\mathcal{F}}$ are the sets of all solutions $y \in \mathcal{C}([0; a_1], \mathbb{R})$ of equations (2.9) and (2.10), respectively.

(II)

(II.1) $n = 1$ and $A \in \mathbb{A}_1$ is an arbitrary and fixed set; thus $A = [0; a_1]$, $a_1 > 0$ ($\mu(A) = a_1$). For $t \in A$, let $A(t)$ be defined by $A(t) = \{\tau : 0 \leq \tau \leq t\}$.

$(E, \mathcal{P}_{\mathcal{A}})$ is a locally convex space. $\mathcal{C}([0; a_1], E)$ denotes a family of all continuous maps $y : [0; a_1] \to E$.

Denote by $(\mathcal{C}([0; a_1], E), \mathcal{D}_{\mathcal{A}})$ a gauge space with a gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ defined on $\mathcal{C}([0; a_1], E)$ as follows:

$$\forall_{\alpha \in \mathcal{A}} \forall_{x,y \in \mathcal{C}([0;a_1],E)} \{D_\alpha(x, y) = \sup_{t \in [0;a_1]} e^{-\theta_\alpha t} P_\alpha[x(t) - y(t)]\}$$

where $\theta_\alpha > 0$, $\alpha \in \mathcal{A}$, are given. If, in particular, $E = \mathbb{R}$, then $\mathcal{D}_{\mathcal{A}} = \{D\}$ and $(\mathcal{C}([0; a_1], \mathbb{R}), D)$ is a gauge space with gauge $D$ defined by

$$\forall_{x,y \in \mathcal{C}([0;a_1],\mathbb{R})} \{D(x, y) = \sup_{t \in [0;a_1]} e^{-\theta t} |x(t) - y(t)|\}$$

where $\theta > 0$ is given. It is obvious that if necessary we can precisely determine values $\theta_\alpha > 0$, $\alpha \in \mathcal{A}$, and $\theta > 0$.

(II.2) A Volterra quadratic integral equation is of the form

$$y(t) = f(t, y(t)) + \int_0^t K(t, \tau, y(h(\tau)))d\tau,\ t \in [0; a_1], \tag{2.12}$$

$y \in \mathcal{C}([0; a_1], E)$. Here $f \in \mathcal{C}([0; a_1] \times E, E)$, $h \in \mathcal{C}([0; a_1], [0; a_1])$ and $K \in \mathcal{BC}([0; a_1] \times [0; a_1] \times E, E)$, the maps $f$, $h$ and $K$ are given maps, and $y \in \mathcal{C}([0; a_1], E)$ is an unknown map to be determined.

A dynamic system $(\mathcal{C}([0; a_1], E), \mathcal{V})$ is defined as follows $(\mathcal{V}y)(t) = f(t, y(t)) + \int_0^t K(t, \tau, y(h(\tau)))d\tau$, $t \in [0; a_1]$. A sequence $(\mathcal{V}^{[m+1]} : m \in \{0\} \cup \mathbb{N})$ satisfies $(\mathcal{V}^{[m+1]}y)(t) = f(t, (\mathcal{V}^{[m]}y)(t)) + \int_0^t K(t, \tau, (\mathcal{V}^{[m]}y)(h(\tau)))d\tau$, where $t \in [0; a_1]$ and $y \in \mathcal{C}([0; a_1], E)$.

$\mathcal{Y}_{\mathcal{V}}$ is a set of all solutions $y \in \mathcal{C}([0; a_1], E)$ of equation (2.12).

(II.3) A Volterra integral equation is of the form

$$y(t) = f(t) + g(t)\int_0^t K(t, \tau, y(h(\tau)))d\tau,\ t \in [0; a_1],\ y \in \mathcal{C}([0; a_1], E). \tag{2.13}$$

Here $f \in \mathcal{C}([0; a_1], E)$, $g \in \mathcal{C}([0; a_1], \mathbb{R})$ with $\max_{t \in [0;a_1]} |g(t)| = \lambda > 0$, $h \in \mathcal{C}([0; a_1], [0; a_1])$, $K \in \mathcal{BC}([0; a_1] \times [0; a_1] \times E, E)$, the maps $f$, $g$, $h$ and $K$ are given maps, and $y \in \mathcal{C}([0; a_1], E)$ is an unknown map to be determined.

A dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is defined as follows

$$(\mathcal{V}y)(t) = f(t) + g(t)\int_0^t K(t,\tau,y(h(\tau)))d\tau, t\in[0;a_1].$$

A sequence $(\mathcal{V}^{[m+1]}: m\in\{0\}\cup\mathbb{N})$ satisfies

$$(\mathcal{V}^{[m+1]}y)(t) = f(t) + g(t)\int_0^t K(t,\tau,(\mathcal{V}^{[m]}y)(h(\tau)))d\tau,$$

where $t\in[0;a_1]$ and $y\in\mathcal{C}([0;a_1],E)$. $\mathcal{Y}_\mathcal{V}$ is a set of all solutions $y\in\mathcal{C}([0;a_1],E)$ of equation (2.13).

(II.4) Define a Volterra quadratic integral equation

$$y(t) = f(t) + g(t,y(t))\int_0^t K(t,\tau,y(h(\tau)))d\tau,\ t\in[0;a_1], \tag{2.14}$$

in which $f\in\mathcal{C}([0;a_1],\mathbb{R})$, $g\in\mathcal{C}([0;a_1]\times\mathbb{R},\mathbb{R})$, $h\in\mathcal{C}([0;a_1],[0;a_1])$ and $K\in\mathcal{BC}([0;a_1]\times[0;a_1]\times\mathbb{R},\mathbb{R})$, the maps $f,g,h$ and $K$ are given maps, and $y\in\mathcal{C}([0;a_1],\mathbb{R})$ is an unknown map to be determined.

We say that $\mathcal{V}$ is a Volterra operator on $\mathcal{C}([0;a_1],\mathbb{R})$ if

$$(\mathcal{V}y)(t) = f(t) + g(t,y(t))\int_0^t K(t,\tau,y(h(\tau)))d\tau,$$

$y\in\mathcal{C}([0;a_1],\mathbb{R})$, $t\in[0;a_1]$. Clearly, $(\mathcal{C}([0;a_1],\mathbb{R}),\mathcal{V})$ is a dynamic system on $\mathcal{C}([0;a_1],\mathbb{R})$. Let $(\mathcal{C}([0;a_1],\mathbb{R}),\mathcal{V}^{[m]})$, $m\in\mathbb{N}$, be a dynamic system, that is $\mathcal{V}:\mathcal{C}([0;a_1],\mathbb{R})\to\mathcal{C}([0;a_1],\mathbb{R})$, $m\in\{0\}\cup\mathbb{N}$. Clearly, a sequence $(\mathcal{V}^{[m+1]}: m\in\{0\}\cup\mathbb{N})$ satisfies

$$(\mathcal{V}^{[m+1]}y)(t) = f(t) + g(t,(\mathcal{V}^{[m]}y)(t))\int_0^t K(t,\tau,(\mathcal{V}^{[m]}y)(h(\tau)))d\tau$$

$t\in[0;a_1]$. For each $y^0\in\mathcal{C}([0;a_1],\mathbb{R})$, a sequence $(\mathcal{V}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})\subset\mathcal{C}([0;a_1],\mathbb{R})$ of iterations starting at $y^0\in\mathcal{C}([0;a_1],\mathbb{R})$ is well defined. Here $\mathcal{V}^{[0]} = I_{\mathcal{C}([0;a_1],\mathbb{R})}$ - identity on $\mathcal{C}([0;a_1],\mathbb{R})$.

$\mathcal{Y}_\mathcal{V}$ is a set of all solutions $y\in\mathcal{C}([0;a_1],\mathbb{R})$ of equation (2.14).

(II.5) Define a Volterra quadratic integral equation

$$y(t) = f(t,y(t)) + g(t,y(t))\int_0^t K(t,\tau,y(h(\tau)))d\tau,\ t\in A, \tag{2.15}$$

in which $f\in\mathcal{C}([0;a_1]\times\mathbb{R},\mathbb{R})$, $g\in\mathcal{C}([0;a_1]\times\mathbb{R},\mathbb{R})$, $h\in\mathcal{C}([0;a_1],[0;a_1])$ and $K\in\mathcal{BC}([0;a_1]\times[0;a_1]\times\mathbb{R},\mathbb{R})$, the maps $f$, $g$, $h$ and $K$ are given maps, and $y\in\mathcal{C}([0;a_1],\mathbb{R})$ is an unknown map to be determined.

We say that $\mathcal{V}$ is a Volterra operator on $\mathcal{C}([0;a_1],\mathbb{R})$ if $(\mathcal{V}y)(t) = f(t,y(t))$ $+g(t,y(t))\int_{A(t)} K(t,\tau,y(h(\tau)))d\tau$, $y\in\mathcal{C}([0;a_1],\mathbb{R})$, $t\in[0;a_1]$. Clearly, $(\mathcal{C}(,\mathbb{R}),\mathcal{V})$ is a dynamic system on $\mathcal{C}([0;a_1],\mathbb{R})$. Let $(\mathcal{C}([0;a_1],\mathbb{R}),\mathcal{V}^{[m]})$, $m\in\mathbb{N}$, be a dynamic system, that is $\mathcal{V}^{[m+1]}:\mathcal{C}([0;a_1],\mathbb{R})\to\mathcal{C}([0;a_1],\mathbb{R})$, $m\in\{0\}\cup\mathbb{N}$. Clearly, a sequence $(\mathcal{V}^{[m+1]}: m\in\{0\}\cup\mathbb{N})$ satisfies $(\mathcal{V}^{[m+1]}y)(t) = f(t,(\mathcal{V}^{[m]}y)(t)) + g(t,(\mathcal{V}^{[m]}y)(t))\int_0^t K(t,\tau,(\mathcal{V}^{[m]}y)(h(\tau)))d\tau$ for $t\in[0;a_1]$. For each $y^0\in\mathcal{C}([0;a_1],\mathbb{R})$, a sequence $(\mathcal{V}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})\subset\mathcal{C}([0;a_1],\mathbb{R})$ of iterations starting at $y^0\in\mathcal{C}([0;a_1],\mathbb{R})$ is well defined. Here $\mathcal{V}^{[0]} = I_{\mathcal{C}([0;a_1],\mathbb{R})}$ - identity on $\mathcal{C}([0;a_1],\mathbb{R})$.

Let $\mathcal{Y}_\mathcal{V}$ be a set of all solutions $y\in\mathcal{C}([0;a_1],\mathbb{R})$ of equation (2.15).

**Remark 2.1 .** We record, for a later use, some observations concerning $(E, \mathcal{P}_{\mathcal{A}})$ and $(\mathcal{C}(A,E), \mathcal{D}_{\mathcal{A}})$:

(a) If $(E, \mathcal{P}_{\mathcal{A}})$ sequentially complete, then $(\mathcal{C}(A,E), \mathcal{D}_{\mathcal{A}})$ is sequentially complete.

(b) If $(E, \mathcal{P}_{\mathcal{A}})$ is Hausdorff, then $(\mathcal{C}(A,E), \mathcal{D}_{\mathcal{A}})$ is Hausdorff.

**Remark 2.2 .** Let $\mathcal{W} : \mathcal{C}(A,E) \to \mathcal{C}(A,E)$, $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{F}$, $A \in \mathbb{A}_n$, $n \in \mathbb{N}$, be as above.

(a) Clearly, any fixed point $y \in \mathcal{C}(A,E)$ of dynamic system $(\mathcal{C}(A,E), \mathcal{W})$, i.e. any point of the set $Fix_{\mathcal{C}(A,E)}(\mathcal{W}) = \{y \in \mathcal{C}(A,E) : y = \mathcal{W}y\}$, is contained in $\mathcal{Y}_{\mathcal{W}}$. Moreover, any periodic point $y \in \mathcal{C}(A,E)$ of $(\mathcal{C}(A,E), \mathcal{W})$, i.e. any point of the set $Per_{\mathcal{C}(A,E)}(\mathcal{W}) = \{y \in \mathcal{C}(A,E) : y = \mathcal{W}^{[q]}y$ for some $q \in \mathbb{N}\}$, is also contained in $\mathcal{Y}_{\mathcal{W}}$.

(b) In general, the dynamic systems $(\mathcal{C}(A,E), \mathcal{W})$ are discontinous in $\mathcal{C}(A,E)$ and do not have fixed points in $\mathcal{C}(A,E)$, but, by some assumptions presented in this paper, $\mathcal{W}$ have periodic points in $\mathcal{C}(A,E)$ and thus the set $\mathcal{Y}_{\mathcal{W}} \subset \mathcal{C}(A,E)$ is nonempty. Moreover, with these assumptions, the sequences $(\mathcal{W}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ starting in $y^0 \in \mathcal{C}(A,E)$ are $\mathcal{D}_{\mathcal{A}}$-convergent in appropiate gauge spaces $(\mathcal{C}(A,E), \mathcal{D}_{\mathcal{A}})$ to some $y \in \mathcal{Y}_{\mathcal{W}}$.

In this context, we provide in Section 3 new tools which imply strategies to obtain in Sections 4 and 5 the new existence, uniqueness and convergence results concerning solutions of integral equations (2.1)-(2.10) and (2.12)-(2.15).

More precisely, Section 4 presents the results concerning Volterra and Fredholm types integral equations (2.1)-(2.10) by using notations and definitions (I), and thus when $A \in \mathbb{A}_n$, $n \in \mathbb{N}$, is arbitrary and when gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha, \alpha \in \mathcal{A}\}$ is defined in (I.1). In addition, when we arrange a deal that $A \in \mathbb{A}_1$ is of form $A = [0; a_1]$, $a_1 > 0$, and a gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha, \alpha \in \mathcal{A}\}$ is defined in (II.1), then, using (II) we defer this case of study of Volterra type integral equations (2.12)-(2.15) to Section 5. Proofs of our results are given in Section 6. Examples are to be found in Section 7.

# 3 Convergence, periodic point and fixed point theorems for Leader type contractions in gauge spaces

Let $(X, \mathcal{D}_{\mathcal{A}})$ be a gauge space (see Definition 2.2). The goal of this section is to exhibit clearly the breadth and importance of the class of Leader type contractions $T : X \to X$ in $(X, \mathcal{D}_{\mathcal{A}})$ and the general convergence, periodic point and fixed point theorems for such contractions. The results hold even in the case when gauges $\mathcal{D}_{\mathcal{A}}$ are not separating on $X$, $(X, \mathcal{D}_{\mathcal{A}})$ are not sequentially complete, $T$ do not have complete graphs and $T$ are not continuous.

We will apply the following definitions.

**Definition 3.1 .** Let $(X, \mathcal{D}_{\mathcal{A}})$ be a gauge space with a gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$, and let $(X, T)$ be a single-valued dynamic system, $T : X \to X$.

(A) $(X, T)$ is said to be a $\mathcal{D}_{\mathcal{A}}$-*admissible in* $w^0 \in X$ if a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$, which is $\mathcal{D}_{\mathcal{A}}$-*sequence* (i.e. $\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} \sup_{n > m} D_\alpha(T^{[m]}(w^0), T^{[n]}(w^0)) = 0\}$), is $\mathcal{D}_{\mathcal{A}}$-converging in $X$ (i.e. $\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} D_\alpha(T^{[m]}(w^0), w) = 0\}$ for some $w \in X$).

(B) Let $M \in 2^X$. $(X, T)$ is said to be a $\mathcal{D}_{\mathcal{A}}$-*admissible on* $M$ iff $(X, T)$ is a $\mathcal{D}_{\mathcal{A}}$-admissible in each $w^0 \in M$.

**Definition 3.2 .** Let $(X, \mathcal{D}_{\mathcal{A}})$ be a gauge space with a gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$. Suppose $(X, T)$ is a single-valued dynamic system, $T : X \to X$, and let $q \in \mathbb{N}$.

(A) Let $w^0 \in X$. We say that a single-valued dynamic system $(X, T^{[q]})$ is a $\mathcal{D}_{\mathcal{A}}$-*closed in* $w^0$, if in the case when a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-converging in $X$ and contains two subsequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ satisfying $\forall_{m \in \mathbb{N}} \{v_m = T^{[q]}(u_m)\}$, then we have that $\exists_{w \in X} \{w = T^{[q]}(w)\}$.

(B) Let $M \in 2^X$. A single-valued dynamic system $(X, T^{[q]})$ is said to be a $\mathcal{D}_{\mathcal{A}}$-*closed on* $M$, if $(X, T^{[q]})$ is a $\mathcal{D}_{\mathcal{A}}$-closed in each $w^0 \in M$.

For details concerning closed maps, see [3, 19].

Useful tools in our considerations contained in Sections 4-6 will be the following two theorems which follow from the general results presented in [29, 30].

**Theorem 3.1** . *Let $(X, \mathcal{D}_{\mathcal{A}})$ be a gauge space with a gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$, and let $(X, T)$ be a single-valued dynamic system, $T : X \to X$. Assume that:*

*(a) $(X, T)$ is $\mathcal{D}_{\mathcal{A}}$-admissible on $X$.*

*(b) $(X, T)$ satisfies*

$$\begin{cases} \forall_{\alpha \in \mathcal{A}} \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{x, y \in X} \forall_{s, l \in \mathbb{N}} \{D_\alpha(T^{[s]}(x), T^{[l]}(y)) \\ < \varepsilon + \eta \Rightarrow [D_\alpha(T^{[s+r]}(x), T^{[l+r]}(y)) < \varepsilon\}. \end{cases} \tag{3.1}$$

*Then the following hold:*

*(A)* ***Convergence of all Picard iterations.*** *For each $w^0 \in X$, there exists $w \in X$ such that a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $w$.*

*(B)* ***Existence of periodic points of all Picard iterations.*** *If there exists $q \in \mathbb{N}$ such that the single-valued dynamic system $(X, T^{[q]})$ is $\mathcal{D}_{\mathcal{A}}$-closed on $X$, then $Fix_X(T^{[q]}) \neq \varnothing$. Moreover, for each $w^0 \in X$, there exists $w \in Fix_X(T^{[q]})$ such that a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $w$.*

*(C)* ***Existence of unique fixed point of all Picard iterations.*** *If $(X, T)$ is $\mathcal{D}_{\mathcal{A}}$-closed on $X$ and the family $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ is separating on $X$, then $\exists_{w \in X} \{Fix_X(T) = \{w\}\}$. Moreover, for each $w^0 \in X$, a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $w$.*

**Theorem 3.2 .** *Let $(X, \mathcal{D}_\mathcal{A})$ be a gauge space with a gauge $\mathcal{D}_\mathcal{A} = \{D_\alpha : \alpha \in \mathcal{A}\}$, and let $(X, T)$ be a single-valued dynamic system, $T : X \to X$. Assume that:*

*(a) There exists $M \in 2^X$ such that $(X, T)$ is $\mathcal{D}_\mathcal{A}$-admissible on $M$.*

*(b) $(X, T)$ and $M$ satisfy*

$$\left\{ \begin{array}{l} \forall_{\alpha \in \mathcal{A}} \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{w^0 \in M} \forall_{s,l \in \mathbb{N}} \{ D_\alpha(T^{[s]}(w^0), T^{[l]}(w^0)) \\ < \varepsilon + \eta \Rightarrow [D_\alpha(T^{[s+r]}(w^0), T^{[l+r]}(w^0)) < \varepsilon \}. \end{array} \right. \tag{3.2}$$

*Then the following hold:*

*(A)* ***Convergence of Picard iterations.*** *For each $w^0 \in M$, there exists $w \in X$ such that $\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} D_\alpha(T^{[m]}(w^0), w) = 0\}$.*

*(B)* ***Existence of periodic points of Picard iterations.*** *If there exist $q \in \mathbb{N}$ and $w^0 \in M$ such that the single-valued dynamic system $(X, T^{[q]})$ is $\mathcal{D}_\mathcal{A}$-closed in $w^0$, then $Fix_X(T^{[q]}) \neq \varnothing$. Moreover, there exists $w \in Fix_X(T^{[q]})$ such that a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_\mathcal{A}$- convergent to $w$.*

**Remark 3.1 .** If $(X, \mathcal{D}_\mathcal{A})$ is sequentially complete, then $(X, T)$ is $\mathcal{D}_\mathcal{A}$-admissible on each $M \in 2^X$, i.e. then hypothesis (a) of Theorems 3.1 and 3.2 automatically holds.

# 4 Convergence, existence and uniqueness theorems for integral equations of Volterra and Fredholm type in locally convex spaces - case (I)

We introduce the following definitions.

**Definition 4.1 .** Let $(E, \mathcal{P}_\mathcal{A})$ be a locally convex space with the topology defined by the family $\mathcal{P}_\mathcal{A} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}(A, E), \mathcal{D}_\mathcal{A})$ be such as in (I.1). Assume that (I.2) or (I.3) or (I.4) or (I.5) or (I.6) is satisfied.

(A) We say that a single-valued dynamic system $(\mathcal{C}(A, E), \mathcal{W})$ is a $\mathcal{D}_\mathcal{A}$-*admissible in* $y^0 \in \mathcal{C}(A, E)$ if a sequence $(\mathcal{W}^{[m]} y^0 : m \in \{0\} \cup \mathbb{N})$, which is $\mathcal{D}_\mathcal{A}$-*sequence* (i.e. $\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} \sup_{n > m} D_\alpha(\mathcal{W}^{[m]} y^0, \mathcal{W}^{[n]} y^0) = 0\}$), is $\mathcal{D}_\mathcal{A}$-converging in $\mathcal{C}(A, E)$ (i.e. $\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} D_\alpha(\mathcal{W}^{[m]} y^0, y) = 0\}$ for some $y \in \mathcal{C}(A, E)$).

(B) Let $M \in 2^{\mathcal{C}(A,E)}$. $(\mathcal{C}(A, E), \mathcal{W})$ is said to be a $\mathcal{D}_\mathcal{A}$-*admissible on* $M$ iff $(\mathcal{C}(A, E), \mathcal{W})$ is a $\mathcal{D}_\mathcal{A}$-admissible in each $y^0 \in M$.

**Remark 4.1 .** If $(E, \mathcal{P}_\mathcal{A})$ is sequentially complete, then $(\mathcal{C}(A, E), \mathcal{W})$ is $\mathcal{D}_\mathcal{A}$-admissible on each $M \in 2^{\mathcal{C}(A,E)}$.

**Definition 4.2 .** Let $(E, \mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}(A, E), \mathcal{D}_{\mathcal{A}})$ be such as in (I.1). Assume that (I.2) or (I.3) or (I.4) or (I.5) or (I.6) is satisfied. Let $q \in \mathbb{N}$.

(A) We say that a single-valued dynamic system $(\mathcal{C}(A, E), \mathcal{W}^{[q]})$ is a $\mathcal{D}_{\mathcal{A}}$-*closed in* $y^0 \in \mathcal{C}(A, E)$, if in the case when a sequence $(\mathcal{W}^{[m]} y^0 : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- converging in $\mathcal{C}(A, E)$ and contains two subsequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ satisfying $\forall_{m \in \mathbb{N}} \{v_m = \mathcal{W}^{[q]} u_m\}$, then we have that $\exists_{y \in \mathcal{C}(A,E)} \{y = \mathcal{W}^{[q]} y\}$.

(B) Let $M \in 2^{\mathcal{C}(A,E)}$. A single-valued dynamic system $(\mathcal{C}(A, E), \mathcal{W}^{[q]})$ is said to be a $\mathcal{D}_{\mathcal{A}}$-*closed on* $M$, if $(\mathcal{C}(A, E), \mathcal{W}^{[q]})$ is a $\mathcal{D}_{\mathcal{A}}$-closed in each $y^0 \in M$.

We prove the following theorems.

**Theorem 4.1 .** *Let $(E, \mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}(A, E), \mathcal{D}_{\mathcal{A}})$ be such as in (I.1). Suppose that $(\mathcal{C}(A, E), \mathcal{W})$ is $\mathcal{D}_{\mathcal{A}}$-admissible on $\mathcal{C}(A, E)$ and assume, moreover, that $(\mathcal{C}(A, E), \mathcal{W})$ satisfies one of the following conditions (i)-(vii):*

*(i) (I.2) or (I.3) is satisfied and, in addition, one of the following conditions holds:*

$$\left\{ \begin{array}{l} \forall_{\alpha \in \mathcal{A}} \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{x, y \in \mathcal{C}(A,E)} \forall_{s, l \in \mathbb{N}} \{ D_\alpha(\mathcal{W}^{[s]} x, \mathcal{W}^{[l]} y) \\ < \varepsilon + \eta \Rightarrow D_\alpha(\mathcal{W}^{[s+r]} x, \mathcal{W}^{[l+r]} y) < \varepsilon \}, \end{array} \right. \tag{4.1}$$

$$\forall_{\alpha \in \mathcal{A}} \forall_{x, y \in \mathcal{C}(A,E)} \forall_{s, l \in \mathbb{N}} \{ \lim_{r \to \infty} D_\alpha(\mathcal{W}^{[s+r]} x, \mathcal{W}^{[l+r]} y) = 0 \}. \tag{4.2}$$

*(ii) (I.4) or (I.5) is satisfied ($E = \mathbb{R}$, $\mathcal{D}_{\mathcal{A}} = \{D\}$) and, in addition, one of the following conditions holds:*

$$\left\{ \begin{array}{l} \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{x, y \in \mathcal{C}(A,\mathbb{R})} \forall_{s, l \in \mathbb{N}} \{ D(\mathcal{W}^{[s]} x, \mathcal{W}^{[l]} y) \\ < \varepsilon + \eta \Rightarrow D(\mathcal{W}^{[s+r]} x, \mathcal{W}^{[l+r]} y) < \varepsilon \}, \end{array} \right. \tag{4.3}$$

$$\forall_{x, y \in \mathcal{C}(A,\mathbb{R})} \forall_{s, l \in \mathbb{N}} \{ \lim_{r \to \infty} D(\mathcal{W}^{[s+r]} x, \mathcal{W}^{[l+r]} y) = 0 \}. \tag{4.4}$$

*(iii) (I.6) is satisfied ($E = \mathbb{R}$, $\mathcal{D}_{\mathcal{A}} = \{D\}$) and, in addition, one of the following conditions holds:*

$$\left\{ \begin{array}{l} \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{x, y \in \mathcal{C}([0;a_1],\mathbb{R})} \forall_{s, l \in \mathbb{N}} \{ D(\mathcal{W}^{[s]} x, \mathcal{W}^{[l]} y) \\ < \varepsilon + \eta \Rightarrow D(\mathcal{W}^{[s+r]} x, \mathcal{W}^{[l+r]} y) < \varepsilon \}, \end{array} \right. \tag{4.5}$$

$$\forall_{x, y \in \mathcal{C}([0;a_1],\mathbb{R})} \forall_{s, l \in \mathbb{N}} \{ \lim_{r \to \infty} D(\mathcal{W}^{[s+r]} x, \mathcal{W}^{[l+r]} y) = 0 \}. \tag{4.6}$$

*(iv) (I.2) is satisfied,*

$$\left\{ \begin{array}{l} \forall_{\alpha \in \mathcal{A}} \exists_{L_\alpha \in [0;\infty)} \forall_{t \in A} \forall_{x, y \in \mathcal{C}(A,E)} \{ P_\alpha[f(t, x(t)) \\ - f(t, y(t))] \leq L_\alpha D_\alpha(x, y) \} \end{array} \right. \tag{4.7}$$

*and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,E)}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A}\{D_\alpha(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\\ <\varepsilon+\eta\Longrightarrow L_\alpha D_\alpha(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y)\\ +\mu(A)P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))]\\ \le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\},\end{array}\right. \tag{4.8}$$

$$\left\{\begin{array}{l}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,E)}\forall_{s,l\in\mathbb{N}}\{L_\alpha D_\alpha(\mathcal{W}^{[s-1]}x,\mathcal{W}^{[l-1]}y)\\ +\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y)(\beta))]<\varepsilon+\eta\\ \Rightarrow L_\alpha D_\alpha(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y)\\ +\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))]<\varepsilon\}.\end{array}\right. \tag{4.9}$$

*(v) (I.3) is satisfied and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,E)}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A}\\ \{D_\alpha(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)<\varepsilon+\eta\\ \Longrightarrow\lambda\mu(A)P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\},\end{array}\right. \tag{4.10}$$

$$\left\{\begin{array}{l}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,E)}\forall_{s,l\in\mathbb{N}}\\ \{\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y)(\beta))]<\varepsilon+\eta\\ \Rightarrow\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))]<\varepsilon\}.\end{array}\right. \tag{4.11}$$

*(vi) (I.4) is satisfied* ($E=\mathbb{R}$, $\mathcal{D}_\mathcal{A}=\{D\}$),

$$\left\{\begin{array}{l}\exists_{Q\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{|g(t,x(t))-g(t,y(t))|\le QD(x,y)\},\\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in A}\forall_{x\in\mathcal{C}(A,\mathbb{R})}\{|K(t,\tau,x(\beta))|\le\kappa\},\\ \exists_{\lambda\in[0;\infty)}\forall_{t\in A}\forall_{x\in\mathcal{C}(A,\mathbb{R})}\{|g(t,x(t))|\le\lambda\}\end{array}\right. \tag{4.12}$$

*and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A}\\ \{D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)<\varepsilon+\eta\Longrightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y)\\ +\lambda\mu(A)|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|\le\frac{\varepsilon}{(\varepsilon+\eta)}D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\},\end{array}\right. \tag{4.13}$$

$$\left\{\begin{array}{l}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,E)}\forall_{s,l\in\mathbb{N}}\{Q\kappa D(\mathcal{W}^{[s-1]}x,\mathcal{W}^{[l-1]}y)\\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y)(\beta))|<\varepsilon+\eta\\ \Rightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y)\\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))\\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|<\varepsilon+\eta\}.\end{array}\right. \tag{4.14}$$



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*(vii) (I.5) is satisfied* ($E=\mathbb{R}$, $\mathcal{D}_{\mathcal{A}}=\{D\}$),

$$\left\{\begin{array}{l} \exists_{L\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{|f(t,x(t))-f(t,y(t))|\leq LD(x,y)\}, \\ \exists_{Q\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{|g(t,x(t))-g(t,y(t))|\leq QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in A}\forall_{x\in\mathcal{C}(A,\mathbb{R})}\{|K(t,\tau,x(\beta))|\leq\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in A}\forall_{x\in\mathcal{C}(A,\mathbb{R})}\{|g(t,x(t))|\leq\lambda\} \end{array}\right. \tag{4.15}$$

*and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A} \\ \{D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)<\varepsilon+\eta \\ \Longrightarrow (L+Q\kappa)D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y)+ \\ \lambda\mu(A)\left|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))\right| \\ \leq\frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\}, \end{array}\right. \tag{4.16}$$

$$\left\{\begin{array}{l} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}} \\ \{(L+Q\kappa)D(\mathcal{W}^{[s-1]}x,\mathcal{W}^{[l-1]}y) \\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y)(\beta))<\varepsilon+\eta \\ \Rightarrow (L+Q\kappa)D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y) \\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|<\varepsilon\}. \end{array}\right. \tag{4.17}$$

*Then the following hold:*

*(A)* ***Convergence property.*** *For each $y^0\in\mathcal{C}(A,E)$, there exists $y\in\mathcal{C}(A,E)$ such that a sequence $(\mathcal{W}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

*(B)* ***Existence of solutions and convergence property.*** *If there exists $q\in\mathbb{N}$ such that the single-valued dynamic system $(\mathcal{C}(A,E),\ \mathcal{W}^{[q]})$ is $\mathcal{D}_{\mathcal{A}}$-closed on $\mathcal{C}(A,E)$, then $\varnothing\neq Fix_{\mathcal{C}(A,E)}(\mathcal{W}^{[q]})\subset\mathcal{Y}_{\mathcal{W}}$. Moreover, for each $y^0\in\mathcal{C}(A,E)$, there exists $y\in Fix_{\mathcal{C}(A,E)}(\mathcal{W}^{[q]})$ such that a sequence $(\mathcal{W}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

