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TOWARDS COERCIVE BOUNDARY ELEMENT METHODS FOR THE WAVE EQUATION

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ABSTRACT. In this note, we discuss the ellipticity of the single layer boundary integral operator for the wave equation in one space dimension. This result not only generalizes the well-known ellipticity of the energetic boundary integral formulation in L^2, but it also turns out to be a particular case of a recent result on the inf-sup stability of boundary integral operators for the wave equation. Instead of the time derivative in the energetic formulation, we use a modified Hilbert transformation, which allows us to stay in Sobolev spaces of the same order. This results in the applicability of standard boundary element error estimates, which are confirmed by numerical results.

1. Introduction

Time-domain boundary integral equations and boundary element methods for the wave equation are well established in the literature; we mention the groundbraking works of Bamberger and Ha Duong [2], Aimi et al. [1], and the review article [4] by Costabel and Sayas. Other works include [6, 8, 9, 11, 12, 13], to mention a few.

The main difficulties in the numerical analysis of these formulations are in the so-called norm gap, coming from continuity and coercivity estimates in different space-time Sobolev norms. When using the energetic boundary element method, a complete stability and error analysis can be done in $L^2(\Sigma)$, see [9], where Σ is the lateral boundary of the space-time domain $Q := \Omega \times (0,T)$.

Using a generalized inf-sup stable variational formulation [20] for the wave equation, in [16] we derived inf-sup stability conditions for all boundary integral operators in related trace spaces. In fact, this work was motivated by our previous result [21] on the spatially one-dimensional case. When replacing the time derivative in the energetic boundary integral formulation by a modified Hilbert transformation [18], the resulting composition with the single layer boundary integral operator becomes elliptic in the natural energy space $[H^{1/2}_{,0}(\Sigma)]'$, similarly to what is known for boundary integral operators for second-order elliptic partial differential equations. Note that $H^{1/2}_{,0}(\Sigma) := [H^1_{,0}(\Sigma), L^2(\Sigma)]_{1/2}$ is defined by interpolation, see, e.g., [10], with $H^1_{,0}(\Sigma) = \{v \in H^1(\Sigma) : v(T) = 0\}$. Analogously, $H^1_{0,}(\Sigma)$ covers zero initial conditions, i.e., $v(0) = 0$.

In this paper, we present a detailed derivation of this new approach, and we discuss the corresponding numerical analysis of a related new boundary element method. In Section 2, we recall the energetic space-time boundary integral formulation [1, 9], and we provide a simplified proof of the ellipticity

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result in $L^2(\Sigma)$. In particular, we obtain that the single layer boundary integral operator $V : L^2(\Sigma) \to H^1_{0,}(\Sigma)$ is an isomorphism. Using duality arguments, we obtain that $V : [H^1_{,0}(\Sigma)]' \to L^2(\Sigma)$ is also an isomorphism. Finally, by an interpolation argument, we conclude that $V : [H^{1/2}_{,0}(\Sigma)]' \to H^{1/2}_{0,}(\Sigma)$ is an isomorphism as well. While this implies an inf-sup stability estimate, as also discussed in [16], in Section 3 we introduce a modified Hilbert transformation $\mathscr{H}_T : H^{1/2}_{0,}(\Sigma) \to H^{1/2}_{,0}(\Sigma)$, see [18], to establish ellipticity of $\mathscr{H}_T V$ in $[H^{1/2}_{,0}(\Sigma)]'$ in Section 4. Although the main result, as given in Lemma 4.1, still involves some unknown constant, Proposition 4.2 gives numerical evidence on the behavior of the ellipticity constant, which agrees with the constant known from the energetic formulation. In Section 5, we present some numerical results which confirm the a priori error estimates, as given in Section 4. In Section 6, we finally draw some conclusions for future work.

2. Energetic space-time boundary integral equation

As in [1], we consider the Dirichlet boundary value problem for the homogeneous wave equation in the one-dimensional spatial domain $\Omega = (0,L)$ with zero initial conditions, and for a given time horizon $T > 0$,

$$\begin{cases} \partial_{tt} u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } (x,t) \in Q := (0,L) \times (0,T), \\ u(x,0) = \partial_t u(x,t)_{|t=0} = 0 & \text{for } x \in (0,L), \\ u(0,t) = g_0(t) & \text{for } t \in (0,T), \\ u(L,t) = g_L(t) & \text{for } t \in (0,T). \end{cases} \tag{2.1}$$

In the one-dimensional case, the fundamental solution of the wave equation is the Heaviside function

$$U^*(x,t) = \frac{1}{2} H(t - |x|),$$

and we can represent the solution u of (2.1) by using the single layer potential

$$u(x,t) = (\widetilde{V} w)(x,t) = \frac{1}{2} \int_0^{t-|x|} w_0(s)\, ds + \frac{1}{2} \int_0^{t-|x-L|} w_L(s)\, ds \quad \text{for } (x,t) \in Q$$

with the density functions $w = (w_0, w_L)$. Note that for any function $z\colon (0,T) \to \mathbb{R}$, we set $z(t) = 0$ for $t < 0$ or $t > T$ in the remainder of this work. To determine the yet unknown density functions (w_0, w_L), we consider the boundary integral equations for $x \to 0$,

$$(V_0 w)(t) := \frac{1}{2} \int_0^t w_0(s)\, ds + \frac{1}{2} \int_0^{t-L} w_L(s)\, ds = g_0(t) \quad \text{for } t \in (0,T), \tag{2.2}$$

and for $x \to L$,

$$(V_L w)(t) := \frac{1}{2} \int_0^{t-L} w_0(s)\, ds + \frac{1}{2} \int_0^t w_L(s)\, ds = g_L(t) \quad \text{for } t \in (0,T). \tag{2.3}$$

We write the boundary integral equations (2.2) and (2.3) in compact form, for $w = (w_0, w_L)$, as

$$(Vw)(t) = \begin{pmatrix} (V_0 w)(t) \\ (V_L w)(t) \end{pmatrix} = \begin{pmatrix} V_{00} & V_{0L} \\ V_{L0} & V_{LL} \end{pmatrix} \begin{pmatrix} w_0 \\ w_L \end{pmatrix} (t) = \begin{pmatrix} g_0(t) \\ g_L(t) \end{pmatrix} = g(t), \quad t \in (0,T). \tag{2.4}$$

In the energetic boundary element method [1], instead of (2.4), the time derivative of (2.4) is considered,

$$\partial_t (Vw)(t) = \partial_t g(t) \quad \text{for } t \in (0,T). \tag{2.5}$$

