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A PROJECTION METHOD FOR VOLTERRA INTEGRAL EQUATIONS IN WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

TERESA DIOGO, LUISA FERMO, AND DONATELLA OCCORSIO

ABSTRACT. This paper is concerned with the numerical treatment of second kind Volterra integral equations whose integrands present diagonal and/or endpoint algebraic singularities. A projection method based on an optimal interpolating operator is developed in the spaces of weighted continuous functions endowed with the supremum norm. In such spaces, the uniqueness of the solution is discussed and suitable conditions are determined to assure the stability and the convergence of the method. Several numerical tests are presented to show the efficiency of the method and the agreement with the theoretical estimates.

1. Introduction

Let us consider the Volterra integral equation

$$f(y)+\int_{-1}^{y} k(x,y)f(x)(y-x)^{\alpha}(1+x)^{\beta}dx=g(y), \quad y\in(-1,1], \tag{1}$$

where $-1<\alpha,\beta\in\mathbb{R}$, the bivariate function k is defined on $D=\{(x,y):-1<x\le y\le 1\}$, the forcing function g is defined on $(-1,1)$ and f is the unknown solution we want to approximate.

The pathology of the equation (1) depends on the kernel which may present algebraic singularities either at $y=x$, and/or at $y=-1$. Such a behavior determines a solution f whose derivatives may have a singularity at the endpoint -1, even if g is smooth (see e.g. [4, Chapter 6], [33]).

Because of this singularity, in order to obtain accurate results, many numerical methods available in the literature (see e.g. [1, 3, 16, 23, 24, 25, 27]) are based on regularization techniques that transform the initial equation into another one, having a smoother solution. On the other hand, such smoothing transformations make the expression of the resulting equation more complex. Moreover, their role to improve the smoothness properties of the solution can be performed till moderate choices of the parameters from which they depend.

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In this paper, in order to skip the use of smoothing transformations, we propose an approach that, according to our knowledge, represents a novelty in the framework of the numerical methods for Volterra Integral Equations (VIEs). We deal with the nonsmooth behavior of the solution f by developing a numerical method in spaces of weighted continuous functions endowed with the uniform norm. To be more precise, we will study the equation in the Banach space C_u, where u is a suitable Jacobi weight $u(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta \geq 0$ which absorbs the endpoint singularities of the considered functions. Such kind of approach has been recently developed for Volterra integral equations [17] and already used in different contexts for other types of integral equations. Among them we mention: second kind Fredholm integral equations [9, 18, 31], integro-differential equations [5, 11, 12], Cauchy singular equations [7, 6, 22, 29, 10], Mellin-type integral equations [14, 15, 26].

The present work is organized as follows. In Section 2, we give an introduction on function spaces and some basic results about a specific interpolation process. Then we determine the conditions under which the existence and uniqueness of the solution of (1) is assured in C_u. In Section 3 a projection method based on Lagrange interpolation polynomials having optimal Lebesgue constants in C_u is introduced. We prove that the method is stable and convergent in Zygmund-type spaces. In such subspaces of C_u we are able to consider functions having unbounded derivatives at $y = \pm 1$. We also provide an error estimate, depending on the smoothness of the known functions and the choice of the space C_u. In Section 4 we present some numerical experiments which validate the theoretical estimates, and compare our method with other approaches [8, 16, 20]. Finally, in Section 5 we collect the proofs of our theoretical results.

2. Basic results and notations

By $\mathscr{C}$ we will denote any positive constant that will have different meanings in different formulas, using the symbol $\mathscr{C} \neq \mathscr{C}(a,b,\ldots)$ to mean that $\mathscr{C}$ is a positive constant independent of the parameters $a,b,\ldots$, and $\mathscr{C} = \mathscr{C}(a,b,\ldots)$ to mean that $\mathscr{C}$ depends on $a,b,\ldots$. If $D,E > 0$ are quantities depending on some parameters, we will write $E \sim D$, if there exists a constant $\mathscr{C}$ independent of E,D such that $\frac{D}{\mathscr{C}} \leq E \leq \mathscr{C}D$.

As usual by $\mathbb{P}_m$ we will denote the space of the algebraic polynomials of degree at most m and the notation k_x (or k_y) will be used for a bivariate function $k(x,y)$ as a function of the only variable y (or x).

Furthermore, we will denote by $v^{a,b}$ a Jacobi weight of real parameters $a,b > -1$, i.e.

$$v^{a,b}(x) := (1-x)^a(1+x)^b, \quad x \in (-1,1).$$

2.1. *Function spaces.* We define $u := v^{\gamma,\delta}$ with $\gamma, \delta \geq 0$, and let C_u be Banach space of all functions f which are continuous on $(-1,1)$ and such that

$$\lim_{x\to\pm1}(fu)(x) = 0, \qquad \text{if} \qquad u(\pm 1) = 0, \qquad \text{respectively}, \tag{2}$$

equipped with the norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{|x|\leq 1} |(fu)(x)|.$$

Somewhere, for brevity, we will set $\|f\|_A := \max_{x\in A}|f(x)|$.

Note that the limit conditions given in (2) are necessary for the validity of the Weierstrass theorem in C_u, according to which functions in C_u can be uniformly approximated by polynomials, i.e.,

$$\lim_{m\to\infty} E_m(f)_u = 0, \qquad E_m(f)_u := \inf_{P\in\mathbb{P}_m} \|(f-P)u\|_\infty.$$

Now, in order to introduce a subspace of C_u, we consider, for any $f\in C_u$ and for an integer $k\geq 1$, the main part of the φ-*modulus of smoothness* [13, p. 90]

$$\Omega_\varphi^k(f,t)_u = \sup_{0<h\leq t} \|u\Delta_{h\varphi}^k f\|_{I_{kh}},$$

where $\varphi(x)=\sqrt{1-x^2}$, $I_{kh}=[-1+(2kh)^2, 1-(2kh)^2]$ and

$$\Delta_{h\varphi}^k f(x) = \sum_{i=0}^k (-1)^i \binom{k}{i} f\left(x+\frac{h\varphi(x)}{2}(k-2i)\right).$$

Then, we define the space of Zygmund-type, of order $\lambda\in\mathbb{R}^+$,

$$Z_\lambda(u) = \left\{ f\in C_u \;:\; \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_u}{t^\lambda} < \infty, r>\lambda \right\},$$

endowed with the norm

$$\|f\|_{Z_\lambda(u)} = \|fu\|_\infty + \sup_{t>0}\frac{\Omega_\varphi^k(f,t)_u}{t^\lambda}.$$

The following equivalence holds true (see e.g. [28, p. 172])

$$\sup_{t>0}\frac{\Omega_\varphi^r(f,t)_u}{t^\lambda} \sim \sup_{n\geq 0}(1+n)^r E_n(f)_u, \tag{3}$$

where the constants implied in the equivalence "$\sim$" are dependent on λ.