*(C)* ***Existence of unique solution and convergence property.*** *If a single-valued dynamic system $(\mathcal{C}(A,E),\mathcal{W})$ is $\mathcal{D}_{\mathcal{A}}$-closed on $\mathcal{C}(A,E)$ and the family $\mathcal{D}_{\mathcal{A}}=\{D_\alpha:\alpha\in\mathcal{A}\}$ is separating on $\mathcal{C}(A,E)$, then there exists $y\in\mathcal{C}(A,E)$ such that $\mathcal{Y}_{\mathcal{W}}=Fix_{\mathcal{C}(A,E)}(\mathcal{W})=\{y\}$. Moreover, for each $y^0\in\mathcal{C}(A,E)$, a sequence $(\mathcal{W}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

**Theorem 4.2 .** *Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}(A,E),\mathcal{D}_{\mathcal{A}})$ be such as in (I.1). Suppose that there exists $M\in 2^{\mathcal{C}(A,E)}$ such that $(\mathcal{C}(A,E),\mathcal{W})$ is $\mathcal{D}_{\mathcal{A}}$-admissible on $M$ and assume that $(\mathcal{C}(A,E),\mathcal{W})$ and $M$ satisfy one of the following conditions (i)-(vii):*

*(i) (I.2) or (I.3) is satisfied, $M\in 2^{\mathcal{C}(A,E)}$ and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D_\alpha(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0) \\ <\varepsilon+\eta\Rightarrow D_\alpha(\mathcal{W}^{[s+r]}y^0,\mathcal{W}^{[l+r]}y^0)<\varepsilon\}, \end{array}\right. \tag{4.18}$$

$$\forall_{\alpha\in\mathcal{A}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty} D_\alpha(\mathcal{W}^{[s+r]}y^0,\mathcal{W}^{[l+r]}y^0)=0\}. \tag{4.19}$$

*(ii) (I.4) or (I.5) is satisfied ($E=\mathbb{R}$, $\mathcal{D}_\mathcal{A}=\{D\}$), $M\in 2^{\mathcal{C}(A,\mathbb{R})}$ and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0) \\ <\varepsilon+\eta\Rightarrow D(\mathcal{W}^{[s+r]}y^0,\mathcal{W}^{[l+r]}y^0)<\varepsilon\}, \end{cases} \tag{4.20}$$

$$\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty} D(\mathcal{W}^{[s+r]}y^0,\mathcal{W}^{[l+r]}y^0)=0\}. \tag{4.21}$$

*(iii) (I.6) is satisfied ($E=\mathbb{R}$, $\mathcal{D}_\mathcal{A}=\{D\}$), $M\in 2^{\mathcal{C}([0;a_1],\mathbb{R})}$ and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0) \\ <\varepsilon+\eta\Rightarrow D(\mathcal{W}^{[s+r]}y^0,\mathcal{W}^{[l+r]}y^0)<\varepsilon\}, \end{cases} \tag{4.22}$$

$$\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty} D(\mathcal{W}^{[s+r]}y^0,\mathcal{W}^{[l+r]}y^0)=0\}. \tag{4.23}$$

*(iv) (I.2) is satisfied, $M\in 2^{\mathcal{C}(A,E)}$, there exists $\{L_\alpha\}_{\alpha\in\mathcal{A}}\in[0;\infty)^\mathcal{A}$ such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in A}\forall_{x,y\in M}\{P_\alpha[f(t,x(t))-f(t,y(t))]\le L_\alpha D_\alpha(x,y)\} \tag{4.24}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A} \\ \{D_\alpha(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow L_\alpha D_\alpha(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\mu(A)P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)\}, \end{cases} \tag{4.25}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{L_\alpha D_\alpha(\mathcal{W}^{[s-1]}y^0,\mathcal{W}^{[l-1]}y^0) \\ +\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s-1]}y^0)(\beta)) \\ -K(t,\tau,(\mathcal{W}^{[l-1]}y^0)(\beta))]<\varepsilon+\eta \\ \Rightarrow L_\alpha D_\alpha(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))]<\varepsilon\}. \end{cases} \tag{4.26}$$

*(v) (I.3) is satisfied, $M\in 2^{\mathcal{C}(A,E)}$ and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A} \\ \{D_\alpha(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow \lambda\mu(A)P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)\}, \end{cases} \tag{4.27}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}} \\ \{\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s-1]}y^0)(\beta)) \\ -K(t,\tau,(\mathcal{W}^{[l-1]}y^0)(\beta))]<\varepsilon+\eta \\ \Rightarrow \lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))]<\varepsilon\}. \end{cases} \tag{4.28}$$



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*(vi) (I.4) is satisfied* ($E=\mathbb{R}$, $\mathcal{D}_{\mathcal{A}}=\{D\}$), $M\in 2^{\mathcal{C}(A,\mathbb{R})}$,

$$\begin{cases} \exists_{Q\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in M}\{|g(t,x(t))-g(t,y(t))|\le QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in A}\forall_{x\in M}\{|K(t,\tau,x(\beta))|\le\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in A}\forall_{x\in M}\{|g(t,x(t))|\le\lambda\} \end{cases} \tag{4.29}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A}\{D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0) \\ <\varepsilon+\eta\Longrightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\lambda\mu(A)|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))|\le\frac{\varepsilon}{(\varepsilon+\eta)}D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)\}, \end{cases} \tag{4.30}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{Q\kappa D(\mathcal{W}^{[s-1]}y^0,\mathcal{W}^{[l-1]}y^0) \\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y^0)(\beta))|<\varepsilon+\eta \\ \Rightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))|<\varepsilon+\eta\}. \end{cases} \tag{4.31}$$

*(vii) (I.5) is satisfied* ($E=\mathbb{R}$, $\mathcal{D}_{\mathcal{A}}=\{D\}$), $M\in 2^{\mathcal{C}(A,\mathbb{R})}$,

$$\begin{cases} \exists_{L\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in M}\{|f(t,x(t))-f(t,y(t))|\le LD(x,y)\}, \\ \exists_{Q\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in M}\{|g(t,x(t))-g(t,y(t))|\le QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in A}\forall_{x\in M}\{|K(t,\tau,x(\beta))|\le\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in A}\forall_{x\in M}\{|g(t,x(t))|\le\lambda\} \end{cases} \tag{4.32}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in A} \\ \{D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow (L+Q\kappa)D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0)+ \\ \lambda\mu(A)\left|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))\right| \\ \le\frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)\}, \end{cases} \tag{4.33}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{(L+Q\kappa)D(\mathcal{W}^{[s-1]}y^0,\mathcal{W}^{[l-1]}y^0) \\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}y^0)(\beta)) \\ -K(t,\tau,(\mathcal{W}^{[l-1]}y^0)(\beta))|<\varepsilon+\eta \\ \Rightarrow (L+Q\kappa)D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))|<\varepsilon\}. \end{cases} \tag{4.34}$$

*Then the following hold:*

*(A)* ***Convergence property.*** *For each $y^0\in M$, there exists $y\in\mathcal{C}(A,E)$ such that a sequence $(\mathcal{W}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

*(B)* ***Existence of solutions and convergence property.*** *If there exist $q\in\mathbb{N}$ and $y^0\in M$ such that the single-valued dynamic system $(\mathcal{C}(A,E),\mathcal{W}^{[q]})$ is $\mathcal{D}_{\mathcal{A}}$-closed in $y^0$, then $\varnothing\ne Fix_{\mathcal{C}(A,E)}(\mathcal{W}^{[q]})\subset\mathcal{Y}_{\mathcal{W}}$. Moreover, there exists $y\in Fix_{\mathcal{C}(A,E)}(\mathcal{W}^{[q]})$ such that a sequence $(y^m=\mathcal{W}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-convergent to $y$.*



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# 5 Convergence, existence and uniqueness theorems for integral equations of Volterra type in locally convex spaces - case (II)

The goal of this section is to exhibit clearly the breadth and importance of the conditions (II.1)-(II.5).

A counterpart of Definition 3.1 is

**Definition 5.1 .** Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}([0;a_1],E),\mathcal{D}_{\mathcal{A}})$ be such as in (II.1). Assume that (II.2) or (II.3) or (II.4) or (II.5) is satisfied.

(A) We say that a single-valued dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is a $\mathcal{D}_{\mathcal{A}}$-*admissible in* $y^0\in\mathcal{C}([0;a_1],E)$ if a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$, which is $\mathcal{D}_{\mathcal{A}}$-*sequence* (i.e. $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m}D_\alpha(\mathcal{V}^{[m]}y^0,\mathcal{V}^{[n]}y^0)=0\}$), is $\mathcal{D}_{\mathcal{A}}$-converging in $\mathcal{C}([0;a_1],E)$ (i.e. $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}D_\alpha(\mathcal{V}^{[m]}y^0,y)=0\}$ for some $y\in\mathcal{C}([0;a_1],E)$).

(B) Let $M\in 2^{\mathcal{C}([0;a_1],E)}$. $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is said to be a $\mathcal{D}_{\mathcal{A}}$-*admissible on* $M$ iff $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is a $\mathcal{D}_{\mathcal{A}}$-admissible in each $y^0\in M$.

**Remark 5.1 .** If $(E,\mathcal{P}_{\mathcal{A}})$ is sequentially complete, then $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is $\mathcal{D}_{\mathcal{A}}$-admissible on each $M\in 2^{\mathcal{C}([0;a_1],E)}$.

**Definition 5.2 .** Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}([0;a_1],E),\mathcal{D}_{\mathcal{A}})$ be such as in (II.1). Assume that (II.2) or (II.3) or (II.4) or (II.5) is satisfied. Let $q\in\mathbb{N}$.

(A) We say that a single-valued dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V}^{[q]})$ is a $\mathcal{D}_{\mathcal{A}}$-*closed in* $y^0\in\mathcal{C}([0;a_1],E)$ if in the case when a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- converging in $\mathcal{C}([0;a_1],E)$ and contains two subsequences $(u_m:m\in\mathbb{N})$ and $(v_m:m\in\mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m=\mathcal{V}^{[q]}u_m\}$, then we have that $\exists_{y\in\mathcal{C}([0;a_1],E)}\{y=\mathcal{V}^{[q]}y\}$.

(B) Let $M\in 2^{\mathcal{C}([0;a_1],E)}$. A single-valued dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V}^{[q]})$ is said to be a $\mathcal{D}_{\mathcal{A}}$-*closed on* $M$ if $(\mathcal{C}([0;a_1],E),\mathcal{V}^{[q]})$ is a $\mathcal{D}_{\mathcal{A}}$-closed in each $y^0\in M$.

We have the following:

**Theorem 5.1 .** *Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}([0;a_1],E),\mathcal{D}_{\mathcal{A}})$ be such as in (II.1). Suppose that $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is $\mathcal{D}_{\mathcal{A}}$-admissible on $\mathcal{C}([0;a_1],E)$ and assume, moreover, that $(\mathcal{C}([0;a_1],E),\mathcal{V})$ satisfies one of the following conditions (i)-(vi):*

*(i) (II.2) or (II.3) is satisfied and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}} \\ \{D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta\Rightarrow D_\alpha(\mathcal{V}^{[s+r]}x,\mathcal{V}^{[l+r]}y)<\varepsilon\}, \end{array}\right. \tag{5.1}$$

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty}D_\alpha(\mathcal{V}^{[s+r]}x,\mathcal{V}^{[l+r]}y)=0\}. \tag{5.2}$$

*(ii) (II.4) or (II.5) is satisfied ($E=\mathbb{R}$, $\mathcal{D}_\mathcal{A}=\{D\}$) and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}} \\ \{D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta\Rightarrow D(\mathcal{V}^{[s+r]}x,\mathcal{V}^{[l+r]}y)<\varepsilon\}, \end{array}\right. \tag{5.3}$$

$$\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty}D(\mathcal{V}^{[s+r]}x,\mathcal{V}^{[l+r]}y)=0\}. \tag{5.4}$$

*(iii) (II.2) is satisfied, there exists $\{L_\alpha\}_{\alpha\in\mathcal{A}}\in[0;\infty)^\mathcal{A}$ such that*

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{t\in[0;a_1]}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\{e^{-\theta_\alpha t}P_\alpha[f(t,x(t)) \\ -f(t,y(t))]\le L_\alpha\cdot D_\alpha(x,y)\}, \end{array}\right. \tag{5.5}$$

*and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta\Longrightarrow L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\frac{1}{e\theta_\alpha}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \text{where } \theta_\alpha\ge 1/a_1 \text{ and } \alpha\in\mathcal{A}, \end{array}\right. \tag{5.6}$$

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}} \\ \{L_\alpha D_\alpha(\mathcal{V}^{[s-1]}x,\mathcal{V}^{[l-1]}y) \\ +\frac{1}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y)(\beta))] \\ <\varepsilon+\eta\Rightarrow L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\frac{1}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]<\varepsilon\} \\ \text{where } \theta_\alpha\ge 1/a_1 \text{ and } \alpha\in\mathcal{A}, \end{array}\right. \tag{5.7}$$

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta\Longrightarrow L_\alpha\cdot D_\alpha(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \text{where } \theta_\alpha>0 \text{ and } \alpha\in\mathcal{A}, \end{array}\right. \tag{5.8}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}} \\ \{L_\alpha D_\alpha(\mathcal{V}^{[s-1]}x,\mathcal{V}^{[l-1]}y) \\ +\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y)(\beta))] \\ <\varepsilon+\eta \Rightarrow L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]<\varepsilon\} \text{ where } \theta_\alpha>0 \text{ and } \alpha\in\mathcal{A}. \end{cases} \tag{5.9}$$

*(iv) (II.3) is satisfied and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta \\ \Longrightarrow \frac{\lambda}{e\theta_\alpha}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \text{where } \theta_\alpha\ge 1/a_1 \text{ and } \alpha\in\mathcal{A}, \end{cases} \tag{5.10}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}} \\ \{\frac{\lambda}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y)(\beta))]<\varepsilon+\eta\Rightarrow \\ \frac{\lambda}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]<\varepsilon\} \\ \text{where } \theta_\alpha\ge 1/a_1 \text{ and } \alpha\in\mathcal{A}, \end{cases} \tag{5.11}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta\Longrightarrow \\ \lambda\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \text{where } \theta_\alpha>0 \text{ and } \alpha\in\mathcal{A}, \end{cases} \tag{5.12}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}} \\ \{\lambda\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y)(\beta))]<\varepsilon+\eta\Rightarrow \\ \lambda\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]<\varepsilon\} \text{ where } \theta_\alpha>0 \text{ and } \alpha\in\mathcal{A}. \end{cases} \tag{5.13}$$

*(v) (II.4) is satisfied ($E=\mathbb{R}$, $\mathcal{D}_\mathcal{A}=\{D\}$),*

$$\begin{cases} \exists_{Q\in[0;\infty)}\forall_{t\in A}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\{e^{-\theta t}|g(t,x(t))-g(t,y(t))| \\ \le QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in A}\forall_{x\in\mathcal{C}([0;a_1],\mathbb{R})}\{|K(t,\tau,x(\beta))|\le\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in A}\forall_{x\in\mathcal{C}([0;a_1],\mathbb{R})}\{|g(t,x(t))|\le\lambda\} \end{cases} \tag{5.14}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)<\varepsilon+\eta \\ \Longrightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y) \\ +\frac{\lambda}{e\theta}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|\leq\frac{\varepsilon}{(\varepsilon+\eta)}D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\} \\ \textit{where } \theta\geq 1/a_1, \end{cases} \tag{5.15}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}} \\ \{Q\kappa D(\mathcal{W}^{[s-1]}x,\mathcal{W}^{[l-1]}y) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y)(\beta))|<\varepsilon+\eta \\ \Rightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|<\varepsilon+\eta\} \textit{ where } \theta\geq 1/a_1, \end{cases} \tag{5.16}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]}\{D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y) \\ <\varepsilon+\eta\Longrightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|\leq\frac{\varepsilon}{(\varepsilon+\eta)}D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)\} \\ \textit{where } \theta>0, \end{cases} \tag{5.17}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],E)}\forall_{s,l\in\mathbb{N}}\{Q\kappa D(\mathcal{W}^{[s-1]}x,\mathcal{W}^{[l-1]}y) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y)(\beta))|<\varepsilon+\eta \\ \Rightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))|<\varepsilon+\eta\} \textit{ where } \theta>0. \end{cases} \tag{5.18}$$

*(vi) (II.5) is satisfied ($E=\mathbb{R}$, $\mathcal{D}_{\mathcal{A}}=\{D\}$),*

$$\begin{cases} \exists_{L\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\{e^{-\theta t}\,|f(t,x(t))-f(t,y(t))| \\ \leq LD(x,y)\}, \\ \exists_{Q\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\{e^{-\theta t}|g(t,x(t))-g(t,y(t))| \\ \leq QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in[0;a_1]}\forall_{x\in\mathcal{C}([0;a_1],\mathbb{R})}\{|K(t,\tau,x(\beta))|\leq\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in A}\forall_{x\in\mathcal{C}([0;a_1],\mathbb{R})}\{|g(t,x(t))|\leq\lambda\} \end{cases} \tag{5.19}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y) \\ <\varepsilon+\eta\Longrightarrow(L+Q\kappa)D(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))|\leq\frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \textit{where } \theta\geq 1/a_1, \end{cases} \tag{5.20}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}} \\ \{(L+Q\kappa)D(\mathcal{V}^{[s-1]}x,\mathcal{V}^{[l-1]}y) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y)(\beta))| \\ <\varepsilon+\eta\Rightarrow(L+Q\kappa)D(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))|<\varepsilon\} \text{ where } \theta\geq 1/a_1, \end{cases} \tag{5.21}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y) \\ <\varepsilon+\eta\Longrightarrow(L+Q\kappa)D(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))|\leq\frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \text{ where } \theta>0, \end{cases} \tag{5.22}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}([0;a_1],\mathbb{R})}\forall_{s,l\in\mathbb{N}} \\ \{(L+Q\kappa)D(\mathcal{V}^{[s-1]}x,\mathcal{V}^{[l-1]}y) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y)(\beta))| \\ <\varepsilon+\eta\Rightarrow(L+Q\kappa)D(\mathcal{V}^{[s+r-1]}x,\mathcal{V}^{[l+r-1]}y) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))|<\varepsilon\} \\ \text{where } \theta>0. \end{cases} \tag{5.23}$$

*Then the following hold:*

*(A)* ***Convergence property.*** *For each $y^0\in\mathcal{C}([0;a_1],E)$, there exists $y\in\mathcal{C}([0;a_1],E)$ such that a sequence $(\mathcal{V}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

*(B)* ***Existence of solutions and convergence property.*** *If there exists $q\in\mathbb{N}$ such that the single-valued dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V}^{[q]})$ is $\mathcal{D}_{\mathcal{A}}$-closed on $\mathcal{C}([0;a_1],E)$, then $\varnothing\neq Fix_{\mathcal{C}([0;a_1],E)}(\mathcal{V}^{[q]})\subset\mathcal{Y}_{\mathcal{V}}$. Moreover, for each $y^0\in\mathcal{C}([0;a_1],E)$, there exists $y\in Fix_{\mathcal{C}([0;a_1],E)}(\mathcal{V}^{[q]})$ such that a sequence $(\mathcal{V}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

*(C)* ***Existence of unique solution and convergence property.*** *If a single-valued dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is $\mathcal{D}_{\mathcal{A}}$-closed on $\mathcal{C}([0;a_1],E)$ and the family $\mathcal{D}_{\mathcal{A}}=\{D_\alpha:\alpha\in\mathcal{A}\}$ is separating on $\mathcal{C}([0;a_1],E)$, then there exists $y\in\mathcal{C}([0;a_1],E)$ such that $\mathcal{Y}_{\mathcal{V}}=Fix_{\mathcal{C}([0;a_1],E)}(\mathcal{V})=\{y\}$. Moreover, for each $y^0\in\mathcal{C}([0;a_1],E)$, a sequence $(\mathcal{V}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

**Theorem 5.2 .** *Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ of continuous seminorms on $E$. Let a gauge space $(\mathcal{C}([0;a_1],E),\mathcal{D}_{\mathcal{A}})$ be such as in (II.1). Suppose that there exists $M\in 2^{\mathcal{C}([0;a_1],E)}$ such that $(\mathcal{C}([0;a_1],E),\mathcal{V})$ is $\mathcal{D}_{\mathcal{A}}$-admissible on $M$ and assume, moreover, that $\mathcal{C}([0;a_1],E),\mathcal{V})$ and $M$ satisfy one of the following conditions (i)-(vi):*

*(i) (II.2) or (II.3) is satisfied,* $M \in 2^{\mathcal{C}([0;a_1],E)}$ *and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D_\alpha(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0) \\ <\varepsilon+\eta \Rightarrow D_\alpha(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)<\varepsilon\}, \end{cases} \tag{5.24}$$

$$\forall_{\alpha\in\mathcal{A}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty} D_\alpha(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)=0\}. \tag{5.25}$$

*(ii) (II.4) or (II.5) is satisfied (*$E=\mathbb{R}$*,* $\mathcal{D}_\mathcal{A}=\{D\}$*),* $M\in 2^{\mathcal{C}([0;a_1],\mathbb{R})}$ *and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0) \\ <\varepsilon+\eta \Rightarrow D(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)<\varepsilon\}, \end{cases} \tag{5.26}$$

$$\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty} D(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)=0\}. \tag{5.27}$$

*(iii) (II.2) is satisfied,* $M\in 2^{\mathcal{C}([0;a_1],E)}$*, there exists* $\{L_\alpha\}_{\alpha\in\mathcal{A}}\in[0;\infty)^{\mathcal{A}}$ *such that*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{t\in[0;a_1]}\forall_{x,y\in M}\{e^{-\theta_\alpha t}P_\alpha[f(t,x(t)) \\ -f(t,y(t))]\le L_\alpha D_\alpha(x,y)\} \end{cases} \tag{5.28}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D_\alpha(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\frac{1}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \text{where } \theta_\alpha\ge 1/a_1 \text{ and } \alpha\in\mathcal{A}, \end{cases} \tag{5.29}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{L_\alpha D_\alpha(\mathcal{V}^{[s-1]}y^0,\mathcal{V}^{[l-1]}y^0) \\ +\frac{1}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y^0)(\beta))] \\ <\varepsilon+\eta\Rightarrow L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\frac{1}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]<\varepsilon\} \text{ where } \theta_\alpha\ge 1/a_1 \text{ and } \alpha\in\mathcal{A}, \end{cases} \tag{5.30}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D_\alpha(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow L_\alpha\cdot D_\alpha(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)\} \\ \text{where } \theta_\alpha>0 \text{ and } \alpha\in\mathcal{A}, \end{cases} \tag{5.31}$$

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{L_\alpha D_\alpha(\mathcal{V}^{[s-1]}y^0,\mathcal{V}^{[l-1]}y^0) \\ +\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y^0)(\beta))] \\ <\varepsilon+\eta\Rightarrow L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]<\varepsilon\} \text{ where } \theta_\alpha>0 \text{ and } \alpha\in\mathcal{A}. \end{cases} \tag{5.32}$$



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*(iv) (II.3) is satisfied,* $M \in 2^{\mathcal{C}([0;a_1],E)}$ *and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D_\alpha(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)<\varepsilon+\eta\Longrightarrow \\ +\frac{\lambda}{e\theta_\alpha}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \textit{where } \theta_\alpha\ge 1/a_1 \textit{ and } \alpha\in\mathcal{A}, \end{array}\right. \tag{5.33}$$

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}} \\ \{\frac{\lambda}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y^0)(\beta))]<\varepsilon+\eta\Rightarrow \\ \frac{\lambda}{e\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]<\varepsilon\} \\ \textit{where } \theta_\alpha\ge 1/a_1 \textit{ and } \alpha\in\mathcal{A}, \end{array}\right. \tag{5.34}$$

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]} \\ \{D_\alpha(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)<\varepsilon+\eta\Longrightarrow \\ \lambda\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]\le\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\} \\ \textit{where } \theta_\alpha>0 \textit{ and } \alpha\in\mathcal{A}, \end{array}\right. \tag{5.35}$$

$$\left\{\begin{array}{l} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}} \\ \{\lambda\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y^0)(\beta))]<\varepsilon+\eta\Rightarrow \\ \lambda\frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha}\sup_{\rho,\sigma,\beta\in[0;a_1]}P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))]<\varepsilon\} \textit{ where } \theta_\alpha>0 \textit{ and } \alpha\in\mathcal{A}. \end{array}\right. \tag{5.36}$$

*(v) (II.4) is satisfied (*$E=\mathbb{R}$*,* $\mathcal{D}_{\mathcal{A}}=\{D\}$*),* $M\in 2^{\mathcal{C}([0;a_1],\mathbb{R})}$*,*

$$\left\{\begin{array}{l} \exists_{Q\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x,y\in M}\{e^{-\theta t}|g(t,x(t))-g(t,y(t))| \\ \le QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in[0;a_1]}\forall_{x\in M}\{|K(t,\tau,x(\beta))|\le\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x\in M}\{|g(t,x(t))|\le\lambda\} \end{array}\right. \tag{5.37}$$

*and, in addition, one of the following conditions holds:*

$$\left\{\begin{array}{l} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]}\{D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0) \\ <\varepsilon+\eta\Longrightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\frac{\lambda}{e\theta}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))|\le\frac{\varepsilon}{(\varepsilon+\eta)}D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)\} \\ \textit{where } \theta\ge 1/a_1, \end{array}\right. \tag{5.38}$$

$$\left\{\begin{array}{l} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{Q\kappa D(\mathcal{W}^{[s-1]}y^0,\mathcal{W}^{[l-1]}y^0) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y^0)(\beta))|<\varepsilon+\eta \\ \Rightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))|<\varepsilon+\eta\} \textit{ where } \theta\ge 1/a_1, \end{array}\right. \tag{5.39}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\forall_{\rho,\sigma,\beta\in[0;a_1]}\{D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y) \\ <\varepsilon+\eta \Longrightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))| \le \frac{\varepsilon}{(\varepsilon+\eta)}D(\mathcal{W}^{[s]}y^0,\mathcal{W}^{[l]}y^0)\} \\ \textit{where } \theta>0, \end{cases} \tag{5.40}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{Q\kappa D(\mathcal{W}^{[s-1]}y^0,\mathcal{W}^{[l-1]}y^0) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l-1]}y^0)(\beta))| <\varepsilon+\eta \\ \Rightarrow Q\kappa D(\mathcal{W}^{[s+r-1]}y^0,\mathcal{W}^{[l+r-1]}y^0) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y^0)(\beta))| <\varepsilon+\eta\} \textit{ where } \theta>0. \end{cases} \tag{5.41}$$

*(vi) (II.5) is satisfied ($E=\mathbb{R}$, $\mathcal{D}_{\mathcal{A}}=\{D\}$), $M\in 2^{\mathcal{C}([0;a_1],\mathbb{R})}$,*

$$\begin{cases} \exists_{L\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x,y\in M}\{e^{-\theta t}\,|f(t,x(t))-f(t,y(t))| \\ \le LD(x,y)\}, \\ \exists_{Q\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x,y\in M}\{e^{-\theta t}|g(t,x(t))-g(t,y(t))| \\ \le QD(x,y)\}, \\ \exists_{\kappa\in[0;\infty)}\forall_{t,\tau,\beta\in[0;a_1]}\forall_{x\in M}\{|K(t,\tau,x(\beta))|\le\kappa\}, \\ \exists_{\lambda\in[0;\infty)}\forall_{t\in[0;a_1]}\forall_{x\in M}\{|g(t,x(t))|\le\lambda\} \end{cases} \tag{5.42}$$

*and, in addition, one of the following conditions holds:*

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow (L+Q\kappa)D(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))| \le \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)\} \\ \textit{where } \theta\ge 1/a_1, \end{cases} \tag{5.43}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{(L+Q\kappa)D(\mathcal{V}^{[s-1]}y^0,\mathcal{V}^{[l-1]}y^0) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l-1]}y^0)(\beta))\} \\ <\varepsilon+\eta \Rightarrow (L+Q\kappa)D(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\frac{\lambda}{e\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))| <\varepsilon\} \textit{ where } \theta\ge 1/a_1, \end{cases} \tag{5.44}$$

$$\begin{cases} \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)<\varepsilon+\eta \\ \Longrightarrow (L+Q\kappa)D(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\ +\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\ -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))| \le \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)\} \\ \textit{where } \theta>0, \end{cases} \tag{5.45}$$

$$
\begin{cases}
\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{(L+Q\kappa)D(\mathcal{V}^{[s-1]}y^0,\mathcal{V}^{[l-1]}y^0) \\
+\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s-1]}y^0)(\beta)) \\
-K(\rho,\sigma,(\mathcal{V}^{[l-1]}y^0)(\beta))\} \\
<\varepsilon+\eta\Rightarrow(L+Q\kappa)D(\mathcal{V}^{[s+r-1]}y^0,\mathcal{V}^{[l+r-1]}y^0) \\
+\lambda\frac{1-e^{-a_1\theta}}{\theta}\sup_{\rho,\sigma,\beta\in[0;a_1]}|K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}y^0)(\beta)) \\
-K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y^0)(\beta))|<\varepsilon\} \text{ where } \theta>0.
\end{cases} \tag{5.46}
$$

*Then the following hold:*

*(A)* ***Convergence property.*** *For each $y^0 \in M$, there exists $y \in \mathcal{C}([0;a_1],E)$ such that a sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$- convergent to $y$.*

*(B)* ***Existence of solutions and convergence property.*** *If there exist $q \in \mathbb{N}$ and $y^0 \in M$ such that the single-valued dynamic system $(\mathcal{C}([0;a_1],E),\mathcal{V}^{[q]})$ is $\mathcal{D}_{\mathcal{A}}$-closed in $y^0$, then $\varnothing \neq Fix_{\mathcal{C}([0;a_1],E)}(\mathcal{V}^{[q]}) \subset \mathcal{Y}_{\mathcal{V}}$. Moreover, there exists $y \in Fix_{\mathcal{C}([0;a_1],E)}(\mathcal{V}^{[q]})$ such that a sequence $(y^m = \mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-convergent to $y$.*

# 6 Proofs

**Proof of Theorem 4.1.**

4.1.1. Since (4.2), (4.4) and (4.6) are special cases of (4.1), (4.3) and (4.5), respectively, and (4.1), (4.3) and (4.5) are special cases of (3.1), the assertions of Theorem 4.1(i)-(iii) are a consequence of Theorem 3.1.

4.1.2. The assertions of Theorem 4.1(iv) hold since conditions (4.8) and (4.9) are special cases of (4.1). Indeed, applying (I.2) and (4.7), we see that, for $\alpha \in \mathcal{A}$ and $m,n,s,l,r \in \mathbb{N}$,

$$
\begin{aligned}
& D_\alpha(\mathcal{W}^{[m]}x,\mathcal{W}^{[n]}y) \\
\leq\ & L_\alpha D_\alpha(\mathcal{W}^{[m-1]}x,\mathcal{W}^{[n-1]}y) \\
& +\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[m-1]}x)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[n-1]}y)(\beta))]
\end{aligned}
$$

and

$$
\begin{aligned}
& D_\alpha(\mathcal{W}^{[s+r]}x,\mathcal{W}^{[l+r]}y) \\
\leq\ & \sup_{t\in A}P_\alpha[f(t,(\mathcal{W}^{[s+r-1]}x)(t))-f(t,(\mathcal{W}^{[l+r-1]}y)(t))] \\
& +\sup_{t\in A}P_\alpha\{\int_A[K(t,\tau,(\mathcal{W}^{[s+r-1]}x)(h(\tau)))-K(t,\tau,(\mathcal{W}^{[l+r-1]}y)(h(\tau)))]d\tau\} \\
\leq\ & L_\alpha D_\alpha(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y)+\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))]\leq\frac{\varepsilon}{\varepsilon+\eta}D_\alpha(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)<\varepsilon.
\end{aligned}
$$

4.1.3. Conditions (4.10) and (4.11) are special cases of (4.1). Indeed, let



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$\alpha \in \mathcal{A}$ and $m, n, s, l, r \in \mathbb{N}$. Then, by (I.3), we have

$$\begin{aligned}
& D_\alpha(\mathcal{W}^{[m]}x, \mathcal{W}^{[n]}y) \leq \\
& \sup_{t\in A} P_\alpha\{g(t)\int_A [K(t,\tau,(\mathcal{W}^{[m-1]}x)(h(\tau))) - K(t,\tau,(\mathcal{W}^{[n-1]}y)(h(\tau)))]d\tau\} \\
\leq \quad & \lambda\mu(A) \sup_{\rho,\sigma,\beta\in A} P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[m-1]}x)(\beta)) - K(\rho,\sigma,(\mathcal{W}^{[n-1]}y)(\beta))]
\end{aligned}$$

and

$$\begin{aligned}
D_\alpha(\mathcal{W}^{[s+r]}x, \mathcal{W}^{[l+r]}y) \quad & \leq \quad \lambda\mu(A) \sup_{\rho,\sigma,\beta\in A} P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\
-K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))] \quad & \leq \quad \frac{\varepsilon}{\varepsilon+\eta} D_\alpha(\mathcal{W}^{[s]}x, \mathcal{W}^{[l]}y) < \varepsilon.
\end{aligned}$$

Therefore, the assertions of Theorem 4.1(v) hold.