We introduce the related energetic bilinear form

$$\begin{aligned} a(w,v) &:= \langle v, \partial_t V w\rangle_{L^2(\Sigma)} \\ &= \frac{1}{2}\int_0^T v_0(t)\frac{d}{dt}\int_0^t w_0(s)\,ds\,dt + \frac{1}{2}\int_0^T v_0(t)\frac{d}{dt}\int_0^{t-L} w_L(s)\,ds\,dt \\ &\quad + \frac{1}{2}\int_0^T v_L(t)\frac{d}{dt}\int_0^{t-L} w_0(s)\,ds\,dt + \frac{1}{2}\int_0^T v_L(t)\frac{d}{dt}\int_0^t w_L(s)\,ds\,dt \\ &= \frac{1}{2}\int_0^T v_0(t)\,w_0(t)\,dt + \frac{1}{2}\int_0^T v_0(t)\,w_L(t-L)\,dt \\ &\quad + \frac{1}{2}\int_0^T v_L(t)\,w_0(t-L)\,dt + \frac{1}{2}\int_0^T v_L(t)\,w_L(t)\,dt\,. \end{aligned}$$

When using both the Cauchy–Schwarz and Hölder inequality, we conclude

$$\begin{aligned} |a(w,v)| &\le \frac{1}{2}\|v_0\|_{L^2(0,T)}\|w_0\|_{L^2(0,T)} + \frac{1}{2}\|v_0\|_{L^2(0,T)}\|w_L\|_{L^2(0,T-L)} \\ &\quad + \frac{1}{2}\|v_L\|_{L^2(0,T)}\|w_0\|_{L^2(0,T-L)} + \frac{1}{2}\|v_L\|_{L^2(0,T)}\|w_L\|_{L^2(0,T)} \\ &\le \frac{1}{2}\|v_0\|_{L^2(0,T)}\Big[\|w_0\|_{L^2(0,T)} + \|w_L\|_{L^2(0,T)}\Big] \\ &\quad + \frac{1}{2}\|v_L\|_{L^2(0,T)}\Big[\|w_0\|_{L^2(0,T)} + \|w_L\|_{L^2(0,T)}\Big] \\ &= \frac{1}{2}\Big[\|v_0\|_{L^2(0,T)} + \|v_L\|_{L^2(0,T)}\Big]\Big[\|w_0\|_{L^2(0,T)} + \|w_L\|_{L^2(0,T)}\Big] \\ &\le \sqrt{\|v_0\|^2_{L^2(0,T)} + \|v_L\|^2_{L^2(0,T)}}\sqrt{\|w_0\|^2_{L^2(0,T)} + \|w_L\|^2_{L^2(0,T)}} = \|v\|_{L^2(\Sigma)}\|w\|_{L^2(\Sigma)} \end{aligned}$$

for all $v = (v_0, v_L), w = (w_0, w_L) \in L^2(\Sigma) := L^2(0,T) \times L^2(0,T)$, where

$$\|z\|_{L^2(\Sigma)} := \Big(\|z_0\|^2_{L^2(0,T)} + \|z_L\|^2_{L^2(0,T)}\Big)^{1/2} \qquad \text{for } z = (z_0, z_L) \in L^2(\Sigma).$$

Moreover, the energetic bilinear form $a(\cdot,\cdot)$ is also $L^2(\Sigma)$-elliptic, see [1, Theorem 2.1]. For later reference, we will give a simplified proof of this result. For this, we introduce

$$n := \min\Big\{m \in \mathbb{N} : T \le mL\Big\}, \tag{2.6}$$

which is the number of time slices $T_j := ((j-1)L, jL)$ for $j = 1,\dots,n$ in the case $T = nL$. In the case $T < nL$, we define the last time slice as $T_n := ((n-1)L, T)$, while all the others remain unchanged.

Theorem 2.1. [1, Theorem 2.1] *For all $w \in L^2(\Sigma)$, there holds the ellipticity estimate*

$$a(w,w) = \langle w, \partial_t V w\rangle_{L^2(\Sigma)} \ge \sin^2\frac{\pi}{2(n+1)}\,\|w\|^2_{L^2(\Sigma)}, \tag{2.7}$$

where the number $n \in \mathbb{N}$ of time slices is defined in (2.6).

Proof. For $w = (w_0, w_L) \in L^2(\Sigma)$, we write

$$\begin{aligned}
&2\,a(w,w)\\
&\quad= \int_0^T [w_0(t)]^2\,dt + \int_0^T w_0(t)\,w_L(t-L)\,dt + \int_0^T w_L(t)\,w_0(t-L)\,dt + \int_0^T [w_L(t)]^2\,dt\\
&\quad= \sum_{j=1}^n \left[\|w_0\|^2_{L^2(T_j)} + \int_{T_j} w_0(t) w_L(t-L)\,dt + \int_{T_j} w_L(t) w_0(t-L)\,dt + \|w_L\|^2_{L^2(T_j)}\right].
\end{aligned}$$

For $t \in T_1$, we have $t - L < 0$, and therefore $w_0(t-L) = w_L(t-L) = 0$ follows. For $j = 2, \ldots, n-1$, we have, using the Cauchy–Schwarz inequality,

$$\begin{aligned}
\int_{T_j} w_0(t) w_L(t-L)\,dt &\le \left(\int_{T_j} [w_0(t)]^2\,dt\right)^{1/2} \left(\int_{T_j} [w_L(t-L)]^2\,dt\right)^{1/2}\\
&\le \left(\int_{T_j} [w_0(t)]^2\,dt\right)^{1/2} \left(\int_{T_{j-1}} [w_L(t)]^2\,dt\right)^{1/2} = \|w_0\|_{L^2(T_j)} \|w_L\|_{L^2(T_{j-1})}.
\end{aligned}$$

Correspondingly, for $j = n$ and $T_n = ((j-1)L, T)$, $T \le nL$, we have

$$\begin{aligned}
\int_{T_n} w_0(t) w_L(t-L)\,dt &\le \left(\int_{T_n} [w_0(t)]^2\,dt\right)^{1/2} \left(\int_{(n-1)L}^T [w_L(t-L)]^2\,dt\right)^{1/2}\\
&= \left(\int_{T_n} [w_0(t)]^2\,dt\right)^{1/2} \left(\int_{(n-2)L}^{T-L} [w_L(t)]^2\,dt\right)^{1/2}\\
&\le \left(\int_{T_n} [w_0(t)]^2\,dt\right)^{1/2} \left(\int_{T_{n-1}} [w_L(t)]^2\,dt\right)^{1/2} = \|w_0\|_{L^2(T_n)} \|w_L\|_{L^2(T_{n-1})}.
\end{aligned}$$

