NON-LIFTABLE ABELIAN AUTOMORPHISM GROUPS OF SMOOTH SURFACES IN $\mathbb{P}^3$

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Abstract. Let X be a smooth hypersurface of degree $d \geq 3$ in the projective space $\mathbb{P}^{n+1}$. If $(n,d) \neq (1,3),(2,4)$ then a finite group G acts on X faithfully is a subgroup of the Projective linear group $\mathrm{PGL}(n+2,\mathbb{C})$. Sufficient conditions are known for G to be lifted to a subgroup of the general linear group $\mathrm{GL}(n+2,\mathbb{C})$. In this paper, we assume that G is a finite abelian subgroup of $\mathrm{PGL}(4,\mathbb{C})$ such that G can not be lifted to $\mathrm{GL}(4,\mathbb{C})$. We determine G by using the action of G on linear subspaces of $\mathbb{P}^3$. As an application, we determine non-liftable abelian groups of $\mathrm{PGL}(4,\mathbb{C})$ acting faithfully on smooth hypersurfaces of degree d in $\mathbb{P}^3$. We give examples of a smooth hypersurface $X \subset \mathbb{P}^3$ and a non-liftable abelian group G such that G acts on X faithfully.

1. Introduction

In this paper, we work over $\mathbb{C}$. For $n \geq 1$, let $\mathrm{GL}(n,\mathbb{C})$ be the general linear group of $\mathbb{C}^n$, and let I_n be the identity matrix of size n. We set $Z(\mathbb{C}^n) := \{A \in \mathrm{GL}(n,\mathbb{C}) \,|\, A = aI_n \text{ for some } a \in \mathbb{C}^*\}$ where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. The projective linear group $\mathrm{PGL}(n,\mathbb{C})$ is the quotient group $\mathrm{GL}(n,\mathbb{C})/Z(\mathbb{C}^n)$. Let $p_n\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$ be the quotient map. For a matrix $A \in \mathrm{GL}(n,\mathbb{C})$, we write $p_n(A)$ as $[A]$. Let $\alpha, \beta \in \mathrm{PGL}(n,\mathbb{C})$ be elements. We say that α is conjugate to β if $\beta = \gamma\alpha\gamma^{-1}$ for some $\gamma \in \mathrm{PGL}(n,\mathbb{C})$. Let G, H be subgroups of $\mathrm{PGL}(n,\mathbb{C})$. We say that G is conjugate to H if $H = \gamma G \gamma^{-1}$ for some $\gamma \in \mathrm{PGL}(n,\mathbb{C})$.

Definition 1.1. *Let G be a subgroup of $\mathrm{PGL}(m,\mathbb{C})$ for $m \geq 2$. We call G liftable (resp. non-liftable) if there is (resp. is not) a subgroup G' of $\mathrm{GL}(m,\mathbb{C})$ such that $p_{m|G'}\colon G' \to G$ is isomorphic.*

Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, and let $\mathrm{Lin}(X)$ be the subgroup of $\mathrm{Aut}(X)$ consisting of the elements induced by $\mathrm{PGL}(n+2,\mathbb{C})$ where $n \geq 1$ and $3 \geq 4$. By [10, Theorem 1 and Theorem 2], $\mathrm{Lin}(X)$ is a finite group for $n \geq 2$ and $d \geq 3$, and $\mathrm{Aut}(X) = \mathrm{Lin}(X)$ for $d \geq 3$ except for the case where $(n,d) = (2,4)$. For a fixed integer $d \geq 4$, the list of groups that appear as automorphism groups of smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ is unknown except for a finite number of n and d pairs ([1,8,9,12,13,14]). Smooth hypersurfaces in $\mathbb{P}^{n+1}$ and their automorphism groups are studied by abelian subgroups of automorphism groups. The case where an abelian subgroup is a cyclic group are studied in [2,3,4,6], and the case where an abelian subgroup is $\mathbb{Z}/\mathbb{Z}^{\oplus n}$ is studied in [7].

Date: July 3, 2024.

2020 *Mathematics Subject Classification.* Primary 20G20, 14J50; Secondary 15B99, 20K01.

Key words and phrases. Automorphism groups; Projective linear group; General linear group; Finite abelian group.

Let G be a finite abelian subgroup of $\mathrm{PGL}(n+2,\mathbb{C})$ such that G acts faithfully on a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. If G is lifted to a subgroup of $\mathrm{GL}(n+2,\mathbb{C})$ and there exists an element $g \in G$ such that $G/\langle g\rangle$ has order coprime to $d-1$, then all possible G are determined ([15, Theorem 4.3]). In this paper, we determine non-liftable finite abelian subgroups of $\mathrm{PGL}(4,\mathbb{C})$.

Theorem 1.2. *Let $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable finite abelian group. Then G is one of the following:*

(i) $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ *and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4,\mathbb{C})$ generated by*

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right] \text{ and } \left[\begin{pmatrix} 0 & C \\ C^{-1} & 0 \end{pmatrix}\right]$$

where $C \in \mathrm{GL}(2,\mathbb{C})$.

(ii) $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$ *and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4,\mathbb{C})$ generated by*

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \text{ and } \left[\begin{pmatrix} 0 & B \\ B^{-1} & 0 \end{pmatrix}\right]$$

such that

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ or } \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

where $a, b \in \mathbb{C}^*$.

(iii) $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$ *and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4,\mathbb{C})$ generated by*

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a^{-1} & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & x & 0 & 0 \\ x^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & y^{-1} & 0 \end{pmatrix}\right], \text{ and}$$

$$\left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

such that $bx = ay$ where $a, b, x, y \in \mathbb{C}$.

(iv) $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/l\mathbb{Z}$ *and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4,\mathbb{C})$ generated by*

$$\left[\begin{pmatrix} e_l I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \text{ and } \left[\begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}\right]$$

where $a, b \in \mathbb{C}^*$.

(v) *$G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2l\mathbb{Z}$ and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4,\mathbb{C})$ generated by*

$$\left[\begin{pmatrix} -e_{2l} & 0 & 0 & 0 \\ 0 & e_{2l} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \text{ and } \left[\begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}\right]$$

where $a, b \in \mathbb{C}^*$.

Theorem 1.2 is followed by Theorem 3.2, Theorem 3.5, Theorem 3.7, Theorem 3.9, and Theorem 3.11. In addition, we determine non-liftable abelian groups of $\mathrm{PGL}(4,\mathbb{C})$ acting faithfully on a smooth hypersurface of degree d in $\mathbb{P}^3$. Let d be an even integer. For the each case (i), (ii), and (iii) of Theorem 1.2, there is a non-liftable abelian group $G \subset \mathrm{PGL}(4,\mathbb{C})$ and a smooth hypersurface $X \subset \mathbb{P}^3$ of degree d such that G acts on X faithfully (Examples 3.8, 3.10, and 3.12). Let $H \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable abelian group. If H is the case (iv) (resp. (v)) of Theorem 3.6 and H acts on $Y \subset \mathbb{P}^3$ faithfully where Y is a smooth hypersurface of degree d, then l divides $d-2$ or d (resp. l divides $2(d-1)$) (Theorem 4.2, Example 4.3, Example 4.4, and Example 4.5). In Section 2 we introduce some preliminary results and notations. Especially, we give sufficient conditions for abelian subgroups of $\mathrm{PGL}(4,\mathbb{C})$ to lift to $\mathrm{GL}(4,\mathbb{C})$. In Section 3, we study non-liftable finite abelian subgroups of $\mathrm{PGL}(4,\mathbb{C})$ by using linear subspaces of $\mathbb{P}^3$ and the sufficient conditions. In section 4, we complete the determination of non-liftable abelian groups of $\mathrm{PGL}(4,\mathbb{C})$ acting faithfully on smooth hypersurfaces of degree d in $\mathbb{P}^3$. The classification is accomplished by Theorems 3.7, 3.9, 3.11, and 4.2, as well as Examples 3.8, 3.10, 3.12, 4.3, 4.4, and 4.5. Finally, please note that the proof methods for the results presented in this paper rely on matrix computations. Therefore, the results hold for any algebraically closed field of characteristic zero.

