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# AN INCREASING NORMALIZED DEPTH FUNCTION

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ABSTRACT. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1,\dots,x_n]$ be the polynomial ring in $n$ variables over $\mathbb{K}$. Assume that $I$ is a squarefree monomial ideal of $S$. For every integer $k \geq 1$, we denote the $k$-th squarefree power of $I$ by $I^{[k]}$. The normalized depth function of $I$ is defined as $g_I(k) = \mathrm{depth}(S/I^{[k]}) - (d_k - 1)$, where $d_k$ denotes the minimum degree of monomials belonging to $I^{[k]}$. Erey, Herzog, Hibi and Saeedi Madani conjectured that for any squarefree monomial ideal $I$, the function $g_I(k)$ is nonincreasing. In this short note, we provide a counterexample for this conjecture. Our example in fact shows that $g_I(2) - g_I(1)$ can be arbitrarily large.

## 1. Introduction

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1,\dots,x_n]$ be the polynomial ring in $n$ variables over $\mathbb{K}$. For any squarefree monomial ideal $I \subset S$ and for any positive integer $k$, the $k$-th squarefree power of $I$ denoted by $I^{[k]}$ is the ideal generated by the squarefree monomials belonging to $I^k$. In [3], Erey, Herzog, Hibi and Saeedi Madani studied the depth of squarefree powers. They introduced the notion of normalized depth function as follows. Let $\nu(I)$ be the largest integer $k$ with $I^{[k]} \neq 0$. For each integer $k = 1,2,\dots,\nu(I)$, we denote the minimum degree of monomials belonging to $I^{[k]}$ by $d_k$. The normalized depth function of $I$ is the function $g_I : \{1,2,\dots,\nu(I)\} \to \mathbb{Z}_{\geq 0}$ defined by

$$g_I(k) = \mathrm{depth}(S/I^{[k]}) - (d_k - 1).$$

The same authors conjectured that for any squarefree monomial ideal $I$, the function $g_I(k)$ is nonincreasing. This conjecture is known to be true in special cases (see e.g., [2], [3], [5]). However, in the next section, we provide a class of ideals disproving the conjecture. Our example indeed shows that the difference $g_I(2) - g_I(1)$ can be arbitrarily large.

## 2. An example

In Theorem 2.2, we introduce a class of ideals $I$ showing that the normalized depth function $g_I(k)$ is not necessarily nonincreasing.

We recall that for any graph $G$ with vertex set $V(G) = \{1,2,\dots,n\}$ and edge set $E(G)$, its edge ideal is defined as

$$I(G) = (x_ix_j \mid \{i,j\} \in E(G)) \subset S.$$

Moreover, a graph $G$ is said to be sequentially Cohen-Macaulay over $\mathbb{K}$ if $S/I(G)$ is sequentially Cohen-Macaulay (one may look at [9, Chapter III] for the definition of sequentially Cohen-Macaulay modules). We say that $G$ is a sequentially Cohen-Macaulay graph if it is sequentially

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Cohen-Macaulay over any field $\mathbb{K}$. A subset $U$ of $V(G)$ is called an independent subset of $G$ if there are no edges among the vertices of $U$. We say that a subset $C \subseteq V(G)$ is a *minimal vertex cover* of $G$ if, first, every edge of $G$ is incident with a vertex in $C$ and, second, there is no proper subset of $C$ with the first property. Note that $C$ is a minimal vertex cover if and only if $V(G) \setminus C$ is a maximal independent subset of $G$. Moreover, it is known by [7, Lemma 9.1.4] that every minimal prime ideal of $I(G)$ is of the form $(x_i \mid i \in C)$ where $C$ is a minimal vertex cover of $G$. Since $I(G)$ is a radical ideal, it follows that the irredundant primary decomposition of $I(G)$ is given by

$$I(G) = \bigcap (x_i \mid i \in C),$$

where the intersection is taken over all minimal vertex covers $C$ of $G$.