Hence, we conclude

$$\begin{aligned}
2\,a(w,w) \geq & \sum_{j=1}^{n}\left[\|w_0\|^2_{L^2(T_j)} + \|w_L\|^2_{L^2(T_j)}\right] \\
& - \sum_{j=2}^{n}\left[\|w_0\|_{L^2(T_j)}\|w_L\|_{L^2(T_{j-1})} + \|w_L\|_{L^2(T_j)}\|w_0\|_{L^2(T_{j-1})}\right] \\
= & \left(\begin{pmatrix} 1 & -\frac{1}{2} & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \|w_0\|_{L^2(T_1)} \\ \|w_L\|_{L^2(T_2)} \\ \|w_0\|_{L^2(T_3)} \\ \|w_L\|_{L^2(T_4)} \\ \vdots \\ \|w_0\|_{L^2(T_{n-1})} \\ \|w_L\|_{L^2(T_n)} \end{pmatrix}, \begin{pmatrix} \|w_0\|_{L^2(T_1)} \\ \|w_L\|_{L^2(T_2)} \\ \|w_0\|_{L^2(T_3)} \\ \|w_L\|_{L^2(T_4)} \\ \vdots \\ \|w_0\|_{L^2(T_{n-1})} \\ \|w_L\|_{L^2(T_n)} \end{pmatrix}\right) \\
& + \left(\begin{pmatrix} 1 & -\frac{1}{2} & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \|w_L\|_{L^2(T_1)} \\ \|w_0\|_{L^2(T_2)} \\ \|w_L\|_{L^2(T_3)} \\ \|w_0\|_{L^2(T_4)} \\ \vdots \\ \|w_L\|_{L^2(T_{n-1})} \\ \|w_0\|_{L^2(T_n)} \end{pmatrix}, \begin{pmatrix} \|w_L\|_{L^2(T_1)} \\ \|w_0\|_{L^2(T_2)} \\ \|w_L\|_{L^2(T_3)} \\ \|w_0\|_{L^2(T_4)} \\ \vdots \\ \|w_L\|_{L^2(T_{n-1})} \\ \|w_0\|_{L^2(T_n)} \end{pmatrix}\right)
\end{aligned}$$

and further,

$$\begin{aligned}
a(w,w) \geq & \frac{\lambda_{\min}}{2}\left[\left(\begin{pmatrix} \|w_0\|_{L^2(T_1)} \\ \|w_L\|_{L^2(T_2)} \\ \|w_0\|_{L^2(T_3)} \\ \|w_L\|_{L^2(T_4)} \\ \vdots \\ \|w_0\|_{L^2(T_{n-1})} \\ \|w_L\|_{L^2(T_n)} \end{pmatrix}, \begin{pmatrix} \|w_0\|_{L^2(T_1)} \\ \|w_L\|_{L^2(T_2)} \\ \|w_0\|_{L^2(T_3)} \\ \|w_L\|_{L^2(T_4)} \\ \vdots \\ \|w_0\|_{L^2(T_{n-1})} \\ \|w_L\|_{L^2(T_n)} \end{pmatrix}\right) + \left(\begin{pmatrix} \|w_L\|_{L^2(T_1)} \\ \|w_0\|_{L^2(T_2)} \\ \|w_L\|_{L^2(T_3)} \\ \|w_0\|_{L^2(T_4)} \\ \vdots \\ \|w_L\|_{L^2(T_{n-1})} \\ \|w_0\|_{L^2(T_n)} \end{pmatrix}, \begin{pmatrix} \|w_L\|_{L^2(T_1)} \\ \|w_0\|_{L^2(T_2)} \\ \|w_L\|_{L^2(T_3)} \\ \|w_0\|_{L^2(T_4)} \\ \vdots \\ \|w_L\|_{L^2(T_{n-1})} \\ \|w_0\|_{L^2(T_n)} \end{pmatrix}\right)\right] \\
= & \frac{\lambda_{\min}}{2}\left[\|w_0\|^2_{L^2(0,T)} + \|w_L\|^2_{L^2(0,T)}\right],
\end{aligned}$$

where

$$\lambda_{\min} = 2\,\sin^2\frac{\pi}{2(n+1)}$$

is the minimal eigenvalue of the involved matrix, which is related to the finite difference approximation of the Laplacian in one dimension.

From the above properties, we conclude that

$$\partial_t V : L^2(\Sigma) \to L^2(\Sigma)$$

defines an isomorphism. Since the time derivative

$$\partial_t : H^1_{0,}(\Sigma) \to L^2(\Sigma)$$

is also an isomorphism, e.g., [18, Sect. 2.1], so is

$$V : L^2(\Sigma) \to H^1_{0,}(\Sigma)\,. \tag{2.8}$$

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Note that, for $u = (u_0, u_L) \in H^1_{0,}(\Sigma) := H^1_{0,}(0,T) \times H^1_{0,}(0,T)$, we have

$$\|u\|^2_{H^1_{0,}(\Sigma)} := \|\partial_t u_0\|^2_{L^2(0,T)} + \|\partial_t u_L\|^2_{L^2(0,T)}.$$

For $\partial_t : H^1_{0,}(0,T) \to L^2(0,T)$, the inverse is given by

$$u(t) = (\partial_t^{-1} f)(t) = \int_0^t f(s)\,ds, \quad t \in (0,T),$$

with $f \in L^2(0,T)$, $u \in H^1_{0,}(0,T)$. Analogously, for $\partial_t : H^1_{,0}(0,T) \to L^2(0,T)$, we find the inverse as

$$u(t) = (\overline{\partial}_t^{-1} f)(t) = -\int_t^T f(s)\,ds, \quad t \in (0,T).$$

For $w, v \in L^2(\Sigma)$ and $u = Vw = (u_0, u_L) \in H^1_{0,}(\Sigma)$, we therefore obtain

$$\langle \overline{\partial}_t^{-1} Vw, v\rangle_{L^2(\Sigma)} = -\int_0^T \int_t^T u_0(s)\,ds\, v_0(t)\,dt - \int_0^T \int_t^T u_L(s)\,ds\, v_L(t)\,dt\,.$$

For $* \in \{0, L\}$ we compute

$$\begin{aligned} -\int_0^T \int_t^T u_*(s)\,ds\, v_*(t)\,dt &= -\int_0^T \int_t^T u_*(s)\,ds\, \partial_t \int_0^t v_*(s)\,ds\,dt \\ &= -\int_t^T u_*(s)\,ds \int_0^t v_*(s)\,ds\Big|_0^T + \int_0^T \partial_t \int_t^T u_*(s)\,ds \int_0^t v_*(s)\,ds\,dt \\ &= -\int_0^T u_*(t) \int_0^t v_*(s)\,ds\,dt, \end{aligned}$$

i.e.,

$$\langle \overline{\partial}_t^{-1} Vw, v\rangle_{L^2(\Sigma)} = -\langle Vw, \partial_t^{-1} v\rangle_{L^2(\Sigma)}\,.$$