2. Preliminary

We prepare a little. Let G be a subgroup of $\mathrm{PGL}(n+2,\mathbb{C})$ for $n \geq 1$. We call G F-liftable if there are a subgroup G' of $\mathrm{GL}(n+2,\mathbb{C})$ and a homogeneous polynomial F such that $p_{n+2|G'} : G' \to G$ is isomorphic, the hypersurface $X \subset \mathbb{P}^{n+1}$ defined by $F = 0$ is smooth, and $A^*F = F$ for $A \in G'$.

Theorem 2.1. ([5, Proposition 1.15]). *For $n \geq 1$ and $d \geq 3$, the automorphism group of every smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ is F-liftable if and only if d and $n+2$ are relatively prime.*

Let $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable finite abelian group such that G acts on a smooth hypersurface of degree d in $\mathbb{P}^3$ faithfully. By Theorem 2.1, d is an even number.

There are sufficient conditions for finite subgroups of $\mathrm{PGL}(n+2,\mathbb{C})$ to lift to $\mathrm{GL}(n+2,\mathbb{C})$ ([12, Theorem 4.8],[5, Proposition 4.7]). The following Lemma 2.2 is a sufficient condition for an abelian subgroup of $\mathrm{PGL}(n+2,\mathbb{C})$ to lift to $\mathrm{GL}(n+2,\mathbb{C})$.

Lemma 2.2. ([7, Lemma 3.1]). *Let $G \subset \mathrm{PGL}(n+2,\mathbb{C})$ be a finite abelian group where $n \geq 1$. If there is an element $g = [A] \in G$ such that A is conjugate to*

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 I_{n+1} \end{pmatrix}$$

where $a_1, a_2 \in \mathbb{C}^*$ are distinct complex numbers, then G is liftable.

In Lemma 2.3, we give a new sufficient condition for $n = 2$. After that, we study group structures of non-liftable abelian finite subgroups of $\mathrm{PGL}(4, \mathbb{C})$ by using linear subspaces of $\mathbb{P}^3$ and the fact that they does not satisfy Lemma 2.2 and Lemma 2.3.

Lemma 2.3. *Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group. If there is an element $g = [A] \in G$ such that A is conjugate to*

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 I_2 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{C}^*$ are pairwise distinct complex numbers, then G is liftable.

Proof. We assume that there is an element $g = [A] \in G$ such that

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 I_2 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{C}^*$ such that $a_i \neq a_j$ for $1 \leq i < j \leq 3$. Let $W_i \subset \mathbb{C}^4$ be the eigenspace of A associated with a_i for $i = 1, 2, 3$. Note that

$$W_1 = \{(x, y, z, w) \in \mathbb{C}^4 \mid y = z = w = 0\},$$
$$W_2 = \{(x, y, z, w) \in \mathbb{C}^4 \mid x = z = w = 0\}, \text{ and}$$
$$W_3 = \{(x, y, z, w) \in \mathbb{C}^4 \mid x = y = 0\}.$$

We take an element $h = [B] \in G$ where $B \in \mathrm{GL}(4, \mathbb{C})$. Since $gh = hg$, $AB = tBA$ for some $t \in \mathbb{C}^*$. Then for $i = 1, 2, 3$,

$$BW_i \in \{W_1, W_2, W_3\}.$$

Since $\dim W_i = 1$ for $i = 1, 2$, and $\dim W_3 = 2$,

$$\{BW_1, BW_2\} = \{W_1, W_2\} \quad \text{and} \quad BW_3 = W_3.$$

As a result,

$$B = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$$

where $C, D \in \mathrm{GL}(2, \mathbb{C})$. We set $C = (c_{ij})_{i,j=1,2}$. Since $AB = tBA$,

$$\begin{pmatrix} a_1 c_{11} & a_1 c_{12} & 0 \\ a_2 c_{21} & a_2 c_{22} & 0 \\ 0 & 0 & a_3 D \end{pmatrix} = \begin{pmatrix} t a_1 c_{11} & t a_2 c_{12} & 0 \\ t a_1 c_{21} & t a_2 c_{22} & 0 \\ 0 & 0 & t a_3 D \end{pmatrix}.$$

Since $D \in \mathrm{GL}(2, \mathbb{C})$ and $a_3 \neq 0$, $t = 1$. Since $a_1 \neq a_2$, $c_{12} = c_{21} = 0$. As a result,

$$B = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Therefore, for a generating set $g_1, \ldots, g_k$ of G, there are complex numbers $a_i, b_i \in \mathbb{C}^*$ and matrices $D_i \in \mathrm{GL}(4, \mathbb{C})$ such that $g_i = \left[\begin{pmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & D_i \end{pmatrix}\right]$ for $i = 1, \ldots, k$. Since $g_i g_j = g_j g_i$, $D_i D_j = D_j D_i$ for $1 \leq i < j \leq k$. Therefore, there is a matrix $M \in \mathrm{GL}(2, \mathbb{C})$ such that $M D_i M^{-1}$ is a diagonal matrix for each $i = 1, \ldots, k$. Since

G is conjugate to a subgroup of $\mathrm{PGL}(4,\mathbb{C})$ generated by diagonal matrices, G is liftable. $\square$

Let G be a group. For an element $g \in G$, let $\mathrm{ord}(g)$ be the order of g. By Lemma 2.2 and Lemma 2.3, we have the following.

Lemma 2.4. *Let G be a non-liftable finite abelian subgroup of $\mathrm{PGL}(4,\mathbb{C})$. We take an element $g = [A] \in G\backslash\{e\}$. Then the matrix A is conjugate to*

$$\begin{pmatrix} a_1 I_2 & 0 \\ 0 & a_2 I_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}^*$ are pairwise distinct complex numbers.

Let G be a finite subgroup of $\mathrm{PGL}(4,\mathbb{C})$. We set $e := [I_4] \in \mathrm{PGL}(4,\mathbb{C})$. For $g = [A] \in G\backslash\{e\}$, we write $r(g)$ as the number of different eigenvalues of the matrix A. Let $A' \in \mathrm{GL}(4,\mathbb{C})$ be a matrix such that $g = [A']$. Since $A = tA'$ for some $t \in \mathbb{C}^*$, the number $r(g)$ does not depend on a matrix representing g. Let $d := \mathrm{ord}(g)$. Then $r(g) \leq \min\{d, 4\}$. We set $r := r(g)$. Let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues of A, and let $W_i \subset \mathbb{C}^4$ be the eigenspace of A associated with λ_i for $i = 1, \dots, r$. We take an element $h = [B] \in G$. Since $gh = hg$, $AB = tBA$ for some $t \in \mathbb{C}^*$. For each $i = 1, \dots, r$

$$BW_i \in \{W_1, \dots, W_r\}.$$

Let j be an integer such that $BW_i = W_j$. Since $B \in \mathrm{GL}(4,\mathbb{C})$, $\dim W_i = \dim W_j$. Let $\mathbb{P}(W_i) \subset \mathbb{P}^3$ be the projective subspace associated with W_i for $i = 1, \dots, r$. For $i = 1, \dots, r$,

$$h(\mathbb{P}(W_i)) \in \{\mathbb{P}(W_1), \dots, \mathbb{P}(W_r)\}.$$

There is the group homomorphism $\Psi_g : G \to \mathcal{S}_r$ such that for $h \in G$ and $i = 1, \dots, r$

$$h(\mathbb{P}(W_i)) = \mathbb{P}(W_{\Psi_g(h)(i)}).$$

Here, $\mathcal{S}_r$ is the symmetric group of degree r.

