# INVERSE NODAL PROBLEM FOR THE INTEGRO-DIFFERENTIAL DIRAC OPERATOR WITH A DELAY IN THE KERNEL

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ABSTRACT. In the present article, we consider an integro-differential Dirac system with an integral delay on a finite interval. We obtain the asymptotical formula for the nodal points of the first components of the eigenfunctions, formulate a uniqueness theorem and prove that the kernel of the Dirac operator can be uniquely determined from a dense subset of the nodal set. We also present examples for reconstructing the kernel by using the nodal points.

## 1. INTRODUCTION

Consider the boundary value problem, denoted by $\mathcal{L} := \mathcal{L}(p, q, H)$, corresponding to the Dirac integro-differential operator of the form

$$\ell y := \mathbf{B}y' + \int_0^x \mathbf{M}(x-t)y(t)dt = \lambda y, \quad 0 < x < \pi, \tag{1.1}$$

with

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{M}(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \ y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix},$$

subject to the boundary conditions

$$y_1(0,\lambda) = 0, \quad y_2(\pi,\lambda) + Hy_1(\pi,\lambda) = 0. \tag{1.2}$$

Here $\lambda$ is a spectral parameter, $H$ is a real number, $p(x), q(x)$ are real-valued functions and $(\pi - x)p(x), (\pi - x)q(x) \in L_2(0,\pi)$.

Inverse problems play an important role in mathematics and have many applications in physics, geophysics, mechanics, electronics, meteorology and other branches of natural sciences and engineering (see, for example [9, 11, 16, 44] and the references therein).

In particular, some aspects of inverse eigenvalue problems for Dirac system had been studied in [33, 36, 38]. Essentially, it is more difficult to investigate the inverse eigenvalue problems for nonlocal operators, in

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particular, integro-differential (see [46] and the references therein) and integral operators (see [45]).

Integro-differential operators have many applications in natural sciences and engineering. For example, consider the initial boundary-value problem consisting of the following hyperbolic integro-differential equation of acoustics

$$\frac{1}{c^2(z)}v_{tt} = \eta_z - \frac{\partial \ln(\rho(z))}{\partial z}\eta, \quad z > 0, \quad t > 0,$$

with initial conditions $v|_{t\leq 0} \equiv 0$, $\eta(0+,t) = \delta'(t)$, where $c(z) > 0$ is the wave velocity, $v(z,t)$ is the acoustic pressure, $\eta(z,t)$ is the stress, $\rho(z)$ is the medium density, $\delta(t)$ is the Dirac delta function, and the functions $v(z,t)$ and $\eta(z,t)$ are related as the following equation with an integral delay

$$\eta(z,t) = v_z(z,t) + \int_0^t k(t-u)v_z(z,u)du.$$

For another example, in electrodynamics, the relationship between $D, B$ (the electric displacement and the magnetic induction, respectively) and $E, H$ (the intensities of the respective fields) can be shown by a system of two integral equaions with a delay in the kernels (for detailes, see [26]). Moreover, since nonlocal integro-differential operators are often more adequate than local differential operators for modeling various phenomena, in the elasticity theory, nonlocal models give more precise description of material properties [1]. Therefore, theoretical study of such models will increase their practical applications. For more applications of integro-differential operators, we refer the interested readers to [25]. We note that the classical methods of inverse problem theory (such as the spectral mappings method and the transformation operator method) are not applicable for these operators.

Each eigenvalue is the square of a natural frequency. The most natural experiment then for finding the nodal positions is to excite the vibrating system at a natural frequency and take measurements of the positions where the system does not vibrate. These positions are the zeros of the eigenfunctions. In the last few decades, in some interesting works, inverse problems were studied by using a new spectral data which are so-called *nodal points*, and their corresponding inverse problems are so-called *inverse nodal problems*. From the physical point of view, the inverse nodal problem corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations.

It seems that the first results of inverse nodal problems was given by McLaughlin [30]. She considered a boundary value problem associated with one-dimensional Schrödinger equations, proposed a new technique and proved that the potential function of the equation can be uniquely determined by given a dense subset nodal points. Then, some numerical schemes for recovering the potential from nodes of boundary value problems with more general boundary conditions, were given in [14].