4.1.4. The assertions of Theorem 4.1(vi) hold since conditions (4.13) and (4.14) are special cases of (4.3). Indeed, if $\alpha \in \mathcal{A}$ and $m, n, s, l, r \in \mathbb{N}$, then, by (I.4) and (4.12), we have

$$\begin{aligned}
& D(\mathcal{W}^{[m]}x, \mathcal{W}^{[n]}y) \\
= \quad & \sup_{t\in A} |(\mathcal{W}^{[m]}x)(t) - (\mathcal{W}^{[n]}y)(t)| \\
\leq \quad & \sup_{t\in A}\{|[g(t,(\mathcal{W}^{[m-1]}x)(t)) - g(t,(\mathcal{W}^{[n-1]}y)(t))] \\
& \cdot\textstyle\int_A K(t,\tau,(\mathcal{W}^{[m-1]}x)(h(\tau))d\tau| \\
& +|g(t,(\mathcal{W}^{[n-1]}y)(t))\textstyle\int_A [K(t,\tau,(\mathcal{W}^{[m-1]}x)(h(\tau)) \\
& -K(t,\tau,(\mathcal{W}^{[n-1]}y)(h(\tau))]d\tau|\} \\
\leq \quad & Q\kappa D(\mathcal{W}^{[m-1]}x, \mathcal{W}^{[n-1]}y) + \lambda\mu(A) \sup_{\rho,\sigma,\beta\in A} |K(\rho,\sigma,(\mathcal{W}^{[m-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{W}^{[n-1]}y)(\beta))|\}\}
\end{aligned}$$

and

$$\begin{aligned}
D(\mathcal{W}^{[s+r]}x, \mathcal{W}^{[l+r]}y) \quad & \leq \quad Q\kappa D(\mathcal{W}^{[s+r-1]}x, \mathcal{W}^{[l+r-1]}y) \\
& \quad\quad +\lambda\mu(A) \sup_{\rho,\sigma,\beta\in A} |K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\
& \quad\quad -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))| \\
& \leq \quad \frac{\varepsilon}{\varepsilon+\eta} D(\mathcal{W}^{[s]}x, \mathcal{W}^{[l]}y) < \varepsilon.
\end{aligned}$$

4.1.5. The assertions of Theorem 4.1(vii) hold since conditions (4.16) and (4.17) are special cases of (4.3). Indeed, if $\alpha \in \mathcal{A}$ and $m, n, s, l, r \in \mathbb{N}$, then, by

(I.5) and (4.15), we have

$$
\begin{aligned}
& D(\mathcal{W}^{[m]}x,\mathcal{W}^{[n]}y)=\sup_{t\in A}|(\mathcal{W}^{[m]}x)(t)-(\mathcal{W}^{[n]}y)(t)| \\
\leq\ & \sup_{t\in A}\{|f(t,(\mathcal{W}^{[m-1]}x)(t))-f(t,(\mathcal{W}^{[n-1]}y)(t))| \\
& +|[g(t,(\mathcal{W}^{[m-1]}x)(t))-g(t,(\mathcal{W}^{[n-1]}y)(t))] \\
& \cdot\textstyle\int_A K(t,\tau,(\mathcal{W}^{[m-1]}x)(h(\tau))d\tau| \\
& +|g(t,(\mathcal{W}^{[n-1]}y)(t))\textstyle\int_A[K(t,\tau,(\mathcal{W}^{[m-1]}x)(h(\tau)) \\
& -K(t,\tau,(\mathcal{W}^{[n-1]}y)(h(\tau))]d\tau|\} \\
\leq\ & (L+Q\kappa)D(\mathcal{W}^{[m-1]}x,\mathcal{W}^{[n-1]}y) \\
& +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[m-1]}x)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[n-1]}y)(\beta))|
\end{aligned}
$$

and

$$
\begin{aligned}
& D(\mathcal{W}^{[s+r]}x,\mathcal{W}^{[l+r]}y)=\sup_{t\in A}|(\mathcal{W}^{[s+r]}x)(t)-(\mathcal{W}^{[l+r]}y)(t)| \\
\leq\ & (L+Q\kappa)D(\mathcal{W}^{[s+r-1]}x,\mathcal{W}^{[l+r-1]}y) \\
& +\lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}|K(\rho,\sigma,(\mathcal{W}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{W}^{[l+r-1]}y)(\beta))| \\
\leq\ & \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{W}^{[s]}x,\mathcal{W}^{[l]}y)<\varepsilon\}.
\end{aligned}
$$

**Proof of Theorem 4.2.**

4.2.1. We see that (4.18), (4.20) and (4.22) are special cases of (3.2). Moreower, (4.19), (4.21) and (4.23) are special cases of (4.18), (4.20) and (4.22), respectively. Consequently, the assertions of Theorem 4.2(i)-(iii) are a consequence of Theorem 3.2.

4.2.2. We have that (4.25) and (4.26) are special cases of (4.18). Indeed, by (I.2) and (4.24), observe that, for $\alpha\in\mathcal{A}$, $m,n\in\mathbb{N}$ and $y^0\in M$,

$$
\begin{aligned}
& D_\alpha(\mathcal{W}^{[m]}y^0,\mathcal{W}^{[n]}y^0) \\
\leq\ & L_\alpha D_\alpha(\mathcal{W}^{[m-1]}y^0,\mathcal{W}^{[n-1]}y^0) \\
& +\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[m-1]}y^0)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[n-1]}y^0)(\beta))].
\end{aligned}
$$

We conclude that the assertions of Theorem 4.2(iv) are a consequence of Theorem 3.2.

4.2.3. Conditions (4.27) and (4.28) are special cases of condition (4.18), since, by (I.3), for $\alpha\in\mathcal{A}$ and $m,n\in\mathbb{N}$,

$$
\begin{aligned}
& D_\alpha(\mathcal{W}^{[m]}y^0,\mathcal{W}^{[n]}y^0) \\
\leq\ & \lambda\mu(A)\sup_{\rho,\sigma,\beta\in A}P_\alpha[K(\rho,\sigma,(\mathcal{W}^{[m-1]}y^0)(\beta))-K(\rho,\sigma,(\mathcal{W}^{[n-1]}y^0)(\beta))].
\end{aligned}
$$

Hence, we conclude that the assertions of Theorem 4.2(v) hold.



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4.2.4. Conditions (4.30) and (4.31) are special cases of condition (4.20). Indeed, using (I.4) and (4.29), for $\alpha \in \mathcal{A}$ and $m, n \in \mathbb{N}$, we have

$$\begin{aligned} & D(\mathcal{W}^{[m]}y^0, \mathcal{W}^{[n]}y^0) \\ \leq \; & Q\kappa D(\mathcal{W}^{[m-1]}y^0, \mathcal{W}^{[n-1]}y^0) + \lambda\mu(A) \sup_{\rho,\sigma,\beta\in A} |K(\rho,\sigma,(\mathcal{W}^{[m-1]}y^0)(\beta)) \\ & -K(\rho,\sigma,(\mathcal{W}^{[n-1]}y^0)(\beta))|. \end{aligned}$$

Therefore, the assertions of Theorem 4.2(vi) hold.

4.2.5. Conditions (4.33) and (4.34) are special cases of condition (4.20). Indeed, if (I.5) is satisfied, then, in view of (4.32), we have that, for $m, n \in \mathbb{N}$ and $y^0 \in M$,

$$\begin{aligned} & D(\mathcal{W}^{[m]}y^0, \mathcal{W}^{[n]}y^0) \\ \leq \; & (L + Q\kappa) D(\mathcal{W}^{[m-1]}y^0, \mathcal{W}^{[n-1]}y^0) \\ & +\lambda\mu(A) \sup_{\rho,\sigma,\beta\in A} |K(\rho,\sigma,(\mathcal{W}^{[m-1]}y^0)(\beta)) - K(\rho,\sigma,(\mathcal{W}^{[n-1]}y^0)(\beta))|. \end{aligned}$$

We deduce from the above that the assertions of Theorem 4.2(vii) hold.

**Proof of Theorem 5.1.**

5.1.1. It is clear that (5.1) and (5.2), in view of (II.1)-(II.3), are special cases of (3.1). Moreover, (5.3) and (5.4), in view of (II.1), (II.4) and (II.5), are also special cases of (3.1). Therefore, the assertions of Theorems 5.1(i) and 5.1(ii) are a consequence of Theorem 3.1.

5.1.2. Observe that the conditions (5.6)-(5.9) are special cases of (5.1). Indeed, we have

$$\sup_{t\in[0;a_1]} te^{-t\theta_\alpha} \leq \frac{1}{e\theta_\alpha} \text{ with } \frac{1}{a_1} \leq \theta_\alpha \text{ for } \alpha \in \mathcal{A}.$$

Consequently, by (II.2) and (5.5), we obtain

$$\begin{aligned}
& D_\alpha(\mathcal{V}^{[s+r]}x, \mathcal{V}^{[l+r]}y) = \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} P_\alpha[(\mathcal{V}^{[s+r]}x)(t) - (\mathcal{V}^{[l+r]}y)(t)] \\
\leq \quad & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha}\{P_\alpha[f(t,(\mathcal{V}^{[s+r-1]}x)(t)) - f(t,(\mathcal{V}^{[l+r-1]}y)(t))] \\
& +P_\alpha\{\int_0^t [K(t,\tau,(\mathcal{V}^{[s+r-1]}x)(h(\tau))) \\
& -K(t,\tau,(\mathcal{V}^{[l+r-1]}y)(h(\tau)))]d\tau\}\} \\
\leq \quad & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha}\{P_\alpha[f(t,(\mathcal{V}^{[s+r-1]}x)(t)) - f(t,(\mathcal{V}^{[ls+r-1]}y)(t))] \\
& +t \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) - K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))]\} \\
\leq \quad & L_\alpha D_\alpha(\mathcal{V}^{[s+r-1]}x, \mathcal{V}^{[l+r-1]}y) \\
& + \sup_{t\in[0;a_1]} te^{-t\theta_\alpha} \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))] \\
\leq \quad & L_\alpha \cdot D_\alpha(\mathcal{V}^{[s+r-1]}x, \mathcal{V}^{[l+r-1]}y) \\
& +\frac{1}{e\theta_\alpha} \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))] \\
\leq \quad & \frac{\varepsilon}{\varepsilon+\eta} D_\alpha(\mathcal{V}^{[s]}x, \mathcal{V}^{[l]}y) < \varepsilon \text{ when } 1/a_1 \leq \theta_\alpha \text{ and } \alpha \in \mathcal{A}.
\end{aligned}$$

This shows that (5.6) is a special case of (5.1).

It is clear, that condition (5.7) implies (5.1). The arguments are very close to those given above.

Now, using (II.2) and (5.5), we obtain

$$
\begin{aligned}
& D_\alpha(\mathcal{V}^{[s+r]}x, \mathcal{V}^{[l+r]}y) \\
= \ & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} P_\alpha[(\mathcal{V}^{[s+r]}x)(t) - (\mathcal{V}^{[l+r]}y)(t)] \\
\leq \ & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} \{P_\alpha[f(t, (\mathcal{V}^{[s+r-1]}x)(t)) - f(t, (\mathcal{V}^{[l+r-1]}y)(t))] \\
& + \int_0^t P_\alpha[K(t,\tau,(\mathcal{V}^{[s+r-1]}x)(h(\tau))) - K(t,\tau,(\mathcal{V}^{[l+r-1]}y)(h(\tau)))]d\tau\} \\
= \ & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} \{P_\alpha[f(t, (\mathcal{V}^{[s+r-1]}x)(t)) - f(t, (\mathcal{V}^{[ls+r-1]}y)(t))] \\
& + \int_0^t P_\alpha[K(t,\tau,(\mathcal{V}^{[s+r-1]}x)(h(\tau))) \\
& -K(t,\tau,(\mathcal{V}^{[l+r-1]}y)(h(\tau)))]e^{-\tau\theta_\alpha}e^{\tau\theta_\alpha}d\tau\} \\
\leq \ & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} \{P_\alpha[f(t, (\mathcal{V}^{[s+r-1]}x)(t)) - f(t, (\mathcal{V}^{[l+r-1]}y)(t))] \\
& + \int_0^t P_\alpha[K(t,\tau,(\mathcal{V}^{[s+r-1]}x)(h(\tau))) \\
& -K(t,\tau,(\mathcal{V}^{[l+r-1]}y)(h(\tau)))]e^{\tau\theta_\alpha}d\tau\} \\
\leq \ & \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} \{P_\alpha[f(t, (\mathcal{V}^{[s+r-1]}x)(t)) - f(t, (\mathcal{V}^{[l+r-1]}y)(t))] \\
& + \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))] \int_0^t e^{\tau\theta_\alpha}d\tau\} \\
\leq \ & L_\alpha \cdot D_\alpha(\mathcal{V}^{[s+r-1]}x, \mathcal{V}^{[l+r-1]}y) + \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))] \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} \int_0^t e^{\tau\theta_\alpha}d\tau \\
= \ & L_\alpha \cdot D_\alpha(\mathcal{V}^{[s+r-1]}x, \mathcal{V}^{[l+r-1]}y) + \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))] \sup_{t\in[0;a_1]} e^{-t\theta_\alpha}\frac{1}{\theta_\alpha}[e^{t\theta_\alpha}-1] \\
\leq \ & L_\alpha \cdot D_\alpha(\mathcal{V}^{[s+r-1]}x, \mathcal{V}^{[l+r-1]}y) \\
& + \frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha} \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[s+r-1]}x)(\beta)) \\
& -K(\rho,\sigma,(\mathcal{V}^{[l+r-1]}y)(\beta))] \\
\leq \ & \frac{\varepsilon}{\varepsilon+\eta} D_\alpha(\mathcal{V}^{[s]}x, \mathcal{V}^{[l]}y) < \varepsilon \text{ when } \theta_\alpha > 0 \text{ and } \alpha \in \mathcal{A}.
\end{aligned}
$$

Therefore, condition (5.8) implies (5.1), since, for $\alpha \in \mathcal{A}$ and $m, n \in \mathbb{N}$,

$$\begin{aligned}
& D_\alpha(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) = \sup_{t\in[0;a_1]} e^{-t\theta_\alpha} P_\alpha[(\mathcal{V}^{[m]}x)(t) - (\mathcal{V}^{[n]}y)(t)] \\
\leq\ & L_\alpha \cdot D_\alpha(\mathcal{V}^{[m-1]}x, \mathcal{V}^{[n-1]}y) \\
& + \frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha} \sup_{\rho,\sigma,\beta\in[0;a_1]} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[m-1]}x)(\beta)) \\
& - K(\rho,\sigma,(\mathcal{V}^{[n-1]}y)(\beta))] \text{ when } \theta_\alpha > 0 \text{ and } \alpha \in \mathcal{A}.
\end{aligned}$$

Likewise, (5.9) implies (5.1). The arguments are very close to those given above.

From this we deduce that the assertions of Theorem 5.1(iii) hold.

5.1.3. Also the assertions of Theorem 5.1(iv) hold since, in view of (II.1) and (II.3), the conditions (5.10)-(5.13) are special cases of (5.1). Indeed, we have

$$\begin{aligned}
& \forall_{\alpha\in\mathcal{A}} \forall_{x,y\in\mathcal{C}([0;a_1],E)} \forall_{m,n\in\mathbb{N}} \forall_{\rho,\sigma,\beta\in[0;a_1]} \{D_\alpha(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) \\
\leq\ & \frac{\lambda}{e\theta_\alpha} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[m-1]}x)(\beta)) \\
& - K(\rho,\sigma,(\mathcal{V}^{[n-1]}y)(\beta))]\} \text{ where } \theta_\alpha \geq 1/a_1
\end{aligned}$$

and

$$\begin{aligned}
& \forall_{\alpha\in\mathcal{A}} \forall_{x,y\in\mathcal{C}([0;a_1],E)} \forall_{m,n\in\mathbb{N}} \forall_{\rho,\sigma,\beta\in[0;a_1]} \{D_\alpha(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) \\
\leq\ & \lambda \frac{1-e^{-a_1\theta_\alpha}}{\theta_\alpha} P_\alpha[K(\rho,\sigma,(\mathcal{V}^{[m-1]}x)(\beta)) \\
& - K(\rho,\sigma,(\mathcal{V}^{[n-1]}y)(\beta))]\} \text{ where } \theta_\alpha > 0.
\end{aligned}$$

5.1.4. As before, observe that the conditions (5.15)-(5.18) are special cases of condition (5.3). In fact, using (II.4) and (5.14), we have

$$\begin{aligned}
& D(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) \\
\leq\ & Q\kappa D(\mathcal{V}^{[m-1]}x, \mathcal{V}^{[n-1]}y) + \frac{\lambda}{e\theta} \sup_{\rho,\sigma,\beta\in[0;a_1]} |K(\rho,\sigma,(\mathcal{V}^{[m-1]}x)(\beta)) \\
& - K(\rho,\sigma,(\mathcal{V}^{[n-1]}y)(\beta))| \text{ for } \theta \geq 1/a_1
\end{aligned}$$

and

$$\begin{aligned}
& D(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) \leq Q\kappa D(\mathcal{V}^{[m-1]}x, \mathcal{V}^{[n-1]}y) \\
& + \lambda \frac{1-e^{-a_1\theta}}{\theta} \sup_{\rho,\sigma,\beta\in[0;a_1]} |K(\rho,\sigma,(\mathcal{V}^{[m-1]}x)(\beta)) \\
& - K(\rho,\sigma,(\mathcal{V}^{[n-1]}y)(\beta))| \text{ for } \theta > 0,
\end{aligned}$$

where $x, y \in \mathcal{C}([0;a_1],\mathbb{R})$ and $m, n \in \mathbb{N}$. Consequently, the assertions of Theorem 5.1(v) hold.

5.1.5. Conditions (5.20)-(5.23) are special cases of condition (5.3). The arguments are very close to those given above. In fact, if $x, y \in \mathcal{C}([0; a_1], \mathbb{R})$ and $m, n \in \mathbb{N}$, then, using (II.5) and (5.19), we have

$$\begin{aligned} & D(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) \\ \leq\ & (L + Q\kappa) D(\mathcal{V}^{[m-1]}x, \mathcal{V}^{[n-1]}y) \\ & + \frac{\lambda}{e\theta} \sup_{\rho,\sigma,\beta \in [0;a_1]} |K(\rho, \sigma, (\mathcal{V}^{[m-1]}x)(\beta)) \\ & - K(\rho, \sigma, (\mathcal{V}^{[n-1]}y)(\beta))| \text{ for } \theta \geq 1/a_1 \end{aligned}$$

and

$$\begin{aligned} & D(\mathcal{V}^{[m]}x, \mathcal{V}^{[n]}y) \\ \leq\ & (L + Q\kappa) D(\mathcal{V}^{[m-1]}x, \mathcal{V}^{[n-1]}y) \\ & + \lambda \frac{1 - e^{-a_1\theta}}{\theta} \sup_{\rho,\sigma,\beta \in [0;a_1]} |K(\rho, \sigma, (\mathcal{V}^{[m-1]}x)(\beta)) \\ & - K(\rho, \sigma, (\mathcal{V}^{[n-1]}y)(\beta))| \text{ for } \theta > 0. \end{aligned}$$

Thus we see that the assertions of Theorem 5.1(vi) hold.

**Proof of Theorem 5.2.**

5.2.1. We will not prove the following facts (a)-(d) since their proofs are based on the strategies presented above and requires only some adaptation of them: (a) Conditions (5.24) and (5.26) are special cases of condition (3.2); (b) Conditions (5.25) and (5.27) are special cases of conditions (5.24) and (5.26), respectively, (c) Conditions (5.29)-(5.32) with (5.28) and conditions (5.33)-(5.36) are special cases of condition (5.24), and (d) Conditions (5.38)-(5.41) with (5.37) and conditions (5.43)-(5.46) with (5.42) are special cases of condition (5.26).

# 7 Examples

Now we provide a list of examples involving famous Volterra and Fredholm integral equations. This rather long list is offered to meet several objectives. First, we seek to bring the reader to conviction that the scope of the theory is broad while providing a spectrum of meaningful applications and, at the same time, generating some insight as regards the fundamental structural assumptions of Theorems 4.1, 4.2, 5.1.and 5.2. Finally, these equations are written in the form compatible with the theory, but a new method of studies of these equations presented here will induce an interesting modification of the classical viewpoint concerning them.

**Example 7.1.** Let $a_1 \in (0; \infty)$ and $\omega \in (1; \infty)$ be such that

$$\frac{2}{5} + \frac{\pi}{4} a_1 \frac{\omega + 1}{\omega^2} < 1. \tag{7.1}$$

Let $A=[0;a_1]$ and $h\in\mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{V}}\subset\mathcal{C}(A,\mathbb{R})$ of solutions of the Volterra quadratic integral equation of the form

$$u(t)=\frac{1}{5}\frac{\sin^2 u(t)}{1+\sin^2 u(t)}+\frac{1}{2}\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)}\int_0^t\frac{d\tau}{\omega+t+\tau+\sin u(h(\tau))},$$

$t\in A$, $u\in\mathcal{C}(A,\mathbb{R})$, is nonempty. Moreover, for each $y^0\in\mathcal{C}(A,\mathbb{R})$, there exists $y\in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V})\subset\mathcal{Y}_{\mathcal{V}}$ such that a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$ is $D$-convergent to $y$ where $\mathcal{V}^{[m+1]}:\mathcal{C}(A,\mathbb{R})\to\mathcal{C}(A,\mathbb{R})$, $m\in\{0\}\cup\mathbb{N}$, is defined by

$$\begin{aligned}(\mathcal{V}^{[m+1]}u)(t) \quad &= \quad \frac{1}{5}\frac{\sin^2(\mathcal{V}^{[m]}u)(t)}{1+\sin^2(\mathcal{V}^{[m]}u)(t)}\\ &\quad +\frac{1}{2}\arctan\frac{\pi[t+|(\mathcal{V}^{[m]}u)(t)|]}{2(1+t^2)}\int_0^t\frac{d\tau}{\omega+t+\tau+\sin(\mathcal{V}^{[m]}u)(h(\tau))},\end{aligned}$$

$t\in A$, $u\in\mathcal{C}(A,\mathbb{R})$, $\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{D(x,y)=\sup_{t\in A}|x(t)-y(t)|\}$, and $(\mathcal{C}(A,\mathbb{R}),D)$ is a gauge space.

The proof proceeds in seven steps.

Define

$$f(t,u(t))=\frac{1}{5}\frac{\sin^2 u(t)}{1+\sin^2 u(t)},\ g(t,u(t))=\frac{1}{2}\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)},$$

$$K(t,\tau,u(h(\tau)))=\frac{1}{\omega+t+\tau+\sin u(h(\tau))},\ u\in\mathcal{C}(A,\mathbb{R}),\ t,\tau\in A.$$

7.1.1. First notice that

$$\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{V}u,\mathcal{V}v)\le\gamma_1 D(u,v)\} \tag{7.2}$$

where

$$\gamma_1=\frac{2}{5}+\frac{\pi}{4}a_1\frac{\omega}{(\omega-1)^2}. \tag{7.3}$$

In fact, we have that

$$\begin{aligned}&D(\mathcal{V}u,\mathcal{V}v)\\ =\ &\sup_{t\in A}|(\mathcal{V}u)(t)-(\mathcal{V}v)(t)|\le\sup_{t\in A}|f(t,u(t))-f(t,v(t))|\\ &+\sup_{t\in A}|[g(t,u(t))-g(t,v(t))]\int_0^t K(t,\tau,v(h(\tau)))d\tau|\\ &+\sup_{t\in A}|g(t,v(t))\int_0^t[K(t,\tau,u(h(\tau)))-K(t,\tau,v(h(\tau)))]d\tau|\end{aligned}$$

and, since

$$\begin{aligned}\sup_{t\in A}|f(t,u(t))-f(t,v(t))| \quad &\le \quad \frac{1}{5}\sup_{t\in A}\frac{\left|\sin^2 u(t)-\sin^2 v(t)\right|}{\left[1+\sin^2 u(t)\right]\left[1+\sin^2 v(t)\right]}\\ &\le \quad \frac{2}{5}\sup_{t\in A}|u(t)-v(t)|=\frac{2}{5}D(u,v),\end{aligned}$$



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$$
\begin{aligned}
& \sup_{t\in A}|g(t,u(t))-g(t,v(t))| \\
\leq\ & \sup_{t\in A}\frac{1}{2}|\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)}-\arctan\frac{\pi[t+|v(t)|]}{2(1+t^2)}| \\
\leq\ & \frac{\pi}{4}\sup_{t\in A}|u(t)-v(t)|=\frac{\pi}{4}D(u,v),
\end{aligned}
$$

$$
\sup_{t\in A}|\int_0^t\frac{d\tau}{\omega+t+\tau+\sin v(h(\tau))}|\leq\frac{a_1}{\omega-1},
$$

$$
\begin{aligned}
& \sup_{t\in A}|\int_0^t[K(t,\tau,u(h(\tau)))-K(t,\tau,v(h(\tau)))]d\tau| \\
\leq\ & \sup_{t\in A}|\int_0^t\frac{[\sin u(h(\tau))-\sin v(h(\tau))]d\tau}{[\omega+t+\tau+\sin u(h(\tau))][\omega+t+\tau+\sin v(h(\tau))]}| \\
\leq\ & \frac{a_1}{(\omega-1)^2}\sup_{\tau\in A}|u(h(\tau))-v(h(\tau))|\leq\frac{a_1}{(\omega-1)^2}\sup_{\beta\in A}|u(\beta)-v(\beta)| \\
=\ & \frac{a_1}{(\omega-1)^2}D(u,v),
\end{aligned}
$$

thus

$$
D(\mathcal{V}u,\mathcal{V}v)\leq(\frac{2}{5}+\frac{\pi}{4}a_1\frac{1}{\omega-1}+\frac{\pi}{4}a_1\frac{1}{(\omega-1)^2})D(u,v)=\gamma_1 D(u,v).
$$

7.1.2. We seek to show that

$$
\forall_{t\in A}\forall_{m\in\{0\}\cup\mathbb{N}}\forall_{u\in\mathcal{C}(A,\mathbb{R})}\{0\leq(\mathcal{V}^{[m+1]}u)(t)<\pi/4\}. \tag{7.4}
$$

Indeed, since $0\leq g(t,u(t))<\pi/4$, $u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, we have

$$
\begin{aligned}
& g(t,(\mathcal{V}^{[m]}u)(t))\ln\frac{\omega+1+2t}{\omega+1+t}=g(t,(\mathcal{V}^{[m]}u)(t))\int_0^t\frac{d\tau}{\omega+t+\tau+1} \\
\leq\ & g(t,(\mathcal{V}^{[m]}u)(t))\int_0^t K(t,\tau,(\mathcal{V}^{[m]}u)(h(\tau)))d\tau \\
=\ & (\mathcal{V}^{[m+1]}u)(t)-f(t,(\mathcal{V}^{[m]}u)(t)) \\
\leq\ & g(t,(\mathcal{V}^{[m]}u)(t))\int_0^t\frac{d\tau}{\omega+t+\tau-1}=g(t,(\mathcal{V}^{[m]}u)(t))\ln\frac{\omega-1+2t}{\omega-1+t},
\end{aligned}
$$

which shows that

$$
\begin{aligned}
0\ =\ & g(t,(\mathcal{V}^{[m]}u)(t))\ln\frac{\omega+1}{\omega+1}\leq(\mathcal{V}^{[m+1]}u)(t)-f(t,(\mathcal{V}^{[m]}u)(t)) \\
\leq\ & g(t,(\mathcal{V}^{[m]}u)(t))\ln(1+\frac{a_1}{\omega-1+a_1})<\frac{\pi}{4}\ln 2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
0\ \leq\ & f(t,(\mathcal{V}^{[m]}u)(t))\leq(\mathcal{V}^{[m+1]}u)(t)\leq\frac{\pi}{4}\ln 2+f(t,(\mathcal{V}^{[m]}u)(t)) \\
\leq\ & \frac{\pi}{4}\ln 2+\frac{1}{10}\leq 0.7854\cdot 0.6932+0.1\leq 0.5445+0.1=0.6445<\frac{\pi}{4}.
\end{aligned}
$$

7.1.3. However, $h \in \mathcal{C}(A,A)$ and therefore $(\mathcal{V}^{[m+1]}u)(h(t))$ also satisfies the above inequality, i.e.

$$\forall_{t\in A}\forall_{m\in\{0\}\cup\mathbb{N}}\forall_{u\in\mathcal{C}(A,\mathbb{R}),h\in\mathcal{C}(A,A)}\{0 \le (\mathcal{V}^{[m+1]}u)(h(t)) < \pi/4\}. \tag{7.5}$$

7.1.4. Suppose now that $s,l,r \in \mathbb{N}$ and $u,v \in \mathcal{C}(A,\mathbb{R})$. It remains to see that

$$D(\mathcal{V}^{[r+s]}u, \mathcal{V}^{[r+l]}v) \le \gamma^r D(\mathcal{V}^{[s]}u, \mathcal{V}^{[l]}v) \tag{7.6}$$

where, in view of (7.1),

$$\gamma = \frac{2}{5} + \frac{\pi}{4}a_1\frac{\omega+1}{\omega^2} < 1. \tag{7.7}$$

In fact, we have

$$\begin{aligned}
&D(\mathcal{V}^{[m+1]}u, \mathcal{V}^{[n+1]}v) = \sup_{t\in A}\left|(\mathcal{V}^{[m+1]}u)(t) - (\mathcal{V}^{[n+1]}v)(t)\right| \\
\le\ & \sup_{t\in A}\frac{1}{5}\frac{\left|\sin^2(\mathcal{V}^{[m]}u)(t) - \sin^2(\mathcal{V}^{[n]}v)(t)\right|}{\left[1+\sin^2(\mathcal{V}^{[m]}u)(t)\right]\left[1+\sin^2(\mathcal{V}^{[n]}v)(t)\right]} \\
&+\sup_{t\in A}\frac{1}{2}|[\arctan\frac{\pi[t+|(\mathcal{V}^{[m]}u)(t)|]}{2(1+t^2)} - \arctan\frac{\pi[t+|(\mathcal{V}^{[m]}v)(t)|]}{2(1+t^2)}] \\
&\cdot\int_0^t \frac{d\tau}{\omega+t+\tau+\sin(\mathcal{V}^{[n]}v)(h(\tau))}| + \sup_{t\in A}|\frac{1}{2}\arctan\frac{\pi}{2}\frac{t+|(\mathcal{V}^{[n]}v)(t)|}{1+t^2} \\
&\cdot\int_0^t \frac{[\sin(\mathcal{V}^{[m]}u)(h(\tau)) - \sin(\mathcal{V}^{[n]}v)(h(\tau))]d\tau}{[\omega+t+\tau+\sin(\mathcal{V}^{[m]}u)(h(\tau))][\omega+t+\tau+\sin(\mathcal{V}^{[n]}v)(h(\tau))]}|
\end{aligned}$$

for each $m,n \ge 1$. Hence, using (7.4) and (7.5), it is routinely verifiable that

$$\begin{aligned}
&D(\mathcal{V}^{[m+1]}u, \mathcal{V}^{[n+1]}v) \\
\le\ & \frac{2}{5}D(\mathcal{V}^{[m]}u, \mathcal{V}^{[n]}v) + \frac{\pi a_1}{4\omega}\sup_{t\in A}\left|(\mathcal{V}^{[m]}u)(t) - (\mathcal{V}^{[n]}v)(t)\right| \\
&+\frac{\pi a_1}{4\omega^2}\sup_{\tau\in A}\left|(\mathcal{V}^{[m]}u)(h(\tau)) - (\mathcal{V}^{[n]}v)(h(\tau))\right| \\
\le\ & (\frac{2}{5}+\frac{\pi a_1}{4\omega})D(\mathcal{V}^{[m]}u, \mathcal{V}^{[n]}v) + \frac{\pi a_1}{4\omega^2}\sup_{\beta\in A}\left|(\mathcal{V}^{[m]}u)(\beta) - (\mathcal{V}^{[n]}v)(\beta)\right| \\
\le\ & (\frac{2}{5}+\frac{\pi a_1}{4\omega}+\frac{\pi a_1}{4\omega^2})D(\mathcal{V}^{[m]}u, \mathcal{V}^{[n]}v) = \gamma D(\mathcal{V}^{[m]}u, \mathcal{V}^{[n]}v)
\end{aligned}$$

for each $m,n \ge 1$. Next, repeat the above construction of $r$ times and starting from $m = r+s-1$ and $n = r+l-1$ we obtain (7.6) with (7.7).

