

GLS HOMOGENIZATION TILDE MAP

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ABSTRACT. In the construction of a cluster algebra on the homogeneous coordinate ring of a partial flag variety by Geiß, Leclerc and Schröer, they defined a special map denoted by “tilde”. This map lifts each element f of the coordinate ring of a Schubert cell uniquely to an element \tilde{f} of the (multi-homogeneous) coordinate ring of the corresponding partial flag variety. The significance of this map appears from its essential role; it lifts the cluster algebra of the coordinate ring of a cell to a cluster algebra living in the coordinate ring of the corresponding partial flag variety. This paper takes a closer look at this map and gives an explicit algorithm to calculate it for the *generalized minors*.

1. INTRODUCTION

Cluster algebras were constructed in 2002 by Fomin and Zelevinsky [4] and they quickly formed a very active area of mathematics. This is due to its connections and relations with many other areas in mathematics like algebraic geometry, Poisson geometry, knot theory, mathematical physics and integrable systems. In Fomin and Zelevinsky’s work, it has been shown that the notion of cluster algebras is strongly related to the notion of total positivity. Consequently, the first explicit relation between cluster algebras and partial flag varieties appeared in the work of Scott [10] in 2006. Two years after, Geiß, Leclerc and Schröer [5] showed that for type A and D_4 the coordinate ring of a partial flag variety contains a cluster algebra induced by a cluster algebra structure of the coordinate ring of the Schubert cell. In fact, they showed the equality for type A and the containment for type D_4 . However, they showed the equality for type D_4 after localizing certain elements. Moreover, they remarked that similar work can be used to generalize their results to the other types of G . Indeed, they conjectured that the coordinate ring of the Schubert cell is a cluster algebra and this structure can be lifted using a special map, called the “tilde” map, to a cluster algebra living in the coordinate ring of the corresponding partial flag variety. In 2011, they proved the conjecture that the coordinate ring of the cell is a cluster algebra for all the simply-laced cases in [7]. Later on, in [8], Goodearl and Yakimov constructed a large class of cluster

Date: February 27, 2024.

2020 Mathematics Subject Classification. Primary 13F60; Secondary 14M15, 13N15.

The research of the author has been supported by NSF grant DMS-2131243.

algebras coming from Poisson algebras. As a consequence of this work, the same conjecture was proved for any semisimple complex algebraic group G . The same tilde map of [5], was used in [9], to show that the cluster algebra of the coordinate ring of the cell can be lifted to a cluster structure living in the coordinate ring of the corresponding partial flag variety for any semisimple complex algebraic group G . Moreover, the same paper proved the equality after localizing some certain elements for any type of G .

In this paper, we take a closer look at the tilde map and give an explicit algorithm to calculate it for the *generalized minors*. This algorithm gives a full description of the cluster algebra of the partial flag variety since the initial seed of the cluster algebra of Goodearl and Yakimov consisted of generalized minors only. The paper is organized as follows: In section 2, we take an overview of cluster algebras by giving the main definitions and notions related to them. After that, we capture the needed notions and results from the Schubert cells and partial flag varieties. In section 4, we describe the tilde map on the generalized minors and then we show how this can be applied to the cluster algebra of [9] in section 5. Finally, we give explicit type A and type B examples in section 6 to show how the results of section 5 work in both of them.

2. CLUSTER ALGEBRAS

We start this section by introducing the notion of a cluster algebra. This is given by the following sequence of definitions.

Definition 2.1. A *seed* is a pair (\mathbf{x}, B) such that $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ is a tuple of algebraically independent variables generating a field isomorphic to the field $\mathbb{C}(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$. Also, B is an $m \times n$ extended skew-symmetrizable matrix, that is, a matrix whose northwestern $n \times n$ submatrix can be transformed to a skew-symmetric matrix by multiplying each row r_i by a nonzero integer d_i . The matrix B is called the *exchange matrix* and the tuple \mathbf{x} is called the *extended cluster*. The variables x_1, \dots, x_n are called *mutable*, while the variables x_{n+1}, \dots, x_m are called *frozen*.

Definition 2.2. Let k be an index of a mutable variable of a seed (\mathbf{x}, B) . A *mutation* at k is a transformation to a new seed (\mathbf{x}', B') in which B' is an $m \times n$ matrix whose entries are

$$(2.1) \quad b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise;} \end{cases}$$

and $\mathbf{x}' = (x'_1, \dots, x'_n, x'_{n+1}, \dots, x'_m)$ is a tuple such that $x'_i = x_i$ for $i \neq k$ and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

The seed (\mathbf{x}', B') obtained by a mutation at k is denoted sometimes by $\mu_k(\mathbf{x}, B)$.