Later on, several authors studied inverse nodal problems for diffusion or Sturm-Liouville operators (see [4, 7, 8, 22, 23, 27, 31, 34, 37, 43]). At the same time, inverse nodal problems associated with Dirac or integro-differential operators are of great interest (see [3, 12, 13, 17, 18, 20, 21, 24, 40, 41, 42]). Note that, for inverse spectral problems (i.e. determining the potential from spectral data consisting of two sets of eigenvalues, or one set of eigenvalues and norming constants), we refer the readers to see [10, 28, 32, 35, 39] and the references therein. Moreover, in [2], the authors considered an integro-differential Dirac system, and studied necessary and sufficient conditions for solvability of the inverse problem from the spectrum, but in the present article, we reconstruct the solution of the inverse problem for Dirac operator from the nodes (the zeros of the eigenfunctions). We mention that in [19] (or [15]), the authors considered a system of Dirac integro-differential equations (or Dirac-type integro-differential operators on a star graph with $d$ segments of the same length, $d \geq 2$, $d \in \mathbb{N}$) as

$$\mathbf{B}Z'(x) + \Omega(x)Z(x) + \int_0^x \mathbf{N}(x,t)Z(t)dt = \lambda Z(x),$$

on the interval $(0, \pi)$ with separated boundary conditions, where

$$\Omega(x) = \begin{pmatrix} V(x) + m & 0 \\ 0 & V(x) - m \end{pmatrix}, \mathbf{N}(x,t) = \begin{pmatrix} N_{11}(x,t) & N_{12}(x,t) \\ N_{21}(x,t) & N_{22}(x,t) \end{pmatrix},$$

$\Omega(x)$ and $N(x,t)$ are real-valued functions in the class of $W_2^1(0,\pi)$ (or $W_1^1(0,\pi)$) and $m$ is a real constant. They obtained the asymptotical formula for the nodal points and provided an algorithm to reconstruct the coefficients of the problem by using the nodal points.

The present paper is organized as follows. In Section 2, first, we study the characteristic function of the boundary value problem (1.1)-(1.2), and obtain the eigenvalues. Then, we present a system of nonlinear integral equations with respect to the functions $p(x), q(x)$ (i.e. the solution of the inverse problem), which plays an important role for investigating the uniqueness theorem for the solution of the inverse problem. In Section 3, by using a set of nodal points of the first components of the eigenfunctions which is dense in the interval $(0, \pi)$, we prove the uniqueness theorem for the solution of the inverse nodal problem, and provide a constructive procedure for solving the inverse problem. Finally, we present examples to reconstruct the kernel $\mathbf{M}(x)$ by using the nodal points.

## 2. The system of nonlinear main equations

Let $y = \phi(x,\lambda) = (\phi_1(x,\lambda), \phi_2(x,\lambda))^T$ be the solution of the system (1.1), satisfying the initial conditions

$$\phi_1(0,\lambda) = 0, \quad \phi_2(0,\lambda) = -1, \tag{2.1}$$

where $T$ is the transposition sign. By substitution it is easy to check that the Cauchy problem (1.1), (2.1) is equivalent to the following integral equation:

$$\phi(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix} + \int_0^x \mathbf{S}(x-t,\lambda)dt \int_0^t \mathbf{M}(t-z)\phi(z,\lambda)dz,$$

where

$$\mathbf{S}(x,\lambda) = \begin{pmatrix} \sin\lambda x & \cos\lambda x \\ \cos\lambda x & -\sin\lambda x \end{pmatrix}.$$