On the other hand, for $z_0 = \partial_t^{-1} w_0$ we have $w_0 = \partial_t z_0$, and hence

$$\int_0^t w_0(s)\,ds = \int_0^t \partial_s z_0(s)\,ds = z_0(t) = \partial_t \int_0^t z_0(s)\,ds,$$

i.e., $Vw_0 = V\partial_t z_0 = \partial_t V z_0$. With this, we conclude

$$\langle \overline{\partial}_t^{-1} Vw, v\rangle_{L^2(\Sigma)} = -\langle V\partial_t \partial_t^{-1} w, \partial_t^{-1} v\rangle_{L^2(\Sigma)} = -\langle \partial_t V \partial_t^{-1} w, \partial_t^{-1} v\rangle_{L^2(\Sigma)} = -a(\partial_t^{-1} w, \partial_t^{-1} v),$$

and, in particular for $v = w$, Theorem 2.1 gives

$$-\langle \overline{\partial}_t^{-1} Vw, w\rangle_{L^2(\Sigma)} = \langle \partial_t V \partial_t^{-1} w, \partial_t^{-1} w\rangle_{L^2(\Sigma)} \geq \sin^2 \frac{\pi}{2(n+1)}\, \|\partial_t^{-1} w\|^2_{L^2(\Sigma)}.$$

For $* \in \{0, L\}$, we define

$$z_*(t) = (\partial_t^{-1} w_*)(t) = \int_0^t w_*(s)\,ds, \quad t \in (0,T),$$

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to compute

$$\begin{aligned}\|\partial_t^{-1} w_*\|^2_{L^2(0,T)} = \|z_*\|^2_{L^2(0,T)} &= \int_0^T z_*(t)\, z_*(t)\, dt = -\int_0^T \partial_t \int_t^T z_*(s)\, ds\, z_*(t)\, dt \\ &= -\int_t^T z_*(s)\, ds\, z_*(t)\Big|_0^T + \int_0^T \int_t^T z_*(s)\, ds\, \partial_t z_*(t)\, dt \\ &= \int_0^T v_*(t)\, w_*(t)\, dt,\end{aligned}$$

where

$$v_*(t) = \int_t^T z_*(s)\, ds \quad \text{for } t \in (0,T), \quad \partial_t v_* = -z_*, \quad v_* \in H^1_{,0}(0,T).$$

From this, we conclude

$$\|\partial_t^{-1} w_*\|_{L^2(0,T)} = \frac{|\langle w_*, v_*\rangle_{(0,T)}|}{\|\partial_t v_*\|_{L^2(0,T)}} \le \sup_{0 \neq \phi \in H^1_{,0}(0,T)} \frac{|\langle w_*, \phi\rangle_{(0,T)}|}{\|\partial_t \phi\|_{L^2(0,T)}} = \|w_*\|_{[H^1_{,0}(0,T)]'}.$$

Indeed, we have

$$\|\partial_t^{-1} w_*\|_{L^2(0,T)} = \|w_*\|_{[H^1_{,0}(0,T)]'},$$

and therefore,

$$-\langle \overline{\partial}_t^{-1} V w, w\rangle_{L^2(\Sigma)} \ge \sin^2 \frac{\pi}{2(n+1)}\, \|w\|^2_{[H^1_{,0}(\Sigma)]'}. \tag{2.9}$$

In fact, by the density of $L^2(\Sigma)$ in $[H^1_{,0}(\Sigma)]'$, the operator

$$-\overline{\partial}_t^{-1} V : [H^1_{,0}(\Sigma)]' \to H^1_{,0}(\Sigma)$$

defines an isomorphism, and so does

$$V : [H^1_{,0}(\Sigma)]' \to L^2(\Sigma). \tag{2.10}$$

Using interpolation, see, e.g., [10], we define $H^{1/2}_{0,}(0,T) := [H^1_{0,}(0,T), L^2(0,T)]_{1/2}$, where $H^1_{0,}(0,T) := \{v \in H^1(0,T) : v(0) = 0\}$. In the same way, we define $H^{1/2}_{,0}(0,T)$ but with zero condition at the final time $t = T$. Hence we can introduce the Sobolev space $H^{1/2}_{0,}(\Sigma) = H^{1/2}_{0,}(0,T) \times H^{1/2}_{0,}(0,T)$ endowed with the Hilbertian norm

$$\|z\|_{H^{1/2}_{0,}(\Sigma)} := \left(\|z_0\|^2_{H^{1/2}_{0,}(0,T)} + \|z_L\|^2_{H^{1/2}_{0,}(0,T)} \right)^{1/2} \quad \text{for } z = (z_0, z_L) \in H^{1/2}_{0,}(\Sigma)$$

and analogously, the Sobolev space $H^{1/2}_{,0}(\Sigma)$. For the single layer boundary integral operator V, we have obtained the mapping properties (2.8) and (2.10), respectively. When applying an interpolation argument, this gives that

$$V : [H^{1/2}_{,0}(\Sigma)]' \to H^{1/2}_{0,}(\Sigma)$$

is an isomorphism as well. Hence, we conclude the inf-sup stability condition

$$c_S\, \|w\|_{[H^{1/2}_{,0}(\Sigma)]'} \le \sup_{0 \neq v \in [H^{1/2}_{0,}(\Sigma)]'} \frac{|\langle Vw, v\rangle_\Sigma|}{\|v\|_{[H^{1/2}_{0,}(\Sigma)]'}} \quad \text{for all } w \in [H^{1/2}_{,0}(\Sigma)]' \tag{2.11}$$

with a constant $c_S > 0$. In fact, (2.11) corresponds to the inf-sup condition in [16, Theorem 5.7], where the test space is slightly larger than used in (2.11). But we will show that $V : [H^{1/2}_{,0}(\Sigma)]' \to H^{1/2}_{0,}(\Sigma)$ in combination with a modified Hilbert transformation [18, 19, 22] even satisfies an ellipticity estimate similar as in (2.7).

