Nonexistence of global solutions for time fractional wave equations in an exterior domain

Jia Wei He

College of Mathematics and Information Science, Guangxi University
Nanning 530004, China

Abstract

This paper concerns two class of initial-boundary value problems of semilinear time fractional wave equations in an exterior domain. Based on a special technique of test function, we determine some critical exponents in the sense of Fujita for corresponding problem, we also obtain the existence and nonexistence of global weak solutions with certain appropriate conditions.

Keywords: Fractional derivative; Nonexistence; Critical exponents; Exterior domain.

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1 Introduction

In this paper, we consider two exterior domain problems for the following semilinear time fractional wave equations

$$\begin{cases} \partial_t^\alpha u - \Delta u = J_t^{1-\gamma}|u|^p, & (t,x) \in (0,\infty)\times D^c, \\ u(t,x) = f(x), & (t,x)\in(0,\infty)\times \partial D, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x\in D^c, \end{cases} \tag{1.1}$$

and the damped fractional wave equations with memory term

$$\begin{cases} \partial_t^\alpha u - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau,x)d\tau + \partial_t^\beta u = J_t^{1-\gamma}|u|^p, & (t,x) \in (0,\infty)\times D^c, \\ u(t,x) = f(x), & (t,x)\in(0,\infty)\times \partial D, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x\in D^c, \end{cases} \tag{1.2}$$

where $D = \overline{B(0,1)}$ is the closed unit ball in $\mathbb{R}^N$, $N \geq 2$, D^c is the exterior domain with obstacle D, i.e., $D^c = \mathbb{R}^N \setminus D$. $p > 1$ and ∂_t^α, ∂_t^β are the Caputo fractional derivative of order $\alpha \in (1,2)$ and $\beta \in (0,1)$ respectively, $J_t^{1-\gamma}$ is the Riemann-Liouville fractional integral of order $1-\gamma$, $u_i \in L^1_{loc}(\overline{D^c})$, $u_i \geq 0$, $i = 0,1$, $f \in L^1(\partial D)$, $\int_{\partial D} f(x)dS_x > 0$, g is a given function which satisfies some condition.

Noting that let $\alpha \to 2$ and $\gamma \to 1$, equation $(1.1)_1$ reduces to semilinear wave equation

$$\partial_{tt} u - \Delta u = |u|^p, \quad (t,x) \in (0,\infty)\times D^c. \tag{1.3}$$

Let $\alpha \to 2$, $\beta \to 1$, $g = 0$ and $\gamma \to 1$, equation $(1.2)_1$ reduces to semilinear damped wave equation

$$\partial_{tt}u - \Delta u + \partial_t u = |u|^p, \quad (t,x) \in (0,\infty) \times D^c. \tag{1.4}$$

Let us mentioned that the global existence and blow-up of solutions to semilinear wave equation

$$\partial_{tt}u - \Delta u = |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{1.5}$$

have been studied extensively, the work of John [9] in $\mathbb{R}^3$ proved that the critical exponent is $p = 1+\sqrt{2}$, Strauss considered in [21] that for each $N \geq 2$, there exists a critical exponent $p_1(N)$ (so called the Strauss critical power) of Fujita type for the equation (1.5), which is the larger positive root of the quadratic equation $(N-1)p^2 - (N+1)p - 2 = 0$. For more details, we refer to the papers [5, 6, 20] and the references therein. Additionally, as for equation (1.3) in an exterior domain, involving with corresponding initial-boundary value conditions have been studied in [29], Zhou and Han considered the blow-up of solutions of more general semilinear wave equations with variable coefficients on exterior domain, they showed the blow-up result when $1 < p < p_1(N)$ for $N \geq 3$. Li and Wang [14] showed a blow-up appearance when $1 < p < p_1(2)$ in the 2D case if $f \equiv 0$, while Lai and Zhou [13] studied the critical case $p = p_1(2)$. Jleli and Samet [8] showed the new results that the weak solutions of such problem must blow-up at finite time in the N-dimensional case:

(a) If $1 < p$ for $N = 2$ or $1 < p < \frac{N}{N-2}$ for $N \geq 3$, $u_i \in L^1_{loc}(\overline{D^c})$, $u_i \geq 0$, $i = 0, 1$ and $f \geq 0$, $f \not\equiv 0$, then the problem admits no global weak solutions.

(b) If $p > \frac{N}{N-2}$ for $N \geq 3$, then the problem have global weak solutions for some $f > 0$, u_0 and u_1.

Moreover, Lai and Yin [12] showed the blow-up solution of damped wave equation (1.4) in a finite time, with critical exponent $p = 1+\frac{N}{2}$ in an exterior domain for the N-dimensional case $N \geq 3$, related results also see [7, 8]. Kafini and Messaoudi [10] obtained a finite-time blow-up result for a nonlinear viscoelastic equation when $\alpha \to 2$ and $\beta \to 1$. For the integer order problems (1.3)-(1.4), there are also many literatures in an exterior domain, we refer the reader to [15–17, 22] and the references therein.

One, we list two motivations of this paper, is that fractional heat equation in an exterior domain has been studied by Jleli and Samet [8] for the case $N \geq 3$

$$\begin{cases} \partial_t^\alpha u - \Delta u = |u|^p, & (t,x) \in (0,\infty) \times D^c, \\ u(t,x) = f(x), & (t,x) \in (0,\infty) \times \partial D, \\ u(0,x) = u_0(x), & x \in D^c, \end{cases} \tag{1.6}$$

where $\alpha \in (0,1)$. The authors established that the critical exponent of (1.6) passes from $1+\frac{2}{N}$ to $\frac{N}{N-2}$, in which the exponent $p = 1+\frac{2}{N}$ is the critical value of heat equation (taking limit $\alpha = 1$, $f = 0$ and domain D^c is rewritten by $\mathbb{R}^N$ in (1.6)) in the sense of Fujita, see e.g. [3, 25]. On the other hand, let $g = 0$ and $\gamma \to 1$ in (1.2), the problem has been considered on a k-time halved space by Agarwal et al. [1], the authors showed that there is no global weak solutions with suitable initial data when $1 < p < 1 + \frac{2\beta}{(N+k)\beta+2(1-\beta)}$. Another point is that time fractional wave equation can be regarded as interpolation between the heat equation and the wave equation, also it governs intermediate processes between diffusion

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and wave propagation, which is usually applied to interpret as the superdiffusion behavior on diffusion process, for example, see [2, 4, 19, 23, 28].