Let $AC(-1,1)$ be the set of absolutely continuous functions in any closed subinterval of $(-1,1)$. Then, for any integer $r>0$, let $W_r(u)$ be the Sobolev-type space of order r defined as

$$W_r(u) = \left\{ f\in C_u : f^{(r-1)}\in AC(-1,1), \quad \|f^{(r)}\varphi^r u\|_\infty < \infty \right\},$$

equipped with the norm

$$\|f\|_{W_r(u)} = \|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty.$$

Finally, in order to estimate $E_m(f)_u$, we recall the weaker version of Jackson's theorem

$$E_m(f)_u \leq \mathscr{C}\int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(f,t)_u}{t}dt, \quad \forall f\in C_u, \quad 0<\mathscr{C}\neq\mathscr{C}(m,f) \tag{4}$$

and the Favard type inequalities (see e.g. [28, p. 172])

$$E_m(f)_u \leq \frac{\mathscr{C}}{m^r}\|f\|_{W_r(u)}, \quad \forall f\in W_r(u), \tag{5}$$

and

$$E_m(f)_u \leq \frac{\mathscr{C}}{m^\lambda}\|f\|_{Z_\lambda(u)}, \quad \forall f\in Z_\lambda(u), \tag{6}$$

where in both cases the constant $\mathscr{C}$ does not depend on m and f, but it depends on r in (5) and on λ in (6).

2.2. *Lagrange interpolation processes.* Let $w := v^{\alpha,\beta}$ be a Jacobi weight with parameters $\alpha,\beta > -1$ and let $\{p_m(w)\}_{m=0}^{\infty}$ be the sequence of the corresponding orthonormal polynomials with positive leading coefficients.

For a given function $f \in C_u$, denote by $\mathscr{L}_m^w(f) \in \mathbb{P}_{m-1}$ the Lagrange polynomial that interpolates f at the zeros $x_1 < x_2 < \cdots < x_m$ of the polynomial $p_m(w)$, i.e.

$$\mathscr{L}_m^w(f,x_i) = f(x_i), \quad i = 1,2,\ldots,m.$$

For our aims, we represent it in the basis $\left\{\varphi_{m,i}(x) = \frac{\ell_{m,i}(x)}{u(x_i)}\right\}_{i=1}^{m}$ where u is the weight of the space and

$$\ell_{m,i}(x) = \frac{p_m(w,x)}{p_m'(w,x_i)(x-x_i)}.$$

Then,

$$\mathscr{L}_m^w(f,x) = \sum_{i=1}^{m} \varphi_{m,i}(x)(fu)(x_i). \tag{7}$$

Denoting by $\|\mathscr{L}_m^w\|_u$ the m-th Lebesgue constant

$$\|\mathscr{L}_m^w\|_u := \|\mathscr{L}_m^w\|_{C_u\to C_u} = \max_{|x|\le 1} \sum_{k=1}^{m} |\varphi_{m,k}(x)|u(x),$$

next Lemma states the conditions under which the above introduced Lagrange process has optimal Lebesgue constants in C_u [28, p.272] (see also [19]).

Lemma 2.1. *Given two Jacobi weights $w = v^{\alpha,\beta}$ and $u = v^{\gamma,\delta}$, then*

$$\|\mathscr{L}_m^w\|_u \sim \log m,$$

if and only if the parameters γ and δ are such that

$$\max\left\{0, \frac{\alpha}{2}+\frac{1}{4}\right\} \le \gamma \le \frac{\alpha}{2}+\frac{5}{4}, \qquad \max\left\{0, \frac{\beta}{2}+\frac{1}{4}\right\} \le \delta \le \frac{\beta}{2}+\frac{5}{4}. \tag{8}$$

Remark 2.2. *Let us note that, under the assumptions of Lemma 2.1, for any $f \in Z_\lambda(u)$, by (6) we deduce*

$$\|[f-\mathscr{L}_m^w(f)]u\|_\infty \le \mathscr{C}\frac{\log m}{m^\lambda}\|f\|_{Z_\lambda(u)}, \quad \mathscr{C} \ne \mathscr{C}(m,f), \tag{9}$$

and by (5), for any $f \in W_r(u)$

$$\|[f-\mathscr{L}_m^w(f)]u\|_\infty \le \mathscr{C}\frac{\log m}{m^r}\|f\|_{W_r(u)}, \quad \mathscr{C} \ne \mathscr{C}(m,f).$$

3. Main results

The aim of this section is to present a projection method for equation (1). To this end, first we introduce the function γ defined as

$$x = \gamma(t,y) = \frac{1+t}{2}y + \frac{t-1}{2}, \tag{10}$$

which maps $[-1,\,1]$ into $[-1,\,y]$ for each fixed y and rewrite equation (1) as

$$f(y) + \mu \int_{-1}^{1} \hat{k}(t,y)\, f(\gamma(t,y))\, w(t) dt = g(y) \tag{11}$$

where

$$\mu = 2^{-(\alpha+\beta+1)}, \quad \hat{k}(t,y) = (1+y)^{\alpha+\beta+1} k(\gamma(t,y),y). \tag{12}$$

In practice, we have transformed the Volterra integral equation (1) into an equation having the same right-hand side and a kernel $\hat{k}$ which may have a low smoothness at $y=-1$, according to the value $\alpha+\beta+1$.

Note that equation (1) has been obtained from (11) simply by making a change of variable in the integral, through a linear transformation admitting inverse $t = \gamma^{-1}(x,y) = \frac{2x-(y-1)}{y+1}$. Consequently, if f satisfies (1) then it is a solution of (11) and vice-versa.

We remark that we need to rewrite equation (1) as in (11) in order to develop a stable and convergent projection method. Specifically, we need to move to the inteval $[-1,1]$ to identify the Jacobi weight w, construct the polynomial $p_m(w)$, introduce the projector $\mathscr{L}_m^w$ through which to project the equation on $\mathbb{P}_{m-1}$.