Remark 2.3. It is not hard to see that the mutation of a seed provides a new seed. Moreover, mutating twice at the same index brings the original seed back. In symbols,

$$\mu_k(\mu_k(\mathbf{x}, B)) = (\mathbf{x}, B).$$

Definition 2.4. Let (\mathbf{x}, B) be a seed. A *cluster algebra (of geometric type)* attached to (\mathbf{x}, B) is the polynomial algebra $\mathcal{A} = \mathbb{C}[x_{n+1}, \dots, x_m][\chi]$, where χ is the set of all possible mutable variables, that is, the mutable variables of the original seed or a seed obtained by a mutation or a sequence of mutations. The seed (\mathbf{x}, B) is called the *initial seed*.

Remark 2.5. By the properties of mutation and algebraically independent sets, it is not hard to see that the cluster algebra attached to some seed is the same cluster algebra attached to any mutation of it.

Definition 2.6. The *rank* of a seed or a cluster algebra attached to it is the number of mutable variables of its initial seed. A cluster algebra is *of finite type* if it has finitely many seeds. Otherwise, it is *of infinite type*.

3. SCHUBERT CELLS AND PARTIAL FLAG VARIETIES

Throughout this paper, let G be a simply-connected semisimple algebraic complex group and let I denote the Dynkin diagram vertex set of G . Also, let K denote a subgroup of the set I whose complement is $J = I \setminus K$. For every G denote by B and B^- a pair of Borel and opposite Borel subgroups in which their unipotent radicals are N and N^- respectively. This first section follows [5, 6].

Remark 3.1. For every unipotent subgroup there are one-parameter root subgroups indexed by the Dynkin diagram set that form a set of distinguished generators of it. Let $x_i(t)$ ($i \in I$, $t \in \mathbb{C}$) denote the ones of N and $y_i(t)$ denote those of N^- .

Example 3.2. If $G = SL_{n+1}$ then $x_i(t) = I + tE_{i,i+1}$, where I is the identity matrix and $E_{i,j}$ is the matrix having 1 in the $i \times j$ entry and 0 elsewhere.

Definition 3.3. A *parabolic subgroup* P of G is a closed subgroup that contains a Borel subgroup.

Example 3.4. (1) Any Borel subgroup of G is parabolic.
 (2) For the pair B and B^- of opposite Borel subgroups, we define P_K to be the subgroup generated by B and $y_k(t)$ ($k \in K$). The subgroup P_K is parabolic and called the *standard parabolic subgroup* associated to B . Similarly, P_K^- is defined to be the subgroup generated by B^- and $x_k(t)$ ($k \in K$) and is parabolic as well.

Remark 3.5. Every parabolic subgroup is conjugate to a unique standard Borel subgroup. This, in many situations, reduces the study of parabolic subgroups to the study of standard parabolic case.

Definition 3.6. A *(partial) flag variety* is a quotient of the form G/P , where P is a parabolic subgroup of G .

Remark 3.7 (Generalized minors). If G is of type A , then a *(flag) minor* is a regular irreducible function of $\mathbb{C}[G]$ defined as follows: For any $I \subset [1, n] := \{1, \dots, n\}$ and any matrix $x \in G$, the minor $\Delta_I(x)$ is defined to be the determinant of the submatrix of x whose rows are indexed by I and columns are indexed by $1, \dots, |I|$. This notion was generalized by Fomin and Zelevinsky in [3] to the notion of *(generalized) minor* $\Delta_{u\varpi_j, w(\varpi_j)}$, where u, w belong to the Weyl group W . The notions of flag minors and generalized minors coincide in type A . However, the generalized minor notion makes sense in any type.

For each simple reflection $s_i \in W$, let

$$\overline{s_i} := \exp(f_i) \exp(-e_i) \exp(f_i) \quad \text{and} \quad \overline{\overline{s_i}} := \exp(-f_i) \exp(e_i) \exp(-f_i).$$

If $w = s_{i_1} \dots s_{i_r}$ with r being the length of w , then define

$$\overline{w} = \overline{s_{i_1}} \dots \overline{s_{i_r}} \quad \text{and} \quad \overline{\overline{w}} = \overline{\overline{s_{i_1}}} \dots \overline{\overline{s_{i_r}}}.$$

Let $G_0 = N^-HN$ be the open set of G consisting of elements having Gaussian decomposition. Indeed, each $x \in G_0$ can be uniquely represented as

$$x = [x]_- [x]_0 [x]_+,$$

where $[x]_- \in N^-$, $[x]_0 \in H$, $[x]_+ \in N$. Fix a choice $\{\varpi_i \mid i \in I\} \subset \mathfrak{h}^*$ of fundamental weights, that is,

$$\varpi_i(h_j) = \delta_{ij}, \quad (i, j \in I).$$

Let $x \mapsto x^{\varpi_i}$ be the corresponding character of H . There is a unique regular function $\Delta_{\varpi_i, \varpi_i}$ on G such that

$$\Delta_{\varpi_i, \varpi_i}(x) = [x]_0^{\varpi_i}.$$

Moreover, $G_0 = \{x \in G \mid \Delta_{\varpi_i, \varpi_i}(x) \neq 0, i \in I\}$. This gives the following definition introduced by Fomin and Zelevinsky in [3].

