Controllability of semi-linear neutral stochastic integro-differential evolution systems with fractional Brownian motion *

Nan Cao and Xianlong Fu†

School of Mathematical Sciences, Shanghai Key Laboratory of PMMP,

East China Normal University, Shanghai 200241, P. R. China

Abstract

This paper focuses on the approximate controllability of semilinear neutral stochastic integro-differential systems with fractional Brownian motion in Hilbert space. The theory of analytic resolvent operators of linear neutral integro-differential equations and Banach fixed point theorem are employed to obtain the main results. In order to improve the regularity of solutions theory of fraction powers and α-norm are also utilized to discuss the problem. An example is presented in the end to illustrate the applications of the obtained results.

Key words: Approximate controllability, neutral integro-differential system, resolvent operator, fractional Brownian motion.

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1 Introduction

As well known, compared with classical partial differential equations, (neutral) partial integro-differential equations can often describe many physical phenomena more accurately in some fields such as fluid dynamics, electronics and chemical kinetics. For this reason in recent years many researchers have developed great interest in partial integro-differential equations. An important way of treating this kind of equations is to transform them into integro-differential evolution equations in abstract spaces and then apply the theory of resolvent operators to study the obtained abstract equations. We refer to the papers [15–17] and the book [30] for the theory of resolvent operators of linear integro-differential evolution equations and the study of abstract linear integro-differential equations via (analytic) resolvent operators. In the past years based on the theory of resolvent operators of linear integro-differential evolution equations, lots of work has been done upon, for instance, existence, regularity, stability and control problems for semilinear integro-differential evolution equations, see [13, 19, 21, 35, 40] among others.

On the other hand, there exist many different kinds of noise or random perturbations which inevitably have different influence on the evolution of various systems. It is of great significance to introduce stochastic effects into the investigation of partial integro-differential equations, and actually in these years a lot of important work has been done for stochastic integro-differential evolution equations and many interesting results has been achieved, see [8, 32, 37] and references therein, where the random disturbances are described by stochastic integrals with respect to semi-martingales, especially by standard Brownian motions (Wiener processes). Particularly, by applying the theory of resolvent operators for linear integro-differential evolution equations founded in [15–17], in [20] the authors discussed the approximate controllability of a stochastic integro-differential evolution equation with infinity delay and Brownian motion in the form of

$$\begin{cases} dy(t) = \left[-Ay(t) + \int_0^t \gamma(t-s)y(s)ds + L(y_t) + f(t,y_t) + Bu(t)\right] dt + g(t,y_t)dW(t), \quad 0 \le t \le T, \\ y_0 = \varphi \in L_p(\Omega; \mathscr{B}_\alpha), \quad t \in (-\infty, 0]. \end{cases}$$

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†The corresponding author, email: xlfu@math.ecnu.edu.cn.

Note that in applications Wiener process is not suitable to represent a noise process if long-range dependence is modelled. It is then desirable to replace Wiener process by fractional Brownian motion (fBm) which has wide physical applicability. fBm is a generalization of standard Brownian motion and is neither semi-martingales nor Markov processes, see [10, 11, 25, 38] and references therein. In the past years, there have also been a lot of studies on the existence and stability of solutions of differential and integro-differential evolution equations with fBm, see [7, 14, 31], for instance. Particularly, approximate controllability problems for various functional differential evolution equations with fBm were investigated respectively by many mathematicians. We mention here the work in Chen [5], Ahmed [1], Ahmed et al. [2] and Tamilalagan and Balasubramaniam [36]. In addition, some kinds of dynamical systems require both Wiener process and fBm to model its dynamics. For example, in financial markets, sometimes two principal random noises influence the prices commonly. The fBm is used to model the noise, which comes from economical background and has a long-range dependence, and, at the same time, Wiener process is used to model another noise called white noise, which is intrinsic to the stock exchange, where millions of agents act independently and behave irrationally sometimes (see [34]). Recently there have also been a lot of studies on the existence and stability of solutions of differential and integro-differential evolution equations driven by winner process and fBm, see [31, 36, 41], for example.

It is observed that, up to now, there are very few papers on approximate controllability of stochastic functional integro-differential evolution systems with fBm. The purpose of this article is to develop the work of [20] to neutral stochastic functional integro-differential evolution systems with both Wiener process and fBm. Precisely, we are going to consider the approximate controllability of systems represented in the following semi-linear neutral integro-differential equations with state-dependent delay of the form

$$\begin{cases} d\left[x(t)+\int_0^t N(t-s)x(s)ds\right] = \left[-Ax(t)+\int_0^t \gamma(t-s)x(s)ds + f(t,x_t)+Bu(t)\right]dt \\ \qquad\qquad + g(t,x_t)dW(t)+h(t,x_t)dB^H(t), \quad t\in[0,T], \\ x_0=\phi\in ML(r,\alpha), \end{cases} \tag{1}$$

where the state variable $x(\cdot)$ takes values in a Hilbert space X and its histories x_t are defined in the usual way by $x_t(\theta) = x(t+\theta)$ for $\theta\in[-r,0]$, belong to the phase space $ML(r,\alpha)$ to be defined later. $N(t)$ in the neutral term is a family of bounded linear operators on X. The (unbounded) linear operator $(-A, D(-A))$ generates an analytic semigroup on X, $\gamma(\cdot)$ is a family of closed linear operators specified later. The control function $u(\cdot)$ is given in the Banach space $L_2([0,T];U)$ with U is also a Hilbert space, and B is a bounded linear operator from U into X. The functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are Lipschitz continuous and uniformly bounded given below. $W(t)$ is a Q-Wiener process defined on a complete probability space and $B^H(t)$ is an fBm with the Hurst parameter $H\in(\frac{1}{2},1)$ to be described in next section.

Eq. (1) is the abstract form of stochastic neutral partial integro-differential equations appearing in heat conduction models. We should stress here that one can not discuss its approximate controllability adopting the theory of resolvent operators for linear integro-differential evolution equations as in [20] since the neutral term is not uniformly bounded. Instead, we will turn to employ the theory of resolvent operators newly established in Henríquez and Dos Santos [9] for the linear neutral integro-differential evolution equations associated to (1):

$$\begin{cases} \frac{d}{dt}\left[x(t)+\int_0^t N(t-s)x(s)ds\right] = Ax(t)+\int_0^t \gamma(t-s)x(s)ds, \ t\in[0,T], \\ x(0)=x_0\in X. \end{cases}$$

In [9], through this theory the authors discussed the existence result for semilinear neutral evolution equations with unbounded delay. Further, in [18], using this theory again Dos Santos et al. investigated the differentiability of solutions of an abstract neutral integro-differential equations with infinite delay. Then, also applying this theory many researches have well studied some other properties of solutions for neutral partial integro-differential equations, see [22, 39, 42], for example.

It can be seen that, since the neutral term is involved in the resolvent operator, the theory of resolvent operators established in [18] will enable us to overcome the difficulty of the non-uniform boundedness of the neutral term and therefore through this we can obtain the sufficient conditions for the approximate controllability. In addition, to improve the regularity of the solution we shall also apply the theory of fractional powers and α-norm. We will first discuss the existence and uniqueness of system solutions by using Banach fixed point theorem. And then we study the approximate controllability of the system (1) under the so-called resolvent condition which is equivalent to the approximate controllability of the corresponding deterministic linear neutral systems and achieve successfully sufficient conditions on approximate controllability for this stochastic system by limit arguments. We would like to mention here that the main results obtained in this paper clearly extend and develop the conclusions appeared in [5], [20], [22], [33] and [39] respectively.

Subsequently, in Section 2, we introduce some fundamental concepts and basic results about resolvent operators and fBm to be used throughout the paper. In Section 3 we first prove the existence and uniqueness of mild solutions for Eq.(1) applying Banach fixed point principle. Following that we establish the approximate controllability result for System (1) via limit arguments. Finally we provide an example in Section 4 to illustrate the applications of the approximate controllability theorem.

2 Preliminaries

Let X and K be two separable Hilbert spaces with the inner products $< \cdot, \cdot >$ and $< \cdot, \cdot >_K$, the norms $\|\cdot\|$ and $\|\cdot\|_K$, respectively. $\mathscr{L}(K;X)$ denotes the Banach space of all bounded linear operators from K into X, its norm is denoted by the same notation $\|\cdot\|$. Particularly, $\mathscr{L}(X;X)$ is abbreviated as $\mathscr{L}(X)$.

Throughout this paper the operator $-A : D(A) \subset X \mapsto X$ is the infinitesimal generator of a compact analytic semigroup $(S(t))_{t\geq 0}$. We denote by Y the Banach space $\big(D(A), \|\cdot\|_Y\big)$ with the graph norm $\|x\|_Y = \|Ax\| + \|x\|$, for $x \in D(A)$. Let $0 \in \rho(A)$, the resolvent set of the operator A, then it is possible to define the fractional power $A^\alpha, 0 < \alpha \leq 1$, as a closed linear operator on its dense domain $D(A^\alpha)$ in X and the expression

$$\| x \|_\alpha = \| A^\alpha x \|, \qquad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Denoting the space $\big(D(A^\alpha), \|\cdot\|_\alpha\big)$ by X_α, then it is well known that for each $0 < \alpha \leq 1$, X_α is a Banach space. Hereafter we denote by $C([0,T]; X_\alpha)$ the Banach space of continuous functions from $[0,T]$ to X_α with the norm

$$\| x \|_C = \sup_{t\in[0,T]} \| A^\alpha x(t) \|, \qquad x \in C\big([0,T]; X_\alpha\big).$$

For the theory of operator semigroups we refer the reader to [12] and [28].

Next we recall some fundamentals and basic facts about the theory of resolvent operators and fractional Brownian motions.

2.1 Resolvent operators

First we introduce briefly the theory of resolvent operators for neutral linear integro-differential equations introduced in [9, 18], which will be the main tool in our later discussion.

Definition 2.1. *A one-parameter family of bounded linear operators $\big(R(t)\big)_{t\geq 0}$ on X is called a resolvent operator for*

$$\begin{cases} \dfrac{d}{dt}\left[x(t) + \displaystyle\int_0^t N(t-s)x(s)ds\right] = -Ax(t) + \displaystyle\int_0^t \gamma(t-s)x(s)ds, \\ x(0) = x_0 \in X, \end{cases}$$

if the following conditions are verified.

(i) The function $R(\cdot) : [0,\infty) \to \mathscr{L}(X)$ is strongly continuous, exponentially bounded and $R(0) = I$.

(ii) For $x \in Y$, $R(t)x \in C^1([0,T], X) \cap C([0,T], Y)$, and

$$\frac{d}{dt}\left[R(t)x + \int_0^t N(t-s)R(s)xds\right] = -AR(t)x + \int_0^t \gamma(t-s)R(s)xds,$$

$$\frac{d}{dt}\left[R(t)x + \int_0^t R(t-s)N(s)xds\right] = -R(t)Ax + \int_0^t R(t-s)\gamma(s)xds,$$

for each $t \geq 0$.

We shall always impose the following hypotheses on the operators appearing in (1).

(V_1) There are constants $M_0 > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that

$$\rho(-A) \supseteq \Lambda_\theta = \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \theta\right\} \cup \{0\} \text{ and } \|R(\lambda, -A)\| \leq M_0|\lambda|^{-1}, \ \ \lambda \in \Lambda_\theta,$$

where $R(\lambda, -A)$ denote the resolvent operator of A. Hence in this case the operator $-A$ generates an analytic semigroup $(S(t))_{t\geq 0}$ on X.