Next theorem states that if the kernel $\hat{k}_t(y) \in Z_\lambda(u)$ and the parameters of the involved Jacobi weights satisfy suitable conditions, equation (11), and then (1), has a unique solution in a certain Zygmund subspace.

Theorem 3.1. *Assume that $\alpha-\gamma>-1,\ \beta-\delta>-1$ and*

$$\sup_{|t|\le 1} \|k_t v^{0,\alpha+\beta+1}\|_{Z_\lambda(u)} < \infty. \tag{13}$$

Then, for any function g belonging to the Zygmund-type space $Z_\lambda(u)$, the equation (1) admits a unique solution $f^* \in Z_\lambda(u)$.

3.1. *The collocation method.* By means of the Lagrange operator $\mathscr{L}_m^w$ defined in (7) we project equation (11) on $\mathbb{P}_{m-1}$. To this end, let us introduce the polynomial sequences $\{g_m\}_m$ and $\{\mathscr{K}_m f\}_m$ defined as

$$g_m = \mathscr{L}_m^w(g),$$

$$(\mathscr{K}_m f)(y) = \mathscr{L}_m^w(\mathscr{K}_m^* f) = \sum_{i=1}^{m} \varphi_{m,i}(x)(\mathscr{K}_m^* f)(x_i)u(x_i), \tag{14}$$

where

$$(\mathscr{K}_m^* f)(y) = \mu \int_{-1}^{1} \mathscr{L}_m^w \left(\hat{k}_y(\cdot) f(\gamma(\cdot,y)),x\right) w(x)dx$$

$$= \mu \sum_{\nu=1}^{m} \lambda_\nu \hat{k}(x_\nu,y) f(\gamma(x_\nu,y)), \tag{15}$$

being $\{\lambda_\nu\}_{\nu=1}^m$ the Christoffel numbers with respect to w.

Setting $\{\gamma_i(x) = \gamma(x,x_i)\}_{i=1}^m$, we have

$$(\mathscr{K}_m^* f)(x_i) = \mu \sum_{\nu=1}^{m} \lambda_\nu \hat{k}(x_\nu,x_i) f(\gamma_i(x_\nu)).$$

Thus, let us consider the following finite-dimensional equation

$$f_m(y) + \mu \sum_{i=1}^{m} \varphi_{m,i}(y) u(x_i) \sum_{\nu=1}^{m} \lambda_\nu \hat{k}(x_\nu,x_i) f_m(\gamma_i(x_\nu)) = g_m(y), \tag{16}$$

where

$$f_m(y) = \sum_{i=1}^{m} \varphi_{m,i}(y) c_i \in \mathbb{P}_{m-1}. \tag{17}$$

An equivalent form of equation (16) is

$$\sum_{i=1}^{m} \varphi_{m,i}(y) \left[c_i + \mu \sum_{\nu=1}^{m} \lambda_\nu u(x_i) \hat{k}(x_\nu,x_i) \sum_{j=1}^{m} \varphi_{m,j}(\gamma_i(x_\nu)) c_j \right]$$

$$= \sum_{i=1}^{m} \varphi_{m,i}(y)(gu)(x_i),$$

from which we get the following system of linear equations

$$\sum_{j=1}^{m} \left[\delta_{ij} + \mu \sum_{\nu=1}^{m} \lambda_\nu \hat{k}(x_\nu,x_i) u(x_i) \frac{\ell_{m,j}(\gamma_i(x_\nu))}{u(x_j)} \right] c_j = (gu)(x_i), \quad i = 1,2,\ldots,m \tag{18}$$

where δ_{ij} denotes the Kronecker symbol.

Denoting by $\mathbf{I}$ the identity matrix of order m, setting

$$\mathbf{c} = [c_1,\ldots,c_m]^T, \quad \mathbf{g} = [g(x_1),\ldots,g(x_m)]^T, \quad \mathbf{D} = \text{diag}[u(x_1),\ldots,u(x_m)]$$

and denoting by $\mathbf{B} \in \mathbb{R}^{m\times m}$ the matrix defined as

$$\mathbf{B}(i,j) = \mu \sum_{\nu=1}^{m} \lambda_\nu \hat{k}(x_\nu,x_i) \ell_{m,j}(\gamma_i(x_\nu)), \qquad 1 \le i,j \le m,$$

the final linear system (18) takes the matrix form

$$(\mathbf{I} + \mathbf{D}\mathbf{B}\mathbf{D}^{-1})\mathbf{c} = \mathbf{D}\mathbf{g}.$$

By solving the previous system, we determine the unique solution of equation (1), according to (17).

Remark 3.2. *Let us mention that in [16] a collocation method was also considered for equation (1) in the spaces of continuous functions. There, a smoothing transformation was first introduced in order to obtain an equivalent equation with better regularity properties. In the present work, we have not applied any regularization techniques. A different interpolation process which is optimal in the weighted spaces where we look for the solution has been considered.*

Now denote by $\mathscr{K} f$ the integral operator in (11), i.e.

$$(\mathscr{K} f)(y) := \mu \int_{-1}^{1} \hat{k}(t,y)\, f(\gamma(t,y))\, w(t)dt. \tag{19}$$

To study the stability and convergence of our method, we need some mapping properties of the sequences $\{\mathscr{K}_m\}_m$ and $\{\mathscr{K}_m^*\}$ which are stated in the next theorems.

Theorem 3.3. *Let $\mathscr{K}_m^*$ be defined in (15). If $0 \le \alpha < \gamma+1$, $0 \le \delta < \beta+1$, and for some real parameter $\lambda > 0$*

$$\sup_{|y|\le 1} \left(v^{\gamma,\alpha+\beta+1+\delta}(y) \|k_y\|_{Z_\lambda(u)} \right) < \infty,$$

then,

$$\|\mathscr{K} - \mathscr{K}_m^*\|_{Z_\lambda(u)\to C_u} \le \frac{\mathscr{C}}{m^\lambda}$$

where $\mathscr{C} \neq \mathscr{C}(m)$.