From [2], we have

$$\phi(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix} + \int_0^x \mathbf{Q}(x,t) \begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt, \tag{2.2}$$

where $\mathbf{Q}(x,t) = (Q_{ij}(x,t))_{i,j=1,2}$ and $Q_{ij}$, $i,j = 1,2$, satisfy the following integral equations:

$$\begin{aligned} Q_{11}(x,t) = &- tq(x-t) - \int_0^{x-t} dz \int_{x-t}^{x} \Big( q(z)Q_{11}(s-z, t-x+s) \\ &+ p(z)Q_{12}(s-z,t-x+s)\Big) ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} Q_{12}(x,t) = &tp(x-t) + \int_0^{x-t} dz \int_{x-t}^{x} \Big( p(z)Q_{11}(s-z, t-x+s) \\ &- q(z)Q_{12}(s-z,t-x+s)\Big) ds, \end{aligned} \tag{2.4}$$

$$Q_{21}(x,t) = Q_{12}(x,t), \quad Q_{22}(x,t) = -Q_{11}(x,t). \tag{2.5}$$

Moreover,

(2.6)
$$\begin{cases} Q_{11}(x,t) = -tq(x-t) + \sum_{n=2}^{\infty} \sum_{i=0}^{n} \frac{a_{ni}t^n}{n!} (p^{*i} * q^{*(n-i)})(x-t), \\ Q_{12}(x,t) = tp(x-t) + \sum_{n=2}^{\infty} \sum_{i=0}^{n} \frac{b_{ni}t^n}{n!} (p^{*i} * q^{*(n-i)})(x-t), \end{cases}$$

where $a_{10} = -b_{11} = -1$, $a_{11} = b_{10} = 0$,

$a_{ni} = -a_{n-1,i} - b_{n-1,i-1}, b_{ni} = a_{n-1,i-1} - b_{n-1,i}, n = 2,3,4,\ldots, i = \overline{0,n},$

$a_{ni} = b_{ni} = 0$ for $i < 0$ or $i > n$, and for any functions $f$ and $g$,

$$(f * g)(x) := \int_0^x f(x-t)g(t)dt, \quad f^{*0} * g = g * f^{*0} := g,$$
$$f^{*1}(x) := f(x), \quad f^{*(k+1)}(x) := (f * f^{*k})(x), \quad k \geq 1.$$

By induction, it is easy to prove that

$$\sum_{i=0}^{n} |a_{ni}|, \sum_{i=0}^{n} |b_{ni}| \leq 2^{n-1}, \quad n = 1, 2, 3, \ldots. \tag{2.7}$$

According to (1.2), the eigenvalues of the boundary value problem $\mathcal{L}$ coincide with the zeros of its characteristic function $\Delta(\lambda) := \phi_2(\pi, \lambda) + H\phi_1(\pi, \lambda)$. From (2.2), we get the following representation for the characteristic function:

$$\Delta(\lambda) = -\cos\lambda\pi + H\sin\lambda\pi + \int_0^\pi \Big(v_1(t)\sin\lambda t + v_2(t)\cos\lambda t\Big)dt, \tag{2.8}$$

where

$$v_m(t) = (-1)^{m+1}\Big(Q_{2m}(\pi, t) + HQ_{1m}(\pi, t)\Big) \in L_2(0, \pi), m = 1, 2. \tag{2.9}$$

Denote

$$\Delta_0(\lambda) = -\cos\lambda\pi + H\sin\lambda\pi,$$
$$G_n = \{\lambda \in \mathbb{C} \ : \ |\lambda| = |\lambda_n^0| + \beta, \ n = 0, \pm1, \pm2, \ldots\},$$
$$G_\delta = \{\lambda \in \mathbb{C} \ : \ |\lambda - \lambda_n^0| \geq \delta, \ n = 0, \pm1, \pm2, \ldots\},$$

where $\delta$ is sufficiently small number ($\delta \ll \beta$), and $\lambda_n^0$ are the zeros of $\Delta_0(\lambda)$, i.e.