To the best of our knowledge, there is still no paper to discuss the current problems, in order to fill this gaps, by applying the test function approach, we establish some critical exponents for problem (1.1) and (1.2) when $N=2$ and $N\geq 3$. We find that there are no global weak solutions of the exponents $1<p<\frac{N\alpha+2(1-\gamma)}{\alpha(N-2)}$ for $N\geq 3$ of problem (1.1) and $1<p<\frac{N\beta+2(1-\gamma)}{\beta(N-2)}$ for $N\geq 3$ of problem (1.2), respectively. Furthermore, passing the limit $\gamma\to 1$ throughout above inequalities, the critical exponents of problem (1.1) or problem (1.2) is still equal to $\frac{N}{N-2}$.

This paper is organized as follows: In Section 2, we recall some useful basic notions of fractional calculus. In Section 3, by means of the method of test function, we prove the main results.

2 Preliminaries

In this section, we first recall some definitions and useful properties of fractional calculus, for more derails, see [11, 27].

Definition 2.1. *Let $u\in L^1(0,T)$. The left-sided Riemann-Liouville fractional integral operator of $\alpha>0$ is defined by*

$$(J_t^\alpha u)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,\quad t>0,$$

the right-sided Riemann-Liouville fractional integral operator of α is defined by

$$(J_T^\alpha u)(t)=\frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}u(s)ds,\quad t>0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *Let $Q_T:=(0,T)\times D^c$. The Caputo fractional derivative of order $\alpha\in(0,1)$ for a function $u\in C^1(Q_T)$ is defined as follows*

$$\partial_t^\alpha u(t,x)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial}{\partial s}u(s,x)ds,$$

while the order $\alpha\in(1,2)$ for a function $u\in C^2(Q_T)$ is defined as follows

$$\partial_t^\alpha u(t,x)=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2}{\partial s^2}u(s,x)ds.$$

Lemma 2.1. *Let $f,g\in C[0,T]$. Then,*

$$\int_0^T (J_t^\alpha f)(t)g(t)dt=\int_0^T f(t)(J_T^\alpha g)(t)dt.$$

Let us introduce a function

$$\xi_T(t)=T^{-\lambda}(T-t)^\lambda,\quad 0\leq t\leq T,\quad \lambda\gg 1.$$

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By a simple calculation, it is easy to check that

$$(J_T^\eta \xi_T)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\eta+1)} T^{-\lambda}(T-t)^{\lambda+\eta}, \quad \eta > 0,\ 0 \le t < T. \tag{2.1}$$

Let $\Phi \in C_0^\infty[0,\infty)$ be a function satisfying

$$0 \le \Phi \le 1; \quad \Phi(\sigma) = \begin{cases} 1, & \text{if } 0 \le \sigma \le 1; \\ 0, & \text{if } 2 \le \sigma. \end{cases}$$

We also introduce

$$\varphi(t,x) = \xi_T(t)\Psi_T(x), \quad (t,x) \in Q_T, \tag{2.2}$$

where

$$\Psi_T(x) = \Upsilon(x)\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right), \quad r > 0,\ \omega \gg 1,\ x \in D^c,$$

and for any $x \in D^c$,

$$\Upsilon(x) = \begin{cases} \ln(|x|), & N = 2; \\ 1 - \dfrac{1}{|x|^{N-2}}, & N \ge 3. \end{cases}$$

For the case $N \ge 3$, noting that $\Upsilon(\cdot)$ is a solution to the exterior problem

$$\begin{aligned} -\Delta\Upsilon(x) &= 0, && \text{in } D^c, \\ \Upsilon(x) &= 0, && \text{on } \partial D, \\ \Upsilon(x) &\to 1, && \text{as } |x| \to \infty. \end{aligned}$$

Additionally, it follows from [8, 18] when $T \to \infty$ that function $\Psi_T(\cdot)$ satisfies

$$\int_{D^c} \Psi_T(x)dx = O(T^{2r}\ln T) \quad \text{for } N = 2; \quad \int_{D^c} \Psi_T(x)dx = O(T^{Nr}) \quad \text{for } N \ge 3. \tag{2.3}$$

For $x \in \partial D$, observe that

$$\nabla\Psi_T(x) = \nabla\left(\Upsilon(x)\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right)\right) = \begin{cases} \Phi^\omega\left(\dfrac{1}{T^{2r}}\right)x, & \text{if } N = 2; \\ (N-2)\Phi^\omega\left(\dfrac{1}{T^{2r}}\right)x, & \text{if } N \ge 3. \end{cases} \tag{2.4}$$

In fact, for $N = 2$, we first have

$$\begin{aligned} \nabla\Psi_T(x) =& \nabla\Upsilon(x)\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right) + \Upsilon(x)\nabla\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right) \\ =& \frac{x}{|x|^2}\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right) + \ln(|x|)\nabla\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right). \end{aligned}$$

Hence, it yields

$$\nabla\Psi_T(x) = \Phi^\omega\left(\frac{1}{T^{2r}}\right)x, \quad \text{for } x \in \partial D.$$

For $N \ge 3$, similarly

$$\nabla\Psi_T(x) = \frac{(N-2)x}{|x|^N}\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right) + \ln(|x|)\nabla\Phi^\omega\left(\frac{|x|^2}{T^{2r}}\right),$$

which shows the second identity of (2.4) for $x \in \partial D$.