Since the order of g is finite, the matrix A is diagonalizable. We may assume that

$$A = \begin{pmatrix} \lambda_1 I_{m_1} & & & \\ & \lambda_2 I_{m_2} & & \\ & & \ddots & \\ & & & \lambda_r I_{m_r} \end{pmatrix}.$$

Then for $h = [B] \in \mathrm{Ker}\,\Psi_g$, there is a matrix $B_i \in \mathrm{GL}(m_i,\mathbb{C})$ for $i = 1, \dots, r$ such that

$$B = \left[\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{pmatrix}\right].$$

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3. Classification of nonliftable abelian subgroups of $\mathrm{PGL}(4,\mathbb{C})$

Lemma 3.1. *Let $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable finite abelian group. If G does not contain an element $g = [A] \in G$ such that A is conjugate to*

$$\begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers, then $|G|$ is an odd number.

Proof. If $|G|$ is an even number, then there is an element $k \in G$ such that $\mathrm{ord}(k) = 2$. Let $C \in \mathrm{GL}(4,\mathbb{C})$ be a matrix such that $k = [C]$. Then $C^2 = uI_4$ for some $u \in \mathbb{C}^*$. This implies that the eigenvalues of C are $\sqrt{u}$ or $-\sqrt{u}$. By Lemma 2.4, this contradicts that G is non-liftable. □

For a positive integer $l \geq 2$, let e_l be a primitive l-th root of unity.

Theorem 3.2. *Let $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable finite abelian group. Then there is an element $g = [A] \in G$ such that A is conjugate to*

$$\begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers.

Proof. We assume that G does not contain an element $g = [A] \in G$ such that A is conjugate to $\begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix}$ where $a, b \in \mathbb{C}^*$ are distinct complex numbers. We take an element $h = [B] \in G \backslash \{e\}$. By Lemma 2.4, we may assume that

$$B = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{pmatrix}$$

where $b_1, b_2, b_3, b_4 \in \mathbb{C}^*$ such that $b_i \neq b_j$ for $1 \leq i < j \leq 4$.

Let $\Psi_h \colon G \to \mathcal{S}_4$ be the group homomorphism defined by h. Let $k \in G \backslash \{e\}$. Since the dimension of the eigenspace of B associated with b_i is one for $1 \leq i \leq 4$, if $k \in \mathrm{Ker}\,\Psi_h$ then k is defined by a diagonal matrix. Then $\mathrm{Ker}\,\Psi_h \subset \mathrm{PGL}(4,\mathbb{C})$ is liftable. Since G is non-liftable, $G \neq \mathrm{Ker}\,\Psi_h$. In particular, $\mathrm{Im}\,\Psi_h$ is not trivial. We take an element $k \in G \backslash \mathrm{Ker}\,\Psi_h$. By Lemma 3.1, $|G|$ is an odd number, and hence $\mathrm{ord}(k)$ is an odd number. Since $\mathrm{Im}\,\Psi_h \subset \mathcal{S}_4$, $k^3 \in \mathrm{Ker}\,\Psi_h$. Then we may assume that there is a matrix $C \in \mathrm{GL}(4,\mathbb{C})$ such that

$$k = [C] \quad \text{and} \quad C = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{C}^*$. Since G is non-liftable, Lemma 2.3, and

$$C^3 = \begin{pmatrix} abcI_3 & 0 \\ 0 & 1 \end{pmatrix},$$

we get that $\mathrm{ord}(k) = 3$, and hence $C^3 = I_4$. Then the eigenvalues of C are 1, e_3, or e_3^2. By Lemma 2.2, Lemma 2.3, and Lemma 2.4, this contradicts the assumption of G. Therefore, G contains an element $g = [A] \in G$ such that A is conjugate to $\begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix}$ where $a, b \in \mathbb{C}^*$ are distinct complex numbers. □

Let G be a group, and let $S \subset G$ be a subset. Let $\langle S \rangle \subset G$ be the subgroup of G generated by S. If $\langle S \rangle = G$, then S is called a generating set of G.

Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group. Let $g = [A] \in G$ be an element such that $A = \begin{pmatrix} e_l I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ where $l := \mathrm{ord}(g)$. Let $W_1, W_2 \subset \mathbb{C}^4$ be the eigenspace of A associated with e_l and 1, respectively. Note that $W_1 = \{(x, y, z, w) \in \mathbb{C}^4 \,|\, z = w = 0\}$ and $W_2 = \{(x, y, z, w) \in \mathbb{C}^4 \,|\, x = y = 0\}$. Let $\mathbb{P}(W_i)$ be the projective subspace associated with W_i for $i = 1, 2$. Since $\dim W_i = 2$, $\mathbb{P}(W_i) \cong \mathbb{P}^1$ for $i = 1, 2$. Let $h \in G$ be an element. If $h \in \mathrm{Ker}\,\Psi_g$, then $h(\mathbb{P}(W_i)) = \mathbb{P}(W_i)$ for $i = 1, 2$. In addition, if $h(\mathbb{P}(W_1)) = \mathbb{P}(W_1)$ or $h(\mathbb{P}(W_2)) = \mathbb{P}(W_2)$, then $h \in \mathrm{Ker}\,\Psi_g$.

Lemma 3.3. *In the above setting, for $h \in G$, if $h_{|\mathbb{P}(W_1)} = \mathrm{id}_{\mathbb{P}(W_1)}$ then $h_{|\mathbb{P}(W_2)} = \mathrm{id}_{\mathbb{P}(W_2)}$.*

Proof. We assume that $h_{|\mathbb{P}(W_1)} = \mathrm{id}_{\mathbb{P}(W_1)}$. Then $h \in \mathrm{Ker}\,\Psi_g$, and there is a matrix $A_{22} \in \mathrm{GL}(2, \mathbb{C})$ such that

$$h = [A] \ \text{ and } \ A = \begin{pmatrix} I_2 & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Since G is a finite group, the order of g is finite. Then A is a diagonalize, and hence A_{22} is a diagonalize. The matrix A_{22} is conjugate to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or aI_2 where $a, b \in \mathbb{C}^*$ are distinct complex numbers. Since G is non-liftable, Lemma 2.2, and Lemma 2.3, we get that A_{22} is conjugate to aI_2, and hence $A_{22} = aI_2$. As a result, $h_{|\mathbb{P}(W_2)} = \mathrm{id}_{\mathbb{P}(W_2)}$. □

We take $h \in \mathrm{Ker}\,\Psi_g$. There are matrices $A_{11}, A_{22} \in \mathrm{GL}(2, \mathbb{C})$ such that $h = \left[\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\right]$. Then

$$\varphi_1 \colon \mathrm{Ker}\,\Psi_g \ni h \longmapsto [A_{11}] \in \mathrm{PGL}(2, \mathbb{C}),$$

and

$$\varphi_2 \colon \mathrm{Ker}\,\Psi_g \ni h \longmapsto [A_{22}] \in \mathrm{PGL}(2, \mathbb{C})$$

are group homomorphisms. By Lemma 3.3, $\mathrm{Ker}\,\varphi_1 = \mathrm{Ker}\,\varphi_2$. We set

$$I_g := \mathrm{Ker}\,\varphi_1 \subset G.$$

If $h \in I_g$, then $A_{11} = aI_2$ and $A_{22} = bI_2$ for $a, b \in \mathbb{C}^*$. Then I_g is a cyclic group and

$$I_g = \left\langle \left[\begin{pmatrix} e_l I_2 & 0 \\ 0 & I_2 \end{pmatrix} \right] \right\rangle$$

where $l := |I_g|$.