3. A modified Hilbert transformation

For $u \in L^2(0,T)$, we consider the Fourier series

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad u_k = \frac{2}{T}\int_0^T u(t)\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)dt,$$

$$u(t) = \sum_{k=0}^{\infty} \overline{u}_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad \overline{u}_k = \frac{2}{T}\int_0^T u(t)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)dt.$$

From [18, Lemma 2.1], we have

$$\|u\|^2_{[H^{1/2}_{,0}(0,T)]'} = \frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{u}_k^2.$$

As in [18], we introduce the transformation operator $\mathscr{H}_T : L^2(0,T) \to L^2(0,T)$ as

$$\mathscr{H}_T u(t) := \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad t \in (0,T), \tag{3.1}$$

which is norm preserving and bijective. By construction, we have that the transformation operator $\mathscr{H}_T : H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ is also an isometric isomorphism, and

$$\langle \partial_t u, \mathscr{H}_T u\rangle_{(0,T)} = \|u\|^2_{H^{1/2}_{0,}(0,T)} \quad \text{for all } u \in H^{1/2}_{0,}(0,T).$$

It is easy to see that

$$|\langle \partial_t u, \mathscr{H}_T z\rangle_{(0,T)}| \le \|u\|_{H^{1/2}_{0,}(0,T)} \|z\|_{H^{1/2}_{0,}(0,T)} \quad \text{for all } u,z \in H^{1/2}_{0,}(0,T). \tag{3.2}$$

The transformation operator $\mathscr{H}_T$, as defined in (3.1), allows a closed representation, see [18, Lemma 2.8], which generalizes the well-known Hilbert transformation, e.g., [3]. Moreover, following [19, Eqn. (2.5)] we conclude the following representation for $u,z \in H^1_{0,}(0,T)$,

$$\langle \partial_t u, \mathscr{H}_T z\rangle_{(0,T)} = -\frac{1}{\pi}\int_0^T \partial_t u(t)\int_0^T \ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi|t-s|}{4T}\right]\partial_s z(s)\,ds\,dt\,.$$

This representation also allows for an efficient evaluation of the bilinear form $\langle \partial_t u, \mathscr{H}_T z\rangle_{(0,T)}$ by using hierarchical matrices, see [19] for a more detailed discussion.

4. A space-time approach in energy spaces

Instead of the boundary integral equation (2.5), we may replace the application of the time derivative by the modified Hilbert transformation $\mathscr{H}_T : H^{1/2}_{0,}(\Sigma) \to H^{1/2}_{,0}(\Sigma)$, i.e., we consider the boundary integral equation to find $w \in [H^{1/2}_{,0}(\Sigma)]'$ such that

$$\mathscr{H}_T V w = \mathscr{H}_T g \quad \text{in } [H^{1/2}_{,0}(\Sigma)]',$$

where $g \in H^{1/2}_{0,}(\Sigma)$ is a given Dirichlet datum. The related bilinear form is given as

$$a_{\mathscr{H}_T}(w,v) := \langle v, \mathscr{H}_T V w\rangle_\Sigma \quad \text{for all } v,w \in [H^{1/2}_{,0}(\Sigma)]'.$$

Recall that for $u = (u_0, u_L) \in H^{1/2}_{0,}(\Sigma)$, we have

$$\partial_t u = (\partial_t u_0, \partial_t u_L) = (v_0, v_L) =: v \in [H^{1/2}_{,0}(\Sigma)]',$$

satisfying

$$\|u\|_{H^{1/2}_{0,}(\Sigma)} = \|v\|_{[H^{1/2}_{,0}(\Sigma)]'}.$$

For $v = \partial_t u$, $w = \partial_t z$ with $u, z \in H^1_{0,}(\Sigma)$, we can write

$$\begin{aligned} a_{\mathscr{H}_T}(w,v) &= \frac{1}{2}\int_0^T v_0(t)\,\mathscr{H}_T\left(\int_0^t w_0(s)\,ds + \int_0^{t-L} w_L(s)\,ds\right)dt \\ &\quad + \frac{1}{2}\int_0^T v_L(t)\,\mathscr{H}_T\left(\int_0^{t-L} w_0(s)\,ds + \int_0^t w_L(s)\,ds\right)dt \\ &= \frac{1}{2}\left[\langle \partial_t u_0, \mathscr{H}_T(z_0 + z_L(\cdot - L))\rangle_{(0,T)} + \langle \partial_t u_L, \mathscr{H}_T(z_0(\cdot - L) + z_L)\rangle_{(0,T)}\right]. \end{aligned}$$

When using (3.2), we obtain

$$\begin{aligned} &|a_{\mathscr{H}_T}(w,v)| \\ &\quad \le \frac{1}{2}\left[\|u_0\|_{H^{1/2}_{0,}(0,T)}\|z_0 + z_L(\cdot - L)\|_{H^{1/2}_{0,}(0,T)} + \|u_L\|_{H^{1/2}_{0,}(0,T)}\|z_0(\cdot - L) + z_L\|_{H^{1/2}_{0,}(0,T)}\right] \\ &\quad \le \frac{1}{2}\left[\|u_0\|_{H^{1/2}_{0,}(0,T)} + \|u_L\|_{H^{1/2}_{0,}(0,T)}\right]\left[\|z_0\|_{H^{1/2}_{0,}(0,T)} + \|z_L\|_{H^{1/2}_{0,}(0,T)}\right] \\ &\quad \le \sqrt{\|u_0\|^2_{H^{1/2}_{0,}(0,T)} + \|u_L\|^2_{H^{1/2}_{0,}(0,T)}}\sqrt{\|z_0\|^2_{H^{1/2}_{0,}(0,T)} + \|z_L\|^2_{H^{1/2}_{0,}(0,T)}} \\ &\quad = \|u\|_{H^{1/2}_{0,}(\Sigma)}\|z\|_{H^{1/2}_{0,}(\Sigma)} \\ &\quad = \|v\|_{[H^{1/2}_{,0}(\Sigma)]'}\|w\|_{[H^{1/2}_{,0}(\Sigma)]'} \end{aligned}$$

for all $v, w \in L^2(\Sigma)$, i.e., the density of $L^2(\Sigma)$ in $[H^{1/2}_{,0}(\Sigma)]'$ yields the boundedness of the bilinear form $a_{\mathscr{H}_T}(\cdot,\cdot)$.

Lemma 4.1. *For $w \in [H^{1/2}_{,0}(\Sigma)]'$, there holds*

$$a_{\mathscr{H}_T}(w,w) = \langle \mathscr{H}_T V w, w\rangle_\Sigma \geq \frac{1}{2}\left(1 - \frac{1}{2}\sup_{m\in\mathbb{N}} \sqrt{\lambda_{\max}(C_m)}\right)\|w\|^2_{[H^{1/2}_{,0}(\Sigma)]'}, \tag{4.1}$$

where $\lambda_{\max}(C_m)$ is the maximal eigenvalue of a symmetric matrix $C_m \in \mathbb{R}^{(m+1)\times(m+1)}$. In the case of $T \leq L$, the matrix C_m is the zero matrix, i.e., $\lambda_{\max}(C_m) = 0$. However, in the case $T > L$, the matrix C_m is defined by the entries

$$c_{\ell i} = \sum_{k=0}^{\infty} b_{k\ell} b_{ki} \quad \text{for } \ell, i = 0,\dots,m,$$

$$b_{kk} = 2\left(1-\frac{L}{T}\right)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{L}{T}\right) \quad \text{for } k\in\mathbb{N}_0,$$

$$b_{k\ell} = \frac{4}{\pi}\frac{\sqrt{2k+1}\sqrt{2\ell+1}}{(k+\ell+1)(k-\ell)}\cos\left((k+\ell+1)\frac{\pi}{2}\frac{L}{T}\right)\sin\left((\ell-k)\frac{\pi}{2}\frac{L}{T}\right)$$

for $k,\ell \in \mathbb{N}_0$, $k-\ell = 2j \neq 0$, $j\in\mathbb{Z}$, and $b_{k\ell} = 0$ else.