3 Main results and proofs

Definition 3.1. *A function u is called to be a local weak solution to problem (1.1), if there exists $T>0$ such that $u\in L^p_{loc}(\overline{Q_T})$ $(p\geq 1)$ satisfying*

$$\int_{Q_T} u\partial_{tt}J_T^{2-\alpha}\varphi dxdt-\int_{Q_T}u\Delta\varphi dxdt=\int_{Q_T}|u|^pJ_T^{1-\gamma}\varphi dxdt-\int_{[0,T]\times\partial D}\partial_\nu\varphi f dS_x dt$$
$$+\int_{D^c}u_1(x)\varphi(0,x)dx-\int_{D^c}u_0(x)\partial_t\varphi(0,x)dx$$

for every function $\varphi\in C^2(Q_T)\cap C(\overline{Q_T})$ with
(i) $\partial_t\varphi(0,\cdot)\in C(\overline{D^c})$, $\partial_{tt}\varphi$, $\Delta\varphi\in C(\overline{Q_T})$;
(ii) $(J_T^{2-\alpha}\varphi)(T,\cdot)=0$, $\partial_t(J_T^{2-\alpha}\varphi)(T,\cdot)=0$;
(iii) $\varphi|_{\partial D}=0$, $\exists R>0$ s.t. $\varphi(\cdot,x)=0$ for $|x|\geq R$;
(iv) $\partial_\nu\varphi\in L^\infty([0,T]\times\partial D)$.
Furthermore, if $T>0$ can be arbitrarily chosen, then u is called a global weak solution to problem (1.1).

Definition 3.2. *A function u is called to be a local weak solution to problem (1.2), if there exists $T>0$ such that $u\in L^p_{loc}(\overline{Q_T})$ $(p\geq 1)$ satisfying*

$$\int_{Q_T} u\partial_{tt}J_T^{2-\alpha}\varphi dxdt-\int_{Q_T}u\Delta\varphi dxdt-\int_{Q_T}u\partial_tJ_T^{1-\beta}\varphi dxdt$$
$$=\int_{Q_T}|u|^pJ_T^{1-\gamma}\varphi dxdt-\int_{[0,T]\times\partial D}\partial_\nu\varphi f dS_x dt+\int_{D^c}(u_0(x)+u_1(x))\varphi(0,x)dx$$
$$-\int_{D^c}u_0(x)\partial_t\varphi(0,x)dx-\int_{Q_T}\int_s^T g(t-s)u(s)ds\Delta\varphi dxdt$$
$$+\int_{[0,T]\times\partial D}^T\int_0^t g(t-s)dsf(x)\partial_\nu\varphi dS_x dt.$$

for every function $\varphi\in C^2(Q_T)\cap C(\overline{Q_T})$ with
(i) $\partial_t\varphi,\partial_{tt}\varphi$, $\Delta\varphi\in C(\overline{Q_T})$;
(ii) $(J_T^{2-\alpha}\varphi)(T,\cdot)=0$, $\partial_t(J_T^{2-\alpha}\varphi)(T,\cdot)=0$;
(iii) $\varphi|_{\partial D}=0$, $\exists R>0$ s.t. $\varphi(\cdot,x)=0$ for $|x|\geq R$;
(iv) $\partial_\nu\varphi\in L^\infty([0,T]\times\partial D)$.
Furthermore, if $T>0$ can be arbitrarily chosen, then u is called a global weak solution to problem (1.1).

In order to determine the critical exponents for our problems, we need the following approximation results.

Lemma 3.1. *For $N=2$, let $m>1$, $m'=\frac{m}{m-1}$. we have*

$$\int_{Q_T}|(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}}|\partial_{tt}(J_T^{2-\alpha}\varphi)(t,x)|^{m'}dxdt=O\big(T^{1+2r-\frac{\alpha m+1-\gamma}{m-1}}\ln T\big),$$

as $T\to\infty$.

Proof. By using (2.2), we have

$$\begin{aligned}&\int_{Q_T}[(J_T^{1-\gamma}\varphi)(t,x)]^{-\frac{1}{m-1}}|\partial_{tt}(J_T^{2-\alpha}\varphi)(t,x)|^{m'}dxdt\\&=\left(\int_{D^c}\Psi_T(x)dx\right)\left(\int_0^T|(J_T^{1-\gamma}\xi_T)(t)|^{-\frac{1}{m-1}}|\partial_{tt}(J_T^{2-\alpha}\xi_T)(t)|^{m'}dt\right).\end{aligned}$$

Since from (2.1)

$$\partial_{tt}(J_T^{2-\alpha}\xi_T)(t)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}T^{-\lambda}(T-t)^{\lambda-\alpha},$$

it follows that

$$\int_0^T|(J_T^{1-\gamma}\xi_T)(t)|^{-\frac{1}{m-1}}|\partial_{tt}(J_T^{2-\alpha}\xi_T)(t)|^{m'}dt=O\left(T^{1-\frac{\alpha m+1-\gamma}{m-1}}\right),\quad \text{for } T>0.$$

Thus, from (2.3), we have the desired result.