$$\lambda_n^0 = n + \frac{1}{\pi}\arctan(\frac{1}{H}).$$

We know from [29] that for $\lambda \in \overline{G_\delta}$, $|\Delta_0(\lambda)| > C_\delta \exp(|\eta|\pi)$, where $\eta = Im\lambda$. Hence,

$$\begin{aligned}
&\lim_{|\lambda|\to\infty} \exp(-|\eta|\pi)(\Delta(\lambda) - \Delta_0(\lambda)) \\
&= \lim_{|\lambda|\to\infty} \exp(-|\eta|\pi)\Big\{\int_0^\pi \Big(Q_{21}(\pi, t) + HQ_{11}(\pi, t)\Big)\sin\lambda t dt \\
&\quad - \int_0^\pi \Big(Q_{22}(\pi, t) + HQ_{12}(\pi, t)\Big)\cos\lambda t dt\Big\} = 0,
\end{aligned}$$

and $|\Delta(\lambda) - \Delta_0(\lambda)| < C_\delta \exp(|\eta|\pi)$, and for sufficiently large $n$ and $\lambda \in G_n$, we have $|\Delta_0(\lambda)| > |\Delta(\lambda) - \Delta_0(\lambda)|$. Therefore, using the Rouche's theorem, it follows that for sufficiently large $n$, the function $\Delta_0(\lambda)$ and $\Delta_0(\lambda) + (\Delta(\lambda) - \Delta_0(\lambda)) = \Delta(\lambda)$ have the same number of zeros inside the contour $G_n$, namely $2n+1$ zeros $\lambda_{-n}, \ldots, \lambda_0, \ldots, \lambda_n$. Hence, the eigenvalues $\lambda_n$ have the form $\lambda_n = \lambda_n^0 + \varepsilon_n$, where $\lim_{n\to\infty} \varepsilon_n = 0$. Substituting $\lambda_n^0 + \varepsilon_n$ for $\lambda_n$ in the last equality and using the fact that $\Delta_0(\lambda_n^0 + \varepsilon_n) = \frac{\partial \Delta_0}{\partial \lambda}(\lambda_n^0)(1 + o(1))\varepsilon_n$, yield that $\{\varepsilon_n\} \in l_2$. Therefore, we have the following theorem.

**Theorem 2.1.** *The eigenvalues of the boundary value problem $\mathcal{L}$ satisfy the asymptotic formula*

$$\lambda_n = n + \frac{1}{\pi} \arctan(\frac{1}{H}) + \varepsilon_n, \tag{2.10}$$

*where* $\{\varepsilon_n\} \in l_2$.

Using the representation (2.8), by the standard methods (see [9]), one can prove that the function $\Delta(\lambda)$ is uniquely determined by its zeros.

It follows from (2.5), (2.6) and (2.9) that

(2.11)

$$\begin{aligned} v_1(\pi - t) =&(\pi - t)p(t) + \sum_{n=2}^{\infty} \sum_{i=0}^{n} \frac{b_{ni}(\pi - t)^n}{n!}(p^{*i} * q^{*(n-i)})(t) \\ &- (\pi - t)Hq(t) + \sum_{n=2}^{\infty} \sum_{i=0}^{n} \frac{a_{ni}(\pi - t)^n}{n!}(p^{*i} * q^{*(n-i)})(t), \end{aligned}$$

(2.12)

$$\begin{aligned} -v_2(\pi - t) =&(\pi - t)q(t) - \sum_{n=2}^{\infty} \sum_{i=0}^{n} \frac{a_{ni}(\pi - t)^n}{n!}(p^{*i} * q^{*(n-i)})(t) \\ &+ (\pi - t)Hp(t) + \sum_{n=2}^{\infty} \sum_{i=0}^{n} \frac{b_{ni}(\pi - t)^n}{n!}(p^{*i} * q^{*(n-i)})(t). \end{aligned}$$

The relations (2.11)-(2.12) can be considered as a system of nonlinear integral equations with respect to the functions $p(x)$, $q(x)$.Using a similar method as in [2], it can be shown that for any functions $v_1(x), v_2(x) \in L_2(0, \pi)$, the system (2.11)-(2.12) has a unique solution $(p(x), q(x))$ such that $(\pi - x)p(x), (\pi - x)q(x) \in L_2(0, \pi)$.