7.1.5. In addition, we see that

$$D(\mathcal{V}^{[m]}u, \mathcal{V}v) \le \gamma_0 D(\mathcal{V}^{[m-1]}u, v) \text{ if } m \ge 2,\ u,v \in \mathcal{C}(A,\mathbb{R}), \tag{7.8}$$

where

$$\gamma_0 = \frac{2}{5} + \frac{\pi}{4}a_1\frac{\omega+1}{\omega(\omega-1)}. \tag{7.9}$$

Indeed, using (7.4) and (7.5), in this case we have

$$\begin{aligned}
& D(\mathcal{V}^{[m]}u, \mathcal{V}v) = \sup_{t\in A}\left|(\mathcal{V}^{[m]}u)(t) - (\mathcal{V}v)(t)\right| \\
\leq\ & \sup_{t\in A}\frac{1}{5}\frac{\left|\sin^2(\mathcal{V}^{[m-1]}u)(t) - \sin^2 v(t)\right|}{\left[1+\sin^2(\mathcal{V}^{[m-1]}u)(t)\right]\left[1+\sin^2 v(t)\right]} \\
& + \sup_{t\in A}\frac{1}{2}|[\arctan\frac{\pi[t+|(\mathcal{V}^{[m-1]}u)(t)|]}{2(1+t^2)} - \arctan\frac{\pi[t+|v(t)|]}{2(1+t^2)}] \\
& \cdot\int_0^t \frac{d\tau}{\omega+t+\tau+\sin v(h(\tau))}| + \sup_{t\in A}|\frac{1}{2}\arctan\frac{\pi[t+|v(t)|]}{2(1+t^2)} \\
& \cdot\int_0^t \frac{[\sin(\mathcal{V}^{[m-1]}u)(h(\tau)) - \sin v(h(\tau))]d\tau}{[\omega+t+\tau+\sin(\mathcal{V}^{[m-1]}u)(h(\tau))][\omega+t+\tau+\sin v(h(\tau))]}| \\
\leq\ & \frac{2}{5}D(\mathcal{V}^{[m-1]}u, v) + \frac{\pi}{4}a_1\frac{1}{\omega-1}\sup_{t\in A}\left|(\mathcal{V}^{[m-1]}u)(t) - v(t)\right| \\
& + \frac{\pi}{4}a_1\frac{1}{\omega(\omega-1)}\sup_{\tau\in A}\left|(\mathcal{V}^{[m]}u)(h(\tau)) - v(h(\tau))\right| \\
\leq\ & (\frac{2}{5}+\frac{\pi}{4}a_1\frac{1}{\omega-1})D(\mathcal{V}^{[m]}u, v) + \frac{\pi}{4}a_1\frac{1}{\omega(\omega-1)}\sup_{\beta\in A}\left|(\mathcal{V}^{[m]}u)(\beta) - v(\beta)\right| \\
\leq\ & (\frac{2}{5}+\frac{\pi}{4}a_1\frac{1}{\omega-1}+\frac{\pi}{4}a_1\frac{1}{\omega(\omega-1)})D(\mathcal{V}^{[m]}u, v) = \gamma_0 D(\mathcal{V}^{[m]}u, v).
\end{aligned}$$

7.1.6. Furthermore, using (7.3) and (7.4), for $r \in \mathbb{N}$,

$$D(\mathcal{V}^{[s+r]}u, \mathcal{V}^{[l+r]}v) \leq \begin{cases} \gamma^r\gamma_0 D(\mathcal{V}^{[s-1]}u, v) & \text{if } s \geq 2,\ l = 1, \\ \gamma^r\gamma_1 D(u, v) & \text{if } s = l = 1. \end{cases} \tag{7.10}$$

7.1.7. We have the following properties: a gauge space $(\mathcal{C}(A,\mathbb{R}), D)$ is sequentially complete, properties (7.6)-(7.10) imply property

$$\begin{aligned}
& \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}x, \mathcal{V}^{[l]}y) \\
<\ & \varepsilon+\eta \Rightarrow D(\mathcal{V}^{[s+r]}x, \mathcal{V}^{[l+r]}y) \leq \gamma^r D(\mathcal{V}^{[s]}x, \mathcal{V}^{[l]}y) \\
\leq\ & \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}x, \mathcal{V}^{[l]}y)\},
\end{aligned}$$

and properties (7.2) and (7.3) imply that the single-valued dynamic system $(\mathcal{C}(A,\mathbb{R}), \mathcal{V})$ is $D$-closed on $\mathcal{C}(A,\mathbb{R})$, $q = 1$. From this we conclude that (I.5) with (2.7) and the assertion (B) of Theorem 4.1(vii) for $\mathcal{W} = \mathcal{V}$ hold.

**Remark 7.1 .** We deduce from (7.3), (7.7) and (7.9) that:
(a) $\gamma < \gamma_0 < \gamma_1$.
(b) If $a_1 = 1$ and $\omega = 2$, then $\gamma < 1 < \gamma_0 < \gamma_1$.
(c) If $a_1 = 4$ and $\omega = 7$, then $\gamma < \gamma_0 < 1 < \gamma_1$.

**Example 7.2.** Let $a_1 \in (0;\infty)$ and $\omega \in (1;\infty)$ be arbitrary and fixed and such that

$$\frac{2}{5} + \frac{\pi a_1}{4}\frac{\omega}{(\omega-1)^2} < 1 \tag{7.11}$$



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or

$$\frac{2}{5}+\frac{\pi a_1}{4}\frac{\omega+1}{\omega^2}<1 \text{ and } \frac{a_1}{\omega-1}\leq \exp(8-\frac{0.4}{\pi})-1. \tag{7.12}$$

Let $A=[0;a_1]$ and $h\in\mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{F}}\subset\mathcal{C}(A,\mathbb{R})$ of solutions of the Fredholm quadratic integral equation of the form

$$u(t)=\frac{1}{5}\frac{\sin^2 u(t)}{1+\sin^2 u(t)}+\frac{1}{2}\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)}\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+\sin u(h(\tau))},$$

$t\in A$, $u\in\mathcal{C}(A,\mathbb{R})$, is nonempty. Moreover, for each $y^0\in\mathcal{C}(A,\mathbb{R})$, there exists $y\in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{F})\subset\mathcal{Y}_{\mathcal{F}}$ such that a sequence $(\mathcal{F}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$ is $D$-convergent to $y$. Here: $\mathcal{F}^{[m+1]}:\mathcal{C}(A,\mathbb{R})\to\mathcal{C}(A,\mathbb{R})$, $m\in\{0\}\cup\mathbb{N}$, is defined by

$$\begin{aligned}(\mathcal{F}^{[m+1]}u)(t)&=\frac{1}{5}\frac{\sin^2(\mathcal{F}^{[m]}u)(t)}{1+\sin^2(\mathcal{F}^{[m]}u)(t)}\\&+\frac{1}{2}\arctan\frac{\pi[t+|(\mathcal{F}^{[m]}u)(t)|]}{2(1+t^2)}\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+\sin(\mathcal{F}^{[m]}u)(h(\tau))},\end{aligned}$$

$t\in A$, $u\in\mathcal{C}(A,\mathbb{R})$, $\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{D(x,y)=\sup_{t\in A}|x(t)-y(t)|\}$, and $(\mathcal{C}(A,\mathbb{R}),D)$ is a gauge space.

The proof proceeds in four steps.

Let

$$f(t,u(t))=\frac{1}{5}\frac{\sin^2 u(t)}{1+\sin^2 u(t)},\ g(t,u(t))=\frac{1}{2}\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)},$$

$$K(t,\tau,u(h(\tau)))=\frac{1}{\omega+t+\tau+\sin u(h(\tau))},\ u\in\mathcal{C}(A,\mathbb{R}),\ t,\tau\in A.$$

7.2.1. If $(a_1,\omega)\in(0;\infty)\times(1;\infty)$ satisfy (7.11), then

$$\forall_{s,l,r\in\mathbb{N}}\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{F}^{[r+s]}u,\mathcal{F}^{[r+l]}v)\leq\lambda^r D(\mathcal{F}^{[s]}u,\mathcal{F}^{[l]}v)\} \tag{7.13}$$

and

$$\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{F}u,\mathcal{F}v)\leq\lambda D(u,v)\} \tag{7.14}$$

where

$$\lambda=\frac{2}{5}+\frac{\pi a_1}{4}\frac{\omega}{(\omega-1)^2}<1. \tag{7.15}$$

In fact, we see that

$$
\begin{aligned}
& D(\mathcal{F}^{[m+1]}u,\mathcal{F}^{[n+1]}v)=\sup_{t\in A}\left|(\mathcal{F}^{[m+1]}u)(t)-(\mathcal{F}^{[n+1]}v)(t)\right| \\
\leq\ & \sup_{t\in A}\frac{1}{5}\frac{\left|\sin^2(\mathcal{F}^{[m]}u)(t)-\sin^2(\mathcal{F}^{[n]}v)(t)\right|}{\left[1+\sin^2(\mathcal{F}^{[m]}u)(t)\right]\left[1+\sin^2(\mathcal{F}^{[n]}v)(t)\right]} \\
& +\sup_{t\in A}\frac{1}{2}|[\arctan\frac{\pi[t+|(\mathcal{F}^{[m]}u)(t)|]}{2(1+t^2)}-\arctan\frac{\pi[t+|(\mathcal{F}^{[m]}v)(t)|]}{2(1+t^2)}] \\
& \cdot\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+\sin(\mathcal{F}^{[n]}v)(h(\tau))}|+\sup_{t\in A}|\frac{1}{2}\arctan\frac{\pi[t+|(\mathcal{F}^{[n]}v)(t)|\}}{2(1+t^2)} \\
& \cdot\int_0^{a_1}\frac{[\sin(\mathcal{F}^{[m]}u)(h(\tau))-\sin(\mathcal{F}^{[n]}v)(h(\tau))]d\tau}{[\omega+t+\tau+\sin(\mathcal{F}^{[m]}u)(h(\tau))][\omega+t+\tau+\sin(\mathcal{F}^{[n]}v)(h(\tau))]}| \\
\leq\ & \frac{2}{5}D(\mathcal{F}^{[m]}u,\mathcal{F}^{[n]}v)+\frac{\pi a_1}{4(\omega-1)}\sup_{t\in A}\left|(\mathcal{F}^{[m]}u)(t)-(\mathcal{F}^{[n]}v)(t)\right| \\
& +\frac{\pi a_1}{4(\omega-1)^2}\sup_{\tau\in A}\left|(\mathcal{F}^{[m]}u)(h(\tau))-(\mathcal{F}^{[n]}v)(h(\tau))\right| \\
\leq\ & (\frac{2}{5}+\frac{\pi a_1}{4(\omega-1)})D(\mathcal{F}^{[m]}u,\mathcal{F}^{[n]}v)+\frac{\pi a_1}{4(\omega-1)^2}\sup_{\beta\in A}\left|(\mathcal{F}^{[m]}u)(\beta)-(\mathcal{F}^{[n]}v)(\beta)\right| \\
\leq\ & (\frac{2}{5}+\frac{\pi a_1}{4(\omega-1)}+\frac{\pi a_1}{4(\omega-1)^2})D(\mathcal{F}^{[m]}u,\mathcal{F}^{[n]}v)=\lambda D(\mathcal{F}^{[m]}u,\mathcal{F}^{[n]}v)
\end{aligned}
$$

for each $m,n\geq 1$ or $m=n=0$. From this we conclude that (7.13) and (7.14) hold. Of course, (7.11) implies (7.15).

7.2.2. Observe that if $a_1\in(0;\infty)$ and $\omega\in(1;\infty)$ are such that

$$\frac{a_1}{\omega-1}\leq\exp\left(4-\frac{0.4}{\pi}\right)-1, \tag{7.16}$$

then

$$\forall_{t\in A}\forall_{m\in\{0\}\cup\mathbb{N}}\forall_{u\in\mathcal{C}(A,\mathbb{R}),h\in\mathcal{C}(A,A)}\{0\leq\mathcal{F}^{[m+1]}u)(h(t)\leq\pi\}. \tag{7.17}$$

Indeed, we see that

$$
\begin{aligned}
0\ \leq\ & g(t,(\mathcal{F}^{[m]}u)(t))\ln(1+\frac{a_1}{\omega+1+a_1})=g(t,(\mathcal{F}^{[m]}u)(t))\ln(\frac{\omega+1+2a_1}{\omega+1+a_1}) \\
\leq\ & g(t,(\mathcal{F}^{[m]}u)(t))\ln\frac{\omega+1+t+a_1}{\omega+1+t}=g(t,(\mathcal{F}^{[m]}u)(t))\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+1} \\
\leq\ & g(t,(\mathcal{F}^{[m]}u)(t))\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+\sin(\mathcal{F}^{[m]}u)(h(\tau))} \\
=\ & (\mathcal{F}^{[m+1]}u)(t)-f(t,(\mathcal{F}^{[m]}u)(t))\leq g(t,(\mathcal{F}^{[m]}u)(t))\int_0^{a_1}\frac{d\tau}{\omega+t+\tau-1} \\
=\ & g(t,(\mathcal{F}^{[m]}u)(t))\ln\frac{\omega-1+t+a_1}{\omega-1+t}\leq g(t,(\mathcal{F}^{[m]}u)(t))\ln\frac{\omega-1+a_1}{\omega-1} \\
=\ & g(t,(\mathcal{F}^{[m]}u)(t))\ln(1+\frac{a_1}{\omega-1})\leq\frac{\pi}{4}\ln(1+\frac{a_1}{\omega-1}).
\end{aligned}
$$

Hence, we conclude that if (7.16) holds, then

$$\begin{aligned} 0 &\leq f(t,(\mathcal{F}^{[m]}u)(t)) \leq (\mathcal{F}^{[m+1]}u)(t) \leq f(t,(\mathcal{F}^{[m]}u)(t)) + \frac{\pi}{4}\ln(1+\frac{a_1}{\omega-1}) \\ &\leq \frac{1}{10}+\frac{\pi}{4}\ln(1+\frac{a_1}{\omega-1}) \leq \pi \end{aligned}$$

so that $\forall_{t\in A}\forall_{m\in\{0\}\cup\mathbb{N}}\{0 \leq (\mathcal{F}^{[m+1]}u)(t) \leq \pi\}$. Since $h \in \mathcal{C}(A,A)$, hence we obtain $\forall_{t\in A}\forall_{m\in\{0\}\cup\mathbb{N}}\{0 \leq (\mathcal{F}^{[m+1]}u)(h(t)) \leq \pi\}$. Therefore, (7.17) holds.

7.2.3. Let $(a_1,\omega) \in (0;\infty)\times(1;\infty)$ satisfy (7.12). Then

$$\forall_{s,l,r\in\mathbb{N}}\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{F}^{[r+s]}u,\mathcal{F}^{[r+l]}v) \leq \lambda^r D(\mathcal{F}^{[s]}u,\mathcal{F}^{[l]}v)\}, \tag{7.18}$$

$$\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{F}u,\mathcal{F}v) \leq \lambda_1 D(u,v)\}, \tag{7.19}$$

$$D(\mathcal{F}^{[m]}u,\mathcal{F}v) \leq \lambda_0 D(\mathcal{F}^{[m-1]}u,v) \text{ if } m \geq 2 \tag{7.20}$$

and

$$D(\mathcal{F}^{[s+r]}u,\mathcal{F}^{[l+r]}v) \leq \begin{cases} \lambda^r\lambda_0 D(\mathcal{F}^{[s-1]}u,v) & \text{if } s\geq 2,\ l=1 \\ \lambda^r\lambda_1 D(u,v) & \text{if } s=l=1 \end{cases} \tag{7.21}$$

where

$$\lambda = \frac{2}{5}+\frac{\pi a_1}{4}\frac{\omega+1}{\omega^2} < 1, \tag{7.22}$$

$$\lambda_1 = \frac{2}{5}+\frac{\pi a_1}{4}\frac{\omega}{(\omega-1)^2} \tag{7.23}$$

and

$$\lambda_0 = \frac{2}{5}+\frac{\pi a_1}{4}\frac{\omega+1}{\omega(\omega-1)}. \tag{7.24}$$

To verify this, we take any $(a_1,\omega) \in (0;\infty)\times(1;\infty)$ satisfying (7.12).
We deduce after some calculations using (7.16) and (7.17) that

$$\begin{aligned} & D(\mathcal{F}^{[m+1]}u,\mathcal{F}^{[n+1]}v) = \sup_{t\in A}\left|(\mathcal{F}^{[m+1]}u)(t)-(\mathcal{F}^{[n+1]}v)(t)\right| \\ \leq\ & \sup_{t\in A}\frac{1}{5}\frac{\left|\sin^2(\mathcal{F}^{[m]}u)(t)-\sin^2(\mathcal{F}^{[n]}v)(t)\right|}{\left[1+\sin^2(\mathcal{F}^{[m]}u)(t)\right]\left[1+\sin^2(\mathcal{F}^{[n]}v)(t)\right]} \\ & +\sup_{t\in A}\frac{1}{2}|[\arctan\frac{\pi[t+|(\mathcal{F}^{[m]}u)(t)|]}{2(1+t^2)}-\arctan\frac{\pi[t+|(\mathcal{F}^{[m]}v)(t)|]}{2(1+t^2)}] \\ & \cdot\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+\sin(\mathcal{F}^{[n]}v)(h(\tau))}|+\sup_{t\in A}|\frac{1}{2}\arctan\frac{\pi[t+|(\mathcal{F}^{[n]}v)(t)|]}{2(1+t^2)} \\ & \cdot\int_0^{a_1}\frac{[\sin(\mathcal{F}^{[m]}u)(h(\tau))-\sin(\mathcal{F}^{[n]}v)(h(\tau))]d\tau}{[\omega+t+\tau+\sin(\mathcal{F}^{[m]}u)(h(\tau))][\omega+t+\tau+\sin(\mathcal{F}^{[n]}v)(h(\tau))]}| \end{aligned}$$

for each $m, n \geq 1$. Thus

$$
\begin{aligned}
& D(\mathcal{F}^{[m+1]}u, \mathcal{F}^{[n+1]}v) \\
\leq\ & \frac{2}{5}D(\mathcal{F}^{[m]}u, \mathcal{F}^{[n]}v) + \frac{\pi a_1}{4\omega}\sup_{t\in A}\left|(\mathcal{F}^{[m]}u)(t) - (\mathcal{F}^{[n]}v)(t)\right| \\
& + \frac{\pi a_1}{4\omega^2}\sup_{\tau\in A}\left|(\mathcal{F}^{[m]}u)(h(\tau)) - (\mathcal{F}^{[n]}v)(h(\tau))\right| \\
\leq\ & (\frac{2}{5} + \frac{\pi a_1}{4\omega})D(\mathcal{F}^{[m]}u, \mathcal{F}^{[n]}v) + \frac{\pi a_1}{4\omega^2}\sup_{\beta\in A}\left|(\mathcal{F}^{[m]}u)(\beta) - (\mathcal{F}^{[n]}v)(\beta)\right| \\
\leq\ & (\frac{2}{5} + \frac{\pi a_1}{4\omega} + \frac{\pi a_1}{4\omega^2})D(\mathcal{F}^{[m]}u, \mathcal{F}^{[n]}v) = \lambda D(\mathcal{F}^{[m]}u, \mathcal{F}^{[n]}v).
\end{aligned}
$$

Hence we derive (7.18) with (7.22).

Now if $u, v \in \mathcal{C}(A, \mathbb{R})$, then

$$
\begin{aligned}
& D(\mathcal{F}u, \mathcal{F}v) = \sup_{t\in A}|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\
\leq\ & \sup_{t\in A}\frac{1}{5}\frac{\left|\sin^2 u(t) - \sin^2 v(t)\right|}{\left[1+\sin^2 u(t)\right]\left[1+\sin^2 v(t)\right]} \\
& + \sup_{t\in A}\frac{1}{2}|[\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)} - \arctan\frac{\pi[t+|v(t)|]}{2(1+t^2)}] \\
& \cdot\int_0^{a_1}\frac{d\tau}{\omega+t+\tau+\sin v(h(\tau)}| + \sup_{t\in A}|\frac{1}{2}\arctan\frac{\pi}{2}\frac{t+|v(t)|}{1+t^2} \\
& \cdot\int_0^{a_1}\frac{[\sin u(h(\tau)) - \sin v(h(\tau))]d\tau}{[\omega+t+\tau+\sin u(h(\tau))][\omega+t+\tau+\sin v(h(\tau))]}| \\
\leq\ & \frac{2}{5}D(u,v) + \frac{\pi a_1}{4(\omega-1)}\sup_{t\in A}|u(t) - v(t)| \\
& + \frac{\pi a_1}{4(\omega-1)^2}\sup_{\tau\in A}|u(h(\tau)) - v(h(\tau))| \\
\leq\ & (\frac{2}{5} + \frac{\pi a_1}{4(\omega-1)})D(u,v) + \frac{\pi a_1}{4(\omega-1)^2}\sup_{\beta\in A}|u(\beta) - v(\beta)| \\
\leq\ & (\frac{2}{5} + \frac{\pi a_1}{4(\omega-1)} + \frac{\pi a_1}{4(\omega-1)^2})D(u,v) = \lambda_1 D(u,v).
\end{aligned}
$$

This becomes (7.19) with (7.23).

Now if $m \geq 2$, then

$$
\begin{aligned}
& D(\mathcal{F}^{[m]}u, \mathcal{F}v) = \sup_{t\in A}\left|(\mathcal{F}^{[m]}u)(t) - (\mathcal{F}v)(t)\right| \\
\leq\ & \frac{2}{5}D(\mathcal{F}^{[m]}u, \mathcal{F}^{[n]}v) + \frac{\pi a_1}{4(\omega-1)}\sup_{t\in A}\left|(\mathcal{F}^{[m-1]}u)(t) - v(t)\right| \\
& + \frac{\pi a_1}{4\omega(\omega-1)}\sup_{\tau\in A}\left|(\mathcal{F}^{[m-1]}u)(h(\tau)) - v(h(\tau))\right|.
\end{aligned}
$$

In consequence

$$
\begin{aligned}
& D(\mathcal{F}^{[m]}u, \mathcal{F}v) \\
\leq \;& (\frac{2}{5} + \frac{\pi a_1}{4(\omega-1)})D(\mathcal{F}^{[m-1]}u, v) + \frac{\pi a_1}{4\omega(\omega-1)} \sup_{\beta \in A} \left| (\mathcal{F}^{[m-1]}u)(\beta) - v(\beta) \right| \\
\leq \;& (\frac{2}{5} + \frac{\pi a_1}{4(\omega-1)} + \frac{\pi a_1}{4\omega(\omega-1)})D(\mathcal{F}^{[m-1]}u, v) = \lambda_0 D(\mathcal{F}^{[m-1]}u, v).
\end{aligned}
$$

Thus (7.20) holds with (7.24).

The proof of (7.21) mimics the one above.

7.2.4. We claim that: a gauge space $(\mathcal{C}(A,\mathbb{R}), D)$ is sequentially complete, property (7.13) with (7.15) in case (7.11) and property (7.18) with (7.20) in case (7.12) imply

$$
\begin{aligned}
& \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{F}^{[s]}x, \mathcal{F}^{[l]}y) \\
< \;& \varepsilon + \eta \Rightarrow D(\mathcal{F}^{[s+r]}x, \mathcal{F}^{[l+r]}y) \leq \lambda^r D(\mathcal{F}^{[s]}x, \mathcal{F}^{[l]}y) \\
\leq \;& \frac{\varepsilon}{\varepsilon+\eta} D(\mathcal{F}^{[s]}x, \mathcal{F}^{[l]}y)\},
\end{aligned}
$$

and property (7.14) with (7.15) in case (7.11) and property (7.19) with (7.23) in case (7.12) imply that the single-valued dynamic system $(\mathcal{C}(A,\mathbb{R}), \mathcal{F})$ is $D$-closed on $\mathcal{C}(A,\mathbb{R})$, $q=1$. This means that (I.5) with (2.8) and the assertion (B) of Theorem 4.1(vii) for $\mathcal{W} = \mathcal{F}$ hold.

**Remark 7.2 .** We have:

(a) For each $(a_1, \omega) \in (0;\infty) \times (1;\infty)$,

$$
\frac{2}{5} + \frac{\pi a_1}{4}\frac{\omega+1}{\omega^2} < \frac{2}{5} + \frac{\pi a_1}{4}\frac{\omega}{(\omega-1)^2}.
$$

Thus $\frac{2}{5} + \frac{\pi a_1}{4}\frac{\omega}{(\omega-1)^2} < 1$ implies $\frac{2}{5} + \frac{\pi a_1}{4}\frac{\omega+1}{\omega^2} < 1$ but not conversely.

(b) If (7.11) holds, then

$$
\frac{a_1}{\omega-1} \leq \exp(4 - \frac{0.4}{\pi}) - 1.
$$

In fact, (7.11) implies

$$
\frac{2}{5} + \frac{\pi a_1}{4}\frac{\omega}{(\omega-1)^2} = \frac{2}{5} + \frac{\pi}{4}\frac{a_1}{\omega-1}\frac{\omega}{\omega-1} < 1.
$$

This gives

$$
\frac{a_1}{\omega-1} < \frac{12}{5\pi}\left(1 - \frac{1}{\omega}\right) < \frac{12}{5\pi} < 0.764.
$$

However

$$
\exp(4 - \frac{0.4}{\pi}) - 1 = \delta \approx 47.07085390482713.
$$



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(c) From (a) and (b) it follows that (7.11) implies (7.12).

(d) Choosing $(a_1, \omega)$ so that $a_1 = 1$ and $\omega = 2$, we see that (7.12) holds since

$$\frac{2}{5} + \frac{\pi a_1}{4} \frac{\omega + 1}{\omega^2} < 0.98905 < 1 \text{ and } \frac{a_1}{\omega - 1} = 1 \leq \exp(4 - \frac{0.4}{\pi}) - 1 = \delta$$

but (7.11) does not hold since

$$\frac{2}{5} + \frac{\pi a_1}{4} \frac{\omega}{(\omega - 1)^2} = 0.4 + \frac{\pi}{2} > 1.$$

(e) Let us observe that

$$\frac{2}{5} + \frac{\pi a_1}{4} \frac{\omega + 1}{\omega^2} < 1 \text{ and } \omega > 1.00821495838332$$

implies (7.12). To verify this, we see that

$$\frac{2}{5} + \frac{\pi a_1}{4} \frac{\omega + 1}{\omega^2} = \frac{2}{5} + \frac{\pi}{4} \frac{a_1}{\omega - 1} \frac{\omega^2 - 1}{\omega^2} < 1$$

implies

$$\frac{a_1}{\omega - 1} < \frac{12}{5\pi} \frac{\omega^2}{\omega^2 - 1}.$$

Next, we calculate that

$$\frac{12}{5\pi} \frac{\omega^2}{\omega^2 - 1} \leq \exp(4 - \frac{0.4}{\pi}) - 1 = \delta$$

implies

$$\omega > \sqrt{\frac{5\pi\delta}{5\pi\delta - 12}} = 1.00821495838332.$$

**Example 7.3.** Let $a_1 \in (0; \infty)$, $\omega \in (1; \infty)$ and $\theta > 0$ be arbitrary and fixed and such that

$$\gamma = \frac{2}{5} + \frac{\pi}{4} \frac{a_1}{\omega} + \frac{\pi}{4} \frac{1 - e^{-\theta a_1}}{\theta \omega^2} < 1. \tag{7.25}$$

Let $A = [0; a_1]$. The set $\mathcal{Y}_{\mathcal{V}} \subset \mathcal{C}(A, \mathbb{R})$ of solutions of the Volterra quadratic integral equation of the form

$$u(t) = \frac{1}{5} \frac{\sin^2 u(t)}{1 + \sin^2 u(t)} + \frac{1}{2} \arctan \frac{\pi[t + |u(t)|]}{2(1 + t^2)} \int_0^t \frac{d\tau}{\omega + t + \tau + \sin u(\tau)},$$

$t \in A$, $u \in \mathcal{C}(A, \mathbb{R})$, is nonempty. Moreover, for each $y^0 \in \mathcal{C}(A, \mathbb{R})$, there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V}) \subset \mathcal{Y}_{\mathcal{V}}$ such that a sequence $(\mathcal{V}^{[m]} y^0 : m \in \{0\} \cup \mathbb{N})$ is *D*-convergent to $y$ where $\mathcal{V}^{[m+1]} : \mathcal{C}(A, \mathbb{R}) \to \mathcal{C}(A, \mathbb{R})$, $m \in \{0\} \cup \mathbb{N}$, is defined by

$$
\begin{aligned}
(\mathcal{V}^{[m+1]}u)(t) \quad &= \quad \frac{1}{5}\frac{\sin^2(\mathcal{V}^{[m]}u)(t)}{1+\sin^2(\mathcal{V}^{[m]}u)(t)} + \frac{1}{2}\arctan\frac{\pi[t+|(\mathcal{V}^{[m]}u)(t)|]}{2(1+t^2)} \\
&\quad \cdot \int_0^t \frac{d\tau}{\omega+t+\tau+\sin(\mathcal{V}^{[m]}u)(\tau)},
\end{aligned}
\tag{7.26}
$$

$\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\{D(x,y) = \sup_{t\in A} e^{-\theta t}|x(t)-y(t)|\}$ and $(\mathcal{C}(A,\mathbb{R}),D)$ is a gauge space.

The proof proceeds in four steps.