Definition 3.8. For $u, v \in W$ and $i \in I$, define the *generalized minor* to be the regular function on G given by

$$\Delta_{u\varpi_i, v\varpi_i}(x) = \Delta_{\varpi_i, \varpi_i}(\overline{\overline{u^{-1}xv}}).$$

Remark 3.9. Let $\Pi_J \cong \mathbb{N}^J$ be the set of elements of the form $\lambda = \sum_{j \in J} a_j \varpi_j$, where $a_j \in \mathbb{N}$. The homogeneous coordinate ring $\mathbb{C}[G/P_K^-]$ is a Π_J graded module defined as

$$\mathbb{C}[G/P_K^-] = \bigoplus_{\lambda \in \Pi_J} L(\lambda),$$

where $L(\lambda)$ is the irreducible G -module whose highest weight is λ (see 2.3 in [5]). Moreover, $\mathbb{C}[G/P_K^-]$ is generated by the subspaces $\bigoplus_{j \in J} L(\varpi_j)$.

Remark 3.10. The multi-degree of a generalized minor $\Delta_{\varpi_j, w(\varpi_j)}$ in $\mathbb{C}[G/P_K^-]$ is ϖ_j . It is known that the affine coordinate ring of the unipotent radical cell N_K of P_K is the homogeneous elements of degree 0 of the localization of the homogeneous coordinate ring of the corresponding flag variety by the elements $\Delta_{\varpi_j, \varpi_j}$ where $j \in J$. In symbols,

$$(3.1) \quad \mathbb{C}[N_K] = \left\{ \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \mid f \in L\left(\sum_{j \in J} a_j \varpi_j\right) \right\}.$$

Equivalently, the generalized minors relate the affine coordinate ring of the cell N_K to the homogeneous coordinate ring of the corresponding flag variety by

$$(3.2) \quad \mathbb{C}[N_K] = \mathbb{C}[G/P_K^-] / (\Delta_{\varpi_j, \varpi_j} - 1)_{j \in J}.$$

Remark 3.11. Naturally, there is a canonical projection map

$$\text{proj}_J : \mathbb{C}[G/P_K^-] \rightarrow \mathbb{C}[N_K]$$

given by the quotient by the ideal generated by $\Delta_{\varpi_j, w(\varpi_j)} - 1$, ($j \in J$). Remarkably, the restriction of the projection map to each homogeneous $L(\lambda)$, ($\lambda \in \Pi_J$) gives an injection $L(\lambda) \hookrightarrow \mathbb{C}[N_K]$.

Remark 3.12. There is a standard partial ordering \preceq on Π_J given by:

$$\lambda \preceq \mu \iff \mu - \lambda \in \mathbb{N}\{\varpi_j \mid j \in J\},$$

that is, $\mu - \lambda$ is an \mathbb{N} -linear combination of the fundamental weights ϖ_j , where $j \in J$.

Remark 3.13. The Chevalley generators of the Lie algebra \mathfrak{g} of G are denoted, as usual, by e_i, f_i, h_i , where i runs over I . The e_i 's generate $\text{Lie}(N) = \mathfrak{n}$. Naturally, N acts from the left and right on $\mathbb{C}[N]$ by these left and right actions:

$$(x \cdot f)(n) = f(nx), \quad (f \in \mathbb{C}[N] \text{ and } x, n \in N),$$

$$(f \cdot x)(n) = f(xn), \quad (f \in \mathbb{C}[N] \text{ and } x, n \in N).$$

Now, if we differentiate these two actions, we get left and right actions of \mathfrak{n} on the coordinate ring $\mathbb{C}[N]$. Throughout this paper, the right action of e_i on $f \in \mathbb{C}[N]$ will be denoted by $e_i^\dagger(f) := f \cdot e_i$.