Lemma 3.4. *In the above setting, we get that $\mathrm{Im}\,\varphi_1 \cong \mathrm{Im}\,\varphi_2$. In particular, let $g_1, \ldots, g_k \in \mathrm{Ker}\,\Psi_g$ be elements such that $\varphi_1(g_1), \ldots, \varphi_1(g_k)$ are a generating set of $\mathrm{Im}\,\varphi_1$. Then $\varphi_2(g_1), \ldots, \varphi_2(g_k)$ are a generating set of $\mathrm{Im}\,\varphi_2$.*

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Proof. Since $\operatorname{Ker}\varphi_1 = \operatorname{Ker}\varphi_2$,

$$\operatorname{Im}\varphi_1 \cong \operatorname{Ker}\Psi_g/\operatorname{Ker}\varphi_1 = \operatorname{Ker}\Psi_g/\operatorname{Ker}\varphi_2 \cong \operatorname{Im}\varphi_2.$$

In particular, since the isomorphism $\operatorname{Im}\varphi_i \cong \operatorname{Ker}\Psi_g/\operatorname{Ker}\varphi_i$ of groups given by φ_i and for $i = 1, 2$, we get that for elements $g_1, \ldots, g_k \in \operatorname{Ker}\Psi_g$ if $\varphi_1(g_1), \ldots, \varphi_1(g_k)$ are a generating set of $\operatorname{Im}\varphi_1$, then $\varphi_2(g_1), \ldots, \varphi_2(g_k)$ are a generating set of $\operatorname{Im}\varphi_2$. $\square$

A finite subgroup of $\mathrm{PGL}(2, \mathbb{C})$ is isomorphic to one of a cyclic group, a dihedral group D_l of order $2l$, the tetrahedral group A_4, the octahedral group S_4, and the icosahedral group A_5 ([11, Chapter X]). Since $\operatorname{Im}\varphi_1 \subset \mathrm{PGL}(2, \mathbb{C})$ is a finite abelian subgroup, $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ or $\mathbb{Z}/k\mathbb{Z}$ where $k \in \mathbb{N}$. Let G' be an abelian subgrop of $\mathrm{PGL}(2, \mathbb{C})$. If $G' \cong \mathbb{Z}/k\mathbb{Z}$, then G' is conjugate to $\left\langle \left[\begin{pmatrix} e_k & 0 \\ 0 & 1 \end{pmatrix}\right] \right\rangle$. If $G' \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, then G' is conjugate to $\left\langle \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right] \right\rangle$ where $a, b \in \mathbb{C}^*$.

Theorem 3.5. *Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group. We assume that there is an element $g = [A] \in G$ such that*

$$A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix} \quad \text{and} \quad \operatorname{Im}\Psi_g \text{ is trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers. We set $l := |I_g|$. Then we have the following:

(i) *$G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/l\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2l\mathbb{Z}$.*

(ii) *If $l = \max\{\operatorname{ord}(h)\}_{h \in G}$, then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/l\mathbb{Z}$ and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4, \mathbb{C})$ generated by*

$$\left[\begin{pmatrix} e_l I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \quad \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}\right]$$

where $a, b \in \mathbb{C}^*$.

(iii) *If $l < \max\{\operatorname{ord}(h)\}_{h \in G}$, then $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2l\mathbb{Z}$ and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4, \mathbb{C})$ generated by*

$$\left[\begin{pmatrix} -e_{2l} & 0 & 0 & 0 \\ 0 & e_{2l} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \quad \text{and} \quad \left[\begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}\right]$$

where $a, b \in \mathbb{C}^*$.

Proof. Since $\operatorname{Im}\Psi_g$ is trivial, $\operatorname{Ker}\Psi_g = G$, and hence there is a short exact sequence:

$$0 \to I_g \to G \to \operatorname{Im}\varphi_1 \to 0.$$

Since $\operatorname{Im}\varphi_1$ is a finite abelian subgroup of $\mathrm{PGL}(2, \mathbb{C})$, $\operatorname{Im}\varphi_1$ is isomorphic to $\mathbb{Z}/k\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ where $k \in \mathbb{N}$. Let $g' := \left[\begin{pmatrix} e_l I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right]$ where $l := |I_g|$. Then $I_g = \langle g' \rangle$.

First, we show that $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$. We assume that $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/k\mathbb{Z}$ where $k \in \mathbb{N}$. By Lemma 3.4, $\operatorname{Im}\varphi_2 \cong \mathbb{Z}/k\mathbb{Z}$. Let $h \in G$ be an element such that

$\langle \varphi_1(h)\rangle = \operatorname{Im}\varphi_1$. Since $\operatorname{Ker}\varphi_1 = \ker\varphi_2$, $G = \langle g', h\rangle$. Since $\operatorname{Ker}\Psi_g = G$, there are matrices $A_1, A_2 \in \mathrm{GL}(2,\mathbb{C})$ such that $h = \left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right]$. Since $\langle \varphi_i(h)\rangle \cong \mathbb{Z}/k\mathbb{Z}$, A_i is conjugate to $\begin{pmatrix} a_i e_k & 0 \\ 0 & a_i \end{pmatrix}$ where $a_i \in \mathbb{C}^*$ for $i = 1, 2$. Let $M_i \in \mathrm{GL}(2,\mathbb{C})$ be a matrix such that

$$M_i A_i M_i^{-1} = \begin{pmatrix} a_i e_k & 0 \\ 0 & a_i \end{pmatrix}$$

for $i = 1, 2$. Then

$$\left[\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\right] g' \left[\begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix}\right] = g',$$

and

$$\left[\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\right] h \left[\begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix}\right] = \left[\begin{pmatrix} a_1 e_k & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 e_k & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}\right].$$

Since $G = \langle g', h\rangle$ and G is conjugate to a subgroup of $\mathrm{PGL}(4,\mathbb{C})$ generated by diagonal matrices, we get that G is liftable. This is a contradiction. Therefore, $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$.

Let $h, k \in G$ be elements such that $\varphi_i(h), \varphi_i(k)$ are a generating set of $\operatorname{Im}\varphi_i$ for $i = 1, 2$. Then $G = \langle g', h, k\rangle$. Since $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, we get that $\max\{\operatorname{ord}(g'')\}_{g''\in G}$ is an even number, and $\operatorname{ord}(h)$ and $\operatorname{ord}(k)$ are even numbers. We set $2u := \operatorname{ord}(h)$ and $2v := \operatorname{ord}(k)$ where $u, v \in \mathbb{N}$.

We assume that $l = \max\{\operatorname{ord}(g'')\}_{g''\in G}$. Since G is an abelian group, $2u$ divides l. We set $l = 2ul'$ where $l' \in \mathbb{N}$. Since $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $h^2 \in I_g$. Since $\operatorname{ord}(h^2) = u$, there is an integer $i \in \mathbb{Z}$ such that

$$h^2 = (g')^{i2l'}.$$

We set

$$h' := h(g')^{-il'}.$$

Then

$$\varphi_1(h) = \varphi_1(h') \text{ and } \operatorname{ord}(h') = 2.$$

In the same way, we see that there is an element $k' \in G$ such that $\varphi_1(h) = \varphi_1(k')$ and $\operatorname{ord}(k') = 2$. Then

$$G = \langle g', h', k'\rangle \cong \mathbb{Z}/l\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}^{\oplus 2}.$$

Let $B_1, B_2, C_1, C_2 \in \mathrm{GL}(2,\mathbb{C})$ be matrices such that $h' = \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\right]$ and $k' = \left[\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}\right]$. Since $\langle \varphi_i(h'), \varphi_i(k')\rangle \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\langle [B_i], [C_i]\rangle$ is conjugate to $\left\langle \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & a_i \\ a_i^{-1} & 0 \end{pmatrix}\right] \right\rangle$ where $a_i \in \mathbb{C}^*$ for $i = 1, 2$. Since $\operatorname{ord}(h') =$

$\mathrm{ord}(k') = 2$, we may assume that

$$h' = \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \text{ and } k' = \left[\begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}\right]$$

where $a, b \in \mathbb{C}^*$.