□

Lemma 3.2. [8] *For $N=2$, let $m>1$, $m'=\frac{m}{m-1}$, we have*

$$\int_{D^c}|\Psi_T(x)|^{-\frac{1}{m-1}}|\Delta\Psi_T(x)|^{m'}dx=O\big(T^{2r(1-m')}\ln T\big),\quad \text{as } T\to\infty.$$

Lemma 3.3. *For $N=2$, let $m>1$, $m'=\frac{m}{m-1}$. we have*

$$\int_{Q_T}|(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}}|\Delta\varphi(t,x)|^{m'}dxdt=O\big(T^{\frac{m-2+\gamma}{m-1}+2r(1-m')}\ln T\big),\quad \text{as } T\to\infty.$$

Proof. By using (2.2), we have

$$\begin{aligned}&\int_{Q_T}|(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}}|\Delta\varphi(t,x)|^{m'}dxdt\\&=\left(\int_{D^c}|\Psi_T(x)|^{-\frac{1}{m-1}}|\Delta\Psi_T(x)|^{m'}dx\right)\left(\int_0^T|(J_T^{1-\gamma}\xi_T)(t)|^{-\frac{1}{m-1}}|\xi_T(t)|^{m'}dt\right).\end{aligned}$$

An elementary calculation shows that

$$\int_0^T|(J_T^{1-\gamma}\xi_T)(t)|^{-\frac{1}{m-1}}|\xi_T(t)|^{m'}dt=O\left(T^{\frac{m-2+\gamma}{m-1}}\right),\quad \text{for } T>0.$$

Thus, by virtue of Lemma 3.2 we have the desired result.

□

Lemma 3.4. *For $N=2$, let $m>1$, $m'=\frac{m}{m-1}$, we have*

$$\int_{Q_T}|(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}}|\partial_t(J_T^{1-\beta}\varphi)(t,x)|^{m'}dxdt=O\big(T^{1+2r-\frac{\beta m+1-\gamma}{m-1}}\ln T\big),$$

as $T\to\infty$.

Proof. By using (2.2), we have

$$\int_{Q_T} |(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}} |\partial_t (J_T^{1-\beta}\varphi)(t,x)|^{m'} dxdt$$
$$= \left(\int_{D^c} \Psi_T(x)dx\right)\left(\int_0^T |(J_T^{1-\gamma}\xi_T)(t)|^{-\frac{1}{m-1}} |\partial_t (J_T^{1-\beta}\xi_T)(t)|^{m'} dt\right).$$

Since

$$\partial_t (J_T^{1-\beta}\xi_T)(t) = -\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)} T^{-\lambda}(T-t)^{\lambda-\beta},$$

it follows that

$$\int_0^T |(J_T^{1-\gamma}\xi_T)(t)|^{-\frac{1}{m-1}} |\partial_t (J_T^{1-\beta}\xi_T)(t)|^{m'} dt = O\left(T^{1-\frac{\beta m+1-\gamma}{m-1}}\right), \quad \text{for } T>0.$$

Thus, we have the desired result.

□

Similarly to Lemma 3.1 and Lemma 3.3, we have the following results.

Lemma 3.5. *For $N \geq 3$, let $m>1$, $m' = \frac{m}{m-1}$, we have*

$$\int_{Q_T} |(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}} |\partial_{tt} (J_T^{2-\alpha}\varphi)(t,x)|^{m'} dxdt = O\big(T^{1+Nr-\frac{\alpha m+1-\gamma}{m-1}}\big),$$

as $T \to \infty$.

Lemma 3.6. *For $N \geq 3$, let $m>1$, $m' = \frac{m}{m-1}$. we have*

$$\int_{Q_T} |(J_T^{1-\gamma}\varphi)(t,x)|^{-\frac{1}{m-1}} |\Delta\varphi(t,x)|^{m'} dxdt = O\big(T^{\frac{m-2+\gamma}{m-1}+r(N-2m')}\big), \quad \text{as } T \to \infty.$$

Next, we show the first result of our problems.

Theorem 3.1. *Assume that $u_i \in L^1_{loc}(\overline{D^c})$, $u_i \geq 0$, $i=1,2$, and $f \in C(\partial D)$, $f \geq 0$, $f \not\equiv 0$. If*

$$p>1, \quad N=2; \quad 1<p<\frac{N\alpha+2(1-\gamma)}{\alpha(N-2)}, \quad N \geq 3.$$

Then, problem (1.1) admits no global weak solution.

Proof. The case of $N=2$. Suppose that u is a global weak solution to problem (1.1). Multiplying (1.1) by the function φ which is defined as in (2.2) and integrating over Q_T, we have

$$\int_{Q_T} \partial_t^\alpha u(t,x)\varphi(t,x)dxdt - \int_{Q_T} \Delta u(t,x)\varphi(t,x)dxdt$$
$$= \frac{1}{\Gamma(1-\gamma)} \int_{Q_T} \int_0^t (t-s)^{-\gamma}|u(s,x)|^p ds\varphi(t,x)dxdt.$$

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By a simple calculation, we determine that

$$\int_{Q_T} \partial_t^\alpha u(t,x)\varphi(t,x)dxdt = - C_1 \int_{D^c} u_1(x)\Psi_T(x)dx \\ - C_2 \int_{D^c} u_0(x)\Psi_T(x)dx + \int_{Q_T} u(t,x)\partial_{tt}(J_T^{2-\alpha}\varphi)(t,x)dxdt,$$

where

$$C_1 = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+3-\alpha)}T^{2-\alpha}, \quad C_2 = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)}T^{1-\alpha}. \tag{3.1}$$

Using Green's formula, we have

$$- \int_{Q_T} \Delta u(t,x)\varphi(t,x)dxdt = - \int_{Q_T} u(t,x)\Delta\varphi(t,x)dxdt + \int_{\partial Q_T} f(x)\partial_\nu\varphi(t,x)dS_x dt.$$

Therefore, it follows that

$$\int_{Q_T} |u(t,x)|^p (J_T^{1-\gamma}\varphi)(t,x)dxdt \\ = -C_1 \int_{D^c} u_1(x)\Psi_T(x)dx - C_2 \int_{D^c} u_0(x)\Psi_T(x)dx + \int_{Q_T} u(t,x)\partial_{tt}(J_T^{2-\alpha}\varphi)(t,x)dxdt \\ - \int_{Q_T} u(t,x)\Delta\varphi(t,x)dxdt + \int_{\partial Q_T} f(x)\partial_\nu\varphi(t,x)dS_x dt.$$