Proof. For $w = (w_0, w_L) \in L^2(\Sigma)$, we consider the Fourier series

$$w_0(t) = \sum_{k=0}^{\infty} \overline{w}_{0,k}\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad \overline{w}_{0,k} = \frac{2}{T}\int_0^T w_0(t)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)dt,$$

$$w_L(t) = \sum_{k=0}^{\infty} \overline{w}_{L,k}\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad \overline{w}_{L,k} = \frac{2}{T}\int_0^T w_L(t)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)dt.$$

In the case $T \leq L$, we explicitly compute

$$\langle \mathscr{H}_T V w, w\rangle_{L^2(\Sigma)} = \frac{T^2}{2}\sum_{k=0}^{\infty}\frac{\overline{w}_{0,k}^2 + \overline{w}_{L,k}^2}{(2k+1)\pi} = \frac{1}{2}\left(\|w_0\|^2_{[H^{1/2}_{,0}(0,T)]'} + \|w_L\|^2_{[H^{1/2}_{,0}(0,T)]'}\right),$$

since there are no coupling terms.

In the case $T > L$, we have the representation

$$\begin{aligned}
&\langle \mathscr{H}_T V w, w\rangle_{L^2(\Sigma)} \\
&= \frac{T^2}{2}\sum_{k=0}^{\infty}\frac{\overline{w}_{0,k}^2+\overline{w}_{L,k}^2}{(2k+1)\pi} + \frac{T^2}{2}\sum_{k=0}^{\infty}\overline{w}_{0,k}\overline{w}_{L,k}\frac{2}{(2k+1)\pi}\left(1-\frac{L}{T}\right)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{L}{T}\right) \\
&\quad + \frac{T^2}{2}\sum_{k-\ell=2j\neq 0}\overline{w}_{0,\ell}\overline{w}_{L,k}\frac{4}{\pi^2}\frac{1}{(k+\ell+1)(k-\ell)}\cos\left((k+\ell+1)\frac{\pi}{2}\frac{L}{T}\right)\sin\left((\ell-k)\frac{\pi}{2}\frac{L}{T}\right) \\
&= \frac{T^2}{2}\sum_{k=0}^{\infty}\left[\widehat{w}_{0,k}^2+\widehat{w}_{L,k}^2\right] + \frac{T^2}{2}\sum_{k=0}^{\infty}2\widehat{w}_{0,k}\widehat{w}_{L,k}\left(1-\frac{L}{T}\right)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{L}{T}\right) \\
&\quad + \frac{T^2}{2}\sum_{k-\ell=2j\neq 0}\widehat{w}_{0,\ell}\widehat{w}_{L,k}\frac{4}{\pi}\frac{\sqrt{2k+1}\sqrt{2\ell+1}}{(k+\ell+1)(k-\ell)}\cos\left((k+\ell+1)\frac{\pi}{2}\frac{L}{T}\right)\sin\left((\ell-k)\frac{\pi}{2}\frac{L}{T}\right),
\end{aligned}$$

where

$$\widehat{w}_{0,k} = \frac{\overline{w}_{0,k}}{\sqrt{(2k+1)\pi}}, \quad \widehat{w}_{L,k} = \frac{\overline{w}_{L,k}}{\sqrt{(2k+1)\pi}}.$$

When using the coefficients $b_{k\ell}$, we write the above result as

$$\langle \mathscr{H}_T V w, w\rangle_{L^2(\Sigma)} = \frac{T^2}{2}\left(\sum_{k=0}^{\infty}\left[\widehat{w}_{0,\ell}^2 + \widehat{w}_{L,k}^2\right] + \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} b_{k\ell}\widehat{w}_{0,\ell}\widehat{w}_{L,k}\right).$$

Following [7, Chapter VIII], we consider the forms

$$B(\widehat{w}_0,\widehat{w}_L) := \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} b_{k\ell}\widehat{w}_{0,\ell}\widehat{w}_{L,k}, \quad B_m(\widehat{w}_0,\widehat{w}_L) := \sum_{k=0}^{m}\sum_{\ell=0}^{m} b_{k\ell}\widehat{w}_{0,\ell}\widehat{w}_{L,k},$$

and for the latter we estimate

$$\begin{aligned}\left|B_m(\widehat{w}_0,\widehat{w}_L)\right| = \left|\sum_{k=0}^{m}\sum_{\ell=0}^{m} b_{k\ell}\widehat{w}_{0,\ell}\widehat{w}_{L,k}\right| &\le \left[\sum_{k=0}^{m}\widehat{w}_{L,k}^2\right]^{1/2}\left[\sum_{k=0}^{m}\left(\sum_{\ell=0}^{m} b_{k\ell}\widehat{w}_{0,\ell}\right)^2\right]^{1/2}\\ &\le \left[\sum_{k=0}^{m}\widehat{w}_{L,k}^2\right]^{1/2}\left[\sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{m} b_{k\ell}\widehat{w}_{0,\ell}\right)^2\right]^{1/2}.\end{aligned}$$

Hence, it remains to consider

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{m} b_{k\ell}\widehat{w}_{0,\ell}\right)^2 &= \sum_{\ell=0}^{m}\sum_{j=0}^{m}\left(\sum_{k=0}^{\infty} b_{k\ell}b_{kj}\right)\widehat{w}_{0,\ell}\widehat{w}_{0,j}\\ &= \sum_{\ell=0}^{m}\sum_{j=0}^{m} c_{\ell j}\widehat{w}_{0,\ell}\widehat{w}_{0,j} \le \lambda_{\max}(C_m)\sum_{\ell=0}^{m}\widehat{w}_{0,\ell}^2.\end{aligned}$$