(V_2) The function $N : [0, \infty) \to \mathscr{L}(X)$ is strongly continuous and $\hat{N}(\lambda)x$ is absolutely convergent for $x \in X$ and $Re(\lambda) > 0$. There exist $\alpha > 0$ and an analytical extension of $\hat{N}(\lambda)$ (still denoted by $\hat{N}(\lambda)$) to Λ_θ such that $\left\|\hat{N}(\lambda)\right\| \leq N_0|\lambda|^{-\alpha}$ for every $\lambda \in \Lambda_\theta$, and so $\left\|\hat{N}(\lambda)x\right\| \leq N_1|\lambda|^{-1}\|x\|_Y$ for each $\lambda \in \Lambda_\theta$ and $x \in D(A)$. Here N_0 and N_1 are constants, the notation $\hat{f}(\lambda)$ represents the Laplace transform of $f(t)$.

(V_3) For all $t \geq 0$, $\gamma(t) : D(\gamma(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(\gamma(t))$ and $\gamma(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists a $b(\cdot) \in L^1_{loc}(R^+)$ such that $\hat{b}(\lambda)$ exists for $Re\lambda > 0$ and $\|\gamma(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\hat{\gamma} : \Lambda_{\frac{\pi}{2}} \to \mathscr{L}(Y, X)$ has an analytical extension (still denoted by $\hat{\gamma}$) to Λ_θ such that $\|\hat{\gamma}(\lambda)x\| \leq \|\hat{\gamma}(\lambda)\| \, \|x\|_Y$ for all $x \in D(A)$, and $\|\hat{\gamma}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

(V_4) There exist a subspace $D \subseteq D(A)$ dense in Y and positive constants $C_i, i = 1, 2$, such that $A(D) \subseteq D(A)$, $\hat{\gamma}(\lambda)(D) \subseteq D(A)$, $\hat{N}(\lambda)(D) \subseteq D(A)$, and $\|A\hat{\gamma}(\lambda)x\| \leq C_1(x)$, $\|\hat{N}(\lambda)x\|_Y \leq C_2|\lambda|^{-\alpha}\|x\|_Y$ for every $x \in D$ and $\lambda \in \Lambda_\theta$.

(V_5) The operator $\hat{N}(\lambda) : X_\alpha \to X_\alpha$ for $\lambda \in \Lambda_\theta$, $0 < \alpha < 1$, and $\left\|\hat{N}(\lambda)\right\|_\alpha \to 0$ as $|\lambda| \to \infty$ uniformly for $\lambda \in \Lambda_\theta$.

Then, it follows from [9, 18] that, under these conditions, there is a resolvent operator $R(t)$ for linear system (1) defined by

$$R(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\vartheta}} e^{\lambda t} G(\lambda) d\lambda, & t > 0, \\ I & t = 0, \end{cases} \tag{2}$$

where $G(\lambda) = \left(\lambda I + \lambda\hat{N}(\lambda) + A - \hat{\gamma}(\lambda)\right)^{-1} \in \mathscr{L}(X)$ satisfying $\|G(\lambda)\| \leq N|\lambda^{-1}|$ for $\lambda \in \Lambda_{r,\vartheta}$ and some $N > 0$. The region $\Lambda_{r,\vartheta} = \left\{\lambda \in \mathbb{C}\backslash\{0\} : |\lambda| > r, |\arg(\lambda)| < \vartheta\right\}$, and $\Gamma_{r,\vartheta} = \bigcup_{i=1}^{3} \Gamma^i_{r,\vartheta}$, the curves $\Gamma^i_{r,\vartheta}$, $i = 1, 2, 3$, are given respectively by

$$\begin{aligned} \Gamma^1_{r,\vartheta} &= \{te^{i\vartheta} : t \geq r\}, \\ \Gamma^2_{r,\vartheta} &= \{re^{i\xi} : -\vartheta \leq \xi \leq \vartheta\}, \\ \Gamma^3_{r,\vartheta} &= \{te^{-i\vartheta} : t \geq r\}, \end{aligned}$$

for $r > 0, \vartheta \in \left(\frac{\pi}{2}, \theta\right)$ being fixed numbers. These curves are oriented so that $\text{Im}(\lambda)$ is increasing. The following theorem summarizes several important properties of the resolvent operator $R(t)$.

Theorem 2.1. *For the resolvent operator $R(t)$ defined in (2), one has*

(i) The resolvent operator $R(t)$ is also analytic and there exist $M, M_\alpha > 0$ such that

$$\|R(t)\| \leq M \quad \text{and} \quad \|A^\alpha R(t)\| \leq \frac{M_\alpha}{t^\alpha}, \quad 0 < t \leq T, \ 0 < \alpha < 1.$$

(ii) If $R(\lambda_0, -A)$ is a compact operator for some $\lambda_0 \in \rho(-A)$, then $R(t)$ is compact for all $t > 0$.

(iii) The family of linear operators $A^\alpha R(t)$ is continuous in $t > 0$ in the uniform operator topology of $\mathscr{L}(X)$.

(iv) If $(\hat{N}(\lambda) - \hat{\gamma}(\lambda))A^{-1}x = A^{-1}(\hat{N}(\lambda) - \hat{\gamma}(\lambda))x$ for $x \in X$, then, for all $\alpha \in [0, 1]$ and $x \in D(A^\alpha)$,

$$A^\alpha R(t)x = R(t)A^\alpha x.$$

Proof. (i) and (ii) are established in [9]. (iii) can be proved readily by (2) and the inequality (2.12) of [9]. As for (iv), the condition obviously implies $AR(t)x = R(t)Ax$ for $x \in D(A)$. Then from the expression of A^α it follows immediately that $A^\alpha R(t)x = R(t)A^\alpha x$ for all $\alpha \in [0, 1]$. □

Remark 1. *Generally speaking, $R(t)$ does not commute with A^α. However, this commutativity can be reached if the condition in Theorem 2.1 (iv) is fulfilled. In this paper, we assume that this condition is always valid so that A^α is always commutative with $R(t)$ for any $0 \le \alpha \le 1$.*

2.2 Fractional Brownian motions

In this part for convenience we state concisely some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian motions. For more details, one can see [4, 26].

Let $\Omega := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a family of right continuous increasing σ-algebras $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

Definition 2.2. *Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H = \{\beta^H(t), t \in \mathbb{R}\}$, with the covariance function*

$$R_H(t,s) = E\left[\beta^H(t)\beta^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad t, s \in \mathbb{R}$$

is called a two-sided one-dimensional fractional Brownian motion (fBm), and H is the Hurst parameter.

Now we introduce the Wiener integral with respect to the one-dimensional β^H.

Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}$-valued step functions on $[0,T]$, that is, $\psi \in \Lambda$ if $\psi(t) = \sum_{i=1}^{n-1} x_i 1_{[s_i, s_{i-1})}(t)$, where $t \in [0,T]$, $x_i \in \mathbb{R}$ and $0 = s_1 < s_2 < \cdots < s_n = T$. For $\psi \in \Lambda$ we define its Wiener integral with respect to β^H as

$$\int_0^T \psi(\sigma)d\beta^H(\sigma) = \sum_{i=1}^{n-1} x_i\left(\beta^H(s_{i+1}) - \beta^H(s_i)\right).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Λ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

Then, the mapping $\psi = \sum_{i=1}^{n-1} x_i 1_{[s_i, s_{i+1})} \mapsto \int_0^T \psi(\sigma)d\beta^H(\sigma)$ is an isometry between Λ and the linear space span $\{\beta^H(t), t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{\text{span}}^{L^2(\Omega)}\left\{\beta^H(t), t \in [0,T]\right\}$ (see [38]). The image of an element $\psi \in \mathcal{H}$ by this isometry is called the Wiener integral of ψ with respect to β^H. Next we give an explicit expression of this integral for $H \in (\frac{1}{2}, 1)$. To do so, consider the kernel

$$K_H(t,s) = c_H s^{1/2-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}du, \quad t > s,$$

where $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$ with $B(\cdot,\cdot)$ denoting the Beta function. Then it is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

Consider the linear operator $K_H^* : \Lambda \to L^2([0,T])$ given by

$$(K_H^*\psi)(s) = \int_s^t \psi(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Then

$$(K_H^* 1_{[0,t]})(s) = K_H(t,s)1_{[0,t]}(s),$$

and K_H^* is an isometry between Λ and $L^2([0,T])$ that can be extended to $\mathcal{H}$ (see [26]).

Denote $W = \left\{W(t), t \in [0,T]\right\}$ with $W(t) = \beta^H\left((K_H^*)^{-1}1_{[0,t]}\right)$, then it turns out that W is a Wiener process and β^H has the Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)dW(s).$$

In addition, for any $\psi \in \mathcal{H}$,

$$\int_0^T \psi(s) d\beta^H(s) = \int_0^T (K_H^* \psi)(t) dW(t)$$

if and only if $K_H^* \psi \in L^2([0,T])$.

Also denoting $L^2_{\mathcal{H}}([0,T]) = \left\{\psi \in \mathcal{H}, K_H^*\psi \in L^2([0,T])\right\}$, since $H > 1/2$, we have (cf. [24]) that

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]).$$

Moreover, the following useful result holds:

Lemma 2.1. *(see [26]) For $\psi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T\int_0^T |\psi(r)||\psi(u)||r-u|^{2H-2} dr du \leq c_H \|\psi\|^2_{L^{1/H}([0,T])}.$$

Next, let us consider a fractional Brownian motion with values in a Hilbert space and give the definition of the corresponding stochastic integral.

Let $(\mathcal{U}, |\cdot|_{\mathcal{U}}, (\cdot,\cdot)_{\mathcal{U}})$ and $(\mathcal{K}, |\cdot|_{\mathcal{K}}, (\cdot,\cdot)_{\mathcal{K}})$ be two separable Hilbert spaces. Let $Q \in \mathscr{L}(\mathcal{K};\mathcal{K})$ be a non-negative self-adjoint operator. Denote by $\mathscr{L}_2^0(\mathcal{K},\mathcal{U})$ the space of all $\xi \in \mathscr{L}(\mathcal{K};\mathcal{U})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, its norm is given by

$$|\xi|^2_{\mathscr{L}_2^0(\mathcal{K},\mathcal{U})} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from $\mathcal{K}$ to $\mathcal{U}$.

Assume that $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. When one considers the series $\sum\limits_{n=1}^{\infty} \beta_n^H(t) e_n,\ t \geq 0,$ where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in $\mathcal{K}$, this series does not necessarily converge in the space $\mathcal{K}$. Thus we consider a $\mathcal{K}$-valued stochastic process $B_Q^H(t)$ given formally by the series

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n, \quad t \geq 0,$$

which is well-defined as a $\mathcal{K}$-valued Q-cylindrical fractional Brownian motion.

Now we present the definition of stochastic integral in B_Q^H as follows.