7.3.1. We see that

$$
\forall_{s,l,r\in\mathbb{N}}\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{V}^{[r+s]}u,\mathcal{V}^{[r+l]}v) \le \gamma^r D(\mathcal{V}^{[s]}u,\mathcal{V}^{[l]}v)\}. \tag{7.27}
$$

Indeed, analogously as in 7.1.2, we prove that

$$
\forall_{t\in A}\forall_{m\in\{0\}\cup\mathbb{N}}\forall_{u\in\mathcal{C}(A,\mathbb{R})}\{0\le\mathcal{V}^{[m+1]}u)(t) < \pi/4\}. \tag{7.28}
$$

Next, using (7.26) and (7.28), we have that, for each $m,n\in\mathbb{N}$ and for $u,v\in\mathcal{C}(A,\mathbb{R})$,

$$
\begin{aligned}
&D(\mathcal{V}^{[m+1]}u,\mathcal{V}^{[n+1]}v) = \sup_{t\in A} e^{-\theta t}\left|(\mathcal{V}^{[m+1]}u)(t)-(\mathcal{V}^{[n+1]}v)(t)\right| \\
\le \quad & \sup_{t\in A}\frac{1}{5}\frac{e^{-\theta t}\left|\sin^2(\mathcal{V}^{[m]}u)(t)-\sin^2(\mathcal{V}^{[n]}v)(t)\right|}{\left[1+\sin^2(\mathcal{V}^{[m]}u)(t)\right]\left[1+\sin^2(\mathcal{V}^{[n]}v)(t)\right]} \\
&+\sup_{t\in A}\frac{1}{2}e^{-\theta t}|[\arctan\frac{\pi[t+|(\mathcal{V}^{[m]}u)(t)|]}{2(1+t^2)} - \arctan\frac{\pi[t+|(\mathcal{V}^{[n]}v)(t)|]}{2(1+t^2)}] \\
&\cdot\int_0^t \frac{d\tau}{\omega+t+\tau+\sin(\mathcal{V}^{[n]}v)(\tau)}| + \sup_{t\in A} e^{-\theta t}|\frac{1}{2}\arctan\frac{\pi[t+|(\mathcal{V}^{[n]}v)(t)|]}{2(1+t^2)} \\
&\cdot\int_0^t \frac{[\sin(\mathcal{V}^{[m]}u)(\tau)-\sin(\mathcal{V}^{[n]}v)(\tau)]d\tau}{[\omega+t+\tau+\sin(\mathcal{V}^{[m]}u)(\tau)][\omega+t+\tau+\sin(\mathcal{V}^{[n]}v)(\tau)]}| \\
\le \quad & (\frac{2}{5}+\frac{\pi a_1}{4\omega})D(\mathcal{V}^{[m]}u,\mathcal{V}^{[n]}v) + \frac{\pi}{4\omega^2}\sup_{t\in A} e^{-\theta t}\int_0^t \left|(\mathcal{V}^{[m]}u)(\tau)-(\mathcal{V}^{[n]}v)(\tau)\right| d\tau.
\end{aligned}
$$

But

$$
\begin{aligned}
& e^{-\theta t}\int_0^t \left|(\mathcal{V}^{[m]}u)(\tau)-(\mathcal{V}^{[n]}v)(\tau)\right| d\tau \\
= \quad & e^{-\theta t}\int_0^t \left|(\mathcal{V}^{[m]}u)(\tau)-(\mathcal{V}^{[n]}v)(\tau)\right| e^{-\theta\tau}e^{\theta\tau}d\tau \\
= \quad & e^{-\theta t}\int_0^t e^{-\theta\tau}\left|(\mathcal{V}^{[m]}u)(\tau)-(\mathcal{V}^{[n]}v)(\tau)\right| e^{\theta\tau}d\tau.
\end{aligned}
$$



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This gives

$$
\begin{aligned}
& \sup_{t\in A} e^{-\theta t}\int_0^t \left|(\mathcal{V}^{[m]}u)(\tau)-(\mathcal{V}^{[n]}v)(\tau)\right| d\tau \\
\leq\ & \sup_{t\in A} e^{-\theta t}\int_0^t \sup_{\beta\in A}[e^{-\theta\beta}\left|(\mathcal{V}^{[m]}u)(\beta)-(\mathcal{V}^{[n]}v)(\beta)\right|]e^{\theta\tau}d\tau \\
=\ & D(\mathcal{V}^{[m]}u,\mathcal{V}^{[n]}v)\sup_{t\in A} e^{-\theta t}\int_0^t e^{\theta\tau}d\tau = D(\mathcal{V}^{[m]}u,\mathcal{V}^{[n]}v)\sup_{t\in A} e^{-\theta t}\frac{e^{\theta t}-1}{\theta} \\
=\ & \frac{1-e^{-\theta a_1}}{\theta}D(\mathcal{V}^{[m]}u,\mathcal{V}^{[n]}v).
\end{aligned}
$$

Therefore, using (7.25),

$$
\begin{aligned}
D(\mathcal{V}^{[m+1]}u,\mathcal{V}^{[n+1]}v) &= \sup_{t\in A} e^{-\theta t}\left|(\mathcal{V}^{[m+1]}u)(t)-(\mathcal{V}^{[n+1]}v)(t)\right| \\
&\leq (\frac{2}{5}+\frac{\pi a_1}{4\omega}+\frac{\pi}{4\omega^2}\frac{1-e^{-\theta a_1}}{\theta})D(\mathcal{V}^{[m]}u,\mathcal{V}^{[n]}v) \\
&= \gamma D(\mathcal{V}^{[m]}u,\mathcal{V}^{[n]}v).
\end{aligned}
$$

Finally, repeat the above construction of $r$ times and starting from $m=r+s-1$ and $n=r+l-1$ we obtain (7.27).

7.3.2. We show that

$$
\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{V}u,\mathcal{V}v)\leq\gamma_1 D(u,v)\} \tag{7.29}
$$

where

$$
\gamma_1=\frac{2}{5}+\frac{\pi}{4}\frac{a_1}{\omega-1}+\frac{\pi}{4}\frac{1-e^{-\theta a_1}}{\theta(\omega-1)^2}.
$$

In fact, we have

$$
\begin{aligned}
& D(\mathcal{V}u,\mathcal{V}v)=\sup_{t\in A} e^{-\theta t}\left|(\mathcal{V}u)(t)-(\mathcal{V}v)(t)\right| \\
\leq\ & \sup_{t\in A}\frac{1}{5}\frac{e^{-\theta t}\left|\sin^2 u(t)-\sin^2 v(t)\right|}{\left[1+\sin^2(\mathcal{V}^{[m]}u)(t)\right]\left[1+\sin^2(\mathcal{V}^{[n]}v)(t)\right]} \\
& +\sup_{t\in A}\frac{1}{2}e^{-\theta t}|[\arctan\frac{\pi[t+|u(t)|]}{2(1+t^2)}-\arctan\frac{\pi[t+|v(t)|]}{2(1+t^2)}] \\
& \cdot\int_0^t\frac{d\tau}{\omega+t+\tau+\sin v(\tau)}|+\sup_{t\in A} e^{-\theta t}|\frac{1}{2}\arctan\frac{\pi[t+|v(t)|]}{2(1+t^2)} \\
& \cdot\int_0^t\frac{[\sin u(\tau)-\sin v(\tau)]d\tau}{[\omega+t+\tau+\sin u(\tau)][\omega+t+\tau+\sin v(\tau)]}| \\
\leq\ & (\frac{2}{5}+\frac{\pi a_1}{4(\omega-1)})D(u,v)+\frac{\pi}{4(\omega-1)^2}\sup_{t\in A} e^{-\theta t}\int_0^t|u(\tau)-v(\tau)|\,d\tau.
\end{aligned}
$$

Since

$$\begin{aligned}
& \sup_{t\in A} e^{-\theta t}\int_0^t |u(\tau)-v(\tau)|\,d\tau = \sup_{t\in A} e^{-\theta t}\int_0^t |u(\tau)-v(\tau)|\,e^{-\theta\tau}e^{\theta\tau}d\tau \\
\leq\ & \sup_{t\in A} e^{-\theta t}\int_0^t \sup_{\beta\in A}[e^{-\theta\beta}\,|u(\beta)-v(\beta)|]e^{\theta\tau}d\tau = D(u,v)\sup_{t\in A} e^{-\theta t}\int_0^t e^{\theta\tau}d\tau \\
=\ & D(u,v)\sup_{t\in A} e^{-\theta t}\frac{e^{\theta t}-1}{\theta} = \frac{1-e^{-\theta a_1}}{\theta}D(u,v),
\end{aligned}$$

this says

$$\begin{aligned}
& D(\mathcal{V}u,\mathcal{V}v) = \sup_{t\in A} e^{-\theta t}\,|(\mathcal{V}u)(t)-(\mathcal{V}v)(t)| \\
\leq\ & (\frac{2}{5}+\frac{\pi a_1}{4(\omega-1)}+\frac{\pi}{4(\omega-1)^2}\frac{1-e^{-\theta a_1}}{\theta})D(u,v) = \gamma_1 D(u,v).
\end{aligned}$$

7.3.3. Moreover, we see that

$$D(\mathcal{V}^{[m]}u,\mathcal{V}v) \leq \gamma_0 D(\mathcal{V}^{[m-1]}u,v) \text{ if } m\geq 2$$

and

$$D(\mathcal{V}^{[s+r]}u,\mathcal{V}^{[l+r]}v) \leq \begin{cases} \gamma^r\gamma_0 D(\mathcal{V}^{[s-1]}u,v) & \text{if } s\geq 2,\ l=1 \\ \gamma^r\gamma_1 D(u,v) & \text{if } s=l=1 \end{cases}$$

where $u,v\in\mathcal{C}(A,\mathbb{R})$, $r\in\mathbb{N}$ and

$$\gamma_0 = \frac{2}{5}+\frac{\pi}{4}\frac{a_1}{\omega-1}+\frac{\pi}{4}\frac{1-e^{-\theta a_1}}{\theta\omega(\omega-1)}.$$

7.3.4. Therefore, a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is a sequentially complete, properties (7.25) and (7.27) imply property

$$\begin{aligned}
& \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)<\varepsilon+\eta \\
\Rightarrow\ & D(\mathcal{V}^{[s+r]}x,\mathcal{V}^{[l+r]}y) \leq \gamma^r D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y) \leq \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y)\},
\end{aligned}$$

and (7.29) implies that $\mathcal{V}$ is $D$-closed on $\mathcal{C}(A,\mathbb{R})$. Hence it follows that (II.5) with (5.22) and the assertion (B) of Theorem 5.1(vi) hold.

**Remark 7.3** . Note that:

(a) $\gamma_1-\gamma = \frac{\pi}{4}\frac{1}{\omega(\omega-1)}\left[a_1+\frac{1-e^{-\theta a_1}}{\theta}\frac{2\omega-1}{\omega(\omega-1)}\right]>0.$

(b) Choosing $(a_1,\omega,\theta)$ so that $a_1=4$, $\omega=7$ and $\theta=0.10602$, we have

$$\begin{aligned}
\frac{2}{5}+\frac{\pi}{4}\frac{a_1}{\omega} &< 0.84879895052<1, \\
\gamma &= \frac{2}{5}+\frac{\pi}{4}\frac{a_1}{\omega}+\frac{\pi}{4}\frac{1-e^{-\theta a_1}}{\theta\omega^2}<0.999983<1, \\
\gamma_1 &= \frac{2}{5}+\frac{\pi}{4}\frac{a_1}{\omega-1}+\frac{\pi}{4}\frac{1-e^{-\theta a_1}}{\theta(\omega-1)^2} \\
&> 1.01373868>1>\gamma, \\
\gamma &< \gamma_0 = \frac{2}{5}+\frac{\pi}{4}\frac{a_1}{\omega-1}+\frac{\pi}{4}\frac{1-e^{-\theta a_1}}{\theta\omega(\omega-1)}<\gamma_1.
\end{aligned}$$



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**Example 7.4.** Let $a_1 \in (0;\infty)$, $A = [0;a_1]$, and $h \in \mathcal{C}(A,A)$. Assume that $f, g \in \mathcal{C}(A,\mathbb{R})$, $\sup_{t\in A}|f(t)| = \lambda$, and $\sup_{t\in A}|g(t)| = \omega$. Suppose also that $\varphi \in \mathcal{C}(A\times\mathbb{R},\mathbb{R})$. With the above notions, for $t \in A$, $u \in \mathcal{C}(A,\mathbb{R})$ and $m \in \{0\}\cup\mathbb{N}$, we set

$$u(t) = f(t) + g(t)u(t)\int_0^{a_1} \arctan\frac{\varphi(t,u(h(\tau)))}{1+\tau^2}d\tau, \tag{7.30}$$

$$(\mathcal{F}^{[m+1]}u)(t) = f(t)\frac{1}{1-g(t)\int_0^{a_1}\arctan\frac{\varphi(t,(\mathcal{F}^{[m]}u)(h(\tau)))}{1+\tau^2}d\tau} \tag{7.31}$$

and

$$M = \mathcal{F}(\mathcal{C}(A,\mathbb{R})); \text{ here } M \in 2^{\mathcal{C}(A,\mathbb{R})}. \tag{7.32}$$

The following three statements (S1)-(S3) hold:
(S1) Assume that $a_1$ and $\mu$ satisfy

$$2 - \pi\mu a_1 > 0. \tag{7.33}$$

Then

$$\forall_{m\in\mathbb{N}}\forall_{u\in\mathcal{C}(A,\mathbb{R})}\{\left\|\mathcal{F}^{[m]}u\right\| = \sup_{t\in A}|(\mathcal{F}^{[m]}u)(t)| \le \lambda\rho\} \tag{7.34}$$

where

$$\rho = \frac{2}{2-\pi\mu a_1}. \tag{7.35}$$

(S2) Assume that $a_1$, $\mu$ and $\varphi$ satisfy

$$1 - \mu\arctan a_1 > 0 \text{ and } \sup_{t,\beta\in A, v\in\mathcal{C}(A,\mathbb{R})}|\varphi(t,v(\beta))| \le 1. \tag{7.36}$$

Then

$$\forall_{m\in\mathbb{N}}\forall_{u\in\mathcal{C}(A,\mathbb{R})}\{\left\|\mathcal{F}^{[m]}u\right\| = \sup_{t\in A}|(\mathcal{F}^{[m]}u)(t)| \le \lambda\rho\} \tag{7.37}$$

where

$$\rho = \frac{1}{1-\mu\arctan a_1}. \tag{7.38}$$

(S3) Assume that assumption (7.33) or (7.36) holds and suppose, in addition, that

$$\exists_{\omega\in(0;\infty)}\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{\sup_{t,\beta\in A}|\varphi(t,u(\beta)) - \varphi(t,v(\beta))| \le \omega D(u,v)\} \tag{7.39}$$

and

$$\gamma = \lambda\rho^2\omega\arctan a_1 < 1. \tag{7.40}$$

Then the set $\mathcal{Y}_{\mathcal{F}} \subset \mathcal{C}(A, \mathbb{R})$ of solutions of the Fredholm quadratic integral equation (7.30) is nonempty. Moreover, for each $y^0 \in M$, there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{F}) \subset \mathcal{Y}_{\mathcal{F}}$ such that a sequence $(\mathcal{F}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A, \mathbb{R})$ of the form $\forall_{u,v \in \mathcal{C}(A,\mathbb{R})}\{D(u, v) = \sup_{t\in A} |u(t) - v(t)|\}$.

The proof proceeds in five steps.

7.4.1. In case (S1), the argument runs as follows: we assume that (7.33) holds and then, in view of (7.31), we see that

$$\begin{aligned}\left\|\mathcal{F}^{[m]}u\right\| &= \sup_{t\in A}\left|(\mathcal{F}^{[m]}u)(t)\right| \\ &\leq \sup_{t\in A}|f(t)|\frac{1}{1-\sup_{t\in A}|g(t)\int_0^{a_1}\arctan\frac{\varphi(t,(\mathcal{F}^{[m-1]}u)(h(\tau)))}{1+\tau^2}d\tau|} \\ &\leq \lambda\frac{1}{1-\mu\frac{\pi}{2}a_1} = \lambda\frac{2}{2-\pi\mu a_1} = \lambda\rho.\end{aligned}$$

Therefore, (7.34) with (7.35) holds.

7.4.2. In case (S2), the argument runs as follows: we assume that (7.36) holds and then, in view of (7.31), we have

$$\begin{aligned}&\left\|\mathcal{F}^{[m]}u\right\| = \sup_{t\in A}\left|(\mathcal{F}^{[m]}u)(t)\right| \\ \leq\ &\sup_{t\in A}|f(t)|\frac{1}{1-\sup_{t\in A}|g(t)\int_0^{a_1}\arctan\frac{\varphi(t,(\mathcal{F}^{[m-1]}u)(h(\tau)))}{1+\tau^2}d\tau|} \\ \leq\ &\frac{\lambda}{1-\mu\sup_{t,\beta\in A,v\in\mathcal{C}(A,\mathbb{R})}|\varphi(t,v(\beta))|\int_0^{a_1}\frac{d\tau}{1+\tau^2}} = \frac{\lambda}{1-\mu\arctan a_1} = \lambda\rho.\end{aligned}$$

Thus, (7.37) with (7.38) holds.

7.4.3. Assume that (S3) holds. Then we see that $\mathcal{F}$ is continuous on $M$ ($M$ is defined by (7.32)). More precisely, we have

$$\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(\mathcal{F}(\mathcal{F}u), \mathcal{F}(\mathcal{F}v)) \leq \gamma D(\mathcal{F}u, \mathcal{F}v)\}. \tag{7.41}$$

Indeed, in view of (7.30)-(7.38) and using arguments such as in 7.4.1 and 7.4.2, we obtain that

$$\begin{aligned}&D(\mathcal{F}(\mathcal{F}u), \mathcal{F}(\mathcal{F}v) = D(\mathcal{F}^{[2]}u, \mathcal{F}^{[2]}v) \\ =\ &\sup_{t\in A}\left|(\mathcal{F}^{[2]}u)(t) - (\mathcal{F}^{[2]}v)(t)\right| \\ =\ &\sup_{t\in A}|f(t)[\frac{1}{1-g(t)\int_0^{a_1}\arctan\frac{\varphi(t,(\mathcal{F}u)(h(\tau)))}{1+\tau^2}d\tau} \\ &-\frac{1}{1-g(t)\int_0^{a_1}\arctan\frac{\varphi(t,(\mathcal{F}v)(h(\tau)))}{1+\tau^2}d\tau}]| \\ \leq\ &\lambda\rho^2\sup_{t,\tau\in A}\int_0^{a_1}\frac{|\varphi(t,(\mathcal{F}u)(h(\tau))) - \varphi(t,(\mathcal{F}v)(h(\tau)))|}{1+\tau^2}d\tau.\end{aligned}$$



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Therefore

$$
\begin{aligned}
& D(\mathcal{F}(\mathcal{F}u), \mathcal{F}(\mathcal{F}v) = D(\mathcal{F}^{[2]}u, \mathcal{F}^{[2]}v) \\
\leq\ & \lambda\rho^2 \sup_{t.\beta\in A} |\varphi(t,(\mathcal{F}u)(\beta)) - \varphi(t,(\mathcal{F}v)(\beta))| \int_0^{a_1} \frac{d\tau}{1+\tau^2} \\
\leq\ & \lambda\rho^2 \arctan a_1 \cdot \sup_{t.\beta\in A} |\varphi(t,(\mathcal{F}u)(\beta)) - \varphi(t,(\mathcal{F}v)(\beta))| \\
\leq\ & \lambda\rho^2\omega \arctan a_1 \cdot D(\mathcal{F}u, \mathcal{F}v) = \gamma \cdot D(\mathcal{F}u, \mathcal{F}v).
\end{aligned}
$$

7.4.4. Assume that (S3) holds. We note that if $s, l, r \in \mathbb{N}$ and $y^0 \in M$, then

$$
D(\mathcal{F}^{[r+s]}y^0, \mathcal{F}^{[r+l]}y^0) \leq \gamma^r D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0). \tag{7.42}
$$

In fact, if $m, n \in \mathbb{N}$ and $u, v \in \mathcal{C}(A, \mathbb{R})$, then we repeat the above line of argument to obtain

$$
\begin{aligned}
& D(\mathcal{F}^{[m+1]}(\mathcal{F}u), \mathcal{F}^{[n+1]}(\mathcal{F}v)) = D(\mathcal{F}^{[m+2]}u, \mathcal{F}^{[n+2]}v) \\
=\ & \sup_{t\in A} \left| (\mathcal{F}^{[m+2]}u)(t) - (\mathcal{F}^{[n+2]}v)(t) \right| \\
\leq\ & \gamma D(\mathcal{F}^{[m+1]}u, \mathcal{F}^{[m+1]}v) = \gamma D(\mathcal{F}^{[m]}(\mathcal{F}u), \mathcal{F}^{[n]}(\mathcal{F}v).
\end{aligned}
$$

In particular, this gives (7.42).

7.4.5. Assume that (S3) holds. We have: a gauge space $(\mathcal{C}(A, \mathbb{R}), D)$ is sequentially complete, properties (7.40) and (7.42) imply property

$$
\begin{aligned}
& \forall_{\varepsilon>0} \exists_{\eta>0} \exists_{r\in\mathbb{N}} \forall_{y^0\in M} \forall_{s,l\in\mathbb{N}} \{ D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) < \varepsilon + \eta \\
\Rightarrow\ & D(\mathcal{F}^{[s+r]}y^0, \mathcal{F}^{[l+r]}y^0) \leq \gamma^r D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) \\
\leq\ & \frac{\varepsilon}{\varepsilon+\eta} D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) \},
\end{aligned}
$$

and (7.41) and (7.40) imply that $\mathcal{F}$ is $D$-closed in each point $y^0$ of $M$. Hence it follows that (I.4) and the assertion (B) of Theorem 4.2(vi) hold for $\mathcal{W} = \mathcal{F}$.

**Example 7.5.** We turn to some examples (see (a)-(d) below) in which 7.4.1-7.4.5 are satisfied.

(a) Assume that

$$
u(t) = \frac{1-t}{e} + \frac{1-t}{2(1+t)} u(t) \int_0^1 \arctan \frac{[1-(t-\frac{1}{2})^2] \sin^2 u(h(\tau))}{1+\tau^2} d\tau.
$$

It is clear that $A = [0;1]$, $f(t) = \frac{1-t}{e}$, $g(t) = \frac{1-t}{2(1+t)}$, $\varphi(t, u(\beta)) = p(t)\sin^2 u(\beta)$ where $p(t) = 1 - (t - \frac{1}{2})^2$, $\sup_{t\in A} |f(t)| = |f(0)| = \frac{1}{e} = \lambda$, $\sup_{t\in A} g(t) = g(0) = \frac{1}{2} = \mu$, $\sup_{t\in A} p(t) = p(\frac{1}{2}) = 1$,

$$
1 - \mu \arctan a_1 = 1 - \frac{1}{2}\frac{\pi}{4} = \frac{8-\pi}{8} > 0,
$$

$$\forall_{v\in\mathcal{C}(A,\mathbb{R})}\{\sup_{t,\beta\in A}|\varphi(t,v(\beta))|=\sup_{t,\beta\in A}p(t)\sin^2 u(\beta)\le 1\}$$

and, in view of (7.36)-(7.40),

$$\begin{aligned}
&\sup_{t.\beta\in A}|\varphi(t,(\mathcal{F}u)(\beta))-\varphi(t,(\mathcal{F}v)(\beta))|\le\sup_{\beta\in A}|\sin^2(\mathcal{F}u)(\beta))-\sin^2(\mathcal{F}v)(\beta))|\\
\le\;&\sup_{\beta\in A}[|\sin(\mathcal{F}u)(\beta)|+|\sin(\mathcal{F}v)(\beta)|]D(\mathcal{F}u,\mathcal{F}v)\\
\le\;&\sup_{\beta\in A}[|(\mathcal{F}u)(\beta)|+|(\mathcal{F}v)(\beta)|]D(\mathcal{F}u,\mathcal{F}v)\le 2\lambda\rho D(\mathcal{F}u,\mathcal{F}v).
\end{aligned}$$

Consequently,

$$\rho=\frac{1}{1-\mu\arctan a_1}=\frac{8}{8-\pi},\ \omega=2\lambda\rho=\lambda\frac{16}{8-\pi}=\frac{1}{e}\frac{16}{8-\pi}$$

and

$$\gamma=\lambda\rho^2\omega\arctan a_1=\lambda^2\frac{64\times 16}{(8-\pi)^3}\frac{\pi}{4}=\frac{1}{e^2}\frac{64\times 16}{(8-\pi)^3}\frac{\pi}{4}<\frac{804.24772}{847.36542}<1.$$

(a) We note that if

$$u(t)=-\frac{t^2}{e^{2t}}-\frac{\ln(1+t)}{2\ln 2}u(t)\int_0^1\arctan[\frac{1}{2}(\frac{1}{3}-t^2)\frac{u^2(h(\tau))}{1+\tau^2}]d\tau,$$

then $A=[0;1]$, $f(t)=-\frac{t^2}{e^{2t}}$, $g(t)=-\frac{\ln(1+t)}{2\ln 2}$, $\varphi(t,u(\beta))=p(t)u^2(\beta)$ where $p(t)=\frac{1}{2}(\frac{1}{3}-t^2)$, $\sup_{t\in[0;1]}|f(t)|=|f(1)|=\frac{1}{e^2}=\lambda$, $\sup_{t\in[0;1]}|g(t)|=|g(1)|=\frac{1}{2}=\mu$ and $\sup_{t\in[0;1]}|p(t)|=|p(1)|=\frac{1}{3}$. By (7.30)-(7.32), (7.36) and (7.37), we then get

$$\begin{aligned}
&\sup_{t.\beta\in A}|\varphi(t,(\mathcal{F}u)(\beta))-\varphi(t,(\mathcal{F}v)(\beta))|\le\frac{1}{3}\sup_{\beta\in A}|(\mathcal{F}u)^2(\beta))-(\mathcal{F}v)^2(\beta))|\\
\le\;&\frac{1}{3}\sup_{\beta\in A}[|(\mathcal{F}u)(\beta)|+|(\mathcal{F}v)(\beta)|]D(\mathcal{F}u,\mathcal{F}v)\le\frac{2}{3}\lambda\rho D(\mathcal{F}u,\mathcal{F}v),
\end{aligned}$$

$$\rho=\frac{2}{2-\pi\mu a_1}=\frac{4}{4-\pi},\ \omega=\frac{2}{3}\lambda\rho$$

and

$$\begin{aligned}
\gamma&=\lambda\rho^2\omega\arctan a_1=\frac{2}{3}\lambda^2\rho^3\frac{\pi}{4}=\frac{2}{3}\frac{1}{e^4}\frac{64}{(4-\pi)^3}\frac{\pi}{4}\\
&=\frac{32\pi}{3e^4(4-\pi)^3}\le\frac{100.53097}{103.60470}<1.
\end{aligned}$$

(c) If

$$u(t)=\frac{27}{2}\frac{t^2}{e^{3t}}+\frac{t}{2(1+t^2)}u(t)\int_0^1\arctan\frac{t|u(h(\tau))|}{(1+|u(h(\tau)|)(1+\tau^2)}d\tau,$$

then $A=[0;1]$,$f(t)=\frac{27}{2}\frac{t^2}{e^{3t}}$, $g(t)=\frac{t}{2(1+t^2)}$, $\varphi(t,u(\beta))=\frac{t|u(\beta)|}{1+|u(\beta|}$, $\sup_{t\in[0;1]} f(t)=f(\frac{2}{3})=\frac{27}{2}\frac{4}{9e^2}=\frac{6}{e^2}=\lambda$, $\sup_{t\in[0;1]}|g(t)|=|g(1)|=\frac{1}{4}=\mu$,

$$\forall_{v\in\mathcal{C}(A,\mathbb{R})}\{\sup_{t,\beta\in A}|\varphi(t,v(\beta))|=\sup_{t,\beta\in A}\frac{t|u(\beta)|}{1+|u(\beta|}\leq 1\}$$

and

$$\begin{aligned}&\sup_{t.\beta\in A}|\varphi(t,(\mathcal{F}u)(\beta))-\varphi(t,(\mathcal{F}v)(\beta))|\\ \leq\ &\sup_{t.\beta\in A}|\,|(\mathcal{F}u)(\beta)|-|(\mathcal{F}v)(\beta)|\,|\leq D(\mathcal{F}u,\mathcal{F}v)\end{aligned}$$

Therefore,

$$\rho=\frac{1}{1-\mu\arctan a_1}=\frac{16}{16-\pi},\ \omega=1$$

and

$$\begin{aligned}\gamma &= \lambda\rho^2\omega\arctan a_1=\lambda\rho^2\frac{\pi}{4}=\frac{6}{e^2}\frac{64\pi}{(16-\pi)^2}\\ &= \frac{384\pi}{e^2(16-\pi)^2}\leq\frac{1206.37158}{1221.69648}<1.\end{aligned}$$

Thus (7.36)-(7.40) hold.

(d) Let

$$u(t)=-\frac{t^2}{e^{2t}}+\frac{1-t}{1+t^2}u(t)\int_0^1\arctan\frac{\ln(1+t)+u^2(h(\tau))}{1+\tau^2}d\tau.$$

We note that $A=[0;1]$, $f(t)=-\frac{t^2}{e^{2t}}$, $g(t)=\frac{1-t}{1+t^2}$, $\varphi(t,u(\beta))=\ln(1+t)+u^2(\beta)$, $\sup_{t\in[0;1]}|f(t)|=|f(1)|=\frac{1}{e^2}=\lambda$, $\sup_{t\in A}|g(t)|=|g(0)|=\frac{1}{2}=\mu$ and, by (7.33)-(7.35), (7.39) and (7.40),

$$\begin{aligned}&\sup_{t.\beta\in A}|\varphi(t,(\mathcal{F}u)(\beta))-\varphi(t,(\mathcal{F}v)(\beta))|\leq\sup_{\beta\in A}|(\mathcal{F}u)^2(\beta))-(\mathcal{F}v)^2(\beta))|\\ \leq\ &\sup_{\beta\in A}[|(\mathcal{F}u)(\beta)|+|(\mathcal{F}v)(\beta)|]D(\mathcal{F}u,\mathcal{F}v)\leq 2\lambda\rho D(\mathcal{F}u,\mathcal{F}v),\end{aligned}$$

$$\rho=\frac{2}{2-\pi\mu a_1}=\frac{4}{4-\pi},\ \omega=2\lambda\rho$$

and

$$\begin{aligned}\gamma &= \lambda\rho^2\omega\arctan a_1=2\lambda^2\rho^3\frac{\pi}{4}=2\frac{1}{e^4}\frac{64}{(4-\pi)^3}\frac{\pi}{4}\\ &= \frac{32\pi}{3e^4(4-\pi)^3}\leq\frac{100.53097}{103.60470}<1.\end{aligned}$$

**Example 7.6.** For $h\in\mathcal{C}(A,A)$, $A=[0;1]$, we set

$$u(t)=\frac{t^2}{e^t}+\frac{1}{2}\frac{1-t^3}{1+t^3}u(t)\int_0^1\arctan[\frac{t^3+1}{2}\frac{u^3(h(\tau))}{1+\tau^2}]d\tau, \tag{7.43}$$

$$(\mathcal{F}^{[m+1]}u)(t) = \frac{t^2}{e^t} \cdot \frac{1}{1 - \frac{1}{2}\frac{1-t^3}{1+t^3}\int_0^1 \arctan[\frac{t^3+1}{2}\frac{(\mathcal{F}^{[m]}u)^3(h(\tau))}{1+\tau^2}]d\tau}, \tag{7.44}$$

$u \in \mathcal{C}(A, \mathbb{R})$, $t \in A$, $m \in \{0\} \cup \mathbb{N}$, and

$$M = \{v \in \mathcal{C}(A, \mathbb{R}) : \|v\| \leq \frac{1}{\sqrt[3]{2}}\}. \tag{7.45}$$

The set $\mathcal{Y}_{\mathcal{F}} \subset \mathcal{C}(A, \mathbb{R})$ of solutions of the Fredholm quadratic integral equation (7.43) is nonempty. Moreover, for each $y^0 \in M$, there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{F}) \subset \mathcal{Y}_{\mathcal{F}}$ such that a sequence $(\mathcal{F}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A, \mathbb{R})$ of the form $\forall_{u,v \in \mathcal{C}(A,\mathbb{R})}\{D(u, v) = \sup_{t \in A} |u(t) - v(t)|\}$.

The proof proceeds in five steps.

7.6.1. Notice that $M$ satisfies

$$\forall_{m \in \mathbb{N}}\{\mathcal{F}^{[m]} : M \to M\}. \tag{7.46}$$

First, we construct the set $M$. Taking into account the equation (7.43), we consider $v \in \mathcal{C}(A, \mathbb{R})$ such that $\|v\| \geq \frac{1}{e} + \frac{1}{2}\|v\|^4$. This becomes

$$\|v\| - \frac{1}{2}\|v\|^4 \geq \frac{1}{e} \tag{7.47}$$

and if we consider the map $\psi$ of the form $\psi(z) = z - \frac{1}{2}z^4$, $z \in [0; \infty)$, then $\psi'(z) = 1 - 2z^3$, $\psi''(z) = -2z^2 < 0$ for $z > 0$, $\psi'(z_0) = 0$ for $z_0 = \frac{1}{\sqrt[3]{2}}$, $\psi$ attains its maximum at $z_0$, and $\psi(z_0) = z_0 - \frac{1}{2}z_0^4 = \frac{1}{\sqrt[3]{2}}(1 - \frac{1}{4}) = \frac{3}{4\sqrt[3]{2}} > 0.59527$ $> 0.36788 > \frac{1}{e}$. From this we conclude that if $\|v\| = \frac{1}{\sqrt[3]{2}}$, then (7.47) holds.