Lemma 3.14 (GLS tilde map). *For any element f of the coordinate ring $\mathbb{C}[N_K]$ there is a unique homogeneous element \tilde{f} in $\mathbb{C}[G/P_K^-]$ whose projection to $\mathbb{C}[N_K]$ is f and whose multi-degree is minimal with respect to the partial ordering \preceq of weights.*

Proof (GLS). For any $\lambda = \sum_{i \in I} a_i \varpi_i$, it is remarked in 2.5 in [5] that the subspace $\text{proj}_I(L(\lambda))$ of $\mathbb{C}[N]$ can be described as

$$\text{proj}_I(L(\lambda)) = \{f \in \mathbb{C}[N] \mid (e_i^\dagger)^{a_i+1} f = 0, i \in I\}.$$

This implies that $\mathbb{C}[N_K]$ can be identified with

$$\{f \in \mathbb{C}[N] \mid e_k^\dagger f = 0, k \in K\} \subset \mathbb{C}[N].$$

Thus, for any $\lambda = \sum_{j \in J} a_j \varpi_j \in \Pi_J$, it follows that

$$\text{proj}_J(L(\lambda)) = \{f \in \mathbb{C}[N_K] \mid (e_j^\dagger)^{a_j+1} f = 0, j \in J\}.$$

Now, for $f \in \mathbb{C}[N_K]$, define

$$(3.3) \quad a_j(f) := \max \left\{ s \mid (e_j^\dagger)^s f \neq 0 \right\},$$

and

$$(3.4) \quad \lambda(f) := \sum_{j \in J} a_j(f) \varpi_j.$$

Obviously, f is an element of $\text{proj}_J(L(\lambda(f)))$ where $\lambda(f)$ is minimal with respect to \preceq , that is, if $\lambda \in \Pi_J$ such that $f \in \text{proj}_J(L(\lambda))$, then $\lambda(f) \preceq \lambda$. On the other hand, as the restriction of proj_J to the piece $L(\lambda(f))$ is injective, this gives the desired uniqueness. That is, the element $\tilde{f} \in L(\lambda(f))$, whose projection $\text{proj}_J(\tilde{f}) = f$. \square

Definition 3.15. For any $f \in \mathbb{C}[N_K]$, the element $\tilde{f} \in \mathbb{C}[G/P_K^-]$ will be called the *lift* (or the *lifting*) of f to $\mathbb{C}[G/P_K^-]$.

Lemma 3.16. For any $f, g \in \mathbb{C}[N_K]$, the lifting commutes with usual multiplication, that is, $\widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g}$. In other words, the following diagram

$$\begin{array}{ccc} \mathbb{C}[N_K] \times \mathbb{C}[N_K] & \xrightarrow{m_{\mathbb{C}[N_K]}} & \mathbb{C}[N_K] \\ \downarrow \widetilde{\times} & & \downarrow \tilde{\cdot} \\ \mathbb{C}[G/P_K^-] \times \mathbb{C}[G/P_K^-] & \xrightarrow{m_{\mathbb{C}[G/P_K^-]}} & \mathbb{C}[G/P_K^-] \end{array}$$

given by

$$\begin{array}{ccc} f \times g & \longmapsto & f \cdot g \\ \downarrow & & \downarrow \\ \tilde{f} \times \tilde{g} & \longmapsto & \widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g} \end{array}$$

commutes. Moreover, if $a_j(f + g) = \max\{a_j(f), a_j(g)\}$ for all $j \in J$, then

$$\widetilde{f + g} = \mu \tilde{f} + \nu \tilde{g},$$

where μ and ν are relatively prime monomials in the variables $\Delta_{\varpi_j, \varpi_j}$, ($j \in J$).

Proof (GLS). The first property follows easily by Leibniz formula and the fact that the endomorphism e_j^\dagger is a derivation of $\mathbb{C}[N_K]$, for all $j \in J$. The additional assumption in the second statement implies the existence of relatively prime monomials μ and ν in $\Delta_{\varpi_j, \varpi_j}$'s such that the multi-degree of each of $\mu\tilde{f}$ and $\nu\tilde{g}$ is the same as the one of $\widetilde{f+g}$. This completes the proof. \square

Definition 3.17. For a generalized minor $\Delta_{\varpi_j, w(\varpi_j)}$, define the *restricted minor* $D_{\varpi_j, w(\varpi_j)}$ to be the restriction of it to N .

Remark 3.18. The restricted minors are the main objects of the cluster structure defined by Goodearl and Yakimov on the coordinate ring of a Schubert cell. Their lift plays a significant role in constructing the cluster structure of a partial flag variety (see [5, 6, 9]). Naively, one might expect that the lift of a restricted minor is always a generalized minor, but this need not be true in general.