Definition 2.3. *Let $\varphi : [0,T] \to \mathscr{L}_2^0(\mathcal{K},\mathcal{U})$ satisfy*

$$\sum_{n=1}^{\infty} \left\| K_H^*(\varphi Q^{\frac{1}{2}} e_n) \right\|_{L^{1/H}([0,T];\mathcal{U})} < \infty.$$

Then its stochastic integral with respect to the fractional Brownian motion B_Q^H is defined as, for $t \geq 0$,

$$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \left(K_H^*(\varphi Q^{\frac{1}{2}} e_n)\right)(s) dW(s).$$

From Lemma 2.1 it was proved in [4] that

Lemma 2.2. *if $\varphi : [0,T] \to \mathscr{L}_2^0(\mathcal{K},\mathcal{U})$ satisfying $\sum\limits_{n=1}^{\infty} \|\varphi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T];\mathcal{U})} < \infty$, then we have, for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$E\left|\int_\beta^\alpha \varphi(s) dB_Q^H(s)\right|^2_{\mathcal{U}} \leq cH(2H-1)(\alpha-\beta)^{2H-1} \sum_{n=1}^{\infty}\int_\beta^\alpha \left|\varphi(s) Q^{\frac{1}{2}} e_n\right|^2_{\mathcal{U}} ds,$$

where $c = c(H)$. If, in addition, $\sum\limits_{n=1}^{\infty} |\varphi Q^{\frac{1}{2}} e_n|_{\mathcal{U}}$ is uniformly convergent for $t \in [0,T]$, then

$$E\left|\int_\beta^\alpha \varphi(s) dB_Q^H(s)\right|^2_{\mathcal{U}} \leq cH(2H-1)(\alpha-\beta)^{2H-1} \int_\beta^\alpha |\varphi(s)|^2_{\mathscr{L}_2^0(\mathcal{K},\mathcal{U})} ds.$$

Subsequently we introduce several function spaces to be used in the later discussion. Denote by $C_\alpha := C([-r,0];X_\alpha)$ the space of all continuous functions from $[-r,0]$ into X_α. Then, denote by $ML(r,\alpha)$ the space of all $\mathscr{F}_0$-measurable functions that belong to $L_2(\Omega;C_\alpha)$, that is, $ML(r,\alpha)$ is the space of $\mathscr{F}_0$-measurable C_α-valued functions $\phi:\Omega\to C_\alpha$ with the norm

$$\mathbb{E}\|x\|^2_{C_\alpha} := \mathbb{E}\left(\sup_{\theta\in[-r,0]}\|A^\alpha\phi(\theta)\|^2\right) < +\infty.$$

Additionally, let $L_2^{\mathscr{F}}([0,T];X)$ be the closed subspace of $L_2([0,T]\times\Omega;X)$ consisting of $\mathscr{F}_t$-adapted processes and $C\big([-r,T];L_2(\Omega;X)\big)$ denotes the Banach space of all continuous maps from $[-r,T]$ into $L_2(\Omega;X)$ satisfying the condition $\sup_{\theta\in[-r,T]}\mathbb{E}\|x\|^2<\infty$.

Finally, we need to state several important lemmas as below which will be employed in proving the main results.

Lemma 2.3. *([29], Proposition 4.15) Let $\varphi\in L_2^{\mathscr{F}}\left(0,T;\mathscr{L}_2^0(K;X)\right)$ with $\varphi(t)k\in X_\alpha$ and $A^\alpha\varphi\in L_2^{\mathscr{F}}\left(0,T;\mathscr{L}_2^0(K;X)\right)$, for $t\geq 0$ and $k\in K$, then*

$$A^\alpha\int_0^T\varphi(s)dW(s) = \int_0^T A^\alpha\varphi(s)dW(s).$$

Lemma 2.4. *([29], Lemma 7.2) For any $t\in[0,T]$ and $p\geq 2$, let $\varphi\in L_p^{\mathscr{F}}(\Omega;L_2(0,T;\mathscr{L}_2^0(K;X)))$, then*

$$\mathbb{E}\left(\sup_{s\in[0,t]}\left\|\int_0^s\varphi(r)dW(r)\right\|^p\right)\leq C_p\mathbb{E}\left(\int_0^t\|\varphi(s)\|^2_{\mathscr{L}_2^0}ds\right)^{\frac{p}{2}}$$

where $C_p = \left(\frac{p}{2}(p-1)\right)^{\frac{p}{2}}\left(\frac{p}{p-1}\right)^{\frac{p^2}{2}}$.

Lemma 2.5. *([8], Lemma 6) Let $p\geq 2$ and $h\in L_p(\Omega;X)$ be fixed, then there exists a function $\tilde{\psi}$ in space $L_p^{\mathscr{F}}(\Omega;L_2(0,T;\mathscr{L}_2^0(K;X)))$ such that*

$$h = \mathbb{E}h + \int_0^T\tilde{\psi}(s)dW(s).$$

3 Approximate controllability

In this section, we discuss the approximate controllability of System (1) and obtain the main results of this paper. The mild solutions of System (1) expressed by the resolvent operator $R(t)$ is defined as follows.

Definition 3.1. *A stochastic process $x(\cdot)$ defined on $[-r,T], 0<r,T<\infty$ is said to be a mild solution of equation (1) if the following conditions are satisfied:*

(i) $x(t,\omega)$ is measurable as a function from $[0,T]\times\Omega$ to X and $x(t)$ is $\mathscr{F}_t$-adapted;

(ii) $\mathbb{E}\|x(t)\|^2<\infty$ for each $t\in[-r,T]$;

(iii) For each $u(t)\in L_2^{\mathscr{F}}([0,T];U)$, the process $x(\cdot)$ satisfies the following integral equation

$$x(t) = \begin{cases} R(t)\phi(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s)+Bu(s)\right]ds \\ \qquad + \displaystyle\int_0^t R(t-s)g(s,x_s)dW(s) + \int_0^t R(t-s)h(s,x_s)dB^H(s), \quad t\in[0,T], \\ \phi(t), \quad t\in(-r,0]. \end{cases}$$

Then, we define the concept of approximate controllability which is the main topic of this paper.

Definition 3.2. *System (1) is said to be approximately controllable on the interval $[0,T]$ for any initial value $\phi\in C_\alpha$, if $\mathscr{R}(T,\phi)$ is dense in X, i.e.*

$$\overline{\mathscr{R}(T,\phi)} = X,$$

where $\mathscr{R}(T,\phi) = \left\{x(T,\phi,u), u(\cdot)\in L^2([0,T];U)\right\}$.

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To study the approximate controllability of System (1) we introduce the following operators

$$\Gamma_s^T = \int_s^T R(T-\tau)BB^*R^*(T-\tau)d\tau,$$

$$\mathcal{R}(\lambda, -\Gamma_s^T) = (\lambda I + \Gamma_s^T)^{-1},$$

where $0 \le s < T$, B^* and $R^*(t)$ denote the adjoints of the operators B and $R(t)$ respectively. Since the operator Γ_s^T is clearly positive, $\mathcal{R}(\lambda, -\Gamma_s^T)$ is well defined for $\lambda > 0$. We will always assume that the operator $\mathcal{R}(\lambda, -\Gamma_T)$ satisfies that

(H_0) $\lambda\mathcal{R}(\lambda, -\Gamma_0^T) \longrightarrow 0$ as $\lambda \longrightarrow 0^+$ in the strong operator topology.

The hypothesis (H_0) is equivalent to the fact that the linear control system corresponding to System (1)

$$\begin{cases} \frac{d}{dt}\left[x(t) + \int_0^t N(t-s)x(s)ds\right] = -Ax(t) + \int_0^t \gamma(t-s)x(s)ds + Bu(t), & t \in [0, T], \\ x(t) = \phi(t), \qquad t \in [-r, 0], \end{cases} \tag{3}$$

is approximately controllable on $[0, T]$. More precisely, we have that

Theorem 3.1. *The following statements are equivalent:*

(i) *The control system* (3) *is approximately controllable on* $[0, T]$.

(ii) *If* $B^*R^*(t)x = 0$ *for all* $t \in [0, T]$, *then* $x = 0$.

(iii) *The condition* (H_0) *holds.*

This theorem can be prove in the similar way as that of ([3], Theorem 2) and ([6], Theorem 4.4.17), so we omit the proof here.

In what follows, to obtain the approximate controllability of System (1), we shall show that, for any $h \in L_2^{\mathscr{F}}(\Omega; X)$, by selecting a proper control u^λ in $L_2^{\mathscr{F}}([0, T]; U)$ (for any given $\lambda \in (0, 1)$), there exists a mild solution $x^\lambda(\cdot, \phi, u^\lambda) : (-r, T] \to L_2(\Omega; X_\alpha)$ for System (1), such that $x^\lambda(T, \phi, u^\lambda) \to h$ in $L_2(\Omega; X)$ as $\lambda \to 0^+$, which by Definition 3.2 means System (1) is approximate controllable on $[0, T]$.

To guarantee the existence of mild solutions, we impose the following restrictions on System (1). Let $\alpha \in (0, 1)$ be given,

(H_1) $R(t)$ is a compact operator for each $t > 0$.

(H_2) $B \in \mathscr{L}(U; X)$, i.e., B is a bounded linear operator from U to X.

(H_3) (i) The nonlinear functions $f : [0, +\infty) \times C_\alpha \to X$, $g : [0, +\infty) \times C_\alpha \to \mathscr{L}_2^0(K; X)$ and $h : [0, +\infty) \times C_\alpha \to \mathscr{L}_2^0(K; X)$ are measurable mappings, satisfying that $f(t, 0)$, $g(t, 0)$ and $h(t, 0)$ are bounded in $X-$norm and $\mathscr{L}_2^0(K; X)-$norm, respectively.

(ii) For any $\xi_1,\ \xi_2 \in C_\alpha$ and $t \in [0, T]$, there exists positive real constant $N_2 > 0$ such that

$$\|f(t, \xi_1) - f(t, \xi_2)\|^2 \vee \|g(t, \xi_1) - g(t, \xi_2)\|^2_{\mathscr{L}_2^0} \vee \|h(t, \xi_1) - h(t, \xi_2)\|^2_{\mathscr{L}_2^0} \le N_2 \|\xi_1 - \xi_2\|^2_{C_\alpha},$$

$$\|f(t, \xi_1)\|^2 \vee \|g(t, \xi_1)\|^2_{\mathscr{L}_2^0} \vee \|h(t, \xi_1)\|^2_{\mathscr{L}_2^0} \le N_2 \left(1 + \|\xi_1\|^2_{C_\alpha}\right).$$

(H_3') (i) The nonlinear functions $f : [0, +\infty) \times C_\alpha \to X$, $g : [0, +\infty) \times C_\alpha \to \mathscr{L}_2^0(K; X)$ and $h : [0, +\infty) \times C_\alpha \to \mathscr{L}_2^0(K; X)$ are measurable mappings satisfying that $f(t, 0)$, $g(t, 0)$ and $h(t, 0)$ are bounded in $X-$norm and $\mathscr{L}_2^0(K; X)-$norm, respectively.