## 3. Inverse nodal problem

In the first part of this section, we obtain the nodal points corresponding to the boundary value problem $\mathcal{L}$, which form a dense subset in the interval $(0, \pi)$. For this, we study some properties of the first component $y_1(x, \lambda_n)$ of the eigenfunction $y(x, \lambda_n)$ of the integro-differential Dirac system (1.1) for sufficiently large $n$. Then, we prove the uniqueness theorem of recovering the kernel $\mathbf{M}(x)$ on $[0, \pi]$, and present an example for reconstructing the solution of the inverse nodal problem.

**Lemma 3.1.** *For sufficiently large $n$, the first component $y_1(x,\lambda_n)$ of the eigenfunction $y(x,\lambda_n)$ of the integro-differential Dirac system* (1.1) *has exactly $n$ nodes in the interval $(0,\pi)$:*

$$x_n^j = \frac{2j\pi + I(x_n^j)}{n} + O(\frac{1}{n}) \tag{3.1}$$

*uniformly with respect to $j \in \mathbb{Z}$, where $c_0 = \frac{1}{\pi}\arctan(\frac{1}{H})$ and*

$$I(x_n^j) = \frac{1}{2}\int_0^{x_n^j} t\Big(p(x_n^j - t) + q(x_n^j - t)\Big)\sin(2c_0 t)dt. \tag{3.2}$$

*Proof.* From (2.2)-(2.4) we see that the function $y_1(x,\lambda_n)$ has the following asymptotic formula for sufficiently large $n$, uniformly in $x$:

$$\begin{aligned} y_1(x,\lambda_n) =& \sin\lambda_n x + \int_0^x \Big\{\Big(-tq(x-t) + \eta_1(x,t)\Big)\sin\lambda_n t \\ &- \Big(tp(x-t) + \eta_2(x,t)\Big)\cos\lambda_n t\Big\}dt, \end{aligned}$$

where

$$\eta_1(x,t) = \sum_{n=2}^{\infty}\sum_{i=0}^{n} \frac{a_{ni}t^n}{n!}(p^{*i} * q^{*(n-i)})(x-t),$$
$$\eta_2(x,t) = \sum_{n=2}^{\infty}\sum_{i=0}^{n} \frac{b_{ni}t^n}{n!}(p^{*i} * q^{*(n-i)})(x-t).$$

From $y_1(x_n^j,\lambda_n) = 0$, we obtain

$$\begin{aligned} \sin(\lambda_n x_n^j) =& \int_0^{x_n^j} \Big\{\Big(tq(x_n^j - t) + \eta_1(x_n^j,t)\Big)\sin\lambda_n t \\ &+ \Big(tp(x_n^j - t) + \eta_2(x_n^j,t)\Big)\cos\lambda_n t\Big\}dt \\ =& \frac{1}{2}\int_0^{x_n^j} t\Big(p(x_n^j - t) + q(x_n^j - t)\Big)\sin(2c_0 t)dt \\ &+ \int_0^{x_n^j}\Big\{\eta_1(x_n^j,t)\Big(\sin\lambda_n t - \frac{1}{2}\sin(2c_0 t)\Big) \\ &\quad + \eta_2(x_n^j,t)\Big(\cos\lambda_n t - \frac{1}{2}\sin(2c_0 t)\Big)\Big\}dt \\ =:& I(x_n^j) + II(x_n^j). \end{aligned}$$

On the other hand, it follows from (2.7) and the relation

$$\frac{2(\pi - t)^n}{n!} \le 2(\pi - t)\exp(2(\pi - t))$$

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that

$$\eta_1(x,t),\eta_2(x,t)\in L_2((0,\pi)\times(0,\pi)).$$

Therefore, from $\sin(\lambda_n x_n^j) = I(x_n^j)+II(x_n^j)$ and using Taylor's expansion for arcsine, we get

$$\lambda_n x_n^j = 2j\pi + I(x_n^j) + O(1)$$

as $n\to\infty$ where

$$I(x_n^j)=\frac{1}{2}\int_0^{x_n^j} t\Big(p(x_n^j-t)+q(x_n^j-t)\Big)\sin(2c_0t)dt.$$

Consequently,

$$x_n^j=\frac{1}{\lambda_n}(2j\pi+I(x_n^j))+O(\frac{1}{\lambda_n}),\quad n\to\infty. \tag{3.3}$$