Geiß, Leclerc and Schröer addressed this example of type A in [5]:

Example 3.19. Consider $G = SL_6$, a group of type A_5 . Take $J = \{1, 3\}$, so $K = \{2, 4, 5\}$. The restricted minor $D_{13,56}$ cannot be written as $D_{12\dots m, i_1 i_2 \dots i_m}$. Therefore, $\tilde{D}_{13,56}$ cannot be a flag minor. However,

$$\begin{aligned} D_{13,56} &= D_{1,2}D_{23,56} - D_{123,256} \\ &= D_{1,2}D_{123,156} - D_{123,256}. \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{D}_{13,56} &= \Delta_{1,2}\Delta_{123,156} - \Delta_{1,1}\Delta_{123,256} \\ &= \Delta_2\Delta_{156} - \Delta_1\Delta_{256}. \end{aligned}$$

4. LIFT DEGREE AND EXPLICIT ALGORITHM

The main goal of this section is to solve the problem of lifting the minors that occurred at the end of the previous section. We start this section by the following proposition:

Proposition 4.1. *Let $f \in \mathbb{C}[N_K]$. Assume that $f = D_{\varpi_j, w(\varpi_j)}$ and $j \in J$. Then $\tilde{f} = \Delta_{\varpi_j, w(\varpi_j)}$.*

Proof. It is obvious that $\Delta_{\varpi_j, w(\varpi_j)}$ is an element that projects to $D_{\varpi_j, w(\varpi_j)}$ such that its degree is ϖ_j . Now, we may assume that there is an element $g \in \mathbb{C}[G/P_K^-]$ whose projection $\text{proj}_J(g) = D_{\varpi_j, w(\varpi_j)}$ and whose degree $\lambda_g \preceq \varpi_j$. By equation (3.4), the only possibility is $\lambda_g = a_j \varpi_j$ and $a_j \leq 1$. This forces λ_g to be ϖ_j , as it is 0 otherwise. By uniqueness, $g = \Delta_{\varpi_j, w(\varpi_j)}$. \square

Lemma 4.2. *Assume that $f, g \in \mathbb{C}[G/P_K^-]$ such that $\text{proj}_J(f) = \text{proj}_J(g)$ and the multi-degree of both is minimal with respect to the partial ordering \preceq of weights. Then $f = g$.*

Proof. Using equation (3.1), there are natural numbers a_j and b_j such that

$$\text{proj}_J(f) = \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \quad \text{and} \quad \text{proj}_J(g) = \frac{g}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{b_j}}.$$

The minimality gives exactly two possibilities, either $f = g$ or $\lambda(f)$ and $\lambda(g)$ are incomparable. The uniqueness of the tilde map makes the latter impossible. Hence, $f = g$ and consequently $a_j = b_j$. \square

This allows to rewrite equation (3.1) as:

$$(4.1) \quad \mathbb{C}[N_K] = \left\{ \text{proj}_J(f) = \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \mid f \in L\left(\sum_{j \in J} a_j \varpi_j\right) \text{ and } a_j \text{ is minimal} \right\}.$$

Corollary 4.3. *If $j \in J$, then the restricted minor $D_{\varpi_j, w\varpi_j}$ is given by*

$$D_{\varpi_j, w\varpi_j} = \frac{\Delta_{\varpi_j, w\varpi_j}}{\Delta_{\varpi_j, \varpi_j}}.$$

Proposition 4.4. *Let $\frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}}$ be an element of $\mathbb{C}[N_K]$ in which a_j is minimal for each j . Then*

$$\widetilde{\left(\frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \right)} = f.$$

Proof. By (4.1) it is clear that $f \in L\left(\sum_{j \in J} a_j \varpi_j\right)$. Obviously,

$$\text{proj}_J(f) = \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}}.$$

The result now follows by the minimality of the a_j 's and the uniqueness of the GLS tilde map. \square

For $f \in \mathbb{C}[N_K]$, recall the definition of $a_j(f)$ given in 3.3.

Definition 4.5. For a restricted minor $D_{\varpi_{i_n}, w(\varpi_{i_n})}$, define the *lift degree* to be the integer d_n in the equation:

$$(4.2) \quad s_{i_1}(s_{i_2} \dots s_{i_n})(\varpi_{i_n}) = s_{i_2} \dots s_{i_n}(\varpi_{i_n}) - d_n \alpha_{i_1},$$

where $w = s_{i_1} s_{i_2} \dots s_{i_n}$ and α_{i_1} is the vertex of the Dynkin diagram indexed by i_1 .

Proposition 4.6. *Assume the setting and notation of Definition (4.5). If $J = \{i_1\}$, then the lift $\widetilde{D}_{\varpi_{i_n}, w(\varpi_{i_n})}$ of the minor $D_{\varpi_{i_n}, w(\varpi_{i_n})}$ is of degree $d_n \varpi_{i_1}$.*

Proof. This follows from the fact that

$$\begin{aligned} a_l(D_{\varpi_{i_n}, w(\varpi_{i_n})}) &= \max \left\{ s \mid (e_l^\dagger)^s D_{\varpi_{i_n}, w(\varpi_{i_n})} \neq 0 \right\} \\ &= \begin{cases} d_n, & \text{if } l = i_1, \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

where a_l is as introduced in Equation (3.3). In fact, the first statement is clear and the second follows from the fact that $\bar{u}x_j(t)\bar{u}^{-1} \in N$, when $\ell(us_i) > \ell(u)$ and $i \neq j$. \square

Corollary 4.7. *Assume the setting and notation of Definition (4.5). If $J = \{i_1\}$, then there is a unique $f \in \mathbb{C}[G/P_K^-]$ of homogeneous degree $d_n \varpi_{i_1}$ such that*

$$D_{\varpi_{i_n}, w(\varpi_{i_n})} = \frac{f}{\Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_n}}.$$

5. AN APPLICATION TO CLUSTER ALGEBRAS

We start this section with some setup and then apply the results of the previous section to the cluster algebra of the homogeneous coordinate ring a partial flag variety defined in [9].