(ii) For any $\xi_1,\ \xi_2 \in C_\alpha$ and $t \in [0, T]$, there exists $N_2 > 0$ such that

$$\|f(t, \xi_1) - f(t, \xi_2)\|^2 \vee \|g(t, \xi_1) - g(t, \xi_2)\|^2_{\mathscr{L}_2^0} \vee \|h(t, \xi_1) - h(t, \varphi_2)\|^2_{\mathscr{L}_2^0} \le N_2 \|\xi_1 - \xi_2\|^2_{C_\alpha},$$

$$\|f(t, \xi_1)\|^2 \vee \|g(t, \xi_1)\|^2_{\mathscr{L}_2^0} \vee \|h(t, \xi_1)\|^2_{\mathscr{L}_2^0} \le N_2.$$

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Let D_T denotes the closed subspace of $C((-r,T];L_2(\Omega;X))$ consisting of measurable and $\mathscr{F}_0-$adapted measurable processes Z with $\|Z\|_{D_T}<\infty$, where

$$\|Z\|_{D_T} := \left(\sup_{t\in[0,T]} \mathbb{E}\,\|Z_t\|_C^2\right)^{\frac{1}{2}} = \left(\sup_{t\in[0,T]} \mathbb{E} \sup_{\theta\in[-r,0]} \|Z_t(\theta)\|^2\right)^{\frac{1}{2}}.$$

Similarly, we define

$$D_{T,\alpha} := \left\{z\in D(A^\alpha):\ A^\alpha z\in D_T\right\},$$

with the norm

$$\|Z\|_{D_{T,\alpha}} := \sup_{t\in[0,T]} \left(\mathbb{E}\,\|Z_t\|_{C_\alpha}^2\right)^{\frac{1}{2}}.$$

We now establish the existence result of mild solutions for System (1), that is

Theorem 3.2. *Let $0<\alpha<\frac{1}{4}$ and assume that the above hypotheses $(H_0)-(H_3)$ are all fulfilled. Then, for any $\phi\in ML(r,\alpha)$, the system (1) admits a unique mild solution on $[-r,T]$.*

We will prove this result by applying Banach fixed point principle. Precisely, for any $\lambda\in(0,1)$, we define the operator P^λ on D_T as, for any function $x(\cdot)\in C((-r,T];L_2(\Omega;X))$ with $x\big|_{[-r,0]}=\phi\in ML(r,\alpha)$,

$$(P^\lambda x)(t) = \begin{cases} R(t)A^\alpha\phi(0) + \displaystyle\int_0^t A^\alpha R(t-s)\Big[f(s,A^{-\alpha}x_s) + Bu^\lambda(s,A^{-\alpha}x)\Big]ds \\ \qquad + \displaystyle\int_0^t A^\alpha R(t-s)g(s,A^{-\alpha}x_s)dW(s) + \int_0^t A^\alpha R(t-s)h(s,A^{-\alpha}x_s)dB^H(s), \quad t\in[0,T], \\ A^\alpha\phi(t), \qquad t\in[-r,0], \end{cases} \tag{4}$$

in which the control function $u^\lambda(t,x)$ is taken as

$$\begin{aligned} u^\lambda(t,A^{-\alpha}x) &= B^*R^*(T-t)\mathcal{R}(\lambda,-\Gamma_0^T)\left(\mathbb{E}h - R(T)\phi(0) + \int_0^t \tilde{\psi}(s)dW(s)\right) \\ &\quad - B^*R^*(T-t)\int_0^t \mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)f(s,A^{-\alpha}x_s)ds \\ &\quad - B^*R^*(T-t)\int_0^t \mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)g(s,A^{-\alpha}x_s)dW(s) \\ &\quad - B^*R^*(T-t)\int_0^t \mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)h(s,A^{-\alpha}x_s)dB^H(s), \end{aligned} \tag{5}$$

where $\phi,\ \tilde{\psi}\in L_2^{\mathscr{F}}\left(\Omega;L_2(0,T;\mathscr{L}_2^0(K;X)\right)$ and $h=\mathbb{E}h+\int_0^T\tilde{\psi}(s)dW(s)$ from Lemma 2.5. Clearly, if the operator P^λ has a fixed point $x(\cdot,\varphi)$ on D_T, then, $y(\cdot,\varphi)=A^{-\alpha}x(\cdot,\varphi)$ will be a mild solution of System (1). Therefore, we are to prove in the sequel that the operator P^λ has a unique fixed point on D_T. For this purpose we need to prove three auxiliary lemmas to make the arguments clearer. Let us begin with two estimates on the control .

Lemma 3.1. *If the hypotheses $(H_0)-(H_3)$ are satisfied, then for the control function $u^\lambda(t,\cdot)$ defined above, there exists a constant $N_3>0$ such that, for all $x,y\in D_{T,\alpha}$,*

$$\mathbb{E}\left\|u^\lambda(t,x)-u^\lambda(t,y)\right\|^2 \le \frac{1}{\lambda^2}N_3\int_0^t \mathbb{E}\|x_s-y_s\|_{C_\alpha}^2 ds,$$

$$\mathbb{E}\left\|u^\lambda(t,x)\right\|^2 \le \frac{1}{\lambda^2}N_3\left(1+\int_0^t \mathbb{E}\|x_s\|_{C_\alpha}^2 ds\right).$$

Proof. We will only verify the first inequality since the proof of the second one is similar. Let x and y be two fixed functions in $D_{T,\alpha}$. Then, from the expression of u^λ, we have

$$\begin{aligned} \mathbb{E}\left\|u^\lambda(t,x)-u^\lambda(t,y)\right\|^2 &\le 3\mathbb{E}\left\|B^*R^*(T-t)\int_0^t \mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)[f(s,x_s)-f(s,y_s)]ds\right\|^2 \\ &\quad + 3\mathbb{E}\left\|B^*R^*(T-t)\int_0^t \mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)[g(s,x_s)-g(s,y_s)]dW(s)\right\|^2 \\ &\quad + 3\mathbb{E}\left\|B^*R^*(T-t)\int_0^t \mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)[h(s,x_s)-h(s,y_s)]dB^H(s)\right\|^2. \end{aligned}$$

According to $(H_2),(H_3)$, Theorem 2.1(i), Lemma 2.2, Lemma 2.4 and Hölder inequality we have

$$\begin{aligned}
\mathbb{E}\left\|u^\lambda(t,x)-u^\lambda(t,y)\right\|^2 \le\ & 3\|B\|^2M^2\Bigg[\frac{1}{\lambda^2}M^2TN_2\mathbb{E}\int_0^t\|x_s-y_s\|^2_{C_\alpha}ds\\
&+C_p\mathbb{E}\left(\int_0^t\left\|\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)\right\|^2\left\|g(s,x_s)-g(s,y_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&+cH(2H-1)T^{2H-1}\mathbb{E}\left(\int_0^t\left\|\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)\right\|^2\left\|h(s,x_s)-h(s,y_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\Bigg]\\
\le\ & \frac{3TN_2}{\lambda^2}\|B\|^2M^4\mathbb{E}\int_0^t\|x_s-y_s\|^2_{C_\alpha}ds+\frac{3C_pN_2}{\lambda^2}\|B\|^2M^4\mathbb{E}\int_0^t\|x_s-y_s\|^2_{C_\alpha}ds\\
&+\frac{3cH(2H-1)N_2}{\lambda^2}T^{2H-1}\|B\|^2M^4\mathbb{E}\int_0^t\|x_s-y_s\|^2_{C_\alpha}ds\\
\le\ & \frac{N_3}{\lambda^2}\mathbb{E}\int_0^t\|x_s-y_s\|^2_{C_\alpha}ds,
\end{aligned}$$

for some constant $N_3>0$. Hence the lemma is proved. □

Based on the preceding estimates we can further obtain the subsequent lemmas.

Lemma 3.2. *Assume that $0<\alpha<\frac{1}{4}$ and $(H_0)-(H_3)$ hold. Then for any $x\in D_T$ with $x\big|_{[-r,0]}=\phi\in ML(r,\alpha)$, $(P^\lambda x)(\cdot)$ is continuous on the interval $[0,T]$ in L_2 sense.*

Proof. Let $0\le t_1<t_2<T$. Then, for any fixed $x\in D_T$,

$$\begin{aligned}
J :=\ & \mathbb{E}\left\|\left(P^\lambda x\right)(t_2)-\left(P^\lambda x\right)(t_1)\right\|^2\\
\le\ & 5\mathbb{E}\left\|[R(t_2)-R(t_1)]A^\alpha\phi(0)\right\|^2+10\mathbb{E}\left\|\int_{t_1}^{t_2}A^\alpha R(t_2-s)f(s,A^{-\alpha}x_s)ds\right\|^2\\
&+10\mathbb{E}\left\|\int_{t_1}^{t_2}A^\alpha R(t_2-s)Bu^\lambda(s,A^{-\alpha}x)ds\right\|^2+10\mathbb{E}\left\|\int_{t_1}^{t_2}A^\alpha R(t_2-s)g(s,A^{-\alpha}x_s)dW(s)\right\|^2\\
&+10\mathbb{E}\left\|\int_{t_1}^{t_2}A^\alpha R(t_2-s)h(s,A^{-\alpha}x_s)dB^H(s)\right\|^2+10\mathbb{E}\left\|\int_0^{t_1}A^\alpha\left[R(t_2-s)-R(t_1-s)\right]f(s,A^{-\alpha}x_s)ds\right\|^2\\
&+10\mathbb{E}\left\|\int_0^{t_1}A^\alpha\left[R(t_2-s)-R(t_1-s)\right]Bu^\lambda(s,A^{-\alpha}x)ds\right\|^2\\
&+10\mathbb{E}\left\|\int_0^{t_1}A^\alpha\left[R(t_2-s)-R(t_1-s)\right]g(s,A^{-\alpha}x_s)dW(s)\right\|^2\\
&+10\mathbb{E}\left\|\int_0^{t_1}A^\alpha\left[R(t_2-s)-R(t_1-s)\right]h(s,A^{-\alpha}x_s)dB^H(s)\right\|^2\\
=\ & \sum_{\iota=1}^{9}I_\iota.
\end{aligned}$$

Using Theorem 2.1 (i), (H_3), and Hölder inequality, we obtain that

$$\begin{aligned}
I_2 \le\ & 10M_\alpha^2\mathbb{E}\left(\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\left\|f(s,A^{-\alpha}x_s)\right\|ds\right)^2\\
\le\ & 10M_\alpha^2\left(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}ds\right)\left(\int_{t_1}^{t_2}\left\|f(s,A^{-\alpha}x_s)\right\|^2ds\right)\\
\le\ & 10M_\alpha^2N_2\frac{(t_2-t_1)^{1-2\alpha}}{1-2\alpha}\int_{t_1}^{t_2}\left(1+\|A^{-\alpha}x_s\|^2_{C_\alpha}\right)ds\\
\le\ & \frac{10N_2M_\alpha^2}{1-2\alpha}(t_2-t_1)^{2-2\alpha}\left(1+\|x\|^2_{D_T}\right),
\end{aligned}$$

and, similarly, from Lemma 3.1,

$$I_3\le\frac{10\|B\|N_2M_\alpha^2}{(1-2\alpha)\lambda^2}(t_2-t_1)^{2-2\alpha}\left(1+(t_2-t_1)\|x\|^2_{D_T}\right).$$

On the other hand, from Lemma 2.4 we have the following estimate for some constant $C_p > 0$,

$$
\begin{aligned}
I_4 &\le 10C_p\mathbb{E}\left(\int_{t_1}^{t_2}\left\|A^\alpha R(t_2-s)g(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 10C_pM_\alpha^2\mathbb{E}\left(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}\left\|g(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 10C_pM_\alpha^2\mathbb{E}\left[\left(\int_{t_1}^{t_2}(t_2-s)^{-4\alpha}ds\right)\left(\int_{t_1}^{t_2}\left(\left\|g(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}\right)^2ds\right)\right]^{\frac{1}{2}}\\
&\le 10C_pN_2M_\alpha^2(1-4\alpha)^{-\frac{1}{2}}(t_2-t_1)^{\frac{1-4\alpha}{2}}\mathbb{E}\left(\int_{t_1}^{t_2}\left(1+\|A^{-\alpha}x_s\|^2_{C_\alpha}\right)^2ds\right)^{\frac{1}{2}}\\
&\le 10C_pN_2M_\alpha^2(1-4\alpha)^{-\frac{1}{2}}(t_2-t_1)^{1-2\alpha}\left(1+\|x\|^2_{D_T}\right),
\end{aligned}
$$

for $1-4\alpha>0$. Thus, by Lemma 2.2, we obtain

$$
\begin{aligned}
I_5 &\le 10cH(2H-1)T^{2H-1}\left(\int_{t_1}^{t_2}\left\|A^\alpha R(t_2-s)h(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 10cH(2H-1)T^{2H-1}M_\alpha^2\left(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}\left\|h(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 10cH(2H-1)T^{2H-1}M_\alpha^2\left[\left(\int_{t_1}^{t_2}(t_2-s)^{-4\alpha}ds\right)\left(\int_{t_1}^{t_2}\left\|h(s,A^{-\alpha}x_s)\right\|^4_{\mathscr{L}_2^0}ds\right)\right]^{\frac{1}{2}}\\
&\le 10cH(2H-1)T^{2H-1}N_2M_\alpha^2(1-4\alpha)^{-\frac{1}{2}}(t_2-t_1)^{\frac{1-4\alpha}{2}}\left(\int_{t_1}^{t_2}\left(1+\|A^{-\alpha}x_s\|^2_{C_\alpha}\right)^2ds\right)^{\frac{1}{2}}\\
&\le 10cH(2H-1)T^{2H-1}N_2M_\alpha^2(1-4\alpha)^{-\frac{1}{2}}(t_2-t_1)^{1-2\alpha}\left(1+\|x\|^2_{D_T}\right).
\end{aligned}
$$