Since $u_i \geq 0$, $i = 0,1$, let

$$I_T = \int_{Q_T} |u(t,x)|^p (J_T^{1-\gamma}\varphi)(t,x)dxdt,$$

it yields

$$I_T - \int_{\partial Q_T} f(x)\partial_\nu\varphi dS_x dt \leq \int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt + \int_{Q_T} |u||\Delta\varphi|dxdt.$$

By the fact that $f \in C(\partial D)$, $f \geq 0$ and $f \not\equiv 0$, it yields $\int_{\partial D} f(x)dS_x > 0$. On the other hand, for any $(t,x) \in [0,T] \times \partial D$, one can find that

$$\partial_\nu\varphi(t,x) = -\langle\nabla\varphi(t,x), x\rangle = -\xi_T(t)\langle\nabla\Psi_T(x), x\rangle,$$

and therefore, it follows form (2.1) and (2.4) that

$$- \int_0^T \int_{\partial D} f(x)\partial_\nu\varphi(t,x)dS_x dt = \Phi^\omega\left(\frac{1}{T^{2r}}\right) \int_0^T \xi_T(t)dt \int_{\partial D} f(x)dS_x := C_f T > 0,$$

for T large enough and this implies that

$$I_T + C_f T \leq \int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt + \int_{Q_T} |u||\Delta\varphi|dxdt.$$

Now, writing

$$\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt = \int_{Q_T} |u|(J_T^{1-\gamma}\varphi)^{\frac{1}{p}}(J_T^{1-\gamma}\varphi)^{-\frac{1}{p}}|\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt.$$

By using Hölder inequality with parameters p and p' $(1/p + 1/p' = 1)$, it yields

$$\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt \leq I_T^{\frac{1}{p}} \left(\int_{Q_T} (J_T^{1-\gamma}\varphi)^{-\frac{1}{p-1}} |\partial_{tt}(J_T^{2-\alpha}\varphi)|^{p'} dxdt \right)^{\frac{1}{p'}}.$$

It follows from Lemma 3.1 that

$$\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt \leq I_T^{\frac{1}{p}} O\left(T^{\frac{(p-1)(1+2r)-\alpha p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right), \quad \text{as } T \to \infty.$$

On the other hand, by using Hölder inequality with the same parameters p and p', in view of Lemma 3.3, we have

$$\begin{aligned} \int_{Q_T} |u||\Delta\varphi|dxdt =& \int_{Q_T} |u|(J_T^{1-\gamma}\varphi)^{\frac{1}{p}} (J_T^{1-\gamma}\varphi)^{-\frac{1}{p}} |\Delta\varphi|dxdt \\ \leq& I_T^{\frac{1}{p}} \left(\int_{Q_T} (J_T^{1-\gamma}\varphi)^{-\frac{1}{p-1}} |\Delta\varphi|^{p'} dxdt \right)^{\frac{1}{p'}} \\ \leq& I_T^{\frac{1}{p}} O\left(T^{\frac{p-2+\gamma-2r}{p}} (\ln T)^{\frac{p-1}{p}} \right), \quad \text{as } T \to \infty. \end{aligned}$$

Therefore, above inequalities imply

$$I_T + C_f T \leq I_T^{\frac{1}{p}} \left[O\left(T^{\frac{(p-1)(1+2r)-\alpha p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right) + O\left(T^{\frac{p-2+\gamma-2r}{p}} (\ln T)^{\frac{p-1}{p}} \right) \right],$$

as $T \to \infty$. Observe that for $r = \frac{\alpha}{2} > 0$, we get

$$\frac{p-2+\gamma-2r}{p} = \frac{(p-1)(1+2r)-(1-\gamma)}{p} - \alpha = \frac{p-2+\gamma-\alpha}{p} := A,$$

which deduce that

$$I_T + C_f T \leq I_T^{\frac{1}{p}} O\left(T^A (\ln T)^B \right),$$

where $B = \frac{p-1}{p}$. Hence, for T large enough, there exists a $C' > 0$ which is independent on T such that

$$I_T + C_f T \leq C' I_T^{\frac{1}{p}} T^A (\ln T)^B. \tag{3.2}$$

Thus, it follows that

$$I_T \geq \left(\frac{C_f}{C'} \right)^p T^{(1-A)p} (\ln T)^{-Bp}.$$

Substituting this into the left-hand side of (3.2), we get

$$I_T \geq \frac{C_f^{p^2}}{C'^{p+p^2}} T^{p^2 - A(p+p^2)} (\ln T)^{-B(p+p^2)}.$$

Repeating the process above, for any $i \in \mathbb{N}$, we have

$$I_T \geq \frac{C_f^{p^i}}{C'^{p+p^2+\cdots+p^i}} T^{p^i - A(p+p^2+\cdots+p^i)} (\ln T)^{-B(p+p^2+\cdots+p^i)}.$$

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Similarly to [8, 26], one can see that

$$I_T \geq \left(\frac{1}{C'}\right)^{\frac{p}{p-1}} \left[C_f C'^{\frac{1}{1-p}} T^{1+\frac{Ap}{1-p}} (\ln T)^{\frac{Bp}{1-p}}\right]^{p^i} [T^{-A}(\ln T)^{-B}]^{\frac{p}{1-p}}. \tag{3.3}$$

Since $p > 1$ and function $x^a(\ln x)^{-1}$ has a minimum value ae for any $a > 0$, $x > 1$, it follows that there exists $T_0 > 0$ such that

$$C_f C'^{\frac{1}{1-p}} T_0^{1+\frac{Ap}{1-p}} (\ln T_0)^{\frac{Bp}{1-p}} > 1,$$

in which

$$1 + \frac{Ap}{1-p} = \frac{\alpha+1-\gamma}{p-1}, \quad \frac{Bp}{1-p} = -1.$$

Taking $T = T_0$ in (3.3), we have

$$I_{T_0} = \infty \quad \text{as } i \to \infty,$$

which contradict (3.2) that is accordant coincident with

$$I_{T_0} \leq [C' T_0^A (\ln T_0)^B]^{\frac{p}{p-1}} < \infty.$$

Hence, we get the desired conclusion.