Definition 5.1. Let $w = s_{i_1} \dots s_{i_n} \in W$. Define the functions

$$\begin{aligned} p(k) &:= \begin{cases} \max\{j < k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ -\infty, & \text{otherwise.} \end{cases} \\ s(k) &:= \begin{cases} \min\{j > k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Also, set

$$S(w) := \{i \in I \mid s_i \leq w\} = \{i \in I \mid i = i_k \text{ for some } k \in [1, m]\}.$$

In [9], it is shown that the work of Goodearl and Yakimov gives a canonical cluster structure in $\mathbb{C}[N_K]$ in which its initial exchange matrix \tilde{B}^w is of size $m \times (m - |S(w)|)$ and its $j \times k$ entry is given by

$$(\tilde{B}^w)_{jk} = \begin{cases} 1, & \text{if } j = p(k), \\ -1, & \text{if } j = s(k), \\ a_{i_j i_k}, & \text{if } j < k < s(j) < s(k), \\ -a_{i_j i_k}, & \text{if } k < j < s(k) < s(j), \\ 0, & \text{otherwise;} \end{cases}$$

where the entry $a_{i_j i_k}$ is the same $i_j \times i_k$ entry of the Cartan matrix of the same type. The initial cluster variables are $D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}$. The frozen ones are exactly those indexed by the k 's that satisfy $s(k) = \infty$.

Remark 5.2. Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed of the cluster algebra $\mathcal{A} = \mathbb{C}[N_K]$. Mutate at k to get the exchange relation

$$x_k x'_k = M_k + L_k,$$

where M_k, L_k are monomials in the variables $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$. As shown in [5], this lifts to the equation

$$\widetilde{x_k x'_k} = \mu_k \widetilde{M_k} + \nu_k \widetilde{L_k},$$

where μ_k and ν_k are relatively prime monomials in the variables $\Delta_{\varpi_j, \varpi_j}$, where j runs in J . This means that μ_k and ν_k can be written as

$$\mu_k = \prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{\alpha_j} \quad \text{and} \quad \nu_k = \prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{\beta_j},$$

where $\min\{\alpha_j, \beta_j\} = 0$ for all j .

Note here that the monomial M_k is the one corresponding to the positive b_{i_k} 's in Definition 2.2, while the monomial L_k is the one corresponding to the negative b_{i_k} 's.

Definition 5.3. Let (\mathbf{x}, B) be a seed of the cluster algebra $\mathcal{A}_J = \mathbb{C}[N_K]$. Define the pair $(\widehat{\mathbf{x}}, \widehat{B})$ as follows:

- The tuple $\widehat{\mathbf{x}}$ consists of the variables \tilde{x} induced by lifting each variable x of \mathbf{x} and consists also of the generalized minors $\Delta_{\varpi_j, \varpi_j}$ modded out in $\mathbb{C}[N_K]$. The variables \tilde{x} preserve the same type of the variables x (mutable or frozen), while the minors $\Delta_{\varpi_j, \varpi_j}$ are frozen.
- The matrix \widehat{B} is given as follows: Extend the matrix B of the initial seed of \mathcal{A}_J by $|J|$ rows labeled by the elements of J such that the entries are

$$\widehat{b}_{jk} = \begin{cases} \beta_j, & \text{if } \beta_j \neq 0; \\ -\alpha_j, & \text{else,} \end{cases}$$

where α_j and β_j are as in Remark 5.2.

The combination of the results of the previous section together with the results of [9] gives us the following theorem:

Theorem 5.4. *Let (\mathbf{x}, B) be the initial seed of the cluster algebra $\mathcal{A} = \mathbb{C}[N_K]$ described in Definition 5.1. Then \mathcal{A} lifts to a cluster algebra $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ whose initial exchange cluster is $(\widehat{\mathbf{x}}, \widehat{B})$. In particular, the extended cluster variables are $\{\widetilde{D}_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}\} \sqcup$*