Also, there holds

$$
\begin{aligned}
I_6 &\le 20\mathbb{E}\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]f(s,A^{-\alpha}x_s)\right\|ds\right)^2\\
&\quad+20\mathbb{E}\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]f(s,A^{-\alpha}x_s)\right\|ds\right)^2\\
&\le 20\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^2ds\right)\left(\int_0^{t_1-\varepsilon}\left\|f(s,A^{-\alpha}x_s)\right\|^2ds\right)\\
&\quad+20\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^2ds\right)\left(\int_{t_1-\varepsilon}^{t_1}\left\|f(s,A^{-\alpha}x_s)\right\|^2ds\right)\\
&\le 20N_2\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^2ds\right)\left(\int_0^{t_1-\varepsilon}\left(1+\|x\|^2_{D_T}\right)ds\right)\\
&\quad+20N_2\left(\int_{t_1-\varepsilon}^{t_1}\left(\frac{M_\alpha}{(t_2-s)^\alpha}+\frac{M_\alpha}{(t_1-s)^\alpha}\right)^2ds\right)\left(\int_{t_1-\varepsilon}^{t_1}\left(1+\|x\|^2_{D_T}\right)ds\right)\\
&\le 20N_2(t_1-\varepsilon)\left(1+\|x\|^2_{D_T}\right)\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^2ds\\
&\quad+20N_2\varepsilon M_\alpha^2\left(\frac{2}{1-2\alpha}\right)\left(1+\|x\|^2_{D_T}\right)\left(-(t_2-t_1)^{1-2\alpha}+(t_2-t_1+\varepsilon)^{1-2\alpha}+\varepsilon^{1-2\alpha}\right).
\end{aligned}
$$

By simple calculations and utilizing Lemma 3.1 we also get

$$
\begin{aligned}
I_7 &\le \frac{20N_3(t_1-\varepsilon)}{\lambda^2}\|B\|^2\left(1+\|x\|^2_{D_T}\right)\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^2ds\\
&\quad+\frac{20N_3\varepsilon}{\lambda^2}\|B\|^2M_\alpha^2\left(\frac{2}{1-2\alpha}\right)\left(1+\|x\|^2_{D_T}\right)\left(-(t_2-t_1)^{1-2\alpha}+(t_2-t_1+\varepsilon)^{1-2\alpha}+\varepsilon^{1-2\alpha}\right).
\end{aligned}
$$

Then it is easy to see that

$$
\begin{aligned}
I_8 &\le 20C_p\mathbb{E}\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]g(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\quad+20C_p\mathbb{E}\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]g(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 20C_p\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^4ds\right)^{\frac12}\mathbb{E}\left(\int_0^{t_1-\varepsilon}\left\|g(s,A^{-\alpha}x_s)\right\|^4_{\mathscr{L}_2^0}ds\right)^{\frac12}\\
&\quad+20C_p\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^4ds\right)^{\frac12}\mathbb{E}\left(\int_{t_1-\varepsilon}^{t_1}\left\|g(s,A^{-\alpha}x_s)\right\|^4_{\mathscr{L}_2^0}ds\right)^{\frac12}\\
&\le 20C_pN_2(t_1-\varepsilon)^{\frac12}\left(1+\|x\|^2_{D_T}\right)\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^4ds\right)^{\frac12}\\
&\quad+20C_pN_2M_\alpha^2\left(\frac{8\varepsilon}{1-4\alpha}\right)^{\frac12}\left(1+\|x\|^2_{D_T}\right)\left(-(t_2-t_1)^{1-4\alpha}+(t_2-t_1+\varepsilon)^{1-4\alpha}+\varepsilon^{1-4\alpha}\right)^{\frac12},
\end{aligned}
$$

for some constant $C_p \ge 0$. Finally, for I_9 we have

$$
\begin{aligned}
I_9 &\le 10cH(2H-1)T^{2H-1}\left(\int_0^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]h(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 20cH(2H-1)T^{2H-1}\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]h(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\quad+20cH(2H-1)T^{2H-1}\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]h(s,A^{-\alpha}x_s)\right\|^2_{\mathscr{L}_2^0}ds\right)\\
&\le 20cH(2H-1)T^{2H-1}\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^4ds\right)^{\frac12}\left(\int_{t_1-\varepsilon}^{t_1}\left\|h(s,A^{-\alpha}x_s)\right\|^4_{\mathscr{L}_2^0}ds\right)^{\frac12}\\
&\quad+20cH(2H-1)T^{2H-1}\left(\int_{t_1-\varepsilon}^{t_1}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^4ds\right)^{\frac12}\left(\int_{t_1-\varepsilon}^{t_1}\left\|h(s,A^{-\alpha}x_s)\right\|^4_{\mathscr{L}_2^0}ds\right)^{\frac12}\\
&\le 20cH(2H-1)T^{2H-1}N_2(t_1-\varepsilon)^{\frac12}\left(1+\|x\|^2_{D_T}\right)\left(\int_0^{t_1-\varepsilon}\left\|A^\alpha\left[R(t_2-s)-R(t_1-s)\right]\right\|^4ds\right)^{\frac12}\\
&\quad+20cH(2H-1)T^{2H-1}N_2M_\alpha^2\left(\frac{8\varepsilon}{1-4\alpha}\right)^{\frac12}\left(1+\|x\|^2_{D_T}\right)\left(-(t_2-t_1)^{1-4\alpha}+(t_2-t_1+\varepsilon)^{1-4\alpha}+\varepsilon^{1-4\alpha}\right)^{\frac12}.
\end{aligned}
$$

Thus, since Theorem 2.1 (*iii*) shows that $A^\alpha R(t)$ is uniformly continuous for $t \in (0, T]$, and in addition $R(t)$ is strongly continuous for $t \ge 0$, we deduce readily that $(P^\lambda x)(t)$ is continuous from the right in $[0, T)$. A similar reasoning manifests that it is also continuous from the left in $(0, T]$. Then the result is proved. □

Lemma 3.3. *Under the hypotheses of Lemma 3.2, the operator P^λ maps D_T into itself. i.e., $P^\lambda(D_T) \subset D_T$.*

Proof. Let $x \in D_T$ and $t \in [0, T]$, then

$$
\begin{aligned}
\mathbb{E}\left\|\left(P^\lambda x\right)_t\right\|^2 \le\ & 5\mathbb{E}\sup_{\theta\in[-r,0]}\left\|R(t+\theta)A^\alpha\phi(0)\right\|^2+5\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta}A^\alpha R(t+\theta-s)f(s,A^{-\alpha}x_s)ds\right\|^2\\
&+5\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta}A^\alpha R(t+\theta-s)Bu^\lambda(s,A^{-\alpha}x)ds\right\|^2\\
&+5\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta}A^\alpha R(t+\theta-s)g(s,A^{-\alpha}x_s)dW(s)\right\|^2\\
&+5\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta}A^\alpha R(t+\theta-s)h(s,A^{-\alpha}x_s)dB^H(s)\right\|^2\\
=\ &\sum_{\iota=10}^{14}I_\iota.
\end{aligned}
$$

Obviously we have
$$I_{10} \leq \ 5M_\alpha^2 \mathbb{E}\|\phi\|^2_{C_\alpha}.$$
Using (H_3), Lemma 3.1 and Theorem 2.1(iv) we get
$$\begin{aligned}
I_{11} \leq & 5M_\alpha^2 \mathbb{E} \sup_{\theta\in[-r,0]} \left(\int_0^{t+\theta} (t+\theta-s)^{-\alpha} \left\| f(s, A^{-\alpha}x_s) \right\| ds \right)^2 \\
\leq & 5M_\alpha^2 \mathbb{E} \sup_{\theta\in[-r,0]} \left(\int_0^{t+\theta} (t+\theta-s)^{-2\alpha} ds \right) \left(\int_0^{t+\theta} \left\| f(s, A^{-\alpha}x_s) \right\|^2 ds \right) \\
\leq & 5N_2 M_\alpha^2 \sup_{\theta\in[-r,0]} \left(\frac{(t+\theta)^{2-2\alpha}}{1-2\alpha} \right) \left(1 + \|x\|^2_{D_T}\right) \\
\leq & 5N_2 M_\alpha^2 \left(\frac{T^{2-2\alpha}}{1-2\alpha} \right) \left(1 + \|x\|^2_{D_T}\right).
\end{aligned}$$
Similarly, there holds
$$I_{12} \leq \frac{5N_2\|B\|M_\alpha^2}{\lambda^2} \left(\frac{T^{2-2\alpha}}{1-2\alpha} \right) \left(1 + T\|x\|^2_{D_T}\right).$$
Further, applying Lemma 2.2 and Lemma 2.4 respectively, we obtain
$$\begin{aligned}
I_{13} \leq & 5C_p \mathbb{E} \sup_{\theta\in[-r,0]} \left(\int_0^{t+\theta} \left\| A^\alpha R(t+\theta-s) g(s, A^{-\alpha}x_s) \right\|^2_{\mathscr{L}_2^0} ds \right) \\
\leq & 5C_p M_\alpha^2 \mathbb{E} \sup_{\theta\in[-r,0]} \left(\int_0^{t+\theta} (t+\theta-s)^{-2\alpha} \left\| g(s, A^{-\alpha}x_s) \right\|^2_{\mathscr{L}_2^0} ds \right) \\
\leq & 5C_p M_\alpha^2 \mathbb{E} \sup_{\theta\in[-r,0]} \left[\left(\int_0^{t+\theta} (t+\theta-s)^{-4\alpha} ds \right) \left(\int_0^{t+\theta} \left\| g(s, A^{-\alpha}x_s) \right\|^4_{\mathscr{L}_2^0} ds \right) \right]^{\frac{1}{2}} \\
\leq & 5C_p N_2 M_\alpha^2 (1-4\alpha)^{-\frac{1}{2}} \left(\sup_{\theta\in[-r,0]} (t+\theta)^{1-2\alpha} \right) \left(1 + \|x\|^2_{D_T}\right) \\
\leq & 5C_p N_2 M_\alpha^2 (1-4\alpha)^{-\frac{1}{2}} T^{1-2\alpha} \left(1 + \|x\|^2_{D_T}\right),
\end{aligned}$$
and
$$\begin{aligned}
I_{14} \leq & 5cH(2H-1)T^{2H-1} \sup_{\theta\in[-r,0]} \left(\int_0^{t+\theta} \left\| A^\alpha R(t+\theta-s) h(s, A^{-\alpha}x_s) \right\|^2_{\mathscr{L}_2^0} ds \right) \\
\leq & 5cH(2H-1)T^{2H-1} M_\alpha^2 \sup_{\theta\in[-r,0]} \left(\int_0^{t+\theta} (t+\theta-s)^{-2\alpha} \left\| h(s, A^{-\alpha}x_s) \right\|^2_{\mathscr{L}_2^0} ds \right) \\
\leq & 5cH(2H-1)T^{2H-1} M_\alpha^2 \sup_{\theta\in[-r,0]} \left[\left(\int_0^{t+\theta} (t+\theta-s)^{-4\alpha} ds \right) \left(\int_0^{t+\theta} \left\| h(s, A^{-\alpha}x_s) \right\|^4_{\mathscr{L}_2^0} ds \right) \right]^{\frac{1}{2}} \\
\leq & 5cH(2H-1)T^{2H-1} N_2 M_\alpha^2 (1-4\alpha)^{-\frac{1}{2}} \left(\sup_{\theta\in[-r,0]} (t+\theta)^{1-2\alpha} \right) \left(1 + \|x\|^2_{D_T}\right) \\
\leq & 5cH(2H-1)T^{2H-1} N_2 M_\alpha^2 (1-4\alpha)^{-\frac{1}{2}} T^{1-2\alpha} \left(1 + \|x\|^2_{D_T}\right).
\end{aligned}$$
The above computations show well that $\|P^\lambda x\|^2_{D_T} < +\infty$. Therefore, we infer that $P^\lambda(D_T) \subset D_T$ and this completes the proof. □

Now we are able to prove Theorem 3.2 to obtain the existence and uniqueness of mild solutions of System (1).