Theorem 3.4. *Let $\mathscr{K}_m^*$ and $\mathscr{K}_m$ be defined in (15) and (14), respectively. Assume that for some $\lambda > 0$*

$$\sup_{|t|\le 1} \|k_t\, v^{0,\alpha+\beta+1}\|_{Z_\lambda(u)} < \infty,$$

and the parameters γ and δ fulfill the following conditions

$$\max\left\{0, \frac{\alpha}{2}+\frac{1}{4}\right\} \le \gamma < \min\left\{\alpha+1, \frac{\alpha}{2}+\frac{5}{4}\right\},$$
$$\max\left\{0, \frac{\beta}{2}+\frac{1}{4}\right\} \le \delta < \min\left\{\beta+1, \frac{\beta}{2}+\frac{5}{4}\right\}.$$

Then,

$$\|\mathscr{K}_m - \mathscr{K}_m^*\|_{Z_\lambda(u)\to C_u} \le \mathscr{C}\, \frac{\log m}{m^\lambda}$$

where $\mathscr{C}$ is independent of m.

By the previous two results the following theorem comes down:

Theorem 3.5. *Under the assumptions of Theorem 3.3 and Theorem 3.4, then*

$$\|\mathscr{K} - \mathscr{K}_m\|_{Z_\lambda(u)\to C_u} \le \mathscr{C} \frac{\log m}{m^\lambda}$$

where $\mathscr{C} \neq \mathscr{C}(m)$.

Next Theorem concludes our study providing the conditions for the convergence and an error estimate of the error.

Theorem 3.6. *Let us assume that the conditions of Theorem 3.3 and Theorem 3.4 are satisfied. Then, for sufficiently large m (say $m > m_0$), the equation (16) has a unique solution $f_m^* \in \mathbb{P}_{m-1}$. Moreover, let f^* be the (unique) solution of (1) and f_m^* the (unique) solution of (16). If the right-hand side $g \in Z_\lambda(u)$, then the following error estimate holds*

$$\|(f^* - f_m^*)u\|_\infty = \mathscr{O}\left(\frac{\log m}{m^\lambda}\right), \tag{20}$$

where the constants implied in "$\mathscr{O}$" do not depend on m.

4. Numerical tests

In this section we present some numerical tests to illustrate the accuracy of the proposed method and to confirm the results obtained in Section 3 about the error estimate and the conditioning of the linear system.

For each example, we fix the space C_u in which look for the solution according to Theorem 3.6, and compute the approximate solution f_m, defined in (17), by solving the system (18). Then, to measure the accuracy we give the discrete weighted errors

$$\varepsilon_m = \max_{i=1,2,\dots,10^3} |(f - f_m)(y_i)u(y_i)|$$

where y_i are equally spaced points in $[-1, 1]$ and f is the exact solution of the considered equation. If f is unknown, we will consider as exact the approximated solution obtained with $m = 700$, i.e. $f \equiv f_{700}$. We will also show the condition numbers in infinity norm of the system (18) (i.e. $\mathrm{cond}(\mathbf{I} + \mathbf{DBD^{-1}})$). Moreover, we compare our results with those obtained by other procedures, providing details on these in each test.

All the computations were carried out using MATLAB R2021a in double precision running on a desktop computer with an Intel Core i7-2600 system (8 cores) under Debian GNU/Linux operating system.

Example 4.1. This equation of Abel's type can be found in [8] (see also [36])

$$f(y) + \int_{-1}^{y} f(x)(y-x)^{-1/2} dx = (1+y)^2 + \frac{16}{15}(1+y)^{5/2}.$$

Its solution is the polynomial $f(y) = (1+y)^2$ which also satisfies the equivalent equation

$$f(y) + \left(\frac{1+y}{2}\right)^{1/2} \int_{-1}^{1} f(\gamma(t,y))\,(1-t)^{-1/2} dt = (1+y)^2 + \frac{16}{15}(1+y)^{5/2},$$

where γ is defined in (10).

According to Theorem 3.4, we can study the equation in the space C_u with $u = v^{0,0}$. From a theoretical point of view, since the exact solution is a polynomial we expect a fast convergence. This is confirmed by the discrete errors reported in the second column of Table 1. In [8], Costarelli & Spigler proposed a collocation method approximating the solution by using different sigmoidal functions and allowing a final linear system of order m. In order to compare our results with theirs, we report, in the fourth and fifth column of Table 1, the relative errors obtained by our method, say it $\widetilde{\varepsilon}_m^{our}$, and those given in [8, Table 3], say it $\widetilde{\varepsilon}_m^{CS}$. About the latter, we have chosen among the proposed three families of sigmoidal functions, that one gives the best results. By "—" we will mean that the machine precision is attained. Moreover, as in [8] the results are only available for $m \geq 10$, in Table 1 we will write "n.a." meaning that the error is not available for the corresponding choice of m.

TABLE 1. Numerical results for Example 4.1

m	ε_m	$\text{cond}(\mathbf{I}+\mathbf{DBD^{-1}})$	$\widetilde{\varepsilon}_m^{our}$	$\widetilde{\varepsilon}_m^{CS}$
4	1.33e-15	3.61e+00	3.68e-15	n.a.
8	2.22e-15	4.62e+00	1.17e-14	n.a.
10	—	—	5.46e-15	9.53e-02
20	—	—	5.12e-15	4.61e-02
30	—	—	—	3.56e-02
40	—	—	—	2.32e-02
50	—	—	—	1.00e-02
500	—	—	—	1.10e-03
1000	—	—	—	5.42e-04

Example 4.2. Let us consider the following integral equation [21]

$$f(y)+\int_{-1}^{y} f(x)\,(y-x)^{-1/2}\,dx=\sqrt{1+y}+\frac{\pi}{2}(1+y).$$

or equivalently

$$f(y)+\left(\frac{1+y}{2}\right)^{1/2}\int_{-1}^{1} f(\gamma(t,y))\,(1-t)^{-1/2}dt=\sqrt{1+y}+\frac{\pi}{2}(1+y),$$

whose exact solution is $f(y)=\sqrt{1+y}$. With $u=v^{0.6,0.9}$, $f\in Z_{2.8}(u)$, the weighted errors behave like $\mathscr{O}\left(\frac{\log m}{m^{2.8}}\right)$ as m increases. Also in this case, the numerical errors reported in the second column of Table 2 agree with the theoretical expectation of convergence speed. In [21] Huang & Wang apply two different approaches for equations of the type (1) in the interval $[0,1]$, with $\beta=0$. They propose an interpolation postprocessing technique to the collocation solution under graded mesh, and an hybrid collocation solution under "looser" graded mesh.