$\{\Delta_{\varpi_j, \varpi_j} \mid j \in J\}$. If $J = \{i_1\}$, then for each $D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}$, there exists a unique $f \in \mathbb{C}[G/P_{\overline{K}}]$ of degree $d_k \varpi_{i_1}$ such that

$$\tilde{D}_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}} = f.$$

Remark 5.5. For a full generality of type A , that is, if J is any nonempty subset of I , one can use the following fact: If Q is a parabolic subgroup between P and G , then

$$w_0 w_{P,0}^{-1} = (w_0 w_{Q,0}^{-1})(w_{Q,0} w_{P,0}^{-1})$$

where $w_{P,0}$ and $w_{Q,0}$ are the longest element of the arbitrary parabolic subgroups P and Q respectively. This will be length-additive, meaning that

$$\dim(G/P) = \dim(G/Q) + \dim(Q/P).$$

So, everything can be reduced to the known cases, for example when P is maximal. Another useful idea, for type A , is to use the Plucker relation together with the fact that the generalized minor is nothing but a flag minor. For instance, see Example 10.3 in [5].

6. EXAMPLES

The results of the previous sections can be used to get these explicit examples.

Example 6.1 (Type A , c.f. [5]). Let G be a semisimple algebraic group of type A_4 . That is, $G = SL_6$. Take $J = \{2\}$ and $K = \{1, 3, 4\}$. We can express the longest word w_0 as

$$w_0 = s_1 s_3 s_4 s_3 s_2 s_3 s_4 s_1 s_2 s_3.$$

The subword $w_K = s_2 s_3 s_4 s_1 s_2 s_3$ generates N_K . By definition, one can easily see that

$$s(1) = 5, \quad s(2) = 6, \quad s(3) = s(4) = s(5) = s(6) = \infty.$$

Therefore, the list of extended cluster variables is

$$\begin{aligned} j = 1 &\implies D_{\varpi_2, s_2 \varpi_2}; && \text{(mutable)} \\ j = 2 &\implies D_{\varpi_3, s_2 s_3 \varpi_3}; && \text{(mutable)} \\ j = 3 &\implies D_{\varpi_4, s_2 s_3 s_4 \varpi_4}; && \text{(frozen)} \\ j = 4 &\implies D_{\varpi_1, s_2 s_3 s_4 s_1 \varpi_1}; && \text{(frozen)} \\ j = 5 &\implies D_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2}; && \text{(frozen)} \\ j = 6 &\implies D_{\varpi_3, s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3}. && \text{(frozen)} \end{aligned}$$

The exchange matrix B is

$$B = \begin{pmatrix} & 1 & 2 \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{pmatrix} .$$

Now, we use the tilde map to lift each extended cluster variable to get the cluster algebra $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ that has the following extended cluster variables

$$\begin{aligned} \widetilde{D}_{\varpi_2, s_2 \varpi_2}; & & (\text{mutable}) \\ \widetilde{D}_{\varpi_3, s_2 s_3 \varpi_3}; & & (\text{mutable}) \\ \widetilde{D}_{\varpi_4, s_2 s_3 s_4 \varpi_4}; & & (\text{frozen}) \\ \widetilde{D}_{\varpi_1, s_2 s_3 s_4 s_1 \varpi_1}; & & (\text{frozen}) \\ \widetilde{D}_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2}; & & (\text{frozen}) \\ \widetilde{D}_{\varpi_3, s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3}. & & (\text{frozen}) \\ \Delta_{\varpi_2, \varpi_2}. & & (\text{frozen}) \end{aligned}$$

Also, from Corollary 4.3 and Corollary 4.7 we get that

$$\begin{aligned} \widetilde{D}_{\varpi_2, s_2 \varpi_2} & = \Delta_{\varpi_2, s_2 \varpi_2} \\ \widetilde{D}_{\varpi_3, s_2 s_3 \varpi_3} & = f_1 \\ \widetilde{D}_{\varpi_4, s_2 s_3 s_4 \varpi_4} & = f_2 \\ \widetilde{D}_{\varpi_1, s_2 s_3 s_4 s_1 \varpi_1} & = f_3 \\ \widetilde{D}_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2} & = \Delta_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2} \\ \widetilde{D}_{\varpi_3, s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3} & = f_4 \end{aligned}$$

such that f_1, f_2, f_3, f_4 are the appropriate unique functions of Corollary 4.7.

Let us now calculate the degree of each of them. For f_1 , consider the equation

$$\begin{aligned} s_2(s_3\varpi_3) &= s_3\varpi_3 - (\alpha_2^\vee, s_3\varpi_3)\alpha_2 \\ &= s_3\varpi_3 - (\alpha_2^\vee, \varpi_3 - \alpha_3)\alpha_2 \\ &= s_2\varpi_2 - \alpha_2. \end{aligned}$$

Hence, the left degree is 1 and by Corollary 4.7 the homogeneous degree of f_1 is ϖ_2 .