Furthermore, using the asymptotic formula (2.10), we have

$$\frac{1}{\lambda_n}=\frac{1}{n}-\frac{c_0+\varepsilon_n}{n^2}+\frac{(c_0+\varepsilon_n)^2}{n^3}+O(\frac{1}{n^4}). \tag{3.4}$$

Substituting (3.4) into (3.3), we arrive at (3.1). □

The equality (3.1) gives the asymptotic expansion for the nodal lengths $l_n^j := x_n^{j+1}-x_n^j$; we have uniformly with respect to $j$

$$\begin{aligned} l_n^j =&\frac{2\pi+I(x_n^{j+1})-I(x_n^j)}{n}+O(\frac{1}{n})\\ =&\frac{2\pi}{n}+O(\frac{1}{n}),\quad n\to\infty. \end{aligned} \tag{3.5}$$

**Corollary 3.2.** *From Lemma 3.1 and* (3.5)*, it follows that the nodal set* $X=\{x_n^j\}$ *is dense in* $(0,\pi)$*.*

Let us now to formulate a uniqueness theorem for the solution of the inverse nodal problem. We also assume that $(\pi-x)\mathbf{M}(x)$ is in the class of $W_2^1[0,\pi]$.

**Theorem 3.3.** *Let* $X$ *be the set of nodal points of* $\mathcal{L}$ *with the kernel* $\mathbf{M}(x)$*, and* $\widetilde{X}$ *be the set of nodal points of* $\mathcal{L}$ *with the kernel* $\widetilde{\mathbf{M}}(x)$*. If* $X=\widetilde{X}$ *and* $H=\widetilde{H}$*, then* $(p(x),q(x))=(\widetilde{p}(x),\widetilde{q}(x))$ *on* $[0,\pi]$*.*

*Proof.* We first prove that the given nodal set $X$ uniquely determines the function $p(x)+q(x)$ on $[0,\pi]$. For each fixed $x\in[0,\pi]$, choose a subsequence $\{x_n^{\ell_{jn}}\}\subset X$ such that $x_n^{\ell_{jn}}\to x$ as $n\to\infty$. Using the asymptotic expansion (3.1), we get

$$nx_n^j-2j\pi=I(x_n^j)+O(\frac{1}{n}),\quad n\to\infty, \tag{3.6}$$

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where $I(x_n^j)$ is defined in (3.2). Moreover, the fact $\lim_{n\to\infty} x_n^{\ell_{jn}} = x$ implies that $\frac{2j\pi}{n} \to x$ and $I(x_n^{\ell_{jn}}) \to I(x)$, where

$$I(x) = \frac{1}{2}\int_0^x t\Big(p(x-t)+q(x-t)\Big)\sin(2c_0 t)dt.$$

From this, it follows that as $n \to \infty$, the limit of the left-hand side of (3.6) exists and

$$f_j(x) := \lim_{n\to\infty}(nx_n^j - 2j\pi) = I(x). \tag{3.7}$$

If $X = \widetilde{X}$, then $f_j(x) = \widetilde{f}_j(x)$, $x \in [0,\pi]$. Hence, by (3.7) we get $I(x) = \widetilde{I}(x)$, $x \in [0,\pi]$. Together with $H = \widetilde{H}$ (i.e. $c_0 = \widetilde{c}_0$) this yields

$$p(x) + q(x) = \widetilde{p}(x) + \widetilde{q}(x), \quad x \in [0,\pi]. \tag{3.8}$$

Next, we are going to show $p(x) = \widetilde{p}(x)$, $q(x) = \widetilde{q}(x)$, on $[0,\pi]$. Multiplying (1.1) with $\lambda = \lambda_n$ by $\widetilde{y}(x,\widetilde{\lambda}_n)$, and $\mathbf{B}\widetilde{y}' + \int_0^x \widetilde{\mathbf{M}}(x-t)\widetilde{y}(t)dt = \lambda\widetilde{y}$ with $\lambda = \widetilde{\lambda}_n$ by $y(x,\lambda_n)$ (in the sense of scalar product in $\mathbb{R}^2$), respectively, subtracting the two equations and integrating from $x_n^j$ to $x_n^{j+1}$, we obtain