Proof. **(Proof of Theorem 3.2)**. As mentioned previously, we prove this theorem by certifying that the operator P^λ has a unique fixed point in D_T and we do this employing the classical Banach fixed point theorem as follows.

In view of Lemma 3.2 and Lemma 3.3, P^λ is a continuous operator on $[0, T]$ that maps D_T into itself. It hence remains to show that there exists a number $n \in \mathbb{N}$ such that $P^{\lambda,n} := \left(P^\lambda\right)^n$ is a contraction.

Let $x, y \in D_T$, for any fixed $t \in [0,T]$, we have that

$$\begin{aligned}
\mathbb{E}\left\|\left(P^{\lambda}x\right)_t - \left(P^{\lambda}y\right)_t\right\|^2 \le\ & 4\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta} A^{\alpha}R(t+\theta-s)\left[f(s,A^{-\alpha}x_s) - f(s,A^{-\alpha}y_s)\right]ds\right\|^2 \\
& + 4\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta} A^{\alpha}R(t+\theta-s)\left[Bu^{\lambda}(s,A^{-\alpha}x) - Bu^{\lambda}(s,A^{-\alpha}y)\right]ds\right\|^2 \\
& + 4\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta} A^{\alpha}R(t+\theta-s)\left[g(s,A^{-\alpha}x_s) - g(s,A^{-\alpha}y_s)\right]dW(s)\right\|^2 \\
& + 4\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta} A^{\alpha}R(t+\theta-s)\left[h(s,A^{-\alpha}x_s) - h(s,A^{-\alpha}y_s)\right]dB^H(s)\right\|^2 \\
=\ & \sum_{\iota=15}^{18} I_{\iota}.
\end{aligned}$$

Then similar to the above computations we can estimate $I_{15} - I_{18}$ separately as

$$\begin{aligned}
I_{15} &\le 4M_{\alpha}^2\mathbb{E}\sup_{\theta\in[-r,0]}\left\|\int_0^{t+\theta}(t+\theta-s)^{-\alpha}\left[f(s,A^{-\alpha}x_s) - f(s,A^{-\alpha}y_s)\right]ds\right\|^2 \\
&\le 4M_{\alpha}^2\mathbb{E}\sup_{\theta\in[-r,0]}\left[\int_0^{t+\theta}(t+\theta-s)^{-2\alpha}ds\cdot\int_0^{t+\theta}\left\|f(s,A^{-\alpha}x_s) - f(s,A^{-\alpha}y_s)\right\|^2 ds\right] \\
&\le 4M_{\alpha}^2N_2\sup_{\theta\in[-r,0]}\left(\frac{(t+\theta)^{1-2\alpha}}{1-2\alpha}\right)\mathbb{E}\int_0^t \|x_s-y_s\|_C^2 ds \\
&\le 4M_{\alpha}^2N_2\left(\frac{T^{1-2\alpha}}{1-2\alpha}\right)\mathbb{E}\int_0^t \|x_s-y_s\|_C^2 ds,
\end{aligned}$$

$$I_{16} \le \frac{4M_{\alpha}^2\|B\|^2N_3}{\lambda^2}\left(\frac{T^{2-2\alpha}}{1-2\alpha}\right)\mathbb{E}\int_0^t \|x_s-y_s\|_C^2 ds,$$

$$\begin{aligned}
I_{17} &\le 4C_p\mathbb{E}\sup_{\theta\in[-r,0]}\left(\int_0^{t+\theta}\left\|A^{\alpha}R(t+\theta-s)\left[g(s,A^{-\alpha}x_s) - g(s,A^{-\alpha}y_s)\right]\right\|_{\mathscr{L}_2^0}^2 ds\right) \\
&\le 4C_pM_{\alpha}^2\mathbb{E}\sup_{\theta\in[-r,0]}\left(\int_0^{t+\theta}(t+\theta-s)^{-4\alpha}ds\right)^{\frac{1}{2}}\left(\int_0^{t+\theta}\left\|g(s,A^{-\alpha}x_s) - g(s,A^{-\alpha}y_s)\right\|_{\mathscr{L}_2^0}^4 ds\right)^{\frac{1}{2}} \\
&\le 4C_pN_2M_{\alpha}^2(1-4\alpha)^{-\frac{1}{2}}\left(\sup_{\theta\in[-r,0]}(t+\theta)^{\frac{1-4\alpha}{2}}\right)\mathbb{E}\int_0^t \|x_s-y_s\|_C^2 ds \\
&\le 4C_pN_2M_{\alpha}^2(1-4\alpha)^{-\frac{1}{2}}T^{\frac{1-4\alpha}{2}}\mathbb{E}\int_0^t \|x_s-y_s\|_C^2 ds,
\end{aligned}$$

and

$$\begin{aligned}
I_{18} &\le 4cH(2H-1)T^{2H-1}\mathbb{E}\sup_{\theta\in[-r,0]}\left(\int_0^{t+\theta}\left\|A^{\alpha}R(t+\theta-s)\left[h(s,A^{-\alpha}x_s) - h(s,A^{-\alpha}y_s)\right]\right\|_{\mathscr{L}_2^0}^2 ds\right) \\
&\le 4cH(2H-1)T^{2H-1}M_{\alpha}^2\mathbb{E}\sup_{\theta\in[-r,0]}\left(\int_0^{t+\theta}(t+\theta-s)^{-4\alpha}ds\right)^{\frac{1}{2}}\left(\int_0^{t+\theta}\left\|h(s,A^{-\alpha}x_s) - h(s,A^{-\alpha}y_s)\right\|_{\mathscr{L}_2^0}^4 ds\right)^{\frac{1}{2}} \\
&\le 4cH(2H-1)N_2M_{\alpha}^2(1-4\alpha)^{-\frac{1}{2}}T^{\frac{4H-4\alpha-1}{2}}\mathbb{E}\int_0^t \|x_s-y_s\|_C^2 ds.
\end{aligned}$$

As a result, there exists a positive real number $Q(\lambda) > 0$ such that

$$\mathbb{E}\left\|\left(P^{\lambda,1}x\right)_t - \left(P^{\lambda,1}y\right)_t\right\|_C^2 \le\ Q(\lambda)\int_0^t \mathbb{E}\|x_s-y_s\|_C^2\, ds, \tag{6}$$

for any $t \in [0,T]$. Then we get

$$\left\|P^{\lambda,1}x - P^{\lambda,1}y\right\|_{D_T}^2 \le\ TQ(\lambda)\|x-y\|_{D_T}^2, \ \text{ for any } x,\ y \in D_T.$$

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Further, it follows from (6) that

$$\begin{aligned}
\left\|P^{\lambda,2}x - P^{\lambda,2}y\right\|_{D_T}^2 &= \sup_{t\in[0,T]} \mathbb{E}\left\|\left(P^{\lambda}(P^{\lambda,1}x)\right)_t - \left(P^{\lambda}(P^{\lambda,1}y)\right)_t\right\|_C^2 \\
&\le \sup_{t\in[0,T]} \int_0^t Q(\lambda)\mathbb{E}\left\|\left(P^{\lambda,1}x\right)_s - \left(P^{\lambda,1}y\right)_s\right\|_C^2 ds \\
&\le \sup_{t\in[0,T]} \int_0^t tQ^2(\lambda)\mathbb{E}\left\|x_s - y_s\right\|_C^2 ds \\
&\le \frac{1}{2}t^2Q^2(\lambda) \sup_{t\in[0,T]} \mathbb{E}\left\|x_t - y_t\right\|_C^2 .
\end{aligned}$$

So we obtain that

$$\left\|P^{\lambda,2}x - P^{\lambda,2}y\right\|_{D_T}^2 \le \frac{1}{2}T^2Q^2(\lambda)\|x-y\|_{D_T}^2, \quad \text{for any } (x,y)\in D_T.$$

Proceeding inductively, we then find that, for any integer $n \ge 1$,

$$\left\|P^{\lambda,n}x - P^{\lambda,n}y\right\|_{D_T}^2 \le \frac{(TQ(\lambda))^n}{n!}\|x-y\|_{D_T}^2 ,$$

from which we deduce that the operator $P^{\lambda,n}$ is a contraction for sufficiently large n, and therefore by Banach fixed point theorem there exists a unique $x(\cdot,\phi)$ such that $P^{\lambda}x(\cdot,\phi) = x(\cdot,\phi)$ in D_T.

Then, by setting $y^{\lambda}(t,\phi) = A^{-\alpha}x^{\lambda}(t,\phi)$ and by virtue of Lemma 2.3 , we conclude that $y^{\lambda}(\cdot,\phi)$ is a mild solution of System (1) on $(-r,T]$.

The uniqueness of mild solutions can be easily proved by Gronwall's inequality and standard arguments. □

We then on the condition to establish the result on the approximate controllability of System (1). That is

Theorem 3.3. *Assume that the assumptions* $(H_0)-(H_2)$ *and* (H_3') *are satisfied and, then the system* (1) *is approximately controllable on* $[0,T]$.

Proof. Let $h \in L_2(\Omega;X)$, and it can be expressed as $h = \mathbb{E}h + \int_0^T \tilde{\psi}(s)dW(s)$ for $\tilde{\psi} \in L_2^{\mathscr{F}}\left(\Omega; L_2(0,T;\mathscr{L}_2^0(K;X)\right)$ by Lemma 2.5. Suppose that $y^{\lambda}(\cdot,\varphi) = A^{-\alpha}x^{\lambda}(\cdot,\varphi)$ is the mild solution of System (1) obtained in Theorem 3.2 under the control function $u^{\lambda}(\cdot)$ given by (5). Then substituting (5) into (4) immediately yields that, for $t \le 0$, $y^{\lambda}(t,\phi) = \phi(t)$ and, for $t \in [0,T]$,

$$\begin{aligned}
y^{\lambda}(t,\phi) &= R(t)\phi(0) + \int_0^t R(t-s)\left[f(s,y_s^{\lambda}) + Bu(s)\right]ds + \int_0^t R(t-s)g(s,y_s^{\lambda})dW(s) + \int_0^t R(t-s)h(s,y_s^{\lambda})dB^H(s) \\
&= R(t)\phi(0) + \int_0^t R(t-s)f(s,y_s^{\lambda})ds + \int_0^t R(t-s)g(s,y_s^{\lambda})dW(s) + \int_0^t R(t-s)h(s,y_s^{\lambda})dB^H(s) \\
&\quad + \int_0^t R(t-s)BB^*R^*(T-s)\mathcal{R}(\lambda,-\Gamma_0^T)\left(\mathbb{E}h - R(T)\phi(0)\right)ds \\
&\quad + \int_0^t R(t-s)BB^*R^*(T-s)\int_0^s \mathcal{R}(\lambda,-\Gamma_\tau^T)\tilde{\psi}(\tau)dW(\tau)ds \\
&\quad - \int_0^t R(t-s)BB^*R^*(T-s)\int_0^s \mathcal{R}(\lambda,-\Gamma_\tau^T)R(T-\tau)f(\tau,y_\tau^{\lambda})d\tau ds \\
&\quad - \int_0^t R(t-s)BB^*R^*(T-s)\int_0^t \mathcal{R}(\lambda,-\Gamma_\tau^T)R(T-\tau)g(\tau,y_\tau^{\lambda})dW(\tau)ds \\
&\quad - \int_0^t R(t-s)BB^*R^*(T-s)\int_0^t \mathcal{R}(\lambda,-\Gamma_\tau^T)R(T-\tau)h(\tau,y_\tau^{\lambda})dB^H(\tau)ds
\end{aligned}$$