Here we want just to highlight the gap between our and their methods, in terms of CPU time necessary to obtain p exact digits. We will refer to the data made available by the authors [21, Table 2]. The results of both the procedures are given in the fifth and sixth column of Table 2 and also graphically synthesized in Figure 1.

In [16], we proposed a projection method based on Lagrange interpolation operator with "additional" nodes, studied in un-weighted spaces of functions. For its nature, that procedure needs the "aid" of the following smoothing transformation

$$\phi_q(z)=2^{1-q}(1+z)^q-1,\qquad q\in\mathbb{N}, \tag{21}$$

widely adopted in numerical methods for solving either Volterra and Fredholm integral equations (see e.g. [30, 18, 32, 34]). Now, in the following two examples we want to compare the results obtained in

TABLE 2. Numerical results for Example 4.2

m	ε_m	$\text{cond}(\mathbf{I}+\mathbf{DBD}^{-1})$	p	CPU in our method	CPU in H & W method
4	8.92e-03	5.01e+00	5	0.0157s	33.3800s
8	1.20e-03	7.67e+00	6	0.0920s	68.2800s
16	1.57e-04	1.10e+01	7	0.5954s	136.8700s
32	1.73e-05	1.48e+01	8	7.9899s	276.5300s
64	1.74e-06	1.89e+01			
128	1.60e-07	2.32e+01			
256	1.47e-08	2.78e+01			
512	1.42e-09	3.26e+01			

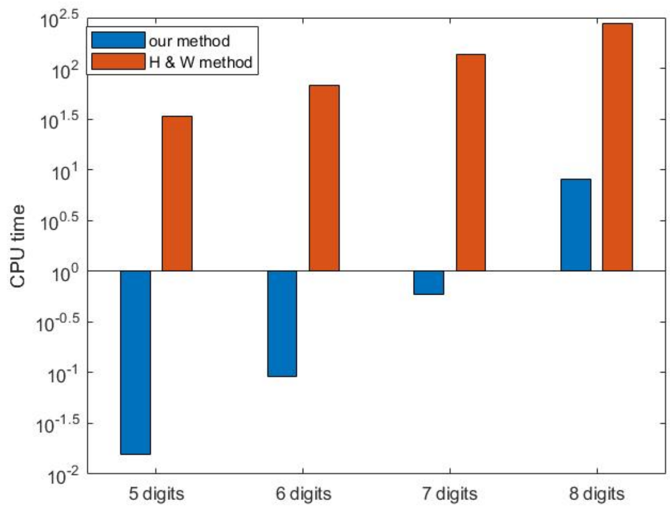

FIGURE 1. CPU time results

[16] and requiring the smoothing transformation (21), with those achieved by the proposed procedure here which takes advantage of the weighted approximation.

Example 4.3. Consider the following equation [16, Ex. 4.3] in C_u, $u = v^{\frac{1}{5},\frac{4}{5}}$

$$f(y) + \int_{-1}^{y} \frac{x}{2+y^2} f(x)\,(y-x)^{-3/4}\,dx = e^{|y|^{9/4}} \cos y, \tag{22}$$

that is

$$f(y) + \left(\frac{1+y}{2}\right)^{1/4} \int_{-1}^{1} \frac{\gamma(t,y)}{2+y^2} f(\gamma(t,y))\,(1-t)^{-3/4}\,dt = e^{|y|^{9/4}} \cos y,$$

where γ is defined in (10).

The right-hand side $g \in Z_{9/4}(u)$, and $k(x,y)$ is a smooth kernel. Therefore, according to Theorem 3.6, the expected order of convergence is $\frac{\log m}{m^{9/4}}$. In Table 3, we give the errors and the condition numbers by the present method and those obtained by the method in [16] with $q = 4$, denoting them ε_m^{old} and cond^{old}. We can immediately note that the proposed method furnishes equivalent results without using smoothing transformations.

TABLE 3. Numerical results for Example 4.3

m	ε_m	$\text{cond}(\mathbf{I}+\mathbf{DBD}^{-1})$	ε_m^{old}	cond^{old}
4	9.68e-01	2.62e+01	1.57e+00	1.32e+01
8	4.96e-02	4.11e+01	3.17e-01	1.51e+01
16	5.60e-03	5.30e+01	1.07e-02	1.37e+01
32	8.90e-04	6.30e+01	4.63e-04	1.35e+01
64	1.47e-04	7.10e+01	6.78e-05	1.31e+01
128	2.46e-05	7.74e+01	1.95e-05	1.30e+01
256	3.91e-06	8.22e+01	1.69e-06	1.28e+01
512	4.33e-07	8.59e+01	2.84e-07	1.27e+01

Example 4.4. Next equation is in [16, Ex.4.4]

$$f(y) + \int_{-1}^{y} |\sin(x+y)|^{\frac{8}{3}} f(x)\,(y-x)^{\frac{1}{5}}\,(1+x)^{\frac{2}{5}} dx = \arctan(1+y^2),$$

or equivalently

$$f(y) + \left(\frac{1+y}{2}\right)^{\frac{8}{5}} \int_{-1}^{1} \left|\sin\left(y\left(\frac{3+t}{2}\right) + \left(\frac{t-1}{2}\right)\right)\right|^{\frac{8}{3}} f(\gamma(t,y))\, v^{\frac{1}{5},\frac{2}{5}}(t)dt = \arctan(1+y^2).$$

According to the assumptions given in Theorem 3.4, we look for the solution in the space C_u, $u(x) = (1-x^2)$. In this case, the equation presents a smooth right-hand side and a kernel such that $\hat{k}_t(y) \in Z_{13/5}(u)$ and $\hat{k}_y(t) \in Z_{8/3}(u)$. Therefore, we expect that the approximate solution converges to the exact one in C_u with a theoretical error of the order $\mathscr{O}\left(\frac{\log m}{m^{-13/5}}\right)$. Table 4 contains our results and those obtained by the method in [16] for $q = 3$. If we compare them with the ones given in [16, Table 5] we can state once again that the proposed approach gives accurate results without the use of a smoothing transformation.