(3.9)

$$\begin{aligned}
0 = \int_{x_n^j}^{x_n^{j+1}} \int_0^x \Big(& p(x-t)y_1(t,\lambda_n)\widetilde{y}_1(x,\widetilde{\lambda}_n) + q(x-t)y_2(t,\lambda_n)\widetilde{y}_1(x,\widetilde{\lambda}_n) \\
&+ q(x-t)y_1(t,\lambda_n)\widetilde{y}_2(x,\widetilde{\lambda}_n) - p(x-t)y_2(t,\lambda_n)\widetilde{y}_2(x,\widetilde{\lambda}_n) \\
&- \widetilde{p}(x-t)\widetilde{y}_1(t,\widetilde{\lambda}_n)y_1(x,\lambda_n) - \widetilde{q}(x-t)\widetilde{y}_2(t,\widetilde{\lambda}_n)y_1(x,\lambda_n) \\
&- \widetilde{q}(x-t)\widetilde{y}_1(t,\widetilde{\lambda}_n)y_2(x,\lambda_n) + \widetilde{p}(x-t)\widetilde{y}_2(t,\widetilde{\lambda}_n)y_2(x,\lambda_n)\Big)dtdx \\
&+ (\widetilde{\lambda}_n - \lambda_n)\int_{x_n^j}^{x_n^{j+1}} \Big(y_1(t,\lambda_n)\widetilde{y}_1(t,\widetilde{\lambda}_n) + y_2(t,\lambda_n)\widetilde{y}_2(t,\widetilde{\lambda}_n)\Big)dt.
\end{aligned}$$

Since $c_0 = \widetilde{c}_0$, we get

$$\widetilde{\lambda}_n - \lambda_n = O(\frac{1}{n}). \tag{3.10}$$

On the other hand, from (2.2) we have

$$\begin{cases}
y_1(x,\lambda) = \sin\lambda x + \int_0^x \Big(Q_{11}(x,t)\sin\lambda t - Q_{12}(x,t)\cos\lambda t\Big)dt, \\
y_2(x,\lambda) = -\cos\lambda x + \int_0^x \Big(Q_{21}(x,t)\sin\lambda t - Q_{22}(x,t)\cos\lambda t\Big)dt.
\end{cases}$$

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Hence, integration by parts gives

$$\begin{aligned} y_1(x,\lambda) =& \sin\lambda x - Q_{11}(x,x)\frac{\cos\lambda x}{\lambda} - Q_{12}(x,x)\frac{\sin\lambda x}{\lambda} \\ &+ \int_0^x \Big(\frac{\partial Q_{11}(x,t)}{\partial x}\frac{\cos\lambda t}{\lambda} + \frac{\partial Q_{12}(x,t)}{\partial x}\frac{\sin\lambda t}{\lambda}\Big)dt, \\ y_2(x,\lambda) =& -\cos\lambda x - Q_{21}(x,x)\frac{\cos\lambda x}{\lambda} - Q_{22}(x,x)\frac{\sin\lambda x}{\lambda} \\ &+ \int_0^x \Big(\frac{\partial Q_{21}(x,t)}{\partial x}\frac{\cos\lambda t}{\lambda} + \frac{\partial Q_{22}(x,t)}{\partial x}\frac{\sin\lambda t}{\lambda}\Big)dt. \end{aligned}$$