We assume that $l < \max\{\mathrm{ord}(g'')\}_{g'' \in G}$. Since $\mathrm{Im}\,\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ and $I_g \cong \mathbb{Z}/l\mathbb{Z}$,

$$2l = \max\{\mathrm{ord}(g'')\}_{g'' \in G}.$$

Since $I_g \cong \mathbb{Z}/l\mathbb{Z}$, we may assume that $\mathrm{ord}(h) = 2l$. Then $I_g = \langle h^2 \rangle$. Since G is an abelian group, $\mathrm{ord}(k) = 2v$ divides $2l$. We set $2l = 2vl'$ where $l' \in \mathbb{N}$. Since $\mathrm{Im}\,\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $k^2 \in I_g$. Since $\mathrm{ord}(k^2) = v$, there is an integer $i \in \mathbb{Z}$ such that

$$k^2 = (h^2)^{il'}.$$

We put

$$k' := kh^{-il'}.$$

Then

$$\varphi_1(k) = \varphi_1(k') \text{ and } \mathrm{ord}(k') = 2.$$

Therefore,

$$G = \langle h, k' \rangle \cong \mathbb{Z}/2l\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

As like the above, since $\langle \varphi_i(h), \varphi_i(k') \rangle \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ and $\mathrm{ord}(h') = 2l$, we may assume that

$$h = \left[\begin{pmatrix} -e_{2l} & 0 & 0 & 0 \\ 0 & e_{2l} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \text{ and } k' = \left[\begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}\right]$$

where $a, b \in \mathbb{C}^*$. □

In Section 4, we study a non-liftable finite abelian group $G \subset \mathrm{PGL}(4, \mathbb{C})$ such that G acts on a smooth hypersurface $Y \subset \mathbb{P}^3$ of degree d faithfully, and G contains an element $g = [A]$ such that

$$A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix} \text{ and } \mathrm{Im}\,\Psi_g \text{ is trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers.

Lemma 3.6. *Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group. We assume that there is an element $g = [A] \in G$ such that $A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix}$ and $\mathrm{Im}\,\Psi_g$ is not trivial where $a, b \in \mathbb{C}^*$ are distinct complex numbers. Then $I_g = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $W_i \subset \mathbb{C}^4$ be the eigenspace of A associated with a_i for $i = 1, 2$. Note that $W_1 = \{(x, y, z, w) \in \mathbb{C}^4 \mid z = w = 0\}$ and $W_2 = \{(x, y, z, w) \in \mathbb{C}^4 \mid x = y = 0\}$. Since $\mathrm{Im}\,\Psi_g$ is not trivial, there is an element $h = [B] \in G$ such that $h(\mathbb{P}(W_1)) = \mathbb{P}(W_2)$ and $h(\mathbb{P}(W_2)) = \mathbb{P}(W_1)$. Then

$$B = \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}$$

where $B_{12}, B_{21} \in \mathrm{GL}(2, \mathbb{C})$. We take an element $k \in I_g \backslash \{e\}$. Then there is a complex number $c \in \mathbb{C}^*$ such that

$$k = [C] \text{ and } C = \begin{pmatrix} cI_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

Since $hk = kh$, there is a complex number $t \in \mathbb{C}^*$ such that $BC = tCB$, i.e.

$$\begin{pmatrix} 0 & B_{12} \\ cB_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & tcB_{12} \\ tB_{21} & 0 \end{pmatrix}.$$

Since $B_{12}, B_{21} \in \mathrm{GL}(2, \mathbb{C})$, we get that $t = c$ and $tc = 1$. Then $c^2 = 1$, i.e. $\mathrm{ord}(k) = 2$. Therefore, $I_g = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$. □

Theorem 3.7. *Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group. We assume that there is an element $g = [A] \in G$ such that*

$$A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix} \text{ and } \mathrm{Im}\,\Psi_g \text{ is not trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers. If $\mathrm{Im}\,\varphi_1$ is trivial, then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4, \mathbb{C})$ generated by

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right] \text{ and } \left[\begin{pmatrix} 0 & C \\ C^{-1} & 0 \end{pmatrix}\right]$$

where $C \in \mathrm{GL}(2, \mathbb{C})$.

Proof. Since $\mathrm{Im}\,\varphi_1$ is trivial, $\mathrm{Ker}\,\Psi_g = I_g$. Then $G/I_g \cong \mathbb{Z}/2\mathbb{Z}$. By Lemma 3.6, $I_g = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$. As a result, $G \cong \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$. Since G is non-liftable, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$. We take an element $h \in G \backslash I_g$. Since $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $G = \langle g, h \rangle$. Since $h \notin I_g = \mathrm{Ker}\,\Psi_g$, there are matrices $C, D \in \mathrm{GL}(2, \mathbb{C})$ such that

$$h = \left[\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\right].$$

Since $h^2 = e$, we may assume that $D = C^{-1}$.

□

Example 3.8. Let $d \in \mathbb{Z}$ be an even number. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d defined by

$$X_0^d + X_1^d + X_2^d + X_3^d = 0.$$

Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right] \text{ and } \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right].$$

Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ and G acts on X faithfully.

Theorem 3.9. *Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group. We assume that there is an element $g = [A] \in G$ such that*

$$A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix} \text{ and } \mathrm{Im}\,\Psi_g \text{ is not trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers. If $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/u\mathbb{Z}$ where $u \in \mathbb{Z}$ and $u \geq 2$, then we have the following:

- *(i) $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$ or there is an element $h \in G$ such that $\operatorname{Im}\Psi_h$ is trivial where $\Psi_h \colon G \to \mathcal{S}_2$ is the group homomorphism defined by h.*
- *(ii) If $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, then G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4, \mathbb{C})$ generated by*

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \quad \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & B \\ B^{-1} & 0 \end{pmatrix}\right]$$

such that

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

where $a, b \in \mathbb{C}^*$.

Proof. By Lemma 3.6, $I_g = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Let $h \in G$ be an element such that $h \notin \operatorname{Ker}\Psi_g$. Then there is a matrix $B \in \mathrm{GL}(4, \mathbb{C})$ such that

$$h = [B] \quad \text{and} \quad B = \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}$$

where $B_{12}, B_{21} \in \mathrm{GL}(2, \mathbb{C})$. Since $\operatorname{Im}\Psi_g \cong \mathbb{Z}/2\mathbb{Z}$, $h^2 \in \operatorname{Ker}\Psi_g$. Since

$$h^2 = [B^2] = \left[\begin{pmatrix} B_{12}B_{21} & 0 \\ 0 & B_{21}B_{12} \end{pmatrix}\right],$$

and $B_{12}B_{21}$ and $B_{21}B_{12}$ have the same eigenvalue, we get that $h^2 \neq g$. Let $k \in \operatorname{Ker}\Psi_g$ be an element such that $\operatorname{Im}\varphi_1 = \langle \varphi_1(k) \rangle$. Since $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/u\mathbb{Z}$, as like the proof of Theorem 3.5, we may assume that

$$k = [C] \quad \text{and} \quad C = \begin{pmatrix} e_u a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & e_u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a \in \mathbb{C}^*$. Since $\operatorname{Im}\varphi_1 = \langle \varphi_1(k) \rangle$, $\operatorname{Ker}\Psi_g = \langle g, k \rangle$. In particular, $\operatorname{Ker}\Psi_g$ is generated by diagonal matrices. Since $\operatorname{Im}\Psi_g = \langle \Psi_g(h) \rangle$, $G = \langle g, h, k \rangle$.