Now we prove (7.46). It is clear that if $v \in M$, then

$$\begin{aligned}
\|\mathcal{F}v\| &= \sup_{t \in A} |\frac{t^2}{e^t} \cdot \frac{1}{1 - \frac{1}{2}\frac{1-t^3}{1+t^3}\int_0^1 \arctan[\frac{t^3+1}{2}\frac{v^3(h(\tau))}{1+\tau^2}]d\tau}| \\
&\leq \frac{1}{e} \cdot \frac{1}{1 - \frac{1}{2}\sup_{\beta \in A}|v^3(\beta)|\int_0^1 \frac{d\tau}{1+\tau^2}} \leq \frac{1}{e}\frac{1}{1 - \frac{\pi}{8}\|v^3\|} \\
&\leq \frac{1}{e} \cdot \frac{1}{1 - \frac{\pi}{16}} = \frac{16}{e(16 - \pi)} < 0.45777 < 0.79370 < \frac{1}{\sqrt[3]{2}}.
\end{aligned}$$

Hence, in view of (7.45), we find (7.46) as desired.

7.6.2. We see that

$$\forall_{u,v \in M}\forall_{m \in \mathbb{N}}\{D(\mathcal{F}^{[m]}u, \mathcal{F}^{[m]}v) \leq \gamma^m D(u, v)\} \tag{7.48}$$

where

$$\gamma = \frac{3}{e}\frac{\pi}{\sqrt[3]{4}}\frac{64}{(16 - \pi)^2} < 1. \tag{7.49}$$

Indeed, in view of (7.43)-(7.46),

$$
\begin{aligned}
& D(\mathcal{F}u,\mathcal{F}v) = \sup_{t\in A}|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)| \\
= & \sup_{t\in A}|\frac{t^2}{e^t}[\frac{1}{1-\frac{1}{2}\frac{1-t^3}{1+t^3}\int_0^1 \arctan[\frac{t^3+1}{2}\frac{u^3(h(\tau))}{1+\tau^2}]d\tau} \\
& -\frac{1}{1-\frac{1}{2}\frac{1-t^3}{1+t^3}\int_0^1 \arctan[\frac{t^3+1}{2}\frac{v^3(h(\tau))}{1+\tau^2}]d\tau}]| \\
\leq & \frac{1}{e}(\frac{16}{16-\pi})^2\sup_{\beta\in A}\int_0^1\frac{|u^3(\beta)-u^3(\beta)|}{1+\tau^2}d\tau \\
\leq & \frac{1}{e}(\frac{16}{16-\pi})^2 D(u,v)\sup_{\beta\in A}|u^2(\beta)+u(\beta)v(\beta)+v^2(\beta)|\int_0^1\frac{d\tau}{1+\tau^2} \\
\leq & \frac{1}{e}(\frac{16}{16-\pi})^2 D(u,v)\frac{3}{\sqrt[3]{4}}\frac{\pi}{4} = \frac{3}{e}\frac{\pi}{\sqrt[3]{4}}\frac{64}{(16-\pi)^2}D(u,v) \\
= & \gamma D(u,v),\ \gamma = \frac{3}{e}\frac{\pi}{\sqrt[3]{4}}\frac{64}{(16-\pi)^2} \leq \frac{603.18579}{713.43679} < 1.
\end{aligned}
$$

7.6.3. Observe that if $s,l,r\in\mathbb{N}$ and $y^0\in M$, then

$$D(\mathcal{F}^{[r+s]}y^0,\mathcal{F}^{[r+l]}y^0) \leq \gamma^r D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0). \tag{7.50}$$

We conclude this from (7.46), (7.48) and (7.49).

7.6.4. Since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete, a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is $D$-admissible on $M$ ($M$ is constructed in 7.6.1). Next, properties (7.49) and (7.50) imply property

$$
\begin{aligned}
& \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0)<\varepsilon+\eta \\
\Rightarrow & D(\mathcal{F}^{[s+r]}y^0,\mathcal{F}^{[l+r]}y^0) \leq \gamma^r D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0) \\
\leq & \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0)\}.
\end{aligned}
$$

In addition, since property (7.48) implies that $\mathcal{F}$ is continuous on $M$, we have that $\mathcal{F}$ is closed in each $y^0\in M$. Therefore, we may use (I.4) and Theorem 4.2(vi) with $\mathcal{W}=\mathcal{F}$.

**Example 7.7.** Assume that $A=[0;1]$ and $h\in\mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{F}}$ of solutions of the Fredholm quadratic integral equation

$$u(t) = \frac{3}{t^2+4}+\frac{u(t)}{1+\ln(1+|u(t)|}\int_0^1\frac{1}{t+3}\frac{1}{\tau+2}u(h(\tau))d\tau, \tag{7.51}$$

$u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, is nonempty. Moreover, for each $y^0\in M$ where

$$M=\{v\in\mathcal{C}(A,\mathbb{R}):\|v\|\leq 3\}, \tag{7.52}$$

there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{F}) \subset \mathcal{Y}_{\mathcal{F}} \subset \mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{F}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$,

$$\begin{aligned} &(\mathcal{F}^{[m+1]}y^0)(t) \\ = \quad & \frac{3}{t^2+4} \cdot \frac{1}{1 - \frac{1}{1+\ln(1+\left|(\mathcal{F}^{[m]}y^0)(t)\right|)} \int_0^1 \frac{1}{t+3} \frac{1}{\tau+2} (\mathcal{F}^{[m]}y^0)(h(\tau)) d\tau}, \end{aligned} \quad (7.53)$$

$t \in A$, $m \in \{0\} \cup \mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v \in \mathcal{C}(A,\mathbb{R})} \{D(u,v) = \sup_{t \in A} |u(t) - v(t)|\}$.

The proof proceeds in four steps.

7.7.1. It is clear that

$$\forall_{m \in \mathbb{N}} \{\mathcal{F}^{[m]} : M \to M\}. \quad (7.54)$$

In fact, based on (7.51) we consider $v \in \mathcal{C}(A,\mathbb{R})$ satisfying $\|v\| \geq \frac{3}{4} + \frac{1}{6}\|v\|^2$, that is $\|v\| - \frac{1}{6}\|v\|^2 \geq \frac{3}{4}$. Putting $\psi(z) = z - \frac{1}{6}z^2$, $z \in [0;\infty)$, we see that $\psi$ attains its maximum at $z_0 = 3$ and $\psi(z_0) = z_0 - \frac{1}{6}z_0^2 = \frac{3}{2} > \frac{3}{4}$ so that $\|v\| = 3$ satisfies $\|v\| - \frac{1}{6}\|v\|^2 \geq \frac{3}{4}$.

Next, we see that if $v \in M$, then, by (7.53),

$$\begin{aligned} \|\mathcal{F}v\| &= \sup_{t \in A} \left| \frac{3}{t^2+4} \cdot \frac{1}{1 - \frac{1}{1+\ln(1+|v(t)|} \int_0^1 \frac{1}{t+3} \frac{1}{\tau+2} v(h(\tau)) d\tau} \right| \\ &\leq \frac{3}{4} \cdot \frac{1}{1 - \frac{1}{6}\sup_{\beta \in A} |v(\beta)|} = \frac{3}{2} < 3. \end{aligned}$$

7.7.2. We claim that

$$\forall_{u,v \in M} \forall_{m \in \mathbb{N}} \{D(\mathcal{F}^{[m]}u, \mathcal{F}^{[m]}v) \leq \frac{1}{2} D(\mathcal{F}^{[m-1]}u, \mathcal{F}^{[m-1]}v)\}. \quad (7.55)$$

To see this, note that

$$\begin{aligned} D(\mathcal{F}u, \mathcal{F}v) &\leq \frac{3}{4} \frac{\frac{1}{6} D(u,v)}{(1 - \frac{1}{6}\sup_{\beta \in A} |u(\beta)|)(1 - \frac{1}{6}\sup_{\beta \in A} |v(\beta)|)} \\ &\leq \frac{3}{4}\frac{4}{6} D(u,v) = \frac{1}{2} D(u,v), \end{aligned}$$

establishing the continuity of $\mathcal{F}$ in $M$.

7.7.3. Observe that if $s, l, r \in \mathbb{N}$ and $y^0 \in M$, then

$$D(\mathcal{F}^{[r+s]}y^0, \mathcal{F}^{[r+l]}y^0) \leq \frac{1}{2^r} D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0). \quad (7.56)$$

This is a consequence of (7.54) and (7.55).

7.7.4. The following implications are obvious: the estimate (7.56) implies

$$\begin{aligned} & \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{y^0 \in M} \forall_{s,l \in \mathbb{N}} \{D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) \\ < \quad & \varepsilon + \eta \Rightarrow D(\mathcal{F}^{[s+r]}y^0, \mathcal{F}^{[l+r]}y^0) \leq \frac{1}{2^r} D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) \\ \leq \quad & \frac{\varepsilon}{\varepsilon+\eta} D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0)\}, \end{aligned}$$

the estimate (7.55) yields that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is closed in each $y^0\in M$ ($M$ is constructed in 7.7.1), and a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. We will notice that (I.4) for $\mathcal{W}=\mathcal{F}$ holds. The above shows that the assertion (B) of Theorem 4.2(vi) holds for $M$ defined in (7.52).

**Example 7.8.** Let $a_1\in(0;\infty)$, $A=[0;a_1]$, and $h\in\mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{V}}$ of solutions of the Volterra quadratic integral equation

$$\begin{aligned} u(t) &= \frac{4t}{e^{2t}}+\frac{1}{2\pi}\arctan[\sqrt{t}+t\sin u(t)] \\ &\quad\cdot\int_0^t\left[\frac{\sin u^2(h(\tau))+3u(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}\right]d\tau, \end{aligned}\tag{7.57}$$

$u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, is nonempty. Moreover, for each $y^0\in M$ where

$$M=\{v\in\mathcal{C}(A,\mathbb{R}):\|v\|\le 2\},\tag{7.58}$$

there exists $y\in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V})\subset\mathcal{Y}_{\mathcal{V}}\subset\mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$,

$$\begin{aligned} &(\mathcal{V}^{[m+1]}y^0)(t) \\ =\;& \frac{4t}{e^{2t}}+\frac{1}{2\pi}\arctan[\sqrt{t}+t\sin(\mathcal{V}^{[m]}y^0)(t)] \\ &\cdot\int_0^t\left[\frac{\sin(\mathcal{V}^{[m]}y^0)^2(h(\tau))+3(\mathcal{V}^{[m]}y^0)(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}\right]d\tau, \end{aligned}$$

$t\in A$, $m\in\{0\}\cup\mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v)=\sup_{t\in A}|u(t)-v(t)|\}$.

The proof proceeds in three steps.

7.8.1. It is clear that

$$\forall_{m\in\mathbb{N}}\{\mathcal{V}^{[m]}:M\to M\}.\tag{7.59}$$

First we construct the set $M$. It is easy to see that

$$\begin{aligned} &\sup_{t\in A}\frac{1}{2\pi}|\arctan[\sqrt{t}+t\sin u(t)] \\ &\cdot\int_0^t[\frac{\sin u^2(h(\tau))+3u(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}]d\tau| \\ \le\;& \frac{1}{2\pi}\sup_{t\in A}\{[\sqrt{t}+t|u(t)|][\sup_{\beta\in A}(\frac{1}{3}|\sin u^2(\beta)|+|u(\beta)|) \\ &\cdot\int_0^t\frac{d\tau}{(1+\tau)(1+t^2)}+\frac{t}{2(1+t^2)}]\} \\ \le\;& \frac{1}{2\pi}\{(\frac{1}{3}\|u\|^2+\|u\|)\sup_{t\in A}[\frac{\ln(1+t)}{1+t^2}(\sqrt{t}+t|u(t)|)] \\ &+\sup_{t\in A}[\frac{t}{2(1+t^2)}(\sqrt{t}+t|u(t)|)]\}. \end{aligned}$$

Clearly,

$$\sup_{t\in A}\frac{t}{1+t^2}\le\sup_{t\in[0;\infty)}\frac{t}{1+t^2}=\frac{1}{2},\ \forall_{t\in[0;\infty)}\{\ln(1+t)\le\sqrt{t}\},\tag{7.60}$$

$$\sup_{t\in A}\frac{\sqrt{t}}{1+t^2}\le\sup_{t\in[0;\infty)}\frac{\sqrt{t}}{1+t^2}=\frac{\sqrt{t_0}}{1+t_0^2}|_{t_0=\frac{1}{\sqrt{3}}}=\frac{3^{3/4}}{4},\tag{7.61}$$

$$\sup_{t\in A}\frac{t\sqrt{t}}{1+t^2}\le\sup_{t\in[0;\infty)}\frac{t\sqrt{t}}{1+t^2}=\frac{t_0\sqrt{t_0}}{1+t_0^2}|_{t_0=\sqrt{3}}=\frac{3^{3/4}}{4}\approx 0.569876764.\tag{7.62}$$

Thus

$$\begin{aligned}&\sup_{t\in A}[\frac{\ln(1+t)}{1+t^2}(\sqrt{t}+t|u(t)|)]\le\sup_{t\in A}[\frac{\sqrt{t}}{1+t^2}(\sqrt{t}+t\,\|u\|)]\\ \le\ &\sup_{t\in A}[\frac{t}{1+t^2}+\frac{t\sqrt{t}}{1+t^2}\,\|u\|)]\le\frac{1}{2}+\frac{3^{3/4}}{4}\,\|u\|\end{aligned}$$

and

$$\begin{aligned}&\sup_{t\in A}[\frac{t}{2(1+t^2)}(\sqrt{t}+t|u(t)|)]\\ \le\ &\sup_{t\in A}[\frac{1}{2}(\frac{t\sqrt{t}}{1+t^2}+\frac{t^2}{1+t^2}\,\|u\|)]\le\frac{1}{2}\frac{3^{3/4}}{4}+\frac{\|u\|}{2},\end{aligned}$$

which implies

$$\begin{aligned}&\sup_{t\in A}\frac{1}{2\pi}|\arctan[\sqrt{t}+t\sin u(t)]\\ &\cdot\int_0^t\left[\frac{\sin u^2(h(\tau))+3u(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}\right]d\tau|\\ \le\ &\frac{1}{2\pi}\{(\frac{1}{3}\,\|u\|^2+\|u\|)(\frac{1}{2}+\frac{3^{3/4}}{4}\,\|u\|)+(\frac{1}{2}\frac{3^{3/4}}{4}+\frac{\|u\|}{2})\}.\end{aligned}\tag{7.63}$$

Moreover, we see that

$$\sup_{t\in A}\frac{4t}{e^{2t}}\le\sup_{t\in[0;\infty)}\frac{4t}{e^{2t}}=\frac{4t_0}{e^{2t_0}}|_{t_0=\frac{1}{2}}=\frac{2}{e}.\tag{7.64}$$

Then, via (7.57), (7.63) and (7.64), inequality

$$\|u\|\ge\frac{2}{e}+\frac{1}{2\pi}\{(\frac{1}{3}\,\|u\|^2+\|u\|)(\frac{1}{2}+\frac{3^{3/4}}{4}\,\|u\|)+(\frac{1}{2}\frac{3^{3/4}}{4}+\frac{\|u\|}{2})\}$$

holds for $\|u\|=2$. In fact, if $\|u\|=2$, then we have

$$\begin{aligned}&2-\frac{1}{2\pi}\{\frac{10}{3}(\frac{1}{2}+\frac{3^{3/4}}{4}2)+(\frac{1}{2}\frac{3^{3/4}}{4}+\frac{2}{2})\}\\ =\ &2-\frac{1}{2\pi}(\frac{8}{3}+\frac{43}{24}3^{3/4})>0.92763939>\frac{2}{e}\approx 0.73575888.\end{aligned}$$

Next, we see that if $v \in M$, then

$$\begin{aligned}
\|\mathcal{V}v\| &= \sup_{t\in A}|\frac{4t}{e^{2t}} + \frac{1}{2\pi}\arctan[\sqrt{t}+t\sin v(t)] \\
&\quad \cdot \int_0^t [\frac{\sin v^2(h(\tau))+3v(h(\tau))}{3(1+\tau)(1+t^2)} + \frac{1}{2(1+t^2)}]d\tau| \\
&\leq \frac{2}{e} + \frac{1}{2\pi}\{(\frac{1}{3}\|u\|^2 + \|u\|)(\frac{1}{2} + \frac{3^{3/4}}{4}\|u\|) + (\frac{1}{2}\frac{3^{3/4}}{4} + \frac{\|u\|}{2})\} \\
&\leq \frac{2}{e} + \frac{1}{2\pi}\{\frac{10}{3}(\frac{1}{2} + \frac{3^{3/4}}{4}2) + (\frac{1}{2}\frac{3^{3/4}}{4} + \frac{2}{2})\} \\
&\approx 0.735758882342885 + 1.074420560114641 \\
&= 1.810179442457526 < 2.
\end{aligned}$$

Therefore, by (7.58), (7.59) holds

7.8.2. We claim that

$$\forall_{u,v\in M}\forall_{m\in\mathbb{N}}\{D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) \leq \gamma^m D(u,v)\} \tag{7.65}$$

where

$$\gamma \approx 0.92583110 < 1. \tag{7.66}$$

To see this, note that

$$\begin{aligned}
& D(\mathcal{V}u, \mathcal{V}v) = \sup_{t\in A}|(\mathcal{V}u)(t) - (\mathcal{V}v)(t)| \\
\leq\ & \frac{1}{2\pi}\sup_{t\in A}|\arctan[\sqrt{t}+t\sin v(t)] \\
& \cdot \int_0^t [\frac{\sin u^2(h(\tau))+3u(h(\tau))}{3(1+\tau)(1+t^2)} - \frac{\sin v^2(h(\tau))+3v(h(\tau))}{3(1+\tau)(1+t^2)}]d\tau \\
& +[\arctan(\sqrt{t}+t\sin u(t)) - \arctan(\sqrt{t}+t\sin v(t))] \\
& \cdot \int_0^t [\frac{\sin v^2(h(\tau))+3v(h(\tau))}{3(1+\tau)(1+t^2)} + \frac{1}{2(1+t^2)}]d\tau| \\
\leq\ & \frac{1}{2\pi}\sup_{t\in A}\{\frac{\pi}{2}\sup_{\beta\in A}[\frac{1}{3}\left|u^2(\beta)-v^2(\beta)\right| + |u(\beta)-v(\beta)|]\int_0^t \frac{d\tau}{(1+\tau)(1+t^2)} \\
& +t\,|u(t)-v(t)|\,[\int_0^t \frac{2d\tau}{(1+\tau)(1+t^2)} + \frac{t}{2(1+t^2)}]\} \\
\leq\ & \frac{1}{2\pi}\sup_{t\in A}\{\frac{\pi}{2}[\frac{4}{3}D(u,v) + D(u,v)]\frac{\ln(1+t)}{1+t^2} \\
& +D(u,v)[\frac{2t\ln(1+t)}{1+t^2} + \frac{t^2}{2(1+t^2)}]\} \\
\leq\ & \frac{1}{2\pi}D(u,v)\sup_{t\in A}\{\frac{7\pi}{3}\frac{\sqrt{t}}{1+t^2} + 2\frac{t\sqrt{t}}{1+t^2} + \frac{t^2}{2(1+t^2)}\}.
\end{aligned}$$

Therefore

$$\begin{aligned} D(\mathcal{V}u, \mathcal{V}v) &\leq \frac{1}{2\pi} D(u,v)\{\frac{7\pi}{3}\frac{3^{3/4}}{4} + 2\frac{3^{3/4}}{4} + \frac{1}{2}\} \\ &= [(\frac{7}{24} + \frac{1}{4\pi})3^{3/4} + \frac{1}{4\pi}\} D(u,v) = \gamma \cdot D(u,v) \end{aligned}$$

where $\gamma \approx 0.92583110$. This, by (7.58) and (7.59), implies (7.65) with (7.66).

7.8.3. Using (7.58), (7.59) and (7.65) with (7.66), we have that if $s, l, r \in \mathbb{N}$ and $y^0 \in M$, then

$$D(\mathcal{V}^{[r+s]}y^0, \mathcal{V}^{[r+l]}y^0) \leq \gamma^r D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0).$$

Hence, we find

$$\begin{aligned} &\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0) \\ < \ &\varepsilon + \eta \Rightarrow D(\mathcal{V}^{[s+r]}y^0, \mathcal{V}^{[l+r]}y^0) \leq \gamma^r D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0) \\ \leq \ &\frac{\varepsilon}{\varepsilon+\eta} D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0)\}. \end{aligned} \tag{7.67}$$

The estimate (7.65) also yields that a dynamic system $(\mathcal{C}(A,\mathbb{R}), \mathcal{V})$ is closed in each $y^0 \in M$ ($M$ is constructed in 7.8.1). In addition, we have that a dynamic system $(\mathcal{C}(A,\mathbb{R}), \mathcal{V})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A,\mathbb{R}), D)$ is sequentially complete. Therefore, from (7.57) and (7.67) it follows that (I.4) for $\mathcal{W} = \mathcal{V}$ holds. We conclude that the assertion (B) of Theorem 4.2(vi) holds.

**Example 7.9.** Let $a_1 \in (0;\infty)$, $A = [0; a_1]$, and $h \in \mathcal{C}(A, A)$. The set $\mathcal{Y}_{\mathcal{V}}$ of solutions of the Volterra quadratic integral equation

$$\begin{aligned} u(t) &= \frac{2t}{e^{2t}} + \frac{1}{3\pi}\arctan[\sqrt{t} + tu(t)] \\ &\quad \cdot \int_0^t \left[\frac{u^{2/3}(h(\tau)) + 3u(h(\tau))}{(1+\tau)(1+t^2)} + \frac{1}{1+t^2}\right] d\tau, \end{aligned} \tag{7.68}$$

$u \in \mathcal{C}(A,\mathbb{R})$, $t \in A$, is nonempty. Moreover, for each $y^0 \in M$ where

$$M = \{v \in \mathcal{C}(A,\mathbb{R}) : \|v\| \leq 2\},$$

there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V}) \subset \mathcal{Y}_{\mathcal{V}} \subset \mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$,

$$\begin{aligned} (\mathcal{V}^{[m+1]}y^0)(t) &= \frac{2t}{e^{2t}} + \frac{1}{3\pi}\arctan[\sqrt{t} + t(\mathcal{V}^{[m]}y^0)(t)] \\ &\quad \cdot \int_0^t [\frac{(\mathcal{V}^{[m]}y^0)^{2/3}(h(\tau)) + 3(\mathcal{V}^{[m]}y^0)(h(\tau))}{(1+\tau)(1+t^2)} + \frac{1}{1+t^2}] d\tau, \end{aligned}$$

$t \in A$, $m \in \{0\} \cup \mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v) = \sup_{t\in A}|u(t) - v(t)|\}$.

The proof proceeds in four steps.
7.9.1. Note that

$$\forall_{m\in\mathbb{N}}\{\mathcal{V}^{[m]}:M\to M\}. \tag{7.69}$$

Indeed, first we construct the set $M$. It is easy to see that

$$\begin{aligned}
&\sup_{t\in A}\frac{1}{3\pi}|\arctan[\sqrt{t}+tu(t)]\\
&\cdot\int_0^t[\frac{u^{2/3}(h(\tau))+3u(h(\tau))}{(1+\tau)(1+t^2)}+\frac{1}{1+t^2}]d\tau|\\
\le\ &\frac{1}{3\pi}\sup_{t\in A}\{[\sqrt{t}+t|u(t)|][\sup_{\beta\in A}(u^{2/3}(\beta)+3|u(\beta)|)\\
&\cdot\int_0^t\frac{d\tau}{(1+\tau)(1+t^2)}+\frac{t}{1+t^2}]\}\\
\le\ &\frac{1}{3\pi}\{(\|u\|^{2/3}+3\|u\|)\sup_{t\in A}[\frac{\ln(1+t)}{1+t^2}(\sqrt{t}+t|u(t)|)]\\
&+\sup_{t\in A}[\frac{t}{1+t^2}(\sqrt{t}+t|u(t)|)]\}.
\end{aligned}$$

Using (7.60)-(7.62) we obtain

$$\begin{aligned}
&\sup_{t\in A}[\frac{\ln(1+t)}{1+t^2}(\sqrt{t}+t|u(t)|)]\le\sup_{t\in A}[\frac{\sqrt{t}}{1+t^2}(\sqrt{t}+t\|u\|)]\\
\le\ &\sup_{t\in A}[\frac{t}{1+t^2}+\frac{t\sqrt{t}}{1+t^2}\|u\|)]\le\frac{1}{2}+\frac{3^{3/4}}{4}\|u\|
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in A}[\frac{t}{1+t^2}(\sqrt{t}+t|u(t)|)]\\
\le\ &\sup_{t\in A}[\frac{t\sqrt{t}}{1+t^2}+\frac{t^2}{1+t^2}\|u\|]\le\frac{3^{3/4}}{4}+\|u\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{t\in A}\frac{1}{3\pi}|\arctan[\sqrt{t}+tu(t)]\\
&\cdot\int_0^t[\frac{u^{2/3}(h(\tau))+3u(h(\tau))}{(1+\tau)(1+t^2)}+\frac{1}{1+t^2}]d\tau|\\
\le\ &\frac{1}{3\pi}\{(\|u\|^{2/3}+3u)(\frac{1}{2}+\frac{3^{3/4}}{4}\|u\|)+(\frac{3^{3/4}}{4}+\|u\|)\}.
\end{aligned} \tag{7.70}$$

Moreover, we see that

$$\sup_{t\in A}\frac{2t}{e^{2t}}\le\sup_{t\in[0;\infty)}\frac{2t}{e^{2t}}=\frac{2t_0}{e^{2t_0}}|_{t_0=\frac{1}{2}}=\frac{1}{e}. \tag{7.71}$$

Then, via (7.68), (7.70) and (7.71), inequality

$$\|u\| \geq \frac{1}{e} + \frac{1}{3\pi}\{(\|u\|^{2/3} + 3\|u\|)(\frac{1}{2} + \frac{3^{3/4}}{4}\|u\|) + (\frac{3^{3/4}}{4} + \|u\|)\}$$

holds for $\|u\| = 2$. In fact, if $\|u\| = 2$, then we have

$$\begin{aligned} & 2 - \frac{1}{3\pi}\{(2^{2/3} + 6)(\frac{1}{2} + \frac{3^{3/4}}{4}2) + (\frac{3^{3/4}}{4} + 2)\} \\ = \; & 2 - \frac{1}{3\pi}[\frac{2^{2/3}}{2} + 5 + (2 \cdot 2^{2/3} + 13)\frac{3^{3/4}}{4}] > 0.67165685 \\ > \; & \frac{1}{e} \approx 0.36787944. \end{aligned}$$

Clearly, if $v \in M$, then (7.69) holds since

$$\begin{aligned} \|\mathcal{V}v\| &= \sup_{t \in A} |\frac{2t}{e^{2t}} + \frac{1}{3\pi}\arctan[\sqrt{t} + tu(t)] \\ &\quad \cdot \int_0^t \left[\frac{u^{2/3}(h(\tau)) + 3u(h(\tau))}{(1+\tau)(1+t^2)} + \frac{1}{1+t^2}\right] d\tau| \\ &\leq \frac{1}{e} + \frac{1}{3\pi}[\frac{2^{2/3}}{2} + 5 + (2 \cdot 2^{2/3} + 13)\frac{3^{3/4}}{4}] \\ &\approx 0.367879441171442 + 1.592753110329102 \\ &= 1.960647522043522 < 2. \end{aligned}$$

7.9.2. Let

$$\gamma = (\frac{2}{3} + \frac{2^{2/3} + 6}{3\pi})\frac{3^{3/4}}{4} + \frac{1}{3\pi} \approx 0.94480074.$$

We claim that:
(a) If $u, v \in M$ and

$$D(u, v) < 1, \tag{7.72}$$

then

$$D^{3/2}(\mathcal{V}u, \mathcal{V}v) \leq \gamma^{3/2} D(u, v) \tag{7.73}$$

and

$$\lim_{m \to \infty} D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) = 0. \tag{7.74}$$

(b) If $u, v \in M$ and

$$D(u, v) \geq 1, \tag{7.75}$$

then

$$D(\mathcal{V}u, \mathcal{V}v) \leq \gamma D(u, v) \tag{7.76}$$

and

$$\lim_{m\to\infty} D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) = 0. \tag{7.77}$$

To see this, note that for arbitrary and fixed $u, v \in M$ we have

$$\begin{aligned}
& D(\mathcal{V}u, \mathcal{V}v) = \sup_{t\in A} |(\mathcal{V}u)(t) - (\mathcal{V}v)(t)| \\
\leq\ & \frac{1}{3\pi} \sup_{t\in A} |\arctan[\sqrt{t} + tv(t)] \\
& \cdot \int_0^t [\frac{u^{2/3}(h(\tau)) + 3u(h(\tau))}{(1+\tau)(1+t^2)} - \frac{v^{2/3}(h(\tau)) + 3v(h(\tau))}{(1+\tau)(1+t^2)}] d\tau \\
& +[\arctan(\sqrt{t} + tu(t)) - \arctan(\sqrt{t} + tv(t))] \\
& \cdot \int_0^t [\frac{v^{2/3}(h(\tau)) + 3v(h(\tau))}{(1+\tau)(1+t^2)} + \frac{1}{1+t^2}] d\tau| \\
\leq\ & \frac{1}{3\pi} \sup_{t\in A} \{ \frac{\pi}{2} \sup_{\beta\in A} [\left| u^{2/3}(\beta) - v^{2/3}(\beta) \right| + 3\,|u(\beta) - v(\beta)|] \int_0^t \frac{d\tau}{(1+\tau)(1+t^2)} \\
& +t\,|u(t) - v(t)|\, [\int_0^t \frac{(2^{2/3}+6)d\tau}{(1+\tau)(1+t^2)} + \frac{t}{1+t^2}]\} \\
\leq\ & \frac{1}{3\pi} \sup_{t\in A} \{ \frac{\pi}{2} \sup_{\beta\in A} [|u(\beta) - v(\beta)|^{2/3} + 3\,|u(\beta) - v(\beta)|] \int_0^t \frac{d\tau}{(1+\tau)(1+t^2)} \\
& +t\,|u(t) - v(t)|\, [\int_0^t \frac{(2^{2/3}+6)d\tau}{(1+\tau)(1+t^2)} + \frac{t}{1+t^2}]\} \\
\leq\ & \frac{1}{3\pi} \sup_{t\in A} \{ \frac{\pi}{2} [D^{2/3}(u,v) + 3D(u,v)] \frac{\ln(1+t)}{1+t^2} \\
& +D(u,v)[\frac{(2^{2/3}+6)t\ln(1+t)}{1+t^2} + \frac{t^2}{1+t^2}]\} \leq \frac{1}{3\pi} \sup_{t\in A} \{ \frac{\pi}{2} [D^{2/3}(u,v) \\
& +3D(u,v)] \frac{\sqrt{t}}{1+t^2} + D(u,v)[(2^{2/3}+6)\frac{t\sqrt{t}}{1+t^2} + \frac{t^2}{1+t^2}]\}.
\end{aligned}$$

Hence, by (7.60)-(7.62), for each $u, v \in M$ we have

$$\begin{aligned}
D(\mathcal{V}u, \mathcal{V}v) &\leq [(\frac{1}{2} + \frac{2^{2/3}+6}{3\pi})\frac{3^{3/4}}{4} + \frac{1}{3\pi}] D(u,v) \\
+\frac{1}{6}\frac{3^{3/4}}{4} D^{2/3}(u,v) &\leq \gamma_1 D(u,v) + \gamma_2 D^{2/3}(u,v)
\end{aligned} \tag{7.78}$$

where

$$\gamma_1 = (\frac{1}{2} + \frac{2^{2/3}+6}{3\pi})\frac{3^{3/4}}{4} + \frac{1}{3\pi} \approx 0.84982128, \tag{7.79}$$

$$\gamma_2 = \frac{1}{6}\frac{3^{3/4}}{4} \approx 0.09497946,\ \gamma = \gamma_1 + \gamma_2 \approx 0.94480074. \tag{7.80}$$

The proofs of cases (a) and (b) are as follows:

*Proof of* (a). Assume that $u, v \in M$ satisfy (7.72). Then

$$D(u,v) < D^{2/3}(u,v) < 1$$

and using (7.78)-(7.80) we obtain

$$D(\mathcal{V}u,\mathcal{V}v) \leq \gamma D^{2/3}(u,v)$$

which implies (7.73).