We first need the following simple lemma.

**Lemma 2.1.** *Let $T$ be a tree with $n$ vertices. Then* $\mathrm{depth}(S/I(T))$ *is equal to the minimum size of a maximal independent subset of $T$.*

*Proof.* It is well-known that any tree is a sequentially Cohen-Macaulay graph (see e.g., [6, Theorem 1.2]). Hence, it follows from [4, Theorem 4] (see also [8, Corollary 3.33]) that $\mathrm{depth}(S/I(T))$ is equal to $n-h$, where $h$ denotes the maximum height of an associated prime of $I(T)$. Thus, using the primary decomposition of $I(T)$ given above, we deduce that $h$ is the maximum size of a minimal vertex cover of $T$. Therefore, $n-h$ is the minimum size of a maximal independent subset of $T$. □

We are now ready to present our example.

**Theorem 2.2.** *Let $n \geq 6$ be an integer and consider the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. For each integer $i$ with $1 \leq i \leq n-4$, set $u_i := x_1x_3x_{i+4}$. Also, set*

$$u_{n-3} := x_1x_4x_5, \quad u_{n-2} := x_2x_3x_4 \quad \text{and} \quad u_{n-1} := x_2x_3x_6.$$

*Let $I$ be the squarefree monomial ideal generated by $u_1, u_2, \dots, u_{n-1}$. Then*

(i) $g_I(1) = 1$*; and*

(ii) $g_I(2) = n-6$.

*In particular,* $g_I(2) = g_I(1) + n - 7$.

*Proof.* (i) One can easily see that $\mathfrak{p} = (x_4, \dots, x_n)$ is a minimal prime ideal of $I$. Thus,

$$\mathrm{depth}(S/I) \leq \dim(S/\mathfrak{p}) = 3. \tag{1}$$

Consider the following short exact sequence.

$$0 \longrightarrow \frac{S}{(I:x_3)} \longrightarrow \frac{S}{I} \longrightarrow \frac{S}{(I,x_3)} \longrightarrow 0$$

It follows from depth lemma [1, Proposition 1.2.9] that

$$\mathrm{depth}(S/I) \geq \min\left\{\mathrm{depth}(S/(I:x_3)), \mathrm{depth}(S/(I,x_3))\right\}. \tag{2}$$

Since $(I,x_3) = (u_{n-3},x_3)$, we have

$$\mathrm{depth}(S/(I,x_3)) = n-2 \geq 4. \tag{3}$$

On the other hand, notice that

$$(I : x_3) = (x_2x_4, x_2x_6) + (x_1x_{i+4} \mid 1 \leq i \leq n-4).$$

In particular, there is a tree $T$ with vertex set $[n] \setminus \{3\}$ such that $(I : x_3) = I(T)$. It is easy to see that $\{1,2\}$ is a maximal independent set in $T$ of minimum size. Since 3 is not a vertex of $T$, Lemma 2.1 implies that

$$\mathrm{depth}(S/(I : x_3)) = 2+1 = 3. \tag{4}$$

We conclude from inequalities (2), (3) and (4) that $\mathrm{depth}(S/I) \geq 3$. This inequality together with inequality (1) implies that $\mathrm{depth}(S/I) = 3$. Equivalently, $g_I(1) = 1$.

(ii) It is obvious that $I^{[2]}$ is the principal ideal generated by $u_{n-3}u_{n-1}$. Thus, $\mathrm{depth}(S/I^{[2]}) = n-1$. In other words, $g_I(2) = n-6$. □

**Remark 2.3.** Note that for the ideal in Theorem 2.2, we have $\nu(I)$. Thus, Theorem 2.2 shows that in general the function $g_I(k)$ can be an increasing function. However, we do not have any example of a graph $G$ for which the function $g_{I(G)}(k)$ is not nonincreasing. So, the conjecture posed in [3] might be true for edge ideals.

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## Declarations

The author declares that there is no conflict of interest for this work.

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