We assume that $u = 2$. Then $|G| = 8$, $|\operatorname{Ker}\Psi_g| = 4$, and $a^4 = 1$. If $8 = \max\{\mathrm{ord}(g'')\}_{g'' \in G}$, then G is cyclic. This contradicts that G is non-liftable. We assume that $4 = \max\{\mathrm{ord}(g'')\}_{g'' \in G}$. By replacing h with hk' where $k' \in \operatorname{Ker}\Psi_g$ such that $\mathrm{ord}(k') = 4$ if necessary, we may assume that $\mathrm{ord}(h) = 4$. Since $h^2 \neq g$, $h^2 \in \operatorname{Ker}\Psi_g$, and $|\operatorname{Ker}\Psi_g| = 4$, we get that $\operatorname{Ker}\Psi_g = \langle g, h^2 \rangle$ and $G = \langle g, h \rangle$. Since $\operatorname{Ker}\Psi_g$ is generated by diagonal matrices, the group homomorphism $\Psi_{h^2} \colon G \to \mathcal{S}_2$ defined by h^2 is trivial. We assume that $2 = \max\{\mathrm{ord}(h)\}_{h \in G}$. Since $|G| = 8$, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$. Since $\mathrm{ord}(k) = 2$, $a^2 = 1$. We may assume that $C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
Since $\mathrm{ord}(h) = 2$, we may assume that $B_{12}B_{21} = B_{21}B_{12} = I_2$. Since $hk = kh$,

$BC = tCB$ for some $t \in \mathbb{C}^*$. Then $B_{12}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-t & 0\\ 0 & t\end{pmatrix}B_{12}$. We set $B_{12} = (b_{ij})$. By the above equation,

$$\begin{pmatrix}-b_{11} & b_{12}\\ -b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix}-tb_{11} & -tb_{12}\\ tb_{21} & tb_{22}\end{pmatrix}.$$

Since $B_{12} \in \mathrm{GL}(2,\mathbb{C})$, if $t = 1$ then $b_{12} = b_{21} = 0$ and if $t \neq 1$ then $t = -1$ and $b_{11} = b_{22} = 0$.

We assume that $u \geq 3$. By replacing h with hk' where $k' \in \operatorname{Ker}\Psi_g$ such that $\mathrm{ord}(k') > 2$ if necessary, we may assume that $\mathrm{ord}(h) > 2$. Since $\operatorname{Im}\Psi_g \cong \mathbb{Z}/2\mathbb{Z}$, $h^2 \in \operatorname{Ker}\Psi_g$. Since $G = \langle h, \operatorname{Ker}\Psi_g\rangle$ and $\operatorname{Ker}\Psi_g$ is generated by diagonal matrices, we get that the group homomorphism $\Psi_{h^2}\colon G \to \mathcal{S}_2$ defined by h^2 is trivial. □

Example 3.10. Let $d \in \mathbb{Z}$ be an even number. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d defined by

$$X_0^d + X_1^d + X_2^d + X_3^d = 0.$$

Let $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix}-I_2 & 0\\ 0 & I_2\end{pmatrix}\right],\quad \left[\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right],\quad \text{and}\quad \left[\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\right].$$

Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$ and G acts on X faithfully.

Theorem 3.11. *Let $G \subset \mathrm{PGL}(4,\mathbb{C})$ be a non-liftable finite abelian group. We assume that there is an element $g = [A] \in G$ such that*

$$A = \begin{pmatrix}aI_2 & 0\\ 0 & bI_2\end{pmatrix} \text{ and } \operatorname{Im}\Psi_g \text{ is not trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers. If $\operatorname{Im}\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$ and G is conjugate to an abelian group $G' \subset \mathrm{PGL}(4,\mathbb{C})$ generated by

$$\left[\begin{pmatrix}-I_2 & 0\\ 0 & I_2\end{pmatrix}\right],\quad \left[\begin{pmatrix}0 & 0 & a & 0\\ 0 & 0 & 0 & b\\ a^{-1} & 0 & 0 & 0\\ 0 & b^{-1} & 0 & 0\end{pmatrix}\right],\quad \left[\begin{pmatrix}0 & x & 0 & 0\\ x^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & y\\ 0 & 0 & y^{-1} & 0\end{pmatrix}\right],\quad \text{and}$$

$$\left[\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right]$$

such that $bx = ay$ where $a, b, x, y \in \mathbb{C}$.

Proof. By Lemma 3.6, $I_g = \langle g\rangle \cong \mathbb{Z}/2\mathbb{Z}$. We take an element $h \in G\backslash\operatorname{Ker}\Psi_g$. As like proof of Theorem 3.11, $h^2 \neq g$. We set

$$h = [B] \text{ and } B = \begin{pmatrix}0 & B_{12}\\ B_{21} & 0\end{pmatrix}$$

where $B_{12}, B_{21} \in \mathrm{GL}(2,\mathbb{C})$. Since $h^2 = [B^2] = \left[\begin{pmatrix}B_{12}B_{21} & 0\\ 0 & B_{21}B_{12}\end{pmatrix}\right]$ and $\mathrm{ord}(h^2)$ is finite, we get that $B_{12}B_{21}$ and $B_{21}B_{12}$ are diagonalizable. Let $S, T \in \mathrm{GL}(2,\mathbb{C})$

be matrices such that $SB_{12}B_{21}S^{-1}$ and $TB_{21}B_{12}T^{-1}$ are diagonal matrices. We set $B' := \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} = \begin{pmatrix} 0 & SB_{12}T^{-1} \\ TB_{21}S^{-1} & 0 \end{pmatrix}$. Then

$$(B')^2 = \begin{pmatrix} SB_{12}B_{21}S^{-1} & 0 \\ 0 & TB_{21}B_{12}T^{-1} \end{pmatrix}.$$

Therefore, we may assume that $B_{12}B_{21}$ and $B_{21}B_{12}$ are diagonal matrices. By multiplying B by a constant if necessary, we may assume that

$$B_{12}B_{21} = B_{21}B_{12} = \begin{pmatrix} e_u & 0 \\ 0 & 1 \end{pmatrix}$$

where $u \in \mathbb{Z}$ such that $\mathrm{ord}(h) = 2u$. Since $\mathrm{Im}\,\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ and $I_g \cong \mathbb{Z}/2\mathbb{Z}$, we get that $|\mathrm{Ker}\,\Psi_g| = 8$. Since $\mathrm{Im}\,\Psi_g \cong \mathbb{Z}/2\mathbb{Z}$, $|G| = 16$. If $16 = \max\{\mathrm{ord}(g'')\}_{g'' \in G}$, then G is cyclic. This contradicts that G is non-liftable. We assume that $8 = \max\{\mathrm{ord}(g'')\}_{g'' \in G}$. By replacing h with hk' where $k' \in \mathrm{Ker}\,\Psi_g$ such that $\mathrm{ord}(k') = 8$ if necessary, we may assume that $\mathrm{ord}(h) = 8$. As like proof of Theorem 3.11, $h^i \neq g$ for $1 \leq i < 8$. Since $h^2 \in \mathrm{Ker}\,\Psi_g$ and $\mathrm{ord}(h) = 8$, $\langle \varphi_1(h^2) \rangle \cong \mathbb{Z}/4\mathbb{Z}$. This contradicts that $\mathrm{Im}\,\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$. Therefore, $4 \geq \max\{\mathrm{ord}(g'')\}_{g'' \in G}$.

We assume that $4 = \max\{\mathrm{ord}(g'')\}_{g'' \in G}$. By replacing h with hk' where $k' \in \mathrm{Ker}\,\Psi_g$ such that $\mathrm{ord}(k') = 4$ if necessary, we may assume that $\mathrm{ord}(h) = 4$. Since $h^2 \in \mathrm{Ker}\,\Psi_g \backslash I_g$ and $\mathrm{Im}\,\varphi_1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, there is an element $k \in \mathrm{Ker}\,\Psi_g$ such that $\mathrm{Im}\,\varphi_1 = \langle \varphi_1(h^2), \varphi_1(k) \rangle$. Let $C \in \mathrm{GL}(4, \mathbb{C})$ be a matrix such that $k = [C]$. Since $B^2 = \begin{pmatrix} e_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathrm{PGL}(2, \mathbb{C}) \supset \mathrm{Im}\,\varphi_1 = \langle \varphi_1(h^2), \varphi_1(k) \rangle \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, we get that $C = \begin{pmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & w & 0 \end{pmatrix}$ where $x, y, z, w \in \mathbb{C}^*$. We assume that $\mathrm{ord}(k) = 4$. Sine $C^2 = \begin{pmatrix} xyI_2 & 0 \\ 0 & zwI_2 \end{pmatrix}$ and Lemma 3.6, $k^2 = g$. By multiplying C by a constant if necessary, we may assume that We may assume that $xy = -1$ and $zw = 1$. As a result, $C = \begin{pmatrix} 0 & x & 0 & 0 \\ -x^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & z^{-1} & 0 \end{pmatrix}$. By Lemma 2.4, C is conjugate to $\begin{pmatrix} e_4 & 0 & 0 & 0 \\ 0 & -e_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Since $\mathrm{PGL}(2, \mathbb{C}) \supset \mathrm{Im}\,\varphi_1 = \langle \varphi_1(h^2), \varphi_1(k) \rangle \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $h^2 = [B^2]$ and $k = [C]$, we get that $B^2C \neq CB^2$. As a result, $\mathrm{Im}\,\Psi_k \cong \mathbb{Z}/4\mathbb{Z}$ where $\Psi_k \colon G \to \mathcal{S}_4$ is the group homomorphism defined by k. Since $|G| = 16$, G is conjugate to $\left\langle \left[\begin{pmatrix} e_4 & 0 & 0 & 0 \\ 0 & -e_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right], [D] \right\rangle$ where D is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ c & 0 & 0 & 0 \end{pmatrix}$,