5. Proofs

In order to prove Theorem 3.1, we need the following Lemmas

Lemma 5.1. *Let $\mathscr{K}$ be defined in (19). Assume that $\gamma < 1+\alpha$, $\delta < 1+\beta$ and*

$$\sup_{(t,y)\in[-1,1]^2} \left(v^{\gamma,\alpha+\beta+1}(y)|k(\gamma(t,y),y)|\right) < \infty. \tag{23}$$

Then,

$$\|(\mathscr{K}f)u\|_\infty \le \mathscr{C}\|fu\|_\infty, \quad \mathscr{C} \neq \mathscr{C}(f).$$

TABLE 4. Numerical results for Example 4.4

m	ε_m	$\mathrm{cond}(\mathbf{I}+\mathbf{DBD}^{-1})$	ε_m^{old}	cond^{old}
4	3.48e-02	1.78e+00	1.80e-01	3.62e+00
8	2.77e-03	2.03e+00	2.66e-02	3.65e+00
16	1.43e-04	2.34e+00	2.39e-03	3.73e+00
32	8.52e-06	2.47e+00	9.68e-05	3.77e+00
64	5.74e-07	2.56e+00	4.45e-06	3.79e+00
128	3.92e-08	2.60e+00	3.53e-07	3.79e+00
256	2.67e-09	2.63e+00	2.81e-08	3.79e+00
512	1.75e-10	2.64e+00	4.29e-10	3.79e+00

Proof. By (19) and (12), for any $y \in [-1,1]$ we have

$$|(\mathscr{K}f)(y)u(y)| \leq \mathscr{C}\|fu\|_\infty v^{\gamma,\alpha+\beta+1+\delta}(y) \int_{-1}^{1} |k(\gamma(t,y),y)| \frac{w(t)}{u(\gamma(t,y))} dt.$$

Hence, noting that

$$\int_{-1}^{1} \frac{w(t)}{u(\gamma(t,y))} dt = 2^\delta (1+y)^{-\delta} \int_{-1}^{1} (1+t)^{\beta-\delta} (1-t)^\alpha \left(\frac{3-t}{2} - y\left(\frac{1+t}{2}\right)\right)^{-\gamma} dt, \tag{24}$$

we have

$$\begin{aligned} |(\mathscr{K}f)(y)u(y)| &\leq \mathscr{C}\|fu\|_\infty \sup_{(t,y)\in[-1,1]^2} \left(v^{\gamma,\alpha+\beta+1}(y)|k(\gamma(t,y),y)|\right) \int_{-1}^{1} \frac{w(t)}{u(t)} dt \\ &\leq \mathscr{C}\|fu\|_\infty, \end{aligned}$$

where the last bound follows if $\alpha-\gamma > -1$, $\beta-\delta > -1$ and under the hypothesis in (23). Thus, we can conclude

$$\|(\mathscr{K}f)u\| \leq \mathscr{C}\|fu\|_\infty.$$

□

Lemma 5.2. *Assuming that*

$$\sup_{t\in[-1,1]} \|k_t v^{0,\alpha+\beta+1}\|_{Z_\lambda(u)} < \infty, \tag{25}$$

it is

$$\lim_m \left(\sup_{f\in Z_\lambda(u)} E_m(\mathscr{K}f)_u \right) = 0.$$

Proof. Let us prove that for any $f \in Z_\lambda(u)$,

$$\Omega_\varphi^r(\mathscr{K}f,t)_u \leq \mathscr{C}t^\lambda \|f\|_{Z_\lambda(u)}. \tag{26}$$

By (19) we have

$$|\Delta_{h\varphi}^r(\mathscr{K}f)(y)u(y)| = \mu\, u(y) \left|\Delta_{h\varphi(y)}^r \left(\int_{-1}^{1} f(\gamma(t,y))\hat{k}(t,y)v^{\alpha,\beta}(t)dt\right)\right|$$

Therefore, taking the supremum on $y \in I_{rh} = [-1+(2rh)^2, 1-(2rh)^2]$ and on $0 < h \le \tau$ we get

$$\Omega_\varphi^r(\mathscr{K} f, \tau)_u \le \mathscr{C} \int_{-1}^{1} \Omega_\varphi^r(f_t \hat{k}_t) v^{\alpha,\beta}(t) dt$$

from which we get

$$\Omega_\varphi^r(\mathscr{K} f, \tau)_u \le \mathscr{C} \sup_{|x|\le 1} \Omega_\varphi^r\left(\hat{k}_t f_t, \tau\right)_u .$$

Now, by the equivalence (3), we get

$$\Omega_\varphi^r(\mathscr{K} f, \tau)_u \le \mathscr{C} \tau^\lambda \sup_{|x|\le 1} \left(\sup_{i\ge 0} (1+i)^\lambda E_i \left(\hat{k}_t f_t\right)_u \right). \tag{27}$$

Consequently, recalling that for any $f_1, f_2 \in C_u$ it is [28, p.384]

$$E_{2m}(f_1 f_2)_u \le \mathscr{C} \left(\|f_1 u\|_\infty E_m(f_2)_u + \|f_2 u\|_\infty E_m(f_1)_u \right), \tag{28}$$

by using (25) and (5) we have, for any $f \in Z_\lambda(u)$

$$E_i\left(\hat{k}_t f_t\right)_u \le \mathscr{C} \left[\|\hat{k}_t u\|_\infty E_{[i/2]}(f)_u + \|f u\|_\infty E_{[i/2]}(\hat{k}_t)_u \right] \le \frac{\mathscr{C}}{i^\lambda} \|f\|_{Z_\lambda(u)}$$

and then by (27) we deduce (26). Now, by (4) we have

$$E_m(\mathscr{K} f)_u \le \mathscr{C} \int_0^{1/m} \frac{\Omega_\varphi^r(\mathscr{K} f, \tau)_u}{\tau} d\tau \le \frac{\mathscr{C}}{m^\lambda} \|f\|_{Z_\lambda(u)},$$

that is

$$\lim_m \left(\sup_{\|f\|_{Z_\lambda(u)}=1} E_m(\mathscr{K} f)_u \right) = 0.$$

□

Proof of Theorem 3.1. First, let us note that if $g \in Z_\lambda(u)$, under the assumption (13), the unknown function f of equation (11) is at least in $Z_\lambda(u)$.