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$$
\begin{aligned}
&= R(t)\phi(0) + \int_0^t R(t-s)f(s,y_s^\lambda)ds + \int_0^t R(t-s)g(s,y_s^\lambda)dW(s) + \int_0^t R(t-s)h(s,y_s^\lambda)dB^H(s)\\
&\quad + \int_0^t R(t-s)BB^*R^*(T-s)\mathcal{R}(\lambda,-\Gamma_0^T)\Big(\mathbb{E}h - R(T)\phi(0)\Big)ds\\
&\quad + \int_0^t\int_s^t R(t-\tau)BB^*R^*(T-\tau)d\tau\,\mathcal{R}(\lambda,-\Gamma_s^T)\tilde{\psi}(s)dW(s)\\
&\quad - \int_0^t\int_s^t R(t-\tau)BB^*R^*(T-\tau)d\tau\,\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)f(s,y_s^\lambda)ds\\
&\quad - \int_0^t\int_s^t R(t-\tau)BB^*R^*(T-\tau)d\tau\,\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)g(s,y_s^\lambda)dW(s)\\
&\quad - \int_0^t\int_s^t R(t-\tau)BB^*R^*(T-\tau)d\tau\,\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)h(s,y_s^\lambda)dB^H(s),
\end{aligned}
$$

which gives at $t = T$ that

$$
\begin{aligned}
y^\lambda(T,\phi) &= R(T)\phi(0) + \Gamma_0^T\mathcal{R}(\lambda,-\Gamma_0^T)\Big(\mathbb{E}h - R(T)\phi(0)\Big) + \int_0^T \left(I - \Gamma_s^T\mathcal{R}(\lambda,-\Gamma_s^T)\right)R(T-s)f(s,y_s^\lambda)ds\\
&\quad + \int_0^T \Big(I - \Gamma_s^T\mathcal{R}(\lambda,-\Gamma_s^T)\Big)R(T-s)g(s,y_s^\lambda)dW(s) + \int_0^T \Big(I - \Gamma_s^T\mathcal{R}(\lambda,-\Gamma_s^T)\Big)R(T-s)h(s,y_s^\lambda)dB^H(s)\\
&\quad + \int_0^T \Gamma_s^T\mathcal{R}(\lambda,-\Gamma_s^T)\tilde{\psi}(s)dW(s).
\end{aligned}
$$

Since $I - \Gamma_s^T\mathcal{R}(\lambda,-\Gamma_s^T) = \lambda\mathcal{R}(\lambda,-\Gamma_s^T)$, we get

$$
\begin{aligned}
y^\lambda(T,\phi) &= R(T)\phi(0) + \Big(I - \lambda\mathcal{R}(\lambda,-\Gamma_0^T)\Big)\Big(\mathbb{E}h - R(T)\phi(0)\Big) + \int_0^T \lambda\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)f(s,y_s^\lambda)ds\\
&\quad + \int_0^T \lambda\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)g(s,y_s^\lambda)dW(s) + \int_0^T \lambda\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)h(s,y_s^\lambda)dB^H(s)\\
&\quad + \int_0^T \Big(I - \lambda\mathcal{R}(\lambda,-\Gamma_s^T)\Big)\tilde{\psi}(s)dW(s)\\
&= h - \lambda\mathcal{R}(\lambda,-\Gamma_0^T)\Big(\mathbb{E}h - R(T)\phi(0)\Big) + \int_0^T \lambda\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)f(s,y_s^\lambda)ds\\
&\quad + \int_0^T \lambda\mathcal{R}(\lambda,-\Gamma_s^T)\left(R(T-s)g(s,y_s^\lambda) - \tilde{\psi}(s)\right)dW(s) + \int_0^T \lambda\mathcal{R}(\lambda,-\Gamma_s^T)R(T-s)h(s,y_s^\lambda)dB^H(s).
\end{aligned}
$$

From Theorem 3.1 the condition (H_0) implies that the linear system (3) is approximately controllable on interval $[0,T]$ and hence we infer that, for all $0 \le s < T$,

$$
\lambda\mathcal{R}(\lambda,-\Gamma_s^T) \longrightarrow 0, \text{ strongly as } \lambda \longrightarrow 0^+, \text{ for any } s \in [0,T).
$$

On the other hand, by assumption (H_3'), we know that $\left\{f(s,y_s^\lambda) : \lambda \in (0,1)\right\}$, $\left\{g(s,y_s^\lambda) : \lambda \in (0,1)\right\}$ and $\left\{h(s,y_s^\lambda) : \lambda \in (0,1)\right\}$ are bounded uniformly in $\lambda \in (0,1)$ in X and $\mathscr{L}_2^0(K;X)$ respectively, which indicates that there are subsequences of $f(s,y_s^\lambda)$, $g(s,y_s^\lambda)$ and $h(s,y_s^\lambda)$, still denoted by themselves, that converge weakly to, say, $f(s)$, $g(s)$ and $h(s)$ in X and $\mathscr{L}_2^0(K;X)$, respectively, for each $s \in [0,T]$. Since, from (H_1), the operator $R(t)$ $(t > 0)$ is compact, it follows immediately that

$$
\begin{aligned}
&R(T-s)f(s,y_s^\lambda) \to R(T-s)f(s),\\
&R(T-s)g(s,y_s^\lambda) \to R(T-s)g(s),\\
&R(T-s)h(s,y_s^\lambda) \to R(T-s)h(s), \quad \text{in} \quad s \in [0,T)\times\Omega.
\end{aligned}
$$

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Thus we deduce that

$$
\begin{aligned}
&\mathbb{E}\left\|y^{\lambda}(T,\phi)-h\right\|^{2}\\
&\leq 8\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{0}^{T})\big(\mathbb{E}h-R(T)\phi(0)\big)\right\|^{2}\\
&\quad+8\left[\mathbb{E}\left(\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})\right\|\left\|R(T-s)\left(f(s,y_{s}^{\lambda})-f(s)\right)\right\|ds\right)^{2}+\mathbb{E}\left(\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})R(T-s)f(s)\right\|ds\right)^{2}\right]\\
&\quad+8\left[\mathbb{E}\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})\right\|^{2}\left\|R(T-s)\left(g(s,y_{s}^{\lambda})-g(s)\right)\right\|_{\mathscr{L}_{2}^{0}}^{2}ds+\mathbb{E}\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})R(T-s)g(s)\right\|_{\mathscr{L}_{2}^{0}}^{2}ds\right]\\
&\quad+8\left[\mathbb{E}\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})\right\|^{2}\left\|R(T-s)\left(h(s,y_{s}^{\lambda})-h(s)\right)\right\|_{\mathscr{L}_{2}^{0}}^{2}ds+\mathbb{E}\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})R(T-s)h(s)\right\|_{\mathscr{L}_{2}^{0}}^{2}ds\right]\\
&\quad+8\mathbb{E}\int_{0}^{T}\left\|\lambda\mathcal{R}(\lambda,-\Gamma_{s}^{T})\tilde{\psi}(s)\right\|_{\mathscr{L}_{2}^{0}}^{2}ds\\
&\to 0\quad\text{as}\quad\lambda\to 0^{+}.
\end{aligned}
$$

Therefore, the system (1) is approximately controllable on $[0,T]$ as desired. □

4 Example

In this section we provide an example to illustrate the applications of the obtained results. We consider the following semi-linear neutral delayed system

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[z(t,x)+\displaystyle\int_{0}^{t}a(t-s)z(s,x)ds\right]=\left[\dfrac{\partial^{2}z(t,x)}{\partial x^{2}}+\displaystyle\int_{0}^{t}b(t-s)\dfrac{\partial^{2}z(t,x)}{\partial x^{2}}ds+F_{1}\Big(t,z\big(t-r_{1}(t),x\big)\Big)+Bu(t,x)\right]dt\\
\qquad\qquad+F_{2}\Big(t,z\big(t,z(t-r_{2}(t),x\big)\Big)d\beta(t)+F_{3}\Big(t,z\big(t,z(t-r_{3}(t),x\big)\Big)d\beta^{H}(t),\ 0<t\leq T,\ 0\leq x\leq\pi\\
z(t,0)=z(t,\pi)=0,\qquad t\in[0,T],\\
z(\theta,x)=\varphi(\theta,x),\qquad -1\leq\theta\leq 0,\ \ 0\leq x\leq\pi,
\end{cases}
\tag{7}
$$

where $r_1(t), r_2(t)$ and $r_3(t)$ are continuous functions with $0<r_1(t), r_2(t), r_3(t)\leq r$, and $\varphi(\cdot,\cdot)$ is $\mathscr{F}_0-$measurable. The functions $a(\cdot)$, $b(\cdot)$ and $F_i(\cdot,\cdot)(i=1,2,3)$ will be described below. $\beta(t)$ denotes a one-dimensional standard Brownian motion, $\beta^H(t)$ denotes fractional Brownian motion.

System (7) arises in the study of heat flow in materials of the so-called retarded type [23, 27]. Here, $z(t,x)$ represents the temperature of the point x at time t, a is a relaxation function. Let $X=L^2([0,\pi])$ and define the operator A as

$$A\xi=-\frac{\partial^2}{\partial x^2}\xi,$$

with the domain

$$D(A)=\Big\{\xi\in X \text{ such that } \xi(0)=\xi(\pi)=0 \text{ with } \xi,\ \frac{\partial}{\partial x}\xi,\ \frac{\partial^2}{\partial x^2}\xi \text{ all in } X,$$
$$\text{and with } \xi \text{ and } \frac{\partial}{\partial x}\xi \text{ both absolutely continuous on } [0,\pi]\Big\}.$$

Then $-A$ is self-adjoint, negative definite and the resolvent operator $\mathcal{R}(\lambda,-A)=(\lambda I+A)^{-1}$ is compact when it exists. Moreover, $-A$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ which is analytic, compact and self-adjoint. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are given by $-n^2,\ n\in\mathbb{N}^+$, with the corresponding normalized eigenvectors $e_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n=1,2,\cdots$. Then the following properties hold:

(i) If $\xi\in D(A)$, then

$$A\xi=\sum_{n=1}^{\infty}n^2\langle\xi,e_n\rangle e_n$$

(ii) For every $\xi \in X$,

$$S(t)\xi = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \xi, e_n\rangle e_n,$$

(iii) For $\alpha \in (0,1)$, the fractional power $A^\alpha : D(A^\alpha) \subseteq X$ of A is given by

$$A^\alpha \xi = \sum_{n=1}^{\infty} n^{2\alpha}\langle \xi, e_n\rangle e_n,$$

where $D(A^\alpha) = \{x \in X : A^\alpha x \in X\}$.