From this, we conclude $\lambda_{\max}(C_m) \ge 0$, and

$$\begin{aligned}\left|B_m(\widehat{w}_0,\widehat{w}_L)\right| &\le \sqrt{\lambda_{\max}(C_m)}\left[\sum_{k=0}^{m}\widehat{w}_{L,k}^2\right]^{1/2}\left[\sum_{\ell=0}^{m}\widehat{w}_{0,\ell}^2\right]^{1/2}\\ &\le \sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\left[\sum_{k=0}^{\infty}\widehat{w}_{L,k}^2\right]^{1/2}\left[\sum_{\ell=0}^{\infty}\widehat{w}_{0,\ell}^2\right]^{1/2}\\ &\le \frac{1}{2}\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\left(\sum_{k=0}^{\infty}\widehat{w}_{L,k}^2 + \sum_{\ell=0}^{\infty}\widehat{w}_{0,\ell}^2\right)\end{aligned}$$

for all $m \in \mathbb{N}$, and therefore

$$\left|B(\widehat{w}_0,\widehat{w}_L)\right| \le \frac{1}{2}\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\left(\sum_{k=0}^{\infty}\widehat{w}_{L,k}^2 + \sum_{\ell=0}^{\infty}\widehat{w}_{0,\ell}^2\right)$$

follows. With this, we finally obtain

$$\begin{aligned}
\langle \mathscr{H}_T V w, w\rangle_{L^2(\Sigma)} &\geq \frac{T^2}{2}\left(1-\frac{1}{2}\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\right)\sum_{k=0}^{\infty}\left[\widehat{w}_{0,k}^2+\widehat{w}_{L,k}^2\right]\\
&= \frac{T^2}{2}\left(1-\frac{1}{2}\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\right)\sum_{k=0}^{\infty}\frac{\overline{w}_{0,k}^2+\overline{w}_{L,k}^2}{(2k+1)\pi}\\
&= \frac{T^2}{4}\left(1-\frac{1}{2}\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\right)\sum_{k=0}^{\infty}\frac{\overline{w}_{0,k}^2+\overline{w}_{L,k}^2}{\frac{\pi}{2}+k\pi}\\
&= \frac{1}{2}\left(1-\frac{1}{2}\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)}\right)\left(\|w_0\|^2_{[H^{1/2}_{,0}(0,T)]'}+\|w_L\|^2_{[H^{1/2}_{,0}(0,T)]'}\right),
\end{aligned}$$

as stated. In both cases $T \leq L$ or $T > L$, the density of $L^2(\Sigma)$ in $[H^{1/2}_{,0}(\Sigma)]'$ yields the assertion.

Proposition 4.2. *Numerical results indicate that*

$$\sup_{m\in\mathbb{N}}\sqrt{\lambda_{\max}(C_m)} = 2-4\sin^2\left(\frac{\pi}{2(n+1)}\right),$$

where n is given in (2.6). Indeed, for $L=1$, $T\in[1,20]$ and $m=20000$, the related results are given in Figure 1. Varying m yields identical results when m is large enough. Then, the ellipticity estimate (4.1) becomes

$$a_{\mathscr{H}_T}(w,w) = \langle \mathscr{H}_T V w, w\rangle_\Sigma \geq \sin^2\left(\frac{\pi}{2(n+1)}\right)\|w\|^2_{[H^{1/2}_{,0}(\Sigma)]'} \quad \textit{for all } w\in[H^{1/2}_{,0}(\Sigma)]', \tag{4.2}$$

where the ellipticity constant is the same as in (2.7), and in (2.9), respectively. Hence, we can think of (4.2) being an interpolation of the ellipticity estimates (2.7) and (2.9).

With the above results, we conclude unique solvability of the variational formulation to find $w\in[H^{1/2}_{,0}(\Sigma)]'$ such that

$$\langle v, \mathscr{H}_T V w\rangle_\Sigma = \langle v, \mathscr{H}_T g\rangle_\Sigma \quad \text{for all } v\in[H^{1/2}_{,0}(\Sigma)]', \tag{4.3}$$

where $g\in H^{1/2}_{0,}(\Sigma)$ is a given Dirichlet datum. Let $W_h\subset[H^{1/2}_{,0}(\Sigma)]'$ be some boundary element space, e.g., of piecewise constant basis functions, which are defined with respect to some decomposition of the lateral boundaries $\{0\}\times(0,T)$ and $\{L\}\times(0,T)$, respectively. The space-time Galerkin boundary element formulation of (4.3) is: Find $w_h\in W_h$ such that

$$\langle v_h, \mathscr{H}_T V w_h\rangle_\Sigma = \langle v_h, \mathscr{H}_T g\rangle_\Sigma \quad \text{for all } v_h\in W_h.$$

When assuming $w\in H^s(\Sigma)$ for some $s\in[0,1]$ and using standard arguments, e.g., [14], we derive an a priori error estimate in the energy norm,

$$\|w-w_h\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq c\,h^{s+\frac{1}{2}}\,\|w\|_{H^s(\Sigma)}.$$

Moreover, using an inverse inequality, we also obtain an error estimate in $L^2(\Sigma)$,

$$\|w-w_h\|_{L^2(\Sigma)} \leq c\,h^s\,\|w\|_{H^s(\Sigma)}. \tag{4.4}$$

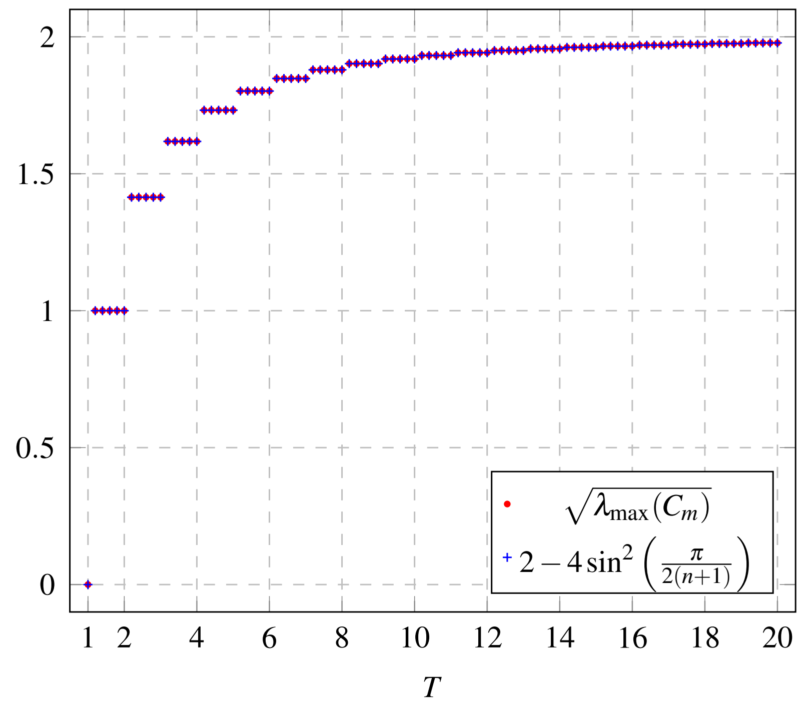

FIGURE 1. Numerical evaluation of $\sqrt{\lambda_{\max}(C_m)}$ for different $T \in [1,20]$, with $L=1$, and $m=20000$.