Together with (2.10) this yields

$$\begin{cases} y_1(t,\lambda_n)\widetilde{y}_1(x,\widetilde{\lambda}_n) = \sin\lambda_n t\sin\widetilde{\lambda}_n x + O(\frac{1}{n}), \\ y_1(t,\lambda_n)\widetilde{y}_2(x,\widetilde{\lambda}_n) = -\sin\lambda_n t\cos\widetilde{\lambda}_n x + O(\frac{1}{n}), \\ \widetilde{y}_1(t,\widetilde{\lambda}_n)y_1(x,\lambda_n) = \sin\widetilde{\lambda}_n t\sin\lambda_n x + O(\frac{1}{n}), \\ \widetilde{y}_1(t,\widetilde{\lambda}_n)y_2(x,\lambda_n) = -\sin\widetilde{\lambda}_n t\cos\lambda_n x + O(\frac{1}{n}), \\ y_2(t,\lambda_n)\widetilde{y}_1(x,\widetilde{\lambda}_n) = -\cos\lambda_n t\sin\widetilde{\lambda}_n x + O(\frac{1}{n}), \\ y_2(t,\lambda_n)\widetilde{y}_2(x,\widetilde{\lambda}_n) = \cos\lambda_n t\cos\widetilde{\lambda}_n x + O(\frac{1}{n}), \\ \widetilde{y}_2(t,\widetilde{\lambda}_n)y_1(x,\lambda_n) = -\cos\widetilde{\lambda}_n t\sin\lambda_n x + O(\frac{1}{n}), \\ \widetilde{y}_2(t,\widetilde{\lambda}_n)y_2(x,\lambda_n) = \cos\widetilde{\lambda}_n t\cos\lambda_n x + O(\frac{1}{n}). \end{cases} \tag{3.11}$$

Substituting (3.10)-(3.11) into (3.9), we get

$$\begin{aligned} &\int_{x_n^j}^{x_n^{j+1}}\int_0^x \Big((\widetilde{p}(x-t)-p(x-t))\cos(\lambda_n(x+t)) \\ &\quad +(\widetilde{q}(x-t)-q(x-t))\sin(\lambda_n(x+t))\Big)dtdx + O(\frac{1}{n}) = 0. \end{aligned}$$

From the above equality and (3.8), we obtain

$$\begin{aligned} &\int_{x_n^j}^{x_n^{j+1}}\int_0^x \Big(\widetilde{p}(x-t)-p(x-t)\Big)\Big(\cos(\lambda_n(x+t))-\sin(\lambda_n(x+t))\Big)dtdx \\ &\quad + O(\frac{1}{n}) = 0 \end{aligned}$$

for all $n$, which means $p(x)=\widetilde{p}(x)$, $x\in[0,\pi]$. Hence, we obtain from (3.8) that $q(x)=\widetilde{q}(x)$ on $[0,\pi]$, which completes the proof. □

Let $X$ be the set of nodal points. For each fixed $x\in(0,\pi)$ we can choose a sequence $\{x_n^{j(n)}\}\subset X$ so that $x_n^{j(n)}$ converges to $x$. Then, the following limit exists and is finite:

$$\lim_{n\to\infty}\Big(x_n^{j(n)} - \frac{2j(n)\pi}{n}\Big) = f(x), \tag{3.12}$$

where $f(x)=I(x)$.

**Example 3.4.** Let $p(x) = q(x)$ and $\{x_n^j\} \subset X$ be the dense subset of nodal points in $(0,\pi)$ given by the following asymptotics:

$$x_n^j = \frac{2j\pi}{n} + \frac{1}{n}\int_0^{\frac{2j\pi}{n}} t\sin(\frac{2j\pi}{n} - t)\sin(2c_0 t)dt + O(\frac{1}{n^2}).$$

where $c_0 = \frac{1}{\pi}\arctan(\frac{1}{H})$. It can be calculated from (3.12) that

$$f(x) = \gamma_1 x\sin(2c_0x) + \gamma_2\Big(\cos(2c_0x) - \cos x\Big),$$

where

$$\gamma_1 = \frac{1}{2}(\frac{1}{1+2c_0} + \frac{1}{1-2c_0}), \quad \gamma_2 = \frac{1}{2}(\frac{1}{(1+2c_0)^2} - \frac{1}{(1-2c_0)^2}).$$

Let $H = 1$, then $c_0 = \frac{1}{4}$. Hence,

$$\gamma_1 = -\frac{2}{3}, \quad \gamma_2 = -\frac{16}{9},$$
$$p(x) = \sin x = q(x).$$

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