We take the linear continuous mapping $B = I$. Then, we impose the following conditions on System (7):

(A_1) The function $a, b \in L^1(\mathbb{R}^+)$, and there is a $\theta \in (\frac{\pi}{2}, \pi)$ such that $|\hat{a}(\lambda)| \leq C/|\lambda|$ for $\lambda \in \Lambda_\theta = \{\lambda \in C \setminus \{0\} : |arg(\lambda)| < \theta\}$, furthermore, the operator $\hat{a}(\lambda) : X_{1/2} \to X_{1/2}$ for $\lambda \in \Lambda_\theta$, and $\frac{\partial \hat{a}(\lambda)}{\partial \lambda} \to 0$ as $|\lambda| \to \infty$ uniformly for $\lambda \in \Lambda_\theta$. In addition, $|\hat{b}(\lambda)| \to 0$ as $\lambda \in \Lambda_\theta$ and $|\lambda| \to \infty$.

(A_2) The functions $F_i(\cdot,\cdot) : [0,T] \times \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous in the second variable and uniformly bounded. i.e., there exist positive constants $K_i > 0$ such that

$$\left|F_i(t,x) - F_i(t,y)\right| \leq K_i |x - y|,$$

for any $(t,x) \in [0,T] \times \mathbb{R}$, $i = 1,2,3$. Moreover, there exist constants $K_4, K_5, K_6 > 0$ such that, for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$\left|F_1(t,x)\right| \leq K_4, \ \left|F_2(t,x)\right| \leq K_5, \ \left|F_3(t,x)\right| \leq K_6.$$

(A_3) The function $\varphi(t,x)$ belongs to $ML(r,\alpha)$.

Now define the map $N(t) : X \to X$, $f(\cdot,\cdot) : [0,T] \times C_\alpha \to X$, $g(\cdot,\cdot) : [0,T] \times C_\alpha \to \mathscr{L}_2^0(\mathbb{R}; X)$, $h(\cdot,\cdot) : [0,T] \times C_\alpha \to \mathscr{L}_2^0(\mathbb{R}; X)$ and the operator $\gamma(t) : D(A) \subset X \to X$ respectively by

$$\begin{aligned}
N(t)x &= a(t)x, \\
\gamma(t) &= b(t)A, \\
f(t,\psi)(x) &= F_1\left(t, \psi(-r_1(t), x)\right), \\
g(t,\psi)(x) &= F_2\left(t, \psi(-r_2(t), x)\right), \\
h(t,\psi)(x) &= F_3\left(t, \psi(-r_3(t), x)\right),
\end{aligned}$$

for $t \in [0,T]$ and $\psi \in C_\alpha$. Then under these notations System (7) is rewritten exactly into the form of (1).

Subsequently we testify that the conditions in Theorem 3.3 are fulfilled for this system. First of all, it follows from (A_1) that the conditions $(V_1) - (V_5)$ are verified with $\hat{N}(\lambda) = \hat{a}(\lambda)I$, $\hat{\gamma}(\lambda) = \hat{b}(\lambda)A$, and hence the corresponding linear system of (7) has a resolvent operator $(R(t))_{t\geq 0}$, which is given by, for $z \in X$,

$$R(t)z = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{r,\vartheta}} e^{\lambda t}\left(\lambda I + \lambda \hat{N}(\lambda) + A - \hat{\gamma}(\lambda)\right)^{-1} z\, d\lambda, & t > 0, \\ I, & t = 0, \end{cases} \tag{8}$$

where $\Gamma_{r,\vartheta}$ is described in Section 2. Moreover, by virtue of Theorem 2.1 we find easily that $R(t)$ is compact for all $t > 0$ since the semigroup $(S(t))_{t\geq 0}$ is so, hence (H_1) holds. Also $B = I$ clearly satisfy (H_2).

On the other hand, it is easy to obtain that, for any $\psi_1(s), \psi_2(s) \in C_\alpha$, $s \in [-r, 0]$,

$$\begin{aligned}
\|\psi_1(s) - \psi_2(s)\|^2 &= \int_0^\pi |\psi_1(s)(x) - \psi_2(s)(x)|^2 dx \\
&\leq \|\psi_1(s) - \psi_2(s)\|_{C_\alpha}^2,
\end{aligned}$$

from which and the assumption (A_2) it follows that the functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are all Lipschitz continuous with respect to the second variable. Moreover, the assumption (A_2) also implies immediately their uniform boundedness so that the condition (H_3') is satisfied as well.

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According to Theorem 3.3, to obtain the approximate controllability for System (7), it remains to verify the condition (H_0). Indeed, from (8), it is easy to see that $R(t) = R^*(t)$, and so for any $x \in X, t \in [0, T]$, we have

$$B^* R^*(t)x = R^*(t)x,$$

Let now $B^* R^*(t)x = 0$, for all $t \in [0, T]$, then,

$$B^* R^*(t)x = R^*(t)x = R(t)x = 0,$$

which from the fact $R(0) = I$ yields that $x = 0$. So (H_0) holds. Consequently, applying Theorem 3.3 we conclude that System (7) is approximately controllable on the interval $[0, T]$.

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References

[1] H. Ahmed, Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space, *Adv. Diff. Equ.*, 113 (2014), 11.

[2] H. Ahmed, M. El-Borai, A. Okb El Bab and M. Elsaid Ramadan, Approximate controllability of noninstantaneous impulsive Hilfer fractional integrodifferential equations with fractional Brownian motion, *Bound. Value Probl.*, 120 (2020), 25.

[3] A. Bashirov and N. Mahmudov, On concepts of controllability for linear deterministic and stochastic systems, *SIAM J. Contr. Optim.*, 37 (1999), 1808-1821.

[4] T. Caraballo, M. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonl. Anal. (TMA)*, 74 (2011), 3671-3684.

[5] M. Chen, Approximate controllability of stochastic equations in a Hilbert space with fractional Brownian motions, *Stoch. Dyn.*, 15 (2015), 16.

[6] R. Curtain and H. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[7] M. Diop, K. Ezzinbi, L. Issaka and K. Ramkumar, Stability for some impulsive neutral stochastic functional integro-differential equations driven by fractional Brownian motion, *Cogent Math. Stat.*, 7 (2020), 24.

[8] J. Dauer and N. Mahmudov, Controllability of stochastic semilinear functional differential equations in Hilbert spaces, *J. Math. Anal. Appl.*, 290 (2004), 373-394.

[9] J. Dos Santos, H. Henríquez and E. Hernández, Existence results for neutral integro-differential equations with unbounded delay, *J. Int. Equ. Appl.*, 23 (2011), 289-330.

[10] T. Duncan, J. Jakubowski and B. Pasik-Duncan, Stochastic integration for fractional Brownian motion in a Hilbert space, *Stoch. Dyn.*, 6 (2006), 53-75.

[11] T. Duncan, B. Pasik-Duncan and B. Maslowski, Fractional Brownian motion and stochastic equations in Hilbert spaces, *Stoch. Dyn.*, 2 (2002), 225-250.

[12] K. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.

[13] K. Ezzinbi, H. Toure and I. Zabsonre, Local existence and regularity of solutions for some partial functional integro-differential equations with infinite delay in Banach spaces, *Nonl. Anal. (TMA)*, 70 (2009), 3378-3389.

4 Apr 2022 08:16:51 PDT
211112-XianlongFu Version 2 - Submitted to J. Integr. Eq. Appl.

[14] M. Garrido-Atienza, K. Lu, and B. Schmalfuss, Random dynamical systems for stochastic partial differential equations driven by a fractional Brownian motion, *Discr. Cont. Dyn. Syst. B.*, 14 (2010), 473-493.

[15] R. Grimmer, Resolvent operator for integral equations in a Banach space, *Trans. Amer. Math. Soc.*, 273 (1983), 333-349.

[16] R. Grimmer and F. Kappel, Series expansions of volterra integrodifferential equations in Banach space, *SIAM J. Math. Anal.*, 15 (1984), 595-604.

[17] R. Grimmer and A. J. Pritchard, Analytic resolvent operators for integral equations in a Banach space, *J. Diff. Equ.*, 50 (1983), 234-259.

[18] H. Henríquez and J. Dos Santos, Differentiability of solutions of abstract neutral integro-differential equations, *J. Int. Equ. Appl.*, 25 (2013), 47-77.

[19] E. Hernández and D. O'Regan, On a new class of abstract neutral integro-differential equations and applications, *Acta. Appl. Math.*, 149 (2017), 125-137.

[20] H. Huang and X. Fu, Approximate controllability of semi-linear stochastic integro-differential equations with infinite delay, *IMA J. Math. Control Inf.*, 37 (2020), 1133-1167.

[21] W. Jedidi, T. Simon and M. Wang, Density solutions to a class of integro-differential equations, *J. Math. Anal. Appl.*, 458 (2018), 134-152.

[22] K. Jeet and N. Sukavanam, Approximate controllability of nonlocal and impulsive neutral integro-differential equations using the resolvent operator theory and an approximating technique, *Appl. Math. Comp.*, 364 (2020), 15.

[23] A. Lunardi, On the linear heat equation with fading memory, *SIAM J. Math. Anal.*, 21 (1990), 1213-1224.

[24] Y. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Topics*, in: Lecture Notes in Mathematics, 1929, 2008.

[25] D. Nualart, Stochastic integration with respect to fractional Brownian motion and applications, *Stoch. Models*, (2002) 3-39.

[26] D. Nualart, *The Malliavin Calculus and Related Topics, second edition.*, Springer-Verlag, Berlin, 2006.

[27] J. Nunziato, On heat conduction in materials with memory, *Quart. Appl. Math.*, 29 (1971), 187-304.

[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[29] G. Prato and J. Zabbczyk, *Stochastic Equations in Infinite Dimensions, Cambridge University Press*, Cambridge, UK, 1992.

[30] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser Verlag, Switzerland, 1993.

[31] Y. Ren, X. Cheng, R. Sakthivel, Impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm, *Appl. Math. Comp.*, 247 (2014), 205-212.

[32] R. Sakthivel, J. Nieto and N. Mahmudov, Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay, *Taiw. J. Math.*, 14 (2010), 1777-1797.

[33] A. Slama and A. Boudaoui, Approximate controllability of retarded impulsive stochastic integro-differential equations driven by fractional Brownian motion, *Filomat*, 33 (2019), 289-306.

[34] G. Shevchenko, Mixed fractional stochastic differential equations with jumps, *Stochastics* , 86 (2014), 203-217.

[35] Z. Taheri, S. Javadi and E. Babolian, Numerical solution of stochastic fractional integro-differential equation by the spectral collocation method, *J. Comp. Appl. Math.*, 321 (2017), 336-347.

[36] P. Tamilalagan and P. Balasubramaniam, Approximate controllability of fractional stochastic differential equations driven by mixed fractional Brownian motion via resolvent operators, *Int. J. Control*, 8 (2017), 1713-1727.

[37] T. Taniguchi, K. Liu and A. Truman, Existence, uniqueness, and asymptotic behavior of mild solutions to stochastic functional differential equations in Hilbert spaces, *J. Diff. Equ.*, 181 (2002), 72-91.

[38] S. Tindel, C. Tudor and F. Viens, Stochastic evolution equations with fractional Brownian motion, *Prob. Theory Rel. Field*, 127 (2003), 186-204.

[39] V. Vijayakumar, Approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces, *IMA J. Math. Control Inf.*, 35 (2018), 297-314.

[40] Q. Wu, L. Hu and Z. Zhang, Convergence and stability of balanced methods for stochastic delay integro-differential equations, *Appl. Math. Comp.*, 237 (2014), 446-460.

[41] X. Zhou, D. Zhou and S. Zhong, Existence and exponential stability in the pth moment for impulsive neutral stochastic integro-differential equations driven by mixed fractional Brownian motion, *J. Inequ. Appl.*, (2019) 2019, 1-19.

[42] J. Zhu and X. Fu, Existence results for neutral integro-differential equations with nonlocal conditions, *J. Int. Equ. Appl.*, 32 (2020), 239-258.