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ c & 0 & 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & 0 & 1 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \end{pmatrix}$ where $a, b, c \in \mathbb{C}^*$. However, there is not a complex number $t \in \mathbb{C}^*$ such that $\begin{pmatrix} e_4 & 0 & 0 & 0 \\ 0 & -e_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} D = tD \begin{pmatrix} e_4 & 0 & 0 & 0 \\ 0 & -e_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This contradicts that G is an abelian group. Therefore, $\mathrm{ord}(k) = 2$. Then $G = \langle h, g, k \rangle \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$. Since $hk = kh$, $BC = tCB$ for $t \in \mathbb{C}^*$. We set $B_{12} = (b_{ij})$. By the above equation,

$$\begin{pmatrix} y^{-1}b_{12} & yb_{11} \\ y^{-1}b_{22} & yb_{21} \end{pmatrix} = \begin{pmatrix} txb_{21} & txb_{22} \\ -tx^{-1}b_{11} & -tx^{-1}b_{12} \end{pmatrix}.$$

Then $yb_{11} = txb_{22}$, $-tyb_{11} = xb_{22}$, $yb_{12} = txyb_{21}$, and $-tb_{12} = xyb_{21}$. Since $B_{12} \in \mathrm{GL}(2, \mathbb{C})$, $t = -t$. This contradicts that $t \in \mathbb{C}^*$. Therefore, $2 = \max\{\mathrm{ord}(g'')\}_{g'' \in G}$. Since $|G| = 16$, $G \cong \mathbb{Z}/2\mathbb{Z}^{oplus4}$. Let $k_1, k_2 \in \mathrm{Ker}\,\Psi_g$ be elements such that $\mathrm{Im}\,\varphi_i = \langle \varphi_i(k_1), \varphi_i(k_2) \rangle$ for $i = 1, 2$. Since $\mathrm{PGL}(2, \mathbb{C}) \supset \mathrm{Im}\,\varphi_i \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ for $i = 1, 2$, there are matrices $C_1, C_2 \in \mathrm{GL}(4, \mathbb{C})$ such that $k_i = [C_i]$ for $i = 1, 2$,

$$C_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}.$$

Note that $G = \langle g, h, k_1, k_2 \rangle$. We set $B_{12} = (b_{ij})$ and $B_{21} = (d_{ij})$. Since $hk_1 = k_1h$, $BC_2 = tC_2B$ for some $t \in \mathbb{C}^*$, i.e.

$$\begin{pmatrix} -b_{11} & b_{12} \\ -b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -tb_{11} & -tb_{12} \\ tb_{21} & tb_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -d_{11} & d_{12} \\ -d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} -td_{11} & -td_{12} \\ td_{21} & td_{22} \end{pmatrix}.$$

Then $t^2 = 1$. We assume that $t = 1$. Then $b_{12} = b_{21} = d_{12} = d_{21} = 0$. Since $h^2 = e$, we may assume that $d_{11} = b_{11}^{-1}$ and $d_{22} = b_{22}^{-1}$. We assume that $t = -1$. Then $b_{11} = b_{22} = d_{11} = d_{22} = 0$ and $b_{12}b_{21} = d_{12}d_{21}$. Since $G = \langle g, h, k_1, hk_2 \rangle$ and $hk_2 = \left[\begin{pmatrix} 0 & 0 & b_{12}y^{-1} & 0 \\ 0 & 0 & 0 & b_{21}y \\ d_{12}x^{-1} & 0 & 0 & 0 \\ 0 & d_{21}x & 0 & 0 \end{pmatrix}\right]$. This results in the case $t = 1$. □

Example 3.12. Let $d \in \mathbb{Z}$ be an even number. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d defined by

$$X_0^d + X_1^d + X_2^d + X_3^d = 0.$$

Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right], \text{ and } \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right].$$

Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$ and G acts on X faithfully.

4. Complete classification of nonliftable abelian automorphism groups

We study a non-liftable finite abelian group $G \subset \mathrm{PGL}(4,\mathbb{C})$ such that G acts on a smooth hypersurface $X \subset \mathbb{P}^3$ of degree d faithfully, and G contains an element $g = [A]$ such that

$$A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix} \text{ and } \mathrm{Im}\,\Psi_g \text{ is trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complex numbers.

Lemma 4.1. *Let X be a smooth hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$, and $g = [A]$ be an automorphism of X where $A \in \mathrm{GL}(4,\mathbb{C})$ such that $A = \begin{pmatrix} e_l I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ where $l := \mathrm{ord}(g)$. Let $W_1 \subset \mathbb{C}^4$ be the eigenspace of A associated with e_l, and let $W_2 \subset \mathbb{C}^4$ be the eigenspace of A associated with 1. Then we have the following.*

- *(i) $\mathbb{P}(W_i) \not\subset X$ for $i = 1, 2$ if and only if l divides d.*
- *(ii) Only one of $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ is included in X if and only if l divides $d-1$.*
- *(iii) $\mathbb{P}(W_i) \subset X$ for $i = 1, 2$ if and only if l divides $d-2$.*

Proof. Let

$$F(X_0, X_1, X_2, X_3) = \sum_{0 \leq i+j \leq d} F_{i,j}(X_2, X_3) X_0^i X_1^j$$

where $F_{i,j}(X_2, X_3) \in \mathbb{C}[X_2, X_3]$ is a homogeneous polynomial of degree $d-(i+j)$ if $F_{i,j}(X_2, X_3) \neq 0$ for $0 \leq i+j \leq d$. Note that $W_1 = \{(x,y,z,w) \in \mathbb{C}^4 \mid z = w = 0\}$ and $W_2 = \{(x,y,z,w) \in \mathbb{C}^4 \mid x = y = 0\}$.

We assume that $\mathbb{P}(W_i) \not\subset X$ for $i = 1, 2$. Since $\mathbb{P}(W_1) \not\subset X$, $F_{i,d-i}(X_2, X_3) \neq 0$ for some $0 \leq i \leq d$. Since $\mathbb{P}(W_2) \not\subset X$, $F_{0,0}(X_2, X_3) \neq 0$. Since

$$A^* F_{i,d-i}(X_2, X_3) X_0^i X_1^{d-1} = e_l^d F_{i,d-i}(X_2, X_3) X_0^i X_1^{d-1}$$

and

$$A^* F_{0,0}(X_2, X_3) = F_{0,0}(X_2, X_3),$$

we get that $(e_l)^d = 1$, and hence l divides d.