The case of $N \geq 3$. Suppose that u is a global weak solution to (1.1). Multiplying (1.1) by the function φ and integrating over Q_T, we have

$$I_T - \int_{\partial Q_T} f(x)\partial_\nu \varphi dS_x dt \leq \int_{Q_T} |u||\partial_{tt}(I_T^{2-\alpha}\varphi)|dxdt + \int_{Q_T} |u||\Delta\varphi|dxdt.$$

By using a similar argument as in the proof of above arguments, we obtain

$$\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt \leq I_T^{\frac{1}{p}} O\left(T^{\frac{(p-1)(1+Nr)-\alpha p-(1-\gamma)}{p}}\right), \quad \text{as } T \to \infty,$$

and

$$\int_{Q_T} |u||\Delta\varphi|dxdt \leq I_T^{\frac{1}{p}} O\left(T^{\frac{p-2+\gamma+rN(p-1)-2rp}{p}}\right), \quad \text{as } T \to \infty.$$

On the other hand, by virtue of (2.4), one has

$$-\int_{Q_T} f\partial_\nu\varphi dS_x dt = (N-2)\Phi^\omega\left(\frac{1}{T^{2r}}\right)\int_0^T \xi_T(t)dt \int_{\partial D} f(x)dS_x := C_f T > 0.$$

Hence, for T large enough, there exists a $C' > 0$ which is independent on T such that

$$I_T + C_f T \leq C' I_T^{\frac{1}{p}} T^A,$$

where we set for $r = \frac{\alpha}{2} > 0$,

$$A = \frac{(p-1)(1+Nr)-(1-\gamma)}{p} - \alpha = \frac{p-2+\gamma+rN(p-1)}{p} - 2r.$$

Using Young's inequality, it yields

$$C' I_T^{\frac{1}{p}} T^A = \left(C' T^A p^{-\frac{1}{p}}\right)\left(p^{\frac{1}{p}} I_T^{\frac{1}{p}}\right) \le C'' T^{Ap'} + I_T,$$

where

$$C'' = \frac{C'^{p'} p^{-\frac{p'}{p}}}{p'} > 0.$$

Thus, we have

$$0 < \frac{C_f}{C''} \le T^{Ap'-1}.$$

From the fact $1 < p < \frac{N\alpha + 2(1-\gamma)}{\alpha(N-2)}$, passing the limit $T \to \infty$, it yields

$$0 < \frac{C_f}{C''} \le 0,$$

which deduce a contradiction, we thus get the conclusion. The proof is completed.

□

As for semilinear exterior Dirichlet problem

$$\begin{cases} -\Delta W_h = W_h, & \text{in } D^c, \\ W_h(x) = h(x), & \text{on } \partial D, \end{cases} \tag{3.4}$$

there is a existence result that was established by Zhang [26], see below lemma 3.7.

Lemma 3.7. *Let $N \ge 3$. If $p > \frac{N}{N-2}$ and $\|h\|_{L^\infty(\partial D)}$ is sufficiently small. Then the problem (3.4) has a positive solution.*

Remark 3.1. *In particular, let $\gamma \to 1$, we give $J_t^{1-\gamma}|u|^p = |u|^p$, by Theorem 3.1, we deduce the critical exponent $1 + \frac{2}{N-2}$ for $N \ge 3$, and then we can give a global positive solution of problem (1.1) (see Corollary 3.1). Moreover, from paper [24] considering the blow-up on $L^p(\mathbb{R}^N)$, the one of critical exponents for following equation*

$$\partial_t^\alpha u(t,x) - \Delta u(t,x) = |u(t,x)|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \quad \alpha \in (1,2),$$

is given by $1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$ with initial values $u_0 \ge 0$, $u_1 \equiv 0$. While on exterior problem by Theorem 3.1, the critical exponent for problem (1.1) shall pass from $1 + \frac{2\alpha}{\alpha N + 2 - 2\alpha}$ to $\frac{N}{N-2}$.

Corollary 3.1. *Let $N \ge 3$, $\gamma = 1$ as in (1.1) and*

$$p^* = 1 + \frac{2}{N-2}.$$

Assume that $u_i \in L^1_{loc}(\overline{D^c})$, $u_i \ge 0$, $i = 1, 2$, and $f \in C(\partial D)$, $f \ge 0$, $f \not\equiv 0$. Then,

(I) if $1 < p < p^*$, problem (1.1) admits no global weak solution.

(II) if $p > p^*$, problem (1.1) admits global weak solutions for some f, u_0 and u_1.

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Proof. The proof of (I) can be found in Theorem 3.1. Next, we check (II). Observe that for $f \in L^\infty(\partial D)$ with $\|f\|_{L^\infty(\partial D)}$ being sufficiently small, by Lemma 3.7, problem (3.4) admits a positive solution W_f. Denote

$$u(t,x) = W_f(x), \quad \text{for } (t,x) \in (0,\infty) \times D^c,$$

then u is a solution of problem (1.1) with $u_0 = W_f$, $u_1 = 0$. Thus, the conclusion holds. □

In the sequel, we show the second result. We present the assumption of function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$1 - \int_0^\infty g(s)ds = l > 0. \tag{3.5}$$

Theorem 3.2. *Assume that $u_i \in L^1_{loc}(\overline{D^c})$, $u_i \geq 0$, $i = 1, 2$, and $f \in C(\partial D)$, $f \geq 0$, $f \not\equiv 0$, g satisfies (3.5). If*

$$p > 1, \quad N = 2; \quad 1 < p < \frac{N\beta + 2(1-\gamma)}{\beta(N-2)}, \quad N \geq 3.$$

Then, problem (1.2) admits no global weak solution.