Now, by Lemma 5.1 and Lemma 5.2 it follows that $\mathscr{K} : Z_\lambda(u) \to C_u$ is compact (see e.g. [35, p.93]). Therefore, by the Theorem of the Fredholm Alternative, equation (1) admits a unique solution in $Z_\lambda(u)$. □

Proof of Theorem 3.3. By the definitions of $\mathscr{K}$ and $\mathscr{K}_m^*$ given in (19) and (15), respectively, we have (see, e.g. [18])

$$\begin{aligned} |[(\mathscr{K} f)(y) - (\mathscr{K}_m^* f)(y)]u(y)| &\le \mu\, u(y) \int_{-1}^{1} |\hat{k}(t,y) f(\gamma(t,y)) - \mathscr{L}_m^w(\hat{k}_y f(\gamma_y), t)|\, w(t) dt \\ &\le \mathscr{C}\, u(y)\, E_{2m-1}(\hat{k}_y f(\gamma_y))_u \int_{-1}^{1} \frac{w(t)}{u(t)} dt. \end{aligned}$$

Hence, taking into account (24) and by applying (28), we can deduce that

$$|[(\mathscr{K} f)(y) - (\mathscr{K}_m^* f)(y)]u(y)| \le \mathscr{C} v^{\gamma,\delta}(y) \left[\|\hat{k}_y u\|_\infty E_{[\frac{2m-1}{2}]}(f(\gamma_y))_u + \|f u\|_\infty E_{[\frac{2m-1}{2}]}(\hat{k}_y)_u \right].$$

Thus, taking the supremum on y by the hypothesis on the kernel we get

$$\|[(\mathscr{K} f)-(\mathscr{K}_m^* f)]u\| \leq \frac{\mathscr{C}}{m^\lambda}\|f\|_{Z_\lambda(u)}.$$

□

Proof of Theorem 3.4. Under the assumptions (8), we get

$$\|(\mathscr{K}_m^*-\mathscr{K}_m)fu\|_\infty \leq \mathscr{C}\log m\, E_{m-1}(\mathscr{K}_m^* f)_u, \tag{29}$$

and by (4)

$$E_{m-1}(\mathscr{K}_m^* f)_u \leq \mathscr{C}\int_0^{\frac{1}{m}} \frac{\Omega_\varphi^r(\mathscr{K}_m^* f,\tau)_u}{\tau}d\tau. \tag{30}$$

Let us prove that for any $f \in Z_\lambda(u)$,

$$\Omega_\varphi^r(\mathscr{K}_m^* f,\tau)_u \leq \mathscr{C}\tau^\lambda\|f\|_{Z_\lambda(u)}. \tag{31}$$

We can assert

$$|\Delta_{h\varphi}^r(\mathscr{K}_m^* f)(y)u(y)| \leq \mathscr{C}u(y)\sum_{\nu=1}^m \lambda_\nu \left|\Delta_{h\varphi(y)}^r\left(f(\gamma(x_\nu,y))\hat{k}(\gamma(x_\nu,y))\right)\right|$$

namely, taking the supremum on $y \in I_{rh} = [-1+(2rh)^2, 1-(2rh)^2]$ and on $0<h\leq\tau$. Therefore

$$\begin{aligned}\Omega_\varphi^r(\mathscr{K}_m^* f,\tau)_u &\leq \mathscr{C}\sum_{\nu=1}^m \lambda_\nu \Omega_\varphi^r\left(\hat{k}_{x_\nu} f(\gamma(x_\nu)),\tau\right)_u\\ &\leq \mathscr{C}\sup_{1\leq\nu\leq m}\Omega_\varphi^r\left(\hat{k}_{x_\nu} f_{x_\nu},\tau)\right)_u\int_{-1}^1 w(t)dt.\end{aligned}$$

Now, recalling the equivalence (3), and using arguments similar to those employed in proving (26), we have at least

$$\Omega_\varphi^r(\mathscr{K}_m^* f,\tau)_u \leq \mathscr{C}\,\tau^\lambda\,\|f\|_{Z_\lambda(u)},$$

i.e. (31) follows. By combining (30) and (29) we get

$$\|(\mathscr{K}_m^*-\mathscr{K}_m)fu\|_\infty \leq \mathscr{C}\frac{\log m}{m^\lambda}\|f\|_{Z_\lambda(u)},$$

and therefore the theorem is proven. □

Proof of Theorem 3.6 We omit the proof of the first part, since it follows from a theorem in [2], pag. 55. Moreover, by (11) and (16), we have

$$\|(f^*-f_m^*)u\|_\infty \leq \mathscr{C}\left[\|(g-g_m)u\|_\infty + \|(\mathscr{K}-\mathscr{K}_m)f^*u\|_\infty\right]$$

from which in view of (9) and by Theorem 3.5, we get

$$\|(f^*-f_m^*)u\|_\infty \leq \mathscr{C}\frac{\log m}{m^\lambda}\left[\|g\|_{Z_\lambda(u)}+\|f^*\|_{Z_\lambda(u)}\right]$$

i.e estimate (20). □

6. Conclusions

A numerical scheme based on Lagrange projection has been proposed to approximate the solution of VIEs of the type of (1). The approximate solution has been obtained by solving a well-conditioned linear system, and the stability and the convergence of the method have been proved, providing estimates of the errors in the weighted uniform norm. This new approach allows us to take into account the singularity of the derivative of the solution, without using any regularization techniques. Moreover, we have shown by some experimental results that our procedure reaches the same accuracy as the one of the method in [16], where a smoothing transformation was used.

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References

[1] S. Allaei, T. Diogo, and M. Rebelo. The Jacobi collocation method for a class of nonlinear Volterra integral equations with weakly singular kernel. *J. Sci. Comput.*, 69:673–695, 2016.

[2] K.E. Atkinson. *The Numerical Solution of Integral Equations of the second kind*. Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 1997.

[3] P. Baratella and P. Orsi. A new approach to the numerical solution of weakly singular Volterra integral equations. *Journal of Computational and Applied Mathematics*, 163:401–418, 2004.

[4] H. Brunner. *Collocation methods for Volterra integral and related functional differential equations*. Cambridge University Press, Cambridge, 2004.

[5] M. R. Capobianco, G. Criscuolo, P. Junghanns, and U. Luther. Uniform convergence of the collocation method for Prandtl's integro-differential equation. *The Anziam Journal*, 42:151–168, 2000.

[6] M.R. Capobianco, P. Junghanns, U. Luther, and G. Mastroianni. Weighted uniform convergence of the quadrature method for Cauchy singular integral equations. In A. Böttcher and I. Gohberg, editors, *Singular Integral Operators and Related Topics*, pages 153–181, Basel, 1996. Birkhäuser Basel.

[7] M.R. Capobianco and M.G. Russo. Uniform convergence estimates for a collocation method for the Cauchy singular integral equation. *Jour. of Integr. Eq. and Appl.*, 9(1):21–44, 1997.