Further, without loss of generality, we may assume that

$$\forall_{m\in\{0\}\cup\mathbb{N}}\{D(\mathcal{V}^{[m]}u,\mathcal{V}^{[m]}v) \neq 0\}. \tag{7.81}$$

Otherwise, $\exists_{m_0\in\{0\}\cup\mathbb{N}}\{D(\mathcal{V}^{[m_0]}u,\mathcal{V}^{[m_0]}v) = 0\}$ and in view of (7.78)-(7.80),

$$\forall_{m\geq m_0}\{D(\mathcal{V}^{[m]}u,\mathcal{V}^{[m]}v) = 0\}. \tag{7.82}$$

Using (7.82) we have that property (7.74) holds in this case.

Let thus (7.81) hold. Since (7.72) and (7.73) imply

$$D(\mathcal{V}u,\mathcal{V}v) < D^{2/3}(\mathcal{V}u,\mathcal{V}v) < 1, \tag{7.83}$$

from (7.78)-(7.80) it follows that

$$D(\mathcal{V}^{[2]}u,\mathcal{V}^{[2]}v) \leq \gamma D^{2/3}(\mathcal{V}u,\mathcal{V}v). \tag{7.84}$$

Using (7.84) and (7.73) we have now that

$$D(\mathcal{V}^{[2]}u,\mathcal{V}^{[2]}v)D(\mathcal{V}u,\mathcal{V}v) \leq \gamma^2 D^{2/3}(u,v)D^{2/3}(\mathcal{V}u,\mathcal{V}v),$$

i.e. $D(\mathcal{V}^{[2]}u,\mathcal{V}^{[2]}v)D^{1/3}(\mathcal{V}u,\mathcal{V}v) \leq \gamma^2 D^{2/3}(u,v)$. Since, by (7.83), $D(\mathcal{V}u,\mathcal{V}v) \leq D^{1/3}(\mathcal{V}u,\mathcal{V}v)$, this gives

$$D(\mathcal{V}^{[2]}u,\mathcal{V}^{[2]}v)D(\mathcal{V}u,\mathcal{V}v) \leq \gamma^2 D^{2/3}(u,v).$$

Repeat the above construction of $m$ times, $m > 2$, we then obtain

$$\textstyle\prod_{k=1}^{m} D(\mathcal{V}^{[k]}u,\mathcal{V}^{[k]}v) \leq \gamma^m D^{2/3}(u,v). \tag{7.85}$$

However, since $\lim_{m\to\infty}\gamma^m D^{2/3}(u,v) = 0$, (7.85) implies that the product $\prod_{k=1}^{\infty} D(\mathcal{V}^{[k]}u,\mathcal{V}^{[k]}v)$ is not convergent and in view of (7.81) we have (7.74) when (7.72) holds.

*Proof of* (b). Suppose that $u, v \in M$ are such that (7.75) holds. Then $D^{2/3}(u,v) \leq D(u,v)$ and (7.78)-(7.80) imply $D(\mathcal{V}u,\mathcal{V}v) \leq \gamma D(u,v)\}$. By (7.75) and (7.76), it is obvious that there exists maximal $m_0 \in \mathbb{N}$ such that

$$\forall_{0\leq k\leq m_0-1}\{D^{2/3}(\mathcal{V}^{[k]}u,\mathcal{V}^{[k]}v) \leq D(\mathcal{V}^{[k]}u,\mathcal{V}^{[k]}v)\}$$

and then (7.78)-(7.80) imply

$$\forall_{0\le k\le m_0-1}\{D(\mathcal{V}^{[k+1]}u,\mathcal{V}^{[k+1]}v)\le \gamma D(\mathcal{V}^{[k]}u,\mathcal{V}^{[k]}v)\le \gamma^k D(u,v)\}.$$

Hence it follows that

$$D(\mathcal{V}^{[m_0]}u,\mathcal{V}^{[m_0]}v)<1. \tag{7.86}$$

Then, for each $m\ge m_0$, adopted to (7.86) analogous consideration as in the proof of (a), we obtain

$$\prod_{k=m_0}^{m} D(\mathcal{V}^{[k]}u,\mathcal{V}^{[k]}v)\le \gamma^{m-m_0}D^{2/3}(\mathcal{V}^{[m_0]}u,\mathcal{V}^{[m_0]}v)$$

and this implies (7.77).

7.9.3. From (7.72), (7.73), (7.75) and (7.76) it follows that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is continuous on $M$ and thus closed in each $y^0\in M$. Next observe that property

$$\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{\lim_{r\to\infty} D(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)=0\}$$

is a consequence of (7.72), (7.74), (7.75) and (7.77). Moreover, a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. Therefore, (I.4) and the assertion (B) of Theorem 4.2(ii) with condition (4.21) hold when $\mathcal{W}=\mathcal{V}$ and $M$ is constructed in 7.9.1.

**Example 7.10.** Let $a_1\in(0;\infty)$, $A=[0;a_1]$, and $h\in\mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{V}}$ of solutions of the Volterra quadratic integral equation

$$\begin{aligned} u(t) &= \frac{3t^2}{\pi(1+t^2)}+\frac{1}{4}\sin[\sqrt{t}+tu(t)] \\ &\quad\cdot\int_0^t\left[\frac{\ln[1+u^2(h(\tau))]+3u(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}\right]d\tau, \end{aligned} \tag{7.87}$$

$u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, is nonempty. Moreover, for each $y^0\in M$, where

$$M=\{u\in\mathcal{C}(A,\mathbb{R}):\|u\|\le 2\}, \tag{7.88}$$

there exists $y\in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V})\subset\mathcal{Y}_{\mathcal{V}}\subset\mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$,

$$\begin{aligned} &(\mathcal{V}^{[m+1]}y^0)(t) \\ =\ &\frac{3t^2}{\pi(1+t^2)}+\frac{1}{4}\sin[\sqrt{t}+t(\mathcal{V}^{[m]}y^0)(t)] \\ &\cdot\int_0^t[\frac{\ln[1+(\mathcal{V}^{[m]}y^0)^2(h(\tau))]+3(\mathcal{V}^{[m]}y^0)(h(\tau))}{3(1+\tau)(1+t^2)} \\ &+\frac{1}{2(1+t^2)}]d\tau, \end{aligned} \tag{7.89}$$

$t \in A$, $m \in \{0\} \cup \mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v \in \mathcal{C}(A,\mathbb{R})}\{D(u,v) = \sup_{t\in A}|u(t)-v(t)|\}$.

The proof proceeds in three steps.

7.10.1. We claim that

$$\forall_{m\in\mathbb{N}}\{\mathcal{V}^{[m]} : M \to M\}. \tag{7.90}$$

The first is the construction of $M$. It is easy to see that

$$\begin{aligned}
& \sup_{t\in A}\frac{1}{4}|\sin[\sqrt{t}+tu(t)] \\
& \times\int_0^t[\frac{\ln[1+u^2(h(\tau))]+3u(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}]d\tau| \\
\leq\ & \frac{1}{4}\sup_{t\in A}\{[\sqrt{t}+t|u(t)|][\sup_{\beta\in A}(\frac{1}{3}|\ln[1+u^2(\beta)]|+|u(\beta)|) \\
& \cdot\int_0^t\frac{d\tau}{(1+\tau)(1+t^2)}+\frac{t}{2(1+t^2)}]\} \\
\leq\ & \frac{1}{4}\sup_{t\in A}\{[\sqrt{t}+t|u(t)|][\sup_{\beta\in A}(\frac{1}{3}|u(\beta)|+|u(\beta)|) \\
& \cdot\int_0^t\frac{d\tau}{(1+\tau)(1+t^2)}+\frac{t}{2(1+t^2)}]\} \\
\leq\ & \frac{1}{4}\{(\frac{1}{3}\|u\|+\|u\|)\sup_{t\in A}[\frac{\ln(1+t)}{1+t^2}(\sqrt{t}+t|u(t)|)] \\
& +\sup_{t\in A}[\frac{t}{2(1+t^2)}(\sqrt{t}+t|u(t)|)]\}.
\end{aligned}$$

Thus, by (7.60)-(7.62),

$$\begin{aligned}
& \sup_{t\in A}[\frac{\ln(1+t)}{1+t^2}(\sqrt{t}+t|u(t)|)] \leq \sup_{t\in A}[\frac{\sqrt{t}}{1+t^2}(\sqrt{t}+t|u(t)|)] \\
\leq\ & \sup_{t\in A}[\frac{t}{1+t^2}+\frac{t\sqrt{t}}{1+t^2}|u(t)|] \leq \frac{1}{2}+\frac{3^{3/4}}{4}\|u\|
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{t\in A}[\frac{t}{2(1+t^2)}(\sqrt{t}+t|u(t)|)] \\
\leq\ & \sup_{t\in A}[\frac{1}{2}(\frac{t\sqrt{t}}{1+t^2}+\frac{t^2}{1+t^2}|u(t)|)] \leq \frac{1}{2}\frac{3^{3/4}}{4}+\frac{\|u\|}{2},
\end{aligned}$$

which implies

$$\begin{aligned}
& \sup_{t\in A}\frac{1}{4}|\sin[\sqrt{t}+tu(t)] \\
& \cdot\int_0^t\left[\frac{\ln[1+u^2(h(\tau))]+3u(h(\tau))}{3(1+\tau)(1+t^2)}+\frac{1}{2(1+t^2)}\right]d\tau| \\
\leq\ & \frac{1}{4}\{(\frac{1}{3}\|u\|^2+\|u\|)(\frac{1}{2}+\frac{3^{3/4}}{4}\|u\|)+(\frac{1}{2}\frac{3^{3/4}}{4}+\frac{\|u\|}{2})\}.
\end{aligned} \tag{7.91}$$

Moreover, we see that

$$\sup_{t\in[0;\infty)} \frac{3t^2}{\pi(1+t^2)} = \frac{3}{\pi}. \tag{7.92}$$

Then, via (7.87), (7.91) and (7.92), inequality

$$\|u\| \geq \frac{3}{\pi} + \frac{1}{4}\{(\frac{1}{3}\|u\| + \|u\|)(\frac{1}{2} + \frac{3^{3/4}}{4}\|u\|) + (\frac{1}{2}\frac{3^{3/4}}{4} + \frac{\|u\|}{2})\}$$

holds for $\|u\| = 2$. In fact, if $\|u\| = 2$, then we have

$$\begin{aligned} & 2 - \frac{1}{4}\{\frac{5}{3}(\frac{1}{2} + \frac{3^{3/4}}{4}2) + (\frac{1}{2}\frac{3^{3/4}}{4} + \frac{2}{2})\} \\ = \; & 2 - \frac{1}{4}(\frac{11}{6} + \frac{23}{24}3^{3/4}) > 0.99553476 > \frac{3}{\pi} \approx 0.95492965. \end{aligned}$$

We also see that, for each $u \in M$,

$$\begin{aligned} \|\mathcal{V}u\| &= \sup_{t\in A} |\frac{3t^2}{\pi(1+t^2)} + \frac{1}{4}\sin[\sqrt{t} + tu(t)] \\ &\quad \cdot \int_0^t \left[\frac{\ln[1+u^2(h(\tau))] + 3u(h(\tau))}{3(1+\tau)(1+t^2)} + \frac{1}{2(1+t^2)}\right] d\tau| \\ &\leq \frac{3}{\pi} + \frac{1}{4}(\frac{11}{6} + \frac{23}{24}3^{3/4}) \\ &\approx 0.954929658551372 + 1.004465232395416 \\ &= 1.959394890946788 < 2. \end{aligned}$$

This gives (7.90).

7.10.2. Let $u, v \in M$. Then there exists $\gamma \in (0;1)$,

$$\gamma = \frac{11}{12}\frac{3^{3/4}}{4} + \frac{1}{8} \approx 0.64738703,$$

such that:

(a) If

$$D(u,v) < 1, \tag{7.93}$$

then

$$D^2(\mathcal{V}u, \mathcal{V}v) \leq \gamma^2 D(u,v) \tag{7.94}$$

and

$$\lim_{m\to\infty} D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) = 0. \tag{7.95}$$

(b) If

$$D(u,v) \geq 1, \tag{7.96}$$

then

$$D(\mathcal{V}u, \mathcal{V}v) \leq \gamma D(u, v) \tag{7.97}$$

and

$$\lim_{m\to\infty} D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) = 0. \tag{7.98}$$

In fact, if $u, v \in M$, then

$$\begin{aligned}
& D(\mathcal{V}u, \mathcal{V}v) = \sup_{t\in A} |(\mathcal{V}u)(t) - (\mathcal{V}v)(t)| \\
\leq\ & \frac{1}{4}\sup_{t\in A} |\sin[\sqrt{t} + tv(t)] \cdot \int_0^t [\frac{\ln[1+u^2(h(\tau))] + 3u(h(\tau))}{3(1+\tau)(1+t^2)} \\
& - \frac{\ln[1+v^2(h(\tau))] + 3v(h(\tau))}{3(1+\tau)(1+t^2)}]d\tau \\
& + [\sin(\sqrt{t} + tu(t)) - \sin(\sqrt{t} + tv(t))] \\
& \cdot \int_0^t [\frac{\ln[1+v^2(h(\tau))] + 3v(h(\tau))}{3(1+\tau)(1+t^2)} + \frac{1}{2(1+t^2)}]d\tau| \\
\leq\ & \frac{1}{4}\sup_{t\in A}\{\sup_{\beta\in A}[\frac{1}{3}\left|\ln[1+u^2(\beta)] - \ln[1+v^2(\beta)]\right| \\
& + |u(\beta) - v(\beta)|] \int_0^t \frac{d\tau}{(1+\tau)(1+t^2)} \\
& + t\,|u(t) - v(t)|\,[\sup_{\beta\in A}|v(\beta)| \int_0^t \frac{d\tau}{(1+\tau)(1+t^2)} + \frac{t}{2(1+t^2)}]\}.
\end{aligned}$$

However,

$$\begin{aligned}
& \left|\ln[1+u^2(\beta)] - \ln[1+v^2(\beta)]\right| \\
=\ & \ln[1 + \frac{\left|u^2(\beta) - v^2(\beta)\right|}{1+\max\{u^2(\beta), v^2(\beta)\}}] \\
\leq\ & \sqrt{\frac{|u^2(\beta) - v^2(\beta)|}{1+\max\{u^2(\beta), v^2(\beta)\}}} \leq \sqrt{|u^2(\beta) - v^2(\beta)|}
\end{aligned}$$

and the above yields

$$\sup_{\beta\in A}\left|\ln[1+u^2(\beta)] - \ln[1+v^2(\beta)]\right| \leq (\|u\| + \|v\|)^{1/2} D^{1/2}(u, v).$$



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Therefore,

$$\begin{aligned}
& D(\mathcal{V}u,\mathcal{V}v) \\
\leq\ & \frac{1}{4}\sup_{t\in A}\{[\frac{1}{3}(\|u\|+\|v\|)^{1/2}D^{1/2}(u,v)+D(u,v)]\frac{\ln(1+t)}{1+t^2} \\
& +D(u,v)[\|v\|\frac{t\ln(1+t)}{1+t^2}+\frac{t^2}{2(1+t^2)}]\} \\
\leq\ & \frac{1}{4}\sup_{t\in A}\{[\frac{1}{3}(\|u\|+\|v\|)^{1/2}D^{1/2}(u,v)+D(u,v)]\frac{\sqrt{t}}{1+t^2} \\
& +D(u,v)[\|v\|\frac{t\sqrt{t}}{1+t^2}+\frac{t^2}{2(1+t^2)}]\}.
\end{aligned}$$

Using (7.60)-(7.62), this gives

$$\begin{aligned}
& D(\mathcal{V}u,\mathcal{V}v) \\
\leq\ & \frac{1}{4}\{[\frac{1}{3}(\|u\|+\|v\|)^{1/2}D^{1/2}(u,v)+D(u,v)]\frac{3^{3/4}}{4} \\
& +D(u,v)[\|v\|\frac{3^{3/4}}{4}+\frac{1}{2}]\} \\
=\ & \frac{1}{6}\frac{3^{3/4}}{4}D^{1/2}(u,v)+\frac{1}{4}(3\frac{3^{3/4}}{4}+\frac{1}{2})D(u,v) \\
=\ & \gamma_1 D^{1/2}(u,v)+\gamma_2 D(u,v) \qquad (7.99)
\end{aligned}$$

where

$$\gamma_1 = \frac{1}{6}\frac{3^{3/4}}{4}\approx 0.09497946, \qquad (7.100)$$

$$\gamma_2 = \frac{1}{4}(3\frac{3^{3/4}}{4}+\frac{1}{2})\approx 0.55240757 \qquad (7.101)$$

$$\gamma = \gamma_1+\gamma_2\approx 0.64738703. \qquad (7.102)$$

Next using (7.89) and (7.99)-(7.102), and by analogous arguments as in Subsections 7.9.2 of Example 7.9, we find (7.93)-(7.98).

7.10.3. From (7.93), (7.94), (7.98) and (7.96) we conclude that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is continuous on $M$ and thus closed in each $y^0\in M$. Next, using (7.89), (7.93), (7.94), (7.96) and (7.198) we conclude that

$$\forall_{s,l\in\mathbb{N}}\forall_{y^0\in M}\{\lim_{r\to\infty}D(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)=0\}.$$

Clearly, a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. Therefore, in view of (7.87) and (I.4), the assertion (B) of Theorem 4.2(ii) with condition (4.21) holds when $\mathcal{W}=\mathcal{V}$ and $M$ is defined by (7.88).



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**Example 7.11.** Let $a_1 \in (0;\infty)$, $A = [0; a_1]$, and $h \in \mathcal{C}(A, A)$. The set $\mathcal{Y}_{\mathcal{V}}$ of solutions of the Volterra quadratic integral equation

$$u(t) = \frac{t^2}{2e^{2t}} + \frac{u^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[|u(h(\tau))|^{1/2} + e^{-u^2(h(\tau))} + e^{t-\tau-1}]d\tau,$$

$u \in \mathcal{C}(A,\mathbb{R})$, $t \in A$, is nonempty. Moreover, for each $y^0 \in M$, where

$$M = \{u \in \mathcal{C}(A,\mathbb{R}) : \|u\| \leq 1\},$$

there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V}) \subset \mathcal{Y}_{\mathcal{V}} \subset \mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$,

$$\begin{aligned}(\mathcal{V}^{[m+1]}y^0)(t) = \frac{t^2}{2e^{2t}} + \frac{(\mathcal{V}^{[m]}y^0)^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[\left|(\mathcal{V}^{[m]}y^0)(h(\tau)\right|^{1/2}\\ +e^{-(\mathcal{V}^{[m]}y^0)^2(h(\tau)} + e^{t-\tau-1}]d\tau,\end{aligned}$$

$t \in A$, $m \in \{0\} \cup \mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v) = \sup_{t\in A} |u(t) - v(t)|\}$.

The proof proceeds in three steps.

7.11.1. We claim that

$$\forall_{m\in\mathbb{N}}\{\mathcal{V}^{[m]} : M \to M\}. \tag{7.103}$$

The first is the construction of $M$. It is easy to see that

$$\begin{aligned}
&\sup_{t\in A}\{|\frac{u^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[u(h(\tau))^{1/2} + e^{-u^2(h(\tau))} + 2e^{t-\tau-1}]d\tau|\}\\
\leq\ &\sup_{t\in A}\{\frac{|u(t)|^2}{2(1+t^2)}[\sup_{\beta\in A}(|u(\beta)|^{1/2} + e^{-|u(\beta|^2}) \times \int_0^t e^{\tau-t}d\tau + \frac{t}{e}]\}\\
\leq\ &\frac{1}{2}\|u\|^2 \sup_{t\in[0;\infty)}[(\|u\|^{1/2}+1)\frac{1-e^{-t}}{1+t^2} + \frac{2t}{e}\frac{1}{1+t^2}]\\
\leq\ &\frac{1}{2}\|u\|^2[(\|u\|^{1/2}+1)\sup_{t\in[0;\infty)}\frac{1-e^{-t}}{1+t^2} + \frac{1}{e}].
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
\sup_{t\in[0;\infty)}\frac{t^2}{2e^{2t}} &= \frac{1}{2e^2} \approx 0.0676764,\\
\sup_{t\in[0;\infty)}\frac{1-e^{-t}}{1+t^2} &\approx \frac{1-e^{-t_0}}{1+t_0^2}|_{t_0=0.72} \approx 0.33801879.
\end{aligned}$$

Consequently, inequality

$$\|u\| \geq \frac{1}{2e^2} + \frac{1}{2}\|u\|^2[(\|u\|^{1/2}+1)\sup_{t\in[0;\infty)}\frac{1-e^{-t}}{1+t^2} + \frac{1}{e}]$$

holds for $\|u\| = 1$. In fact, if $\|u\| = 1$, then we have

$$
\begin{aligned}
& 1 - \frac{1}{2}[2 \sup_{t\in[0;\infty)} \frac{1-e^{-t}}{1+t^2} + \frac{1}{e}] > 0.07001134 \\
> \ & \frac{1}{2e^2} \approx 0.06766764.
\end{aligned}
$$

Next we see that, for each $u \in M$,

$$
\begin{aligned}
\|\mathcal{V}u\| &= \sup_{t\in A} |\frac{t^2}{2e^{2t}} + \frac{u^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[|u(h(\tau))|^{1/2} + e^{-u^2(h(\tau))} + e^{t-\tau-1}]d\tau| \\
&\leq \frac{1}{2e^2} + \sup_{t\in[0;\infty)} \frac{1-e^{-t}}{1+t^2} + \frac{1}{2e} \approx 0.067667641618306 \\
&\quad +0.338018798761873 + 0.183938720585721 \\
&= 0.5896251609659 < 1.
\end{aligned}
$$

This implies (7.103).

7.11.2. Let $u, v \in M$. Then there exists $\gamma \in (0; 1)$,

$$\gamma = (\frac{5}{2} + \frac{1}{\sqrt{2e}}) \sup_{t\in A} \frac{1-e^{-t}}{1+t^2} + \frac{1}{2e} \approx 0.536888840552378,$$

such that:

(a) If

$$D(u, v) < 1, \tag{7.104}$$

then

$$D^2(\mathcal{V}u, \mathcal{V}v) \leq \gamma^2 D(u, v) \tag{7.105}$$

and

$$\lim_{m\to\infty} D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) = 0. \tag{7.106}$$

(b) If

$$D(u, v) \geq 1, \tag{7.107}$$

then

$$D(\mathcal{V}u, \mathcal{V}v) \leq \gamma D(u, v) \tag{7.108}$$

and

$$\lim_{m\to\infty} D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) = 0. \tag{7.109}$$

In fact, if $u, v \in M$, then

$$\begin{aligned}
& D(\mathcal{V}u, \mathcal{V}v) = \sup_{t\in A} |(\mathcal{V}u)(t) - (\mathcal{V}v)(t)| \\
\leq \quad & \sup_{t\in A} |\frac{u^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[|u(h(\tau))|^{1/2} + e^{-u^2(h(\tau))} + e^{t-\tau-1}]d\tau \\
& -\frac{v^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[|v(h(\tau))|^{1/2} + e^{-v^2(h(\tau))} + e^{t-\tau-1}]d\tau| \\
\leq \quad & \sup_{t\in A} |\frac{u^2(t)-v^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[|v(h(\tau))|^{1/2} + e^{-v^2(h(\tau))} + e^{t-\tau-1}]d\tau \\
& +\frac{v^2(t)}{2(1+t^2)} \int_0^t e^{\tau-t}[|u(h(\tau))|^{1/2} + e^{-u^2(h(\tau))} \\
& - |v(h(\tau))|^{1/2} - e^{-v^2(h(\tau))}]d\tau| \\
\leq \quad & 2D(u,v) \sup_{t\in A} \int_0^t [e^{\tau-t}(1+1) + \frac{1}{e}]\frac{d\tau}{2(1+t^2)} \\
& + \sup_{\beta\in A}[|\,|u(\beta)|^{1/2} - |v(\beta)|^{1/2}\,| + |e^{-u^2(\beta)} - e^{-v^2(\beta)}|] \sup_{t\in A}\int_0^t \frac{e^{\tau-t}d\tau}{2(1+t^2)} \\
\leq \quad & 2D(u,v) \sup_{t\in A}[2\frac{1-e^{-t}}{2(1+t^2)} + \frac{t}{2e(1+t^2)}] \\
& + \sup_{\beta\in A}[|u(\beta)-v(\beta)|^{1/2} + \sqrt{\frac{2}{e}}|u(\beta)-v(\beta)|] \sup_{t\in A}\frac{1-e^{-t}}{2(1+t^2)} \\
\leq \quad & D(u,v)[2\sup_{t\in A}\frac{1-e^{-t}}{1+t^2} + \frac{1}{2e}] \\
& +\frac{1}{2}[D^{1/2}(u,v) + \sqrt{\frac{2}{e}}D(u,v)] \sup_{t\in A}\frac{1-e^{-t}}{1+t^2} \\
= \quad & \gamma_1 D^{1/2}(u,v) + \gamma_2 D(u,v)
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}\sup_{t\in A}\frac{1-e^{-t}}{1+t^2} \approx \frac{1}{2}\cdot 0.338018798761873 \\
&= 0.169009399380936, \qquad (7.110)
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= (2+\frac{1}{\sqrt{2e}})\sup_{t\in A}\frac{1-e^{-t}}{1+t^2} + \frac{1}{2e} \\
&\approx 0.144970159007868 + 0.183939720585721 \\
&= 0.367879441171442, \qquad (7.111)
\end{aligned}$$

$$\gamma = \gamma_1 + \gamma_2 \approx 0.536888840552378. \qquad (7.112)$$

Next using (7.109)-(7.112) and by analogous arguments as in 7.9.2 of Example 9.9, we find (7.104)-(7.109).

7.11.3. From (7.104), (7.105), (7.107) and (7.108) we conclude that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is continuous on $M$ and thus closed in each $y^0 \in M$. Next, using (7.109), (a) and (b), we conclude that

$$\forall_{s,l\in\mathbb{N}}\forall_{y^0\in M}\{\lim_{r\to\infty} D(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)=0\}.$$

Clearly, a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. Therefore, (I.4) and the assertion (B) of Theorem 4.2(ii) with condition (4.21) hold when $\mathcal{W}=\mathcal{V}$ and $M$ is constructed in 7.11.1.

**Example 7.12.** Let $A=[0;1]$ and $h\in\mathcal{C}(A,A)$. Fix $\mu\in(0;1)$ and let $\lambda\in\mathbb{R}$ satisfies

$$0<|\lambda|\le\frac{\pi\mu(3-\mu)^2}{52}. \tag{7.113}$$

The set $\mathcal{Y}_{\mathcal{F}}$ of solutions of the Fredholm integral equation

$$u(t)=\sin(\pi t)+\lambda\int_0^1\cos(\pi t)\sin(\pi\tau)u^3(h(\tau))d\tau,$$

$u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, is nonempty. Moreover, for each $y^0\in M_{\mu,\lambda}$, where

$$M_{\mu,\lambda}=\{u\in\mathcal{C}(A,\mathbb{R}):\|u\|\le\sqrt{\frac{\pi\mu}{6\,|\lambda|}}\},$$

there exists $y\in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{F})\subset\mathcal{Y}_{\mathcal{F}}\subset\mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{F}^{[m]}y^0: m\in\{0\}\cup\mathbb{N})$,

$$(\mathcal{F}^{[m+1]}y^0)(t)=\sin(\pi t)+\lambda\int_0^1\cos(\pi t)\sin(\pi\tau)(\mathcal{F}^{[m]}y^0)^3(h(\tau))d\tau,$$

$t\in A$, $m\in\{0\}\cup\mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v)=\sup_{t\in A}|u(t)-v(t)|\}$.

The proof proceeds in three steps.

7.12.1. It is clear that

$$\forall_{m\in\mathbb{N},\mu\in(0;1)}\{\mathcal{F}^{[m]}:M_{\mu,\lambda}\to M_{\mu,\lambda}\}. \tag{7.114}$$

To establish (7.114), notice that

$$\begin{aligned}&\sup_{t\in A}|\lambda\int_0^1\cos(\pi t)\sin(\pi\tau)u^3(h(\tau))d\tau|\\ \le\ &|\lambda|\cdot|\int_0^1\sin(\pi\tau)d\tau|\cdot\sup_{\beta\in A}|u^3(\beta)|=\frac{2\,|\lambda|}{\pi}\,\|u\|^3,\ u\in\mathcal{C}(A,\mathbb{R}),\end{aligned}$$

$$|\int_0^1\sin(\pi\tau)d\tau|=\frac{2}{\pi},\quad\sup_{t\in[0;\infty)}|\sin(\pi t)|=1.$$



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Hence, if $u \in M_{\mu,\lambda}$, then

$$\begin{aligned}
\|\mathcal{F}v\| &= \sup_{t\in A} |\sin(\pi t) + \lambda \int_0^1 \cos(\pi t)\sin(\pi\tau)u^3(h(\tau))d\tau| \\
&\leq 1 + \frac{2|\lambda|}{\pi}\sup_{\beta\in A}|u^3(\beta)| \leq 1 + \frac{2|\lambda|}{\pi}\|u\|^3 \leq 1 + \frac{2|\lambda|}{\pi}(\sqrt{\frac{\pi\mu}{6|\lambda|}})^3 \\
&= 1 + \frac{2|\lambda|}{\pi}\frac{\pi\mu}{6|\lambda|}\sqrt{\frac{\pi\mu}{6|\lambda|}} = 1 + \frac{\mu}{3}\sqrt{\frac{\pi\mu}{6|\lambda|}}.
\end{aligned}$$

However

$$1 + \frac{\mu}{3}\sqrt{\frac{\pi\mu}{6|\lambda|}} \leq \sqrt{\frac{\pi\mu}{6|\lambda|}}. \tag{7.115}$$

In fact, (7.113) implies

$$1 \leq \frac{(3-\mu)^2}{9}\frac{\pi\mu}{6|\lambda|}.$$

Then

$$1 \leq (1 - \frac{\mu}{3})\sqrt{\frac{\pi\mu}{6|\lambda|}} = \frac{3-\mu}{3}\sqrt{\frac{\pi\mu}{6|\lambda|}}$$

and this gives (7.115). Therefore, (7.114) holds.

7.12.2. We claim that

$$\forall_{u,v\in M_{\mu,\lambda}}\forall_{m\in\mathbb{N}}\{D(\mathcal{F}^{[m]}u, \mathcal{F}^{[m]}v) \leq \mu^m D(u,v)\}. \tag{7.116}$$

To see this, note that

$$\begin{aligned}
&D(\mathcal{F}u, \mathcal{F}v) = \sup_{t\in A}|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\
\leq\ & \sup_{t\in A}|\lambda\int_0^1 \cos(\pi t)\sin(\pi\tau)[u^3(h(\tau)) - v^3(h(\tau))]d\tau| \\
\leq\ & \frac{2|\lambda|}{\pi}\sup_{\beta\in A}|u^3(\beta) - v^3(\beta)| \\
\leq\ & \frac{2|\lambda|}{\pi}D(u,v)[\|u\|^2 + \|u\|\,\|v\| + \|v\|^2] \\
\leq\ & \frac{2|\lambda|}{\pi}3\mu\frac{\pi}{6|\lambda|}D(u,v) = \mu D(u,v).
\end{aligned}$$

This, by (7.114), gives (7.116).

7.12.3. Observe that the estimate (7.116) implies

$$\begin{aligned}
&\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M_\mu}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) \\
<\ & \varepsilon + \eta \Rightarrow D(\mathcal{F}^{[s+r]}y^0, \mathcal{F}^{[l+r]}y^0) \leq \mu^r D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0) \\
\leq\ & \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{F}^{[s]}y^0, \mathcal{F}^{[l]}y^0)\},
\end{aligned}$$



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From (7.116) it follows also that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is continuous on $M_{\mu,\lambda}$ and thus closed in each $y^0 \in M_{\mu,\lambda}$. In addition, we see that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is $D$-admissible on $M_{\mu,\lambda} \subset \mathcal{C}(A,\mathbb{R})$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. The above shows that (I.3) is satisfied and that the assertion (B) of Theorem 4.2(i) when $\mathcal{W}=\mathcal{F}$ holds.