Proof. The case of $N = 2$. Suppose that u is a global weak solution to problem (1.2). Multiplying (1.2) by the function φ and integrating over Q_T, we have

$$\begin{aligned}\int_{Q_T} \partial_t^\alpha u(t,x)\varphi(t,x)dxdt - \int_{Q_T} \Delta u(t,x)\varphi(t,x)dxdt + \int_{Q_T} (g * \Delta u)(t,x)\varphi(t,x)dxdt \\ + \int_{Q_T} \partial_t^\beta u(t,x)\varphi(t,x)dxdt \\ = \frac{1}{\Gamma(1-\gamma)} \int_{Q_T} \int_0^t (t-s)^{-\gamma}|u(s,x)|^p ds\varphi(t,x)dxdt,\end{aligned}$$

where $*$ is the finite convolution defined by

$$(a * b)(t) = \int_0^t a(t-s)b(s)ds, \quad t > 0.$$

On the other hand, by applying Green's formula, we have

$$\begin{aligned}\int_{Q_T} (g * \Delta u)(t,x)\varphi(t,x)dxdt = &\int_0^T \int_0^t \int_{D^c} g(t-s)u(s,x)\xi_T(t)\Delta\Psi(x)dxdsdt \\ &- \int_0^T \int_0^t g(t-s) \int_{\partial D} f(x)\xi_T(t)\partial_\nu \Psi(x)dxdsdt.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\int_{Q_T} |u(t,x)|^p (J_T^{1-\gamma}\varphi)(t,x)dxdt \\ &= -C_1 \int_{D^c} u_1(x)\Psi(x)dx - C_2 \int_{D^c} u_0(x)\Psi(x)dx + \int_{Q_T} u(t,x)\partial_{tt}(J_T^{2-\alpha}\varphi)(t,x)dxdt \\ &- \int_{Q_T} u(t,x)\Delta\varphi(t,x)dxdt + \int_{\partial Q_T} f(x)\partial_\nu \varphi(t,x)dxdt\end{aligned}$$

$$
\begin{aligned}
&- C_3 \int_{D^c} u_0(x)\Psi(x)dx - \int_{Q_T} u(t,x)\partial_t(J_T^{1-\beta}\varphi)(t,x)dxdt \\
&+ \int_0^T \int_0^t \int_{D^c} g(t-s)u(s,x)\xi_T(t)\Delta\Psi(x)dxdsdt \\
&- \int_0^T \int_0^t g(t-s)\int_{\partial D} f(x)\xi_T(t)\partial_\nu\Psi(x)dxdsdt,
\end{aligned}
$$

where C_1, C_2 is defined in (3.1) and $C_3 := T^{1-\beta}\Gamma(\lambda+1)/\Gamma(\lambda+1-\beta)$. Since $u_i \geq 0$, $i = 0, 1$, let

$$
I_T = \int_{Q_T} |u(t,x)|^p (J_T^{1-\gamma}\varphi)(t,x)dxdt,
$$

it yields

$$
\begin{aligned}
&I_T - \int_{\partial Q_T} f(x)\partial_\nu\varphi dS_x dt + \int_0^T \int_0^t g(s)ds \int_{\partial D} f(x)\partial_\nu\varphi dS_x dt \\
&\leq \int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt + \int_{Q_T} |u||\partial_t(J_T^{1-\beta}\varphi)|dxdt + \int_{Q_T} |u||\Delta\varphi|dxdt \\
&\quad + \int_0^T \int_0^t \int_{D^c} g(t-s)|u(s,x)|\Delta\varphi(t,x)|dxdsdt.
\end{aligned}
$$

On the other hand, by virtue of the assumption (3.5) and the monotone decreasing function $\xi_T(\cdot)$, we have

$$
\begin{aligned}
&\int_0^T \int_0^t \int_{D^c} g(t-s)|u(s,x)||\Delta\varphi(t,x)|dxdsdt \\
&= \int_0^T \int_s^T g(t-s)\xi_T(t)dt \int_{D^c} |u(s,x)||\Delta\Psi(x)|dxds \\
&\leq \int_0^T \int_0^{T-s} g(z)dz\xi_T(s) \int_{D^c} |u(s,x)||\Delta\Psi(x)|dxds \\
&\leq l \int_{Q_T} |u||\Delta\varphi|dxdt.
\end{aligned}
$$

Hence, it from $\int_{\partial D} f(x)dS_x > 0$ follows that

$$
-\int_0^T \int_{\partial D} f(x)\partial_\nu\varphi(t,x)dS_x dt := C_f T > 0,
$$

which implies that

$$
\begin{aligned}
I_T + C_f T \leq &\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt + \int_{Q_T} |u||\partial_t(J_T^{1-\beta}\varphi)|dxdt \\
&+ (1+l)\int_{Q_T} |u||\Delta\varphi|dxdt.
\end{aligned}
$$

Now, writing

$$
\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt = \int_{Q_T} |u|(J_T^{1-\gamma}\varphi)^{\frac{1}{p}}(J_T^{1-\gamma}\varphi)^{-\frac{1}{p}}|\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt.
$$

By using Hölder inequality with conjugate parameters p and p', it yields

$$\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt \leq I_T^{\frac{1}{p}} \left(\int_{Q_T} (J_T^{1-\gamma}\varphi)^{-\frac{1}{p-1}} |\partial_{tt}(J_T^{2-\alpha}\varphi)|^{p'} dxdt \right)^{\frac{1}{p'}}.$$

It follows from Lemma 3.1 that

$$\int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt \leq I_T^{\frac{1}{p}} O\left(T^{\frac{(p-1)(1+2r)-\alpha p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right), \quad \text{as } T \to \infty.$$