[8] D. Costarelli and R. Spigler. Solving Volterra integral equations of the second kind by sigmoidal functions approximation. *Journal of Integral Equations and Applications*, 25(2):193–222, 2013.

[9] M.C. De Bonis and G. Mastroianni. Projection methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations. *SIAM J. Numer. Anal.*, 44(4):1351–1374, 2006.

[10] M.C. De Bonis and G. Mastroianni. Direct methods for CSIE in weighted Zygmund spaces with uniform norm. *Riv. Math. Univ. Parma*, 2:29–55, 2011.

[11] M.C. De Bonis and D. Occorsio. Quadrature methods for integro-differential equations of Prandtl's type in weighted spaces of continuous functions. *Applied Mathematics and Computation*, 393:art.125721, 2021.

[12] M.C. De Bonis, D. Occorsio, and W. Themistoclakis. Filtered interpolation for solving Prandtl's integro-differential equations. *Numerical Algorithms*, 88:679–709, 2021.

[13] Z. Ditzian and W. Totik. *Moduli of smoothness*. SCMG Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo, 1987.

[14] L. Fermo and C. Laurita. A Nyström method for a boundary integral equation related to the Dirichlet problem on domains with corners. *Numerische Mathematik*, 130(1):35–71, 2015.

[15] L. Fermo and C. Laurita. A Nyström method for mixed boundary value problems in domains with corners. *Applied Numerical Mathematics*, 149:65–82, 2020.

[16] L. Fermo and D. Occorsio. A projection method with smoothing transformation for second kind Volterra integral equations. *Dolomites Research Notes on Approximation*, 14:12–26, 2021.

[17] L. Fermo and D. Occorsio. Weakly singular linear Volterra integral equations: A Nystrm method in weighted spaces of continuous functions. *Journal of Computational and Applied Mathematics*, 406, 2022.

[18] L. Fermo and M.G. Russo. Numerical methods for Fredholm integral equations with singular right-hand sides. *Adv. Comput. Math.*, 33:305–330, 2010.

[19] Mastroianni G. and Russo M. G. Lagrange Interpolation in some weighted uniform spaces. *Facta Univ. Ser. Math. Inform.*, 12:185–201, 1997.

[20] H. Guo, H. Cai, and X. Zhang. A Jacobi-collocation method for second kind Volterra integral equations with a smooth kernel. *Abstract and Applied Analysis*, 7:1–10, 2014.

[21] Q. Huang and M. Wang. Superconvergence of interpolated collocation solutions for weakly singular volterra integral equations of the second kind. *Computational and Applied Mathematics*, 40(3), 2021.

[22] P. Junghanns and U. Luther. Cauchy singular integral equations in spaces of continuous functions and methods for their numerical solution. *J. of Comput. and Appl. Math.*, 77:201–237, 1997.

[23] M. Kolk and A. Pedas. Numerical solution of Volterra integral equations with weakly singular kernels which may have a boundary singularity. *Mathematical Modelling and Analysis*, 14(1):79–89, 2009.

[24] M. Kolk and A. Pedas. Numerical solution of Volterra integral equations with singularities. *Front. Math. China*, 8:239–259, 2013.

[25] M. Kolk, A. Pedas, and G. Vainikko. High-order methods for Volterra integral equations with general weak singularities. *Numerical Functional Analysis and Optimization*, 30(9-10):1002–1024, 2009.

[26] C. Laurita. A new stable numerical method for Mellin integral equations in weighted spaces with uniform norm. *Calcolo*, 57(3), 2020.

[27] X. Li, T. Tang, and C. Xu. Numerical solutions for weakly singular Volterra integral equations using Chebyshev and Legendre pseudo-spectral Galerkin methods. *J. Sci. Comput.*, 67:43–64, 2016.

[28] G. Mastroianni and G. V. Milovanovic. *Interpolation Processes Basic Theory and Applications*. Springer Monographs in Mathematics. Springer Verlag, Berlin, 2009.

[29] G. Mastroianni, M.G. Russo, and W. Themistoclakis. *Numerical Methods for Cauchy Singular Integral Equations in Spaces of Weighted Continuous Functions*, pages 311–336. Birkhäuser Basel, Basel, 2006.

[30] G. Monegato and L. Scuderi. High order methods for weakly singular integral equations with nonsmooth input functions. *Math. Comput.*, 224(67):1493–1515, 1989.

[31] D. Occorsio and M.G. Russo. A mixed collocation scheme for solving second kind Fredholm integral equations in $[-1,1]$. *Electronic Transactions on Numerical Analysis (ETNA)*, to appear, 2021.

[32] A. Pedas and G. Vainikko. Smoothing transformation and piecewise polynomial collocation for weakly singular Volterra integral equations. *Computing*, 73:271–293, 2004.

[33] A. Pedas and G. Vainikko. Integral equations with diagonal and boundary singularities of the kernel. *Zeitschrift fur Analysis und ihre Anwendung*, 25(4):487–516, 2006.

[34] L. Scuderi. A collocation method for the generalized airfoil equation for an airfol with a flap. *SIAM J. Numer. Anal.*, 35:1725–1739, 1998.

[35] A. F. Timan. *Theory of Approximation of functions of real variable*. Dover, New York, 1994.

[36] S.A. Yousefi. Numerical solution of Abel's integral equation by using Legendre wavelets. *Applied Mathematics and Computation*, 175(1):574–580, 2006.

CENTER FOR COMPUTATIONAL AND STOCHASTIC MATHEMATICS (CEMAT-IST/ULISBOA), DEPARTMENT OF MATHEMATICS, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBON, PORTUGAL
E-mail address: tdiogo@math.ist.utl.pt

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF CAGLIARI, VIA OSPEDALE 72, 09124 CAGLIARI, ITALY
E-mail address: fermo@unica.it

DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE AND ECONOMICS,, UNIVERSITY OF BASILICATA, VIA DELL'ATENEO LUCANO 10, 85100 POTENZA, ITALY AND ISTITUTO PER LE APPLICAZIONI DEL CALCOLO "MAURO PICONE", NAPLES BRANCH, NATIONAL RESEARCH COUNCIL (C.N.R.) OF ITALY, VIA P. CASTELLINO, 111, 80131 NAPOLI, ITALY.
E-mail address: donatella.occorsio@unibas.it