**Example 7.13.** Let $A=[-\frac{1}{2};\frac{1}{2}]$ and $h \in \mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{F}} \subset \mathcal{C}(A,\mathbb{R})$ of solutions of the Fredholm integral equation

$$u(t)=t+\frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{t\tau}u^4(h(\tau))d\tau,\ u \in \mathcal{C}(A,\mathbb{R}),\ t \in A,$$

is nonempty. Moreover, denote

$$\lambda=\sup_{t \in A}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{t\tau}d\tau=\frac{e^{t_0\frac{1}{2}}-e^{-t_0\frac{1}{2}}}{t_0}|_{t_0=\frac{1}{2}}=1.010449267232673,$$

let $\mu_0 \in (0;1)$ be provided by

$$\frac{\mu_0(4-\mu_0)^3}{24}=\lambda,\ (\mu_0 \approx 0.6383821), \tag{7.117}$$

let $\mu \in (\mu_0;1)$ and suppose that $M_\mu$ is the set:

$$M_\mu=\{u \in \mathcal{C}(A,\mathbb{R}):\|u\| \leq \sqrt[3]{\frac{\mu}{3\lambda}}\}.$$

Then, for each $y^0 \in M_\mu$, there exists $y \in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{F}) \subset \mathcal{Y}_{\mathcal{F}}$ such that a sequence $(\mathcal{F}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$,

$$(\mathcal{F}^{[m+1]}y^0)(t)=t+\frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{t\tau}(\mathcal{F}^{[m]}y^0)^4(h(\tau))d\tau,$$

$t \in A$, $m \in \{0\}\cup\mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v \in \mathcal{C}(A,\mathbb{R})}\{D(u,v)=\sup_{t \in A}|u(t)-v(t)|\}$.

The proof proceeds in three steps.

7.13.1. It is clear that

$$\forall_{m \in \mathbb{N}}\{\mathcal{F}^{[m]}: M_\mu \to M_\mu\}. \tag{7.118}$$

Indeed, if $u \in M_\mu$, then we have

$$\begin{aligned}\|\mathcal{F}v\| &= \sup_{t \in A}|t+\frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{t\tau}u^4(h(\tau))d\tau| \\ &\leq \frac{1}{2}+\frac{3}{4}\lambda\sup_{\beta \in A}|u(\beta)|^4 \leq \frac{1}{2}+\frac{3}{4}\lambda(\sqrt[3]{\frac{\mu}{3\lambda}})^4=\frac{1}{2}+\frac{\mu}{4}\sqrt[3]{\frac{\mu}{3\lambda}}.\end{aligned} \tag{7.119}$$

Now, let us observe that a map $\psi$ of the form

$$\psi(z) = \frac{z(4-z)^3}{24}$$

is increasing on $(0;1)$ and

$$\psi(0) = 0 < \frac{z(4-z)^3}{24} < \frac{27}{24} = \psi(1) = 1.125 \text{ for } z \in (0;1).$$

Using this in conjugation with (7.117), we obtain

$$\lambda < \frac{\mu(4-\mu)^3}{24} \text{ for } \mu \in (\mu_0;1),\ (\mu_0 \approx 0.6383821).$$

We can then rewrite the above as

$$8 < (4-\mu)^3 \frac{\mu}{3\lambda}.$$

That is, if $\mu \in (\mu_0;1)$, then

$$\frac{1}{2} < \frac{4-\mu}{4}\sqrt[3]{\frac{\mu}{3\lambda}} = (1-\frac{\mu}{4})\sqrt[3]{\frac{\mu}{3\lambda}}.$$

Using this and (7.119), we are led to

$$\|\mathcal{F}u\| \leq \frac{1}{2} + \frac{\mu}{4}\sqrt[3]{\frac{\mu}{3\lambda}} < \sqrt[3]{\frac{\mu}{3\lambda}}.$$

Therefore, (7.118) holds.

7.13.2. We claim that

$$\forall_{u,v\in M_\mu}\forall_{m\in\mathbb{N}}\{D(\mathcal{F}^{[m]}u,\mathcal{F}^{[m]}v) \leq \mu^m D(u,v)\}. \tag{7.120}$$

Indeed, for $u, v \in M_\mu$, we obtain

$$\begin{aligned} D(\mathcal{F}u,\mathcal{F}v) &= \sup_{t\in A}|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\ &\leq \frac{3}{4}\sup_{t\in A}|\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{t\tau}[u^4(h(\tau)) - v^4(h(\tau))]d\tau| = \frac{3}{4}\lambda\sup_{\beta\in A}|u^4(\beta) - v^4(\beta)| \\ &\leq \frac{3}{4}\lambda D(u,v)[\|u\|^3 + \|u\|^2\|v\| + \|u\|\|v\|^2 + \|v\|^3] \\ &\leq \frac{3}{4}\lambda 4D(u,v)4\frac{\mu}{3\lambda} = \mu D(u,v). \end{aligned}$$

This, in view of (7.118), implies (7.120).

7.13.3. Let $\mu \in (\mu_0;1)$ be arbitrary and fixed. Observe that the estimate (7.120) implies

$$\begin{aligned} &\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M_\mu}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0) \\ < \;& \varepsilon+\eta \Rightarrow D(\mathcal{F}^{[s+r]}y^0,\mathcal{F}^{[l+r]}y^0) \leq \mu^r D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0) \\ \leq \;& \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{F}^{[s]}y^0,\mathcal{F}^{[l]}y^0)\}. \end{aligned}$$

From (7.120) we also deduce that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is continuous on $M_\mu$ and thus closed in each $y^0\in M_\mu$. In addition, we see that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is $D$-admissible on $M_\mu$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. Using (I.3) we therefore have that the assertion (B) of Theorem 4.2(i) when $\mathcal{W}=\mathcal{F}$ holds.

**Example 7.14.** Let $A=[0;1]$ and $h\in\mathcal{C}(A,A)$. The set $\mathcal{Y}_{\mathcal{V}}$ of solutions of the Volterra integral equation

$$u(t)=\frac{t^2}{4e^t}+\int_0^t\frac{(t-\tau)^2}{4}e^{\tau-t}u(h(\tau))d\tau,$$

$u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, is nonempty. Moreover, for each $y^0\in M$, where

$$M=\{u\in\mathcal{C}(A,\mathbb{R}):\|u\|\leq\frac{1}{e(4-e)}\},$$

there exists $y\in Fix_{\mathcal{C}(A,\mathbb{R})}(\mathcal{V})\subset\mathcal{Y}_{\mathcal{V}}\subset\mathcal{C}(A,\mathbb{R})$ such that a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$,

$$(\mathcal{V}^{[m+1]}y^0)(t)=\frac{t^2}{4e^t}+\int_0^t\frac{(t-\tau)^2}{4}e^{\tau-t}(\mathcal{V}^{[m]}y^0)(h(\tau))d\tau,$$

$t\in A$, $m\in\{0\}\cup\mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v)=\sup_{t\in A}|u(t)-v(t)|\}$.

The proof proceeds in three steps.

7.14.1. It is clear that

$$\forall_{m\in\mathbb{N}}\{\mathcal{V}^{[m]}:M\to M\}.$$

Indeed, if $u\in M$, then we have

$$\begin{aligned}\|\mathcal{V}u\| &= \sup_{t\in A}|\frac{t^2}{4e^t}+\int_0^t\frac{(t-\tau)^2}{4}e^{\tau-t}u(h(\tau))d\tau|\leq\frac{1}{4e}+\frac{e}{4}\sup_{\beta\in A}|u(\beta)|\int_0^t d\tau\\ &\leq \frac{1}{4e}+\frac{e}{4}\|u\|\leq\frac{1}{4e}+\frac{e}{4}\frac{1}{e(4-e)}=\frac{4-e+e}{4e(4-e)}=\frac{1}{e(4-e)}.\end{aligned}$$

7.14.2. We claim that

$$\forall_{u,v\in M}\forall_{m\in\mathbb{N}}\{D(\mathcal{V}^{[m]}u,\mathcal{V}^{[m]}v)\leq(\frac{e}{4})^mD(u,v)\}.\tag{7.121}$$

Indeed, for $u,v\in M$, we obtain

$$\begin{aligned}&D(\mathcal{V}u,\mathcal{V}v)\\ =\ &\sup_{t\in A}|(\mathcal{V}u)(t)-(\mathcal{V}v)(t)|\leq\sup_{t\in A}|\int_0^t\frac{(t-\tau)^2}{4}e^{\tau-t}[u(h(\tau))-v(h(\tau))]d\tau|\\ \leq\ &\frac{e}{4}\sup_{\beta\in A}|u(\beta)-v(\beta)|\sup_{t\in A}|\int_0^t d\tau|=\frac{e}{4}D(u,v).\end{aligned}$$

By 7.14.1, this gives (7.121).

7.14.3. Observe that the estimate (7.121) implies

$$\begin{aligned}&\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)\\ < \ &\varepsilon+\eta\Rightarrow D(\mathcal{V}^{[s+r]}y^0,\mathcal{V}^{[l+r]}y^0)\le(\frac{e}{4})^r D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)\\ \le \ &\frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}y^0,\mathcal{V}^{[l]}y^0)\}.\end{aligned}$$

From (7.121) we deduce also that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is continuous on $M$ and thus closed in each $y^0\in M$. Obviously, a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{V})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete and $M\subset\mathcal{C}(A,\mathbb{R})$. The above shows that the assertion (B) of Theorem 4.2(i) with condition (I.3) and when $\mathcal{W}=\mathcal{V}$ holds.

**Example 7.15.** Let $A=[0;1]$. The Fredholm integral equation

$$u(t)=\arctan t+\int_0^1\frac{u(\tau)\arctan t}{1-\tau^2}d\tau,$$

$u\in\mathcal{C}(A,\mathbb{R})$, $t\in A$, has a unique solution $y\in\mathcal{C}(A,\mathbb{R})$, $\mathcal{Y}_{\mathcal{F}}=\{y\}$, of the form

$$y(t)=\frac{32}{32-\pi^2}\arctan t,\ t\in A. \tag{7.122}$$

Moreover, for each $y^0\in\mathcal{C}(A,\mathbb{R})$, a sequence $(\mathcal{F}^{[m]}y^0 : m\in\{0\}\cup\mathbb{N})$,

$$(\mathcal{F}^{[m+1]}y^0)(t)=\arctan t+\int_0^1\frac{(\mathcal{F}^{[m]}y^0)(\tau)\arctan t}{1-\tau^2}d\tau,$$

$t\in A$, $m\in\{0\}\cup\mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v)=\sup_{t\in A}|u(t)-v(t)|\}$.

The proof proceeds in four steps.

7.15.1. We claim that

$$\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\forall_{m\in\mathbb{N}}\{D(\mathcal{F}^{[m]}u,\mathcal{F}^{[m]}v)\le(\frac{\pi}{4})^{2m}D(u,v)\}. \tag{7.123}$$

Indeed, for $u,v\in\mathcal{C}(A,\mathbb{R})$, we obtain

$$\begin{aligned}&D(\mathcal{F}u,\mathcal{F}v)=\sup_{t\in A}|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)|\\ \le \ &\sup_{t\in A}|\arctan t\int_0^1\frac{[u(\tau)-v(\tau)]d\tau}{1-\tau^2}|\le\frac{\pi}{4}D(u,v)\int_0^1\frac{d\tau}{1-\tau^2}\\ = \ &(\frac{\pi}{4})^2D(u,v).\end{aligned}$$

This implies (7.123).

7.15.2. Observe that the estimate (7.123) implies

$$\begin{aligned}
& \forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{F}^{[s]}u,\mathcal{F}^{[l]}v) \\
< \ & \varepsilon+\eta \Rightarrow D(\mathcal{F}^{[s+r]}u,\mathcal{F}^{[l+r]}v) \leq (\frac{\pi}{4})^{2r}D(\mathcal{F}^{[s]}u,\mathcal{F}^{[l]}v) \\
\leq \ & \frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{F}^{[s]}u,\mathcal{F}^{[l]}y)\}.
\end{aligned}$$

Next from (7.123) we deduce that a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is continuous on $\mathcal{C}(A,\mathbb{R})$ and thus closed in each $y^0 \in \mathcal{C}(A,\mathbb{R})$. Also a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is $D$-admissible on $\mathcal{C}(A,\mathbb{R})$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete. The above shows that (I.3) and the assertion (B) of Theorem 4.1(i) when $\mathcal{W}=\mathcal{F}$ hold.

7.15.3. We claim that $y$, given by (7.122), satisfies $\mathcal{F}y=y$. To see this, notice that

$$\begin{aligned}
& \arctan t+\int_0^1\frac{y(\tau)\arctan t}{1-\tau^2}d\tau=[1+\frac{32}{32-\pi^2}\int_0^1\frac{\arctan\tau}{1-\tau^2}d\tau]\arctan t \\
= \ & [1+\frac{32}{32-\pi^2}\frac{1}{2}(\frac{\pi}{4})^2]\arctan t=\frac{32}{32-\pi^2}\arctan t.
\end{aligned}$$

7.15.4. We also note that

$$\forall_{y^0\in\mathcal{C}(A,\mathbb{R})}\{\lim_{m\to\infty}D(\mathcal{F}^{[m]}y^0,y)=0\}. \tag{7.124}$$

In fact, let $y^0\in\mathcal{C}(A,\mathbb{R})$. Observe that $\exists_{\gamma_1,\gamma_2\in\mathbb{R}}\forall_{t\in A}\{\gamma_1\leq y^0(t)\leq\gamma_2\}$. Consequently,

$$(\mathcal{F}y^0)(t)=[1+\int_0^1\frac{y^0(\tau)}{1-\tau^2}d\tau]\arctan t,\ t\in A,$$

satisfies $B_1^1\arctan t\leq(\mathcal{F}y^0)(t)\leq B_2^1\arctan t$, $t\in A$, where

$$B_i^1=1+\int_0^1\frac{\gamma_i}{1-\tau^2}d\tau=1+\gamma_i\frac{\pi}{4},\ i=1,2.$$

Next, we see that

$$(\mathcal{F}^{[2]}y^0)(t)=[1+\int_0^1\frac{(\mathcal{F}y^0)(\tau)}{1-\tau^2}d\tau]\arctan t,\ t\in A,$$

satisfies $B_1^2\arctan t\leq(\mathcal{F}^{[2]}y^0)(t)\leq B_2^2\arctan t$, $t\in A$, where

$$\begin{aligned}
B_i^2 &= 1+B_i^1\int_0^1\frac{\arctan\tau}{1-\tau^2}d\tau=1+(1+\gamma_i\frac{\pi}{4})\int_0^1\frac{\arctan\tau}{1-\tau^2}d\tau \\
&= 1+(1+\gamma_i\frac{\pi}{4})\frac{1}{2}(\frac{\pi}{4})^2=1+\frac{1}{2}(\frac{\pi}{4})^2+\frac{1}{2}\gamma_i(\frac{\pi}{4})^3,\ i=1,2.
\end{aligned}$$

Clearly,

$$(\mathcal{F}^{[3]}y^0)(t)=[1+\int_0^1\frac{(\mathcal{F}^{[2]}y^0)(\tau)}{1-\tau^2}d\tau]\arctan t,\ t\in A,$$

satisfies $B_1^3 \arctan t \leq (\mathcal{F}^{[2]}y^0)(t) \leq B_2^3 \arctan t$, $t \in A$, where

$$\begin{aligned}
B_i^3 &= 1 + B_i^2 \int_0^1 \frac{\arctan \tau}{1-\tau^2} d\tau = 1 + [1 + \frac{1}{2}(\frac{\pi}{4})^2 + \frac{1}{2}\gamma_i(\frac{\pi}{4})^3]\frac{1}{2}(\frac{\pi}{4})^2 \\
&= 1 + \frac{1}{2}(\frac{\pi}{4})^2 + \frac{1}{2^2}(\frac{\pi}{4})^4 + \frac{1}{2^2}\gamma_i(\frac{\pi}{4})^5,\ i = 1, 2.
\end{aligned}$$

In general, we have that

$$(\mathcal{F}^{[m]}y^0)(t) = [1 + \int_0^1 \frac{(\mathcal{F}^{[m-1]}y^0)(\tau)}{1-\tau^2} d\tau] \arctan t,\ t \in A,\ m \geq 2,$$

satisfies $B_1^m \arctan t \leq (\mathcal{F}^{[m]}y^0)(t) \leq B_2^m \arctan t$, $t \in A$, where

$$\begin{aligned}
B_i^m &= 1 + B_i^{m-1} \int_0^1 \frac{\arctan \tau}{1-\tau^2} d\tau = 1 + B_i^{m-1}\frac{1}{2}(\frac{\pi}{4})^2 \\
&= 1 + \frac{1}{2}(\frac{\pi}{4})^2 + \ldots + \frac{1}{2^{m-1}}(\frac{\pi}{4})^{2(m-1)} + \frac{1}{2^{m-1}}\gamma_i(\frac{\pi}{4})^{2(m-1)+1},\ i = 1, 2.
\end{aligned}$$

However,

$$\textstyle\sum_{m=1}^{\infty} \frac{1}{2^{m-1}}(\frac{\pi}{4})^{2(m-1)} = 1 + \frac{1}{2}(\frac{\pi}{4})^2 \frac{1}{1 - \frac{1}{2}(\frac{\pi}{4})^2} = \frac{32}{32-\pi^2}.$$

Therefore,

$$\lim_{m\to\infty} B_i^m \arctan t = \frac{32}{32-\pi^2} \arctan t,\ i = 1, 2,\ t \in A.$$

This implies (7.124).

**Example 7.16.** Let $A = [0; \frac{\pi}{2}]$. The Fredholm integral equation

$$u(t) = \cos t + \int_0^{\frac{\pi}{2}} u(\tau) \cos t \sin \tau d\tau,\ u \in \mathcal{C}(A, \mathbb{R}),\ t \in A,$$

has a unique solution $y \in \mathcal{C}(A, \mathbb{R})$, $\mathcal{Y}_{\mathcal{F}} = \{y\}$, of the form

$$y(t) = 2\cos t,\ t \in A. \tag{7.125}$$

Moreover, for each $y^0 \in \mathcal{C}(A, \mathbb{R})$, a sequence $(\mathcal{F}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$,

$$(\mathcal{F}^{[m+1]}y^0)(t) = \cos t + \int_0^{\frac{\pi}{2}} (\mathcal{F}^{[m]}y^0)(\tau) \cos t \sin \tau d\tau,$$

$t \in A$, $m \in \{0\} \cup \mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A, \mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u, v) = \sup_{t\in A} |u(t) - v(t)|\}$.

The proof proceeds in four steps.

7.16.1. We claim that

$$\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\forall_{m\in\mathbb{N}}\{D(\mathcal{F}^{[m]}u, \mathcal{F}^{[m]}v) \leq D(u, v)\}. \tag{7.126}$$

Indeed, for $u, v \in \mathcal{C}(A, \mathbb{R})$, we obtain

$$\begin{aligned} D(\mathcal{F}u, \mathcal{F}v) &= \sup_{t \in A} |(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\ &\leq |\int_0^{\frac{\pi}{2}} \sin \tau d\tau| \sup_{\beta \in A} |u(\beta) - v(\beta)| \leq D(u, v). \end{aligned}$$

This implies (7.126).

7.16.2. Now observe that

$$\forall_{u,v \in \mathcal{C}(A,\mathbb{R})} \{ \lim_{m \to \infty} D(\mathcal{F}^{[m]}u, \mathcal{F}^{[m]}v) = 0 \}. \tag{7.127}$$

In fact, if $w \in \mathcal{C}(A, \mathbb{R})$, then $\exists_{\gamma_1, \gamma_2 \in \mathbb{R}} \forall_{t \in A} \{ \gamma_1 \leq w(t) \leq \gamma_2 \}$,

$$\begin{aligned} (1 + \gamma_1) \cos t &= [1 + \gamma_1 \int_0^{\frac{\pi}{2}} \sin \tau d\tau] \cos t \\ &\leq (\mathcal{F}w)(t) = \cos t + \int_0^{\frac{\pi}{2}} w(\tau) \cos t \sin \tau d\tau \\ &\leq [1 + \gamma_2 \int_0^{\frac{\pi}{2}} \sin \tau d\tau] \cos t = (1 + \gamma_2) \cos t, \end{aligned}$$

$$\begin{aligned} & (1 + \frac{1}{2} + \frac{1}{2}\gamma_1) \cos t = [1 + (1 + \gamma_1) \int_0^{\frac{\pi}{2}} \cos \tau \sin \tau d\tau] \cos t \\ \leq \ & (\mathcal{F}^{[2]}w)(t) = [1 + \int_0^{\frac{\pi}{2}} (\mathcal{F}w)(\tau) \sin \tau d\tau] \cos t \\ \leq \ & [1 + (1 + \gamma_2) \int_0^{\frac{\pi}{2}} \cos \tau \sin \tau d\tau] \cos t = (1 + \frac{1}{2} + \frac{1}{2}\gamma_2) \cos t \end{aligned}$$

and, for $m \geq 3$,

$$\begin{aligned} & (1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{m-1}} + \frac{1}{2^{m-1}}\gamma_1) \cos t \\ \leq \ & (\mathcal{F}^{[m]}w)(t) \leq (1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{m-1}} + \frac{1}{2^{m-1}}\gamma_2) \cos t. \end{aligned} \tag{7.128}$$

Let $u, v \in \mathcal{C}(A, \mathbb{R})$. Then $\exists_{\alpha_1, \alpha_2 \in \mathbb{R}} \forall_{t \in A} \{ \alpha_1 \leq u(t) \leq \alpha_2 \}$, $\exists_{\beta_1, \beta_2 \in \mathbb{R}} \forall_{t \in A} \{ \beta_1 \leq v(t) \leq \beta_2 \}$ and using (7.128) we, therefore, have

$$\frac{1}{2^{m-1}}(\alpha_1 - \beta_2) \cos t \leq (\mathcal{F}^{[m]}u)(t) - (\mathcal{F}^{[m]}v)(t) \leq \frac{1}{2^{m-1}}(\alpha_2 - \beta_1) \cos t, \ t \in A.$$

This implies (7.127).

7.16.3. Thus (I.3), condition (4.2) and the assertion (B) of Theorem 4.1(i) when $\mathcal{W} = \mathcal{F}$ hold. In fact, (7.126) say that $\mathcal{F}$ is continuous on $\mathcal{C}(A, \mathbb{R})$ which implies that $\mathcal{F}$ is closed in each $y^0 \in \mathcal{C}(A, \mathbb{R})$. Furthermore, (7.127) implies that

$$\forall_{u,v \in \mathcal{C}(A,\mathbb{R})} \forall_{s,l \in \mathbb{N}} \{ \lim_{r \to \infty} D(\mathcal{F}^{[r+s]}u, \mathcal{F}^{[r+l]}v) = 0 \}.$$

Also a dynamic system $(\mathcal{C}(A,\mathbb{R}),\mathcal{F})$ is $D$-admissible on $\mathcal{C}(A,\mathbb{R})$ since a gauge space $(\mathcal{C}(A,\mathbb{R}),D)$ is sequentially complete.

7.16.4. We prove that there exists a unique $y\in\mathcal{C}(A,\mathbb{R})$ such that

$$\forall_{y^0\in\mathcal{C}(A,\mathbb{R})}\{\lim_{m\to\infty} D(\mathcal{F}^{[m]}y^0,y)=0\}.$$

We determine this map $y$. If $y^0\in\mathcal{C}(A,\mathbb{R})$, then $\exists_{\gamma_1,\gamma_2\in\mathbb{R}}\forall_{t\in A}\{\gamma_1\leq y^0(t)\leq\gamma_2\}$ and using (7.128) we have

$$\begin{aligned}
&(1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{m-1}}+\frac{1}{2^{m-1}}\gamma_1)\cos t\\
\leq\ &(\mathcal{F}^{[m]}y^0)(t)\leq(1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{m-1}}+\frac{1}{2^{m-1}}\gamma_2)\cos t.
\end{aligned}$$

However $\forall_{t\in A}\forall_{i=1,2}\{\lim_{m\to\infty}(1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{m-1}}+\frac{1}{2^{m-1}}\gamma_i)\cos t=2\cos t\}$ and thus $\mathcal{Y}_{\mathcal{F}}=\{y\}$ where $y$ is of the form (7.125).

**Example 7.17.** Let $A=[0;1]$, $\alpha\in(0;1)$, $\beta\in(1;\infty)$ and $h\in\mathcal{C}(A,A)$. Suppose that $n\in\mathbb{N}$ satisfies

$$n>\frac{26\beta^3}{\Gamma(\alpha+1)}. \tag{7.129}$$

The set $\mathcal{Y}_{\mathcal{V}}^+$, $\mathcal{Y}_{\mathcal{V}}^+\subset\{y\in\mathcal{C}(A,\mathbb{R}):y(t)\geq 0 \text{ for } t\in A\}$, of nonnegative solutions of the quadratic integral equation of fractional order

$$\begin{aligned}
u(t)\ =\ &\beta^{t^2-3}+\frac{1}{4}\sin^2(tu(t))\\
&+\sqrt[n]{u(t)}\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau+u^2(h(\tau))]d\tau,\ t\in A,
\end{aligned}\tag{7.130}$$

is nonempty. Moreover, for each $y^0\in M$, where

$$M=\{u\in\mathcal{C}(A,\mathbb{R}):\frac{1}{\beta^3}\leq u(t)\leq 2 \text{ for } t\in A\}, \tag{7.131}$$

there exists $y\in\mathcal{Y}_{\mathcal{V}}^+$ such that a sequence $(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})$,

$$\begin{aligned}
(\mathcal{V}^{[m+1]}y^0)(t)\ =\ &\beta^{t^2-3}+\frac{1}{4}\sin^2(t(\mathcal{V}^{[m]}y^0)(t))\\
&+\sqrt[n]{(\mathcal{V}^{[m]}y^0)(t)}\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau+(\mathcal{V}^{[m]}y^0)^2(h(\tau))]d\tau,
\end{aligned}$$

$t\in A$, $m\in\{0\}\cup\mathbb{N}$, is $D$-convergent to $y$; $D$ is a gauge on $\mathcal{C}(A,\mathbb{R})$ of the form $\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{D(u,v)=\sup_{t\in A}|u(t)-v(t)|\}$.

The proof proceeds in four steps.

7.17.1. The followings are obvious:

$$\forall_{t\in A}\{\frac{1}{\beta^3}\leq\beta^{t^2-3}\leq\frac{1}{\beta^2}\}, \tag{7.132}$$

$$\forall_{t\in A}\forall_{u,v\in\mathcal{C}(A,\mathbb{R})}\{\frac{1}{4}|\sin^2(tu(t))-\sin^2(tu(t))|\leq\frac{1}{2}D(u,v)\},\tag{7.133}$$

$$\begin{aligned}\forall_{t\in A}\forall_{u,v\in M}\{|\sqrt[n]{u(t)}-\sqrt[n]{v(t)}| &\leq \sup_{z\in M}\frac{1}{n}(\|z\|)^{\frac{1}{n}-1}D(u,v)\\ &= \frac{1}{n}\beta^{3(1-\frac{1}{n})}D(u,v)\},\end{aligned}\tag{7.134}$$

$$\forall_{t\in A}\forall_{u\in M}\{\sqrt[n]{u(t)}\leq\frac{2}{n}\beta^{3(1-\frac{1}{n})}\},\tag{7.135}$$

$$\forall_{t\in A}\forall_{u\in M}\{\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau+u^2(h(\tau))]d\tau\leq\frac{5}{\Gamma(\alpha+1)}\}.\tag{7.136}$$

7.17.2. We claim that

$$\forall_{m\in\mathbb{N}}\{\mathcal{V}^{[m]}:M\to M\}.\tag{7.137}$$

Let $u\in M$. By (7.131), (7.135) and (7.136), we get

$$\begin{aligned}&\sup_{t\in A}|\frac{1}{4}\sin^2(tu(t))\\ &+\sqrt[n]{u(t)}\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau+u^2(h(\tau))]d\tau|\\ \leq\ &\frac{1}{4}+\frac{2}{n}\beta^{3(1-\frac{1}{n})}\frac{5}{\Gamma(\alpha+1)}<\frac{1}{4}+\frac{10}{n}\frac{\beta^3}{\Gamma(\alpha+1)}.\end{aligned}$$

Hence, in view of (7.130), (7.131) and (7.132), observe that

$$2-[\frac{1}{4}+\frac{10}{n}\frac{\beta^3}{\Gamma(\alpha+1)}]=\frac{7}{4}-\frac{10}{n}\frac{\beta^3}{\Gamma(\alpha+1)}>\frac{1}{\beta^2}$$

when

$$\frac{10}{n}\frac{\beta^3}{\Gamma(\alpha+1)}<\frac{7}{4}-\frac{1}{\beta^2},\tag{7.138}$$

i.e. when

$$n>\frac{40\beta^5}{(7\beta^2-4)\Gamma(\alpha+1)}.\tag{7.139}$$

Observe that (7.129) implies (7.139).

Next, by (7.131), (7.132), (7.135), (7.136) and (7.138), we see that, for each $u \in M$,

$$\begin{aligned}
\frac{1}{\beta^3} &\leq (\mathcal{V}u)(t) \leq \sup_{t\in A}\{\beta^{t^2-3} + \frac{1}{4}\sin^2(tu(t)) \\
&\quad + \sqrt[n]{u(t)}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau + u^2(h(\tau))]d\tau\} \\
&\leq \frac{1}{\beta^2} + \frac{1}{4} + \frac{10}{n}\beta^{3(1-\frac{1}{n})}\frac{1}{\Gamma(\alpha+1)} < \frac{1}{\beta^2} + \frac{1}{4} + \frac{10}{n}\frac{\beta^3}{\Gamma(\alpha+1)} \\
&\leq \frac{1}{\beta^2} + \frac{1}{4} + \frac{7}{4} - \frac{1}{\beta^2} \leq 2.
\end{aligned}$$

This implies (7.137).

7.17.3. We claim that

$$\forall_{u,v\in M}\forall_{m\in\mathbb{N}}\{D(\mathcal{V}^{[m]}u, \mathcal{V}^{[m]}v) \leq y^m D(u,v)\}, \tag{7.140}$$

where

$$\gamma = \frac{1}{2} + \frac{13}{n}\frac{\beta^3}{\Gamma(\alpha+1)} < 1. \tag{7.141}$$

Indeed, if $u, v \in M$, then, by (7.133)-(7.136), we obtain

$$\begin{aligned}
&D(\mathcal{V}u, \mathcal{V}v) \\
= \ &\sup_{t\in A}|(\mathcal{V}u)(t) - (\mathcal{V}v)(t)| \leq \sup_{t\in A}|\frac{1}{4}[\sin^2(tu(t)) - \sin^2(tv(t))] \\
&+ \sqrt[n]{v(t)}\{\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau + u^2(\lambda(\tau))]d\tau - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau + v^2(\lambda(\tau))]d\tau\} \\
&+[\sqrt[n]{u(t)} - \sqrt[n]{v(t)}] \cdot \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\tau + v^2(\lambda(\tau))]d\tau| \\
\leq \ &\frac{1}{2}D(u,v) + \frac{2}{n}\beta^{3(1-\frac{1}{n})}\frac{4D(u,v)}{\Gamma(\alpha+1)} + \frac{1}{n}\beta^{3(1-\frac{1}{n})} \cdot D(u,v)\frac{5}{\Gamma(\alpha+1)} \\
= \ &[\frac{1}{2} + \frac{13}{n}\frac{\beta^3}{\Gamma(\alpha+1)}]D(u,v) = \gamma D(u,v).
\end{aligned}$$

Clearly, (7.129) implies (7.141).

7.17.4. Assumption (7.129) implies (7.141) and, consequently, from (7.140) it follows that

$$\begin{aligned}
&\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}}\{D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0) \\
< \ &\varepsilon + \eta \Rightarrow D(\mathcal{V}^{[s+r]}y^0, \mathcal{V}^{[l+r]}y^0) \leq \gamma^r D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0) \\
\leq \ &\frac{\varepsilon}{\varepsilon+\eta}D(\mathcal{V}^{[s]}y^0, \mathcal{V}^{[l]}y^0)\}.
\end{aligned}$$

Further, from (7.140) and (7.137) we deduce that a dynamic system $(M, \mathcal{V})$ is continuous on $M$ and thus closed in each $y^0 \in M$. Obviously, a dynamic system $(M, \mathcal{V})$ is $D$-admissible on $M$ since a gauge space $(\mathcal{C}(A, \mathbb{R}), D)$ is sequentially complete and $M \subset \mathcal{C}(A, \mathbb{R})$. The above shows that (I.6), the condition (4.22) and the assertion (B) of Theorem 4.2(iii) when $\mathcal{W} = \mathcal{V}$ hold.

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