Moreover,

$$\int_{Q_T} |u||\partial_{t}(J_T^{1-\beta}\varphi)|dxdt \leq I_T^{\frac{1}{p}} \left(\int_{Q_T} (J_T^{1-\gamma}\varphi)^{-\frac{1}{p-1}} |\partial_{t}(J_T^{1-\beta}\varphi)|^{p'} dxdt \right)^{\frac{1}{p'}}.$$

It follows from Lemma 3.4 that

$$\int_{Q_T} |u||\partial_{t}(J_T^{1-\beta}\varphi)|dxdt \leq I_T^{\frac{1}{p}} O\left(T^{\frac{(p-1)(1+2r)-\beta p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right), \quad \text{as } T \to \infty.$$

On the other hand, by using Hölder inequality with the same parameters p and p', in view of Lemma 3.3, we have

$$\begin{aligned}\int_{Q_T} |u||\Delta\varphi|dxdt &= \int_{Q_T} |u|(J_T^{1-\gamma}\varphi)^{\frac{1}{p}}(J_T^{1-\gamma}\varphi)^{-\frac{1}{p}}|\Delta\varphi|dxdt \\ &\leq I_T^{\frac{1}{p}} \left(\int_{Q_T} (J_T^{1-\gamma}\varphi)^{-\frac{1}{p-1}} |\Delta\varphi|^{p'} dxdt \right)^{\frac{1}{p'}} \\ &\leq I_T^{\frac{1}{p}} O\left(T^{\frac{p-2+\gamma-2r}{p}} (\ln T)^{\frac{p-1}{p}} \right), \quad \text{as } T \to \infty.\end{aligned}$$

Therefore, we have

$$\begin{aligned} I_T + C_f T \leq & I_T^{\frac{1}{p}} \Bigg[O\left(T^{\frac{(p-1)(1+2r)-\alpha p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right) + O\left(T^{\frac{(p-1)(1+2r)-\beta p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right) \\ & + O\left(T^{\frac{p-2+\gamma-2r}{p}} (\ln T)^{\frac{p-1}{p}} \right) \Bigg] \\ \leq & I_T^{\frac{1}{p}} O\left(T^{\frac{(p-1)(1+2r)-\beta p-(1-\gamma)}{p}} (\ln T)^{\frac{p-1}{p}} \right) + O\left(T^{\frac{p-2+\gamma-2r}{p}} (\ln T)^{\frac{p-1}{p}} \right) \Bigg] \end{aligned}$$

as $T \to \infty$. Observe that for $r = \frac{\beta}{2} > 0$, we get

$$\frac{p-2+\gamma-2r}{p} = \frac{(p-1)(1+2r)-(1-\gamma)}{p} - \beta = \frac{p-2+\gamma-\beta}{p} := A,$$

which deduce that

$$I_T + C_f T \leq I_T^{\frac{1}{p}} O\left(T^A (\ln T)^B \right),$$

where $B = \frac{p-1}{p}$. Hence, for T large enough, there exists a $C' > 0$ which is independent on T such that

$$I_T + C_f T \leq C' I_T^{\frac{1}{p}} T^A (\ln T)^B.$$

Repeating the process in Theorem 3.1, we get the conclusion.

The case of $N \geq 3$. Suppose that u is a global weak solution to problem (1.2). Multiplying (1.2) by the function φ and integrating over Q_T, we have

$$\begin{aligned} &I_T - \int_{\partial Q_T} f(x)\partial_\nu \varphi dS_x dt + \int_0^T \int_0^t g(s)ds \int_{\partial D} f(x)\partial_\nu \varphi dS_x dt \\ &\leq \int_{Q_T} |u||\partial_{tt}(J_T^{2-\alpha}\varphi)|dxdt + \int_{Q_T} |u||\partial_t(J_T^{1-\beta}\varphi)|dxdt + \int_{Q_T} |u||\Delta\varphi|dxdt \\ &\quad + \int_0^T \int_0^t \int_{D^c} g(t-s)|u(s,x)|\Delta\varphi(t,x)|dxdsdt. \end{aligned}$$

By using a similar argument as in the proof of above arguments, for T large enough, there exists a $C' > 0$ which is independent on T such that

$$I_T + C_f T \leq C' I_T^{\frac{1}{p}} T^A,$$

where

$$A = \frac{(p-1)(1+Nr) - (1-\gamma)}{p} - \beta.$$

Using Young's inequality, it yields

$$C' I_T^{\frac{1}{p}} T^A = \left(C' T^A p^{-\frac{1}{p}}\right)\left(p^{\frac{1}{p}} I_T^{\frac{1}{p}}\right) \leq C'' T^{Ap'} + I_T,$$

where C'' is a positive constant. Thus, we have

$$0 < \frac{C}{C''} \leq T^{Ap'-1}.$$

Passing the limit $T \to \infty$, we thus deduce a contradiction by fact $1 < p < \frac{N\beta + 2(1-\gamma)}{\beta(N-2)}$. The proof is completed.

□

Similarly to Corollary 3.1, one can check the following conclusion.

Corollary 3.2. *Let* $N \geq 3$, $\gamma = 1$ *as in (1.2) and*

$$p^* = 1 + \frac{2}{N-2}.$$

Assume that $u_i \in L^1_{loc}(\overline{D^c})$, $u_i \geq 0$, $i = 1, 2$, *and* $f \in C(\partial D)$, $f \geq 0$, $f \not\equiv 0$, g *satisfies (3.5). Then,*

(I) if $1 < p < p^*$, *problem (1.2) admits no global weak solution.*

(II) if $p > p^*$, *problem (1.2) admits global weak solutions for some* f, u_0 *and* u_1.

Remark 3.2. *It is notice that if* $g = 0$ *in problem (1.2), we establish the same results for damped fractional wave equations which will be derived from Theorem 3.2 and Corollary 3.2. Also, the existence results are somewhat rudimentary in this paper since we mainly obtain the blow-up of solutions.*

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