In what follows, we assume that $\mathbb{P}(W_2) \subset X$. Then $F_{0,0}(X_2, X_3) = 0$, and $[0:0:1:0], [0:0:0:1] \in X$. Since X is smooth at $[0:0:1:0]$ and $[0:0:0:1]$, and $F_{0,0}(X_2, X_3) = 0$, we get that $F_{1,0}(X_2, X_3) \neq 0$ or $F_{0,1}(X_2, X_3) \neq 0$. For simplicity, we assume that $F_{1,0}(X_2, X_3) \neq 0$. If $\mathbb{P}(W_2) \not\subset X$, then $F_{i,d-i}(X_2, X_3) \neq 0$ for some $0 \leq i \leq d$. Since

$$A^* F_{i,d-i}(X_2, X_3) X_0^i X_1^{d-1} = e_l^d F_{i,d-i}(X_2, X_3) X_0^i X_1^{d-1}$$

and

$$A^* F_{1,0}(X_2, X_3) X_0 = e_l F_{1,0}(X_2, X_3) X_0,$$

we get that $(e_l)^d = e_l$, and hence l divides $d-1$. If $\mathbb{P}(W_2) \subset X$, then $F_{i,d-i}(X_2, X_3) = 0$ for $0 \leq i \leq d$. Since X is smooth at $[1:0:0:0]$, $F_{d-1,0}(X_2, X_3) \neq 0$. Since

$$A^* F_{d-1,0}(X_2, X_3) X_0^{d-1} = e_l^{d-1} F_{d-1,0}(X_2, X_3) X_0^{d-1}$$

and

$$A^* F_{1,0}(X_2, X_3) X_0 = e_l F_{1,0}(X_2, X_3) X_0,$$

we get that $(e_l)^{d-1} = e_l$, and hence l divides $d-2$. $\square$

Theorem 4.2. *Let d be an even number. Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group such that G acts on a smooth hypersurface $X \subset \mathbb{P}^3$ of degree d faithfully. We assume that there is an element $g = [A] \in G$ such that*

$$A = \begin{pmatrix} aI_2 & 0 \\ 0 & bI_2 \end{pmatrix} \text{ and } \operatorname{Im} \Psi_g \text{ is trivial}$$

where $a, b \in \mathbb{C}^*$ are distinct complec numbers. Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/k\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2l\mathbb{Z}$ where $i = 1$ or 2, k divides $d-2$ or d, and l divides $d-1$.

Proof. For simplicity, we assume that $\mathrm{ord}(g) = l$. By Theorem 2.1, d is an even number. By Lemma 4.1, l divides $d-2$, $d-1$, or d. Since $d-1$ is an odd number, if l divides $d-1$ then $l < \max\{\mathrm{ord}(h)\}_{h \in G}$, and hence $2l$ divides $2(d-1)$.

We only show that if $l < \max\{\mathrm{ord}(h)\}_{h \in G}$ and l divides $d-2$ (resp. d) then $2l$ divides $d-2$ (resp. d). We assume that $l < \max\{\mathrm{ord}(h)\}_{h\in G}$. By Theorem 3.5, $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}$, and we may assume that $G = \langle g := [\alpha], h := [\beta] \rangle$ where

$$\alpha = \begin{pmatrix} -e_{2l} & 0 & 0 & 0 \\ 0 & e_{2l} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{pmatrix}$$

and $a, b \in \mathbb{C}^*$. Let $W_1 = \{(x, y, z, w) \in \mathbb{C}^4 \mid z = w = 0\}$ and $W_2 = \{(x, y, z, w) \in \mathbb{C}^4 \mid x = y = 0\}$. Let $F(X_0, X_1, X_2, X_3)$ be the defining equation of X.

We assume that l divides $d-2$. By Lemma 4.1, $\mathbb{P}(W_i) \subset X$ for $i = 1, 2$. Then $[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1] \in X$. Since X is smooth at $[1:0:0:0], [0:1:0:0], [0:0:1:0]$, and $[0:0:0:1]$, $F(X_0, X_1, X_2, X_3)$ has $(a_iX_0 + b_iX_1)X_i^{d-1}$ and $(c_jX_2 + b_jX_3)X_j^{d-1}$ terms for $i = 2, 3$ and $j = 0, 1$ where either a_i or b_i is not 0 for $i = 2, 3$ and either c_j or d_j is not 0 for $j = 0, 1$. Since $\beta^* X_0X_2^{d-1} = a^{-1}b^{1-d}X_1X_3^{d-1}$ and $[\beta] \in \mathrm{Aut}(X)$, if $F(X_0, X_1, X_2, X_3)$ has a $X_0X_2^{d-1}$ term then $F(X_0, X_1, X_2, X_3)$ has a $X_1X_3^{d-1}$ term. Then $(-e_{2l}) \times (-1)^{d-1} = e_{2l}$, and hence $e_{2l}^{d-1} = e_{2l}$. As a result, $2l$ divides $d-2$. Since $\beta^* X_1X_2^{d-1} = ab^{1-d}X_0X_3^{d-1}$ and $[\beta] \in \mathrm{Aut}(X)$, if $F(X_0, X_1, X_2, X_3)$ has a $X_1X_2^{d-1}$ term then $F(X_0, X_1, X_2, X_3)$ has a $X_0X_3^{d-1}$ term. Then $e_{2l} \times (-1)^{d-1} = -e_{2l}$, and hence $e_{2l}^{d-1} = e_{2l}$. As a result, $2l$ divides $d-2$.

We assume that l divides d. Let

$$F(X_0, X_1, X_2, X_3) = \sum_{0 \le i+j \le d} F_{i,j}(X_2, X_3) X_0^i X_1^j$$

where $F_{i,j}(X_2, X_3) \in \mathbb{C}[X_2, X_3]$ is a homogeneous polynomial of degree $d-(i+j)$ if $F_{i,j}(X_2, X_3) \neq 0$ for $0 \le i+j \le d$. By Lemma 4.1, $\mathbb{P}(W_i) \not\subset X$ for $i = 1, 2$. Then $F_{i,d-i}(X_2, X_3) \neq 0$ for some $0 \le i \le d$. and $F_{0,0}(X_2, X_3) \neq 0$. Since $\beta^* X_0^i X_1^j = a^{-i}a^j X_0^j X_1^i$ and $[\beta] \in \mathrm{Aut}(X)$, $F_{j,i}(X_2, X_3) \neq 0$. Since $F_{i,d-i}(X_2, X_3), F_{d-i,i}(X_2, X_3) \in \mathbb{C}^*$ and $[\alpha] \in \mathrm{Aut}(X)$, we get that $(-1)^i (e_{2l})^{d-i} = (-1)^{d-i}(e_{2l})^i$. As a result, $e_{2l}^{d-2i} = 1$, i.e. $2l$ divides $d-2i$ for some $0 \le i \le d$. Since l divides d, $2l$ divides d. □

Example 4.3. Let d be an even number. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d defined by

$$X_0^{d-1}X_2 + X_1^{d-1}X_3 + X_0X_2^{d-1} + X_1X_3^{d-1} = 0.$$

Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} e_{d-2}I_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \quad \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/(d-2)\mathbb{Z}$ and G acts on X faithfully.

Let $H \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} -e_{d-2} & 0 & 0 & 0 \\ 0 & e_{d-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Then $H \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(d-2)\mathbb{Z}$ and G acts on X faithfully.

Example 4.4. Let d be an even number. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d defined by

$$X_0^{d-1}X_2 + X_1^{d-1}X_3 + X_2^d + X_3^d = 0.$$

Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} -e_{d-1} & 0 & 0 & 0 \\ 0 & e_{d-1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Then $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2(d-1)\mathbb{Z}$ and G acts on X faithfully.

Example 4.5. Let d be an even number. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d defined by

$$X_0^d + X_1^d + X_2^d + X_3^d = 0.$$

Let $G \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} e_dI_2 & 0 \\ 0 & I_2 \end{pmatrix}\right], \quad \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/d\mathbb{Z}$ and G acts on X faithfully.

Let $H \subset \mathrm{PGL}(4, \mathbb{C})$ be a non-liftable finite abelian group generated by

$$\left[\begin{pmatrix} -e_d & 0 & 0 & 0 \\ 0 & e_d & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \quad \text{and} \quad \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Then $H \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ and G acts on X faithfully.

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