5. Numerical results

Instead of the boundary integral equation (2.4) of the indirect approach, we consider, as in [17], the boundary integral equation of the direct approach

$$Vw = (\frac{1}{2}I + K)g \quad \text{on } \Sigma, \tag{5.1}$$

including the double layer boundary integral operator K on the right hand side. In this case, the unknown w is the spatial normal derivative $\partial_{n_x} u$ of the solution u of (2.1).

For a boundary element approximation, consider a decomposition of the lateral boundary

$$\Sigma = \bigcup_{i=1}^{N_0+N_L} \overline{\tau}_i$$

into $N_0 + N_L$ boundary elements τ_i with maximal mesh size $h = \max_i |\tau_i|$. Here, N_0 is the number of boundary elements for the boundary $\{0\} \times (0,T)$ and N_L is the number of boundary elements for the boundary $\{L\} \times (0,T)$. The conforming ansatz space of piecewise constant functions

$$S_h^0(\Sigma) := S_{h_0}^0(0,T) \times S_{h_L}^0(0,T) \subset [H_{,0}^{1/2}(\Sigma)]'$$

is used to define an approximate solution $w_h \in S_h^0(\Sigma)$. Then, the Galerkin discretization of (5.1) to find $w_h \in S_h^0(\Sigma)$ such that

$$\langle v_h, \mathscr{H}_T V w_h \rangle_{L^2(\Sigma)} = \langle v_h, \mathscr{H}_T (\frac{1}{2} I + K) Q_h g \rangle_{L^2(\Sigma)} \quad \text{for all } v_h \in S_h^0(\Sigma) \tag{5.2}$$

is equivalent to the global linear system

$$V_h \underline{w} = \underline{g} \tag{5.3}$$

with the related system matrix $V_h \in \mathbb{R}^{(N_0+N_L)\times(N_0+N_L)}$, the right-hand side $\underline{g} \in \mathbb{R}^{N_0+N_L}$ and the vector of unknown coefficients $\underline{w} \in \mathbb{R}^{N_0+N_L}$ of $w_h \in S_h^0(\Sigma)$. Here, for an easier implementation, we approximate the right-hand side $g \in H_{0,}^{1/2}(\Sigma)$ by $Q_h g$, where Q_h is the L^2 projection on the space of piecewise linear, continuous functions fulfilling homogeneous initial conditions for $t = 0$. The assembling of the matrix $V_h \in \mathbb{R}^{(N_0+N_L)\times(N_0+N_L)}$ and the right-hand side $\underline{g} \in \mathbb{R}^{N_0+N_L}$, i.e., the realization of $\mathscr{H}_T$, is done as proposed in [22, Subsection 2.2]. The integrals for computing the projection $Q_h g$ are calculated by using high-order quadrature rules. The global linear system (5.3) is solved by a direct solver.

In the numerical examples, we consider the spatial domain $\Omega = (0,3)$, i.e., $L = 3$, and the time interval $(0,6)$, i.e., $T = 6$. The lateral boundaries $\{0\} \times (0,T)$ and $\{L\} \times (0,T)$ are discretized uniformly into $N_0 = N_L = 2^{\ell+1}$ boundary elements each, $\ell = 3,4,5,\ldots,12$.

In the first example, we consider the smooth solution

$$u_1(x,t) = \begin{cases} \frac{1}{2}(t-x-2)^3(x-t)^3 & \text{for } x \le t \le 2+x, \\ 0 & \text{otherwise.} \end{cases}$$

Due to $w_1 = \partial_{n_x} u_1 \in H^1(\Sigma)$ and using the error estimate (4.4), we expect a linear order of convergence, as confirmed by the numerical results given in Table 1.

ℓ	N_0+N_L	$\|w_1 - w_{1,h}\|_{L^2(\Sigma)}$	eoc
3	32	4.48 –1	
4	64	2.11 –1	1.09
5	128	1.04 –1	1.02
6	256	5.18 –2	1.01
7	512	2.59 –2	1.00
8	1024	1.29 –2	1.00
9	2048	6.47 –3	1.00
10	4096	3.23 –3	1.00
11	8192	1.62 –3	1.00
12	16384	8.09 –4	1.00

TABLE 1. Numerical results for the boundary element method (5.2) in the case $w_1 \in H^1(\Sigma)$.

As a second example, we consider the singular solution

$$u_2(x,t) = \begin{cases} \frac{1}{2}\,|\sin(\pi(x-t))| & \text{for } x \le t, \\ 0 & \text{otherwise,} \end{cases}$$

where we have $w_2 \in H^s(\Sigma)$ for $s < \frac{1}{2}$. Hence, using (4.4), we expect the reduced order $\frac{1}{2}$ of convergence when considering the error in $L^2(\Sigma)$. This is confirmed by the numerical results as given in Table 2.

ℓ	N_0+N_L	$\|w_2 - w_{2,h}\|_{L^2(\Sigma)}$	eoc
3	32	2.59 +0	0.34
4	64	1.75 +0	0.56
5	128	1.21 +0	0.53
6	256	8.45 –1	0.52
7	512	5.93 –1	0.51
8	1024	4.18 –1	0.51
9	2048	2.95 –1	0.50
10	4096	2.08 –1	0.50
11	8192	1.47 –1	0.50
12	16384	1.04 –1	0.50

TABLE 2. Numerical results for the boundary element method (5.2) in the case $w_2 \in H^s(\Sigma)$, $s < \frac{1}{2}$.

6. Conclusions

In this note, we have shown that the single layer boundary integral operator of the wave equation in one space dimension is elliptic in the energy space $[H^{1/2}_{,0}(\Sigma)]'$, when composed with some modified Hilbert transformation. This result corresponds to the well-known ellipticity results for boundary integral operators related to second-order elliptic partial differential equations. In the multi-dimensional case, instead of ellipticity, a related inf-sup stability condition can be established to ensure unique solvability of the boundary integral equation, see [16] for a more detailed discussion. More recent work [5] considers the wave single layer boundary integral operator on flat objects in combination with the standard Hilbert transformation. The relation with the present approach follows from [15] showing that the standard and the modified Hilbert transformations differ by a compact perturbation. It is obvious that we can extend this approach also to the hypersingular boundary integral operator, and to the double layer boundary integral operator. Ellipticity of boundary integral operators is an important ingredient in the a priori and a posteriori error analysis of boundary element methods, in the construction of appropriate preconditioners, and in the coupling with finite element methods. It goes without saying that this proposed new approach requires more work in the numerical analysis, and in the implementation of the proposed scheme, including the composition of the single layer boundary integral operator and the modified Hilbert transformation, which are both non-local. Nevertheless, this work may give some more insight into the numerical analysis of existing boundary element methods for the wave equation, and it presents an alternative approach for a reliable and efficient numerical solution of the wave equation.

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