Consequently,

$$f_1 = \frac{\Delta_{\varpi_3, s_2 s_3 \varpi_3} \Delta_{\varpi_2, \varpi_2}}{\Delta_{\varpi_3, \varpi_3}}$$

Analogously, one can get that the homogeneous degree of f_2, f_3, f_4 is ϖ_2 as well. Therefore,

$$\begin{aligned} f_2 &= \frac{\Delta_{\varpi_4, s_2 s_3 s_4 \varpi_4} \Delta_{\varpi_2, \varpi_2}}{\Delta_{\varpi_4, \varpi_4}}, \\ f_3 &= \frac{\Delta_{\varpi_4, s_2 s_3 s_4 s_1 \varpi_1} \Delta_{\varpi_2, \varpi_2}}{\Delta_{\varpi_1, \varpi_1}}, \\ f_4 &= \frac{\Delta_{\varpi_3, s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3} \Delta_{\varpi_2, \varpi_2}}{\Delta_{\varpi_3, \varpi_3}}. \end{aligned}$$

Now, the lift of the exchange relations can be calculated easily. For instance, mutating at $k = 1$, we get

$$\begin{aligned} D_{\varpi_2, s_2 \varpi_2} D'_1 &= D_{\varpi_3, s_2 s_3 \varpi_3} D_{\varpi_1, s_2 s_3 s_4 s_1 \varpi_1} + D_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2} \\ &= \frac{f_1}{\Delta_{\varpi_2, \varpi_2}} \frac{f_3}{\Delta_{\varpi_2, \varpi_2}} + \frac{\Delta_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2}}{\Delta_{\varpi_2, \varpi_2}} \\ &= \frac{f_1 f_3 + \Delta_{\varpi_2, \varpi_2} \Delta_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2}}{\Delta_{\varpi_2, \varpi_2}^2}. \end{aligned}$$

Thus, by Lemma 3.16 and Proposition 4.4

$$\widetilde{D_{\varpi_2, s_2 \varpi_2} D'_1} = f_1 f_3 + \Delta_{\varpi_2, \varpi_2} \Delta_{\varpi_2, s_2 s_3 s_4 s_1 s_2 \varpi_2}.$$

The other mutation exchange relations can be lifted similarly. Finally, we get that the exchange matrix attached to the cluster algebra $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ is

$$\widehat{B} = \begin{pmatrix} & 1 & 2 \\ 0 & -1 & \\ 1 & 0 & \\ \hline 0 & 1 & 3 \\ 1 & 0 & 4 \\ -1 & 1 & 5 \\ 0 & -1 & 6 \\ \hline -1 & 0 & \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 1 \in J \end{matrix} .$$

Example 6.2 (Type B , c.f. [9]). Let G be a semisimple algebraic group of type B_3 , say $G = SO_{2(3)+1} = SO_7$, $J = \{3\}$ and $K = I \setminus J = \{1, 2\}$. Consider the longest word

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_3 .$$

In [9], it was shown that $\mathbb{C}[G/P_K^-]$ has a cluster structure whose initial extended cluster is given by the variables

$$\begin{array}{ll} \widetilde{D}_{\varpi_3, s_3 \varpi_3}; & \text{(mutable)} \\ \widetilde{D}_{\varpi_2, s_3 s_2 \varpi_2}; & \text{(mutable)} \\ \widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}; & \text{(mutable)} \\ \widetilde{D}_{\varpi_1, s_3 s_2 s_1 \varpi_1}; & \text{(frozen)} \\ \widetilde{D}_{\varpi_2, s_3 s_2 s_1 s_3 s_2 \varpi_2}; & \text{(frozen)} \\ \widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3}; & \text{(frozen)} \\ \Delta_{\varpi_3, \varpi_3}. & \text{(frozen)} \end{array}$$

Now, using Corollary 4.3 and Corollary 4.7 it is easily seen that

$$\begin{array}{ll} \widetilde{D}_{\varpi_3, s_3 \varpi_3} & = \Delta_{\varpi_3, s_3 \varpi_3} \\ \widetilde{D}_{\varpi_2, s_3 s_2 \varpi_2} & = f \\ \widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3} & = \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3} \\ \widetilde{D}_{\varpi_1, s_3 s_2 s_1 \varpi_1} & = g \\ \widetilde{D}_{\varpi_2, s_3 s_2 s_1 s_3 s_2 \varpi_2} & = h \\ \widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3} & = \Delta_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3} \end{array}$$

where f, g, h are the unique suitable functions gotten by Corollary 4.7. Moreover, the lifting of the mutation exchange relations of the cluster algebra \mathcal{A} becomes obvious

now. In fact, exchange relation induced by mutating at 1 is

$$D_{\varpi_3, s_3 \varpi_3} D'_1 = D_{\varpi_2, s_3 s_2 \varpi_2} + D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}.$$

Now, to find the lift degree of $D_{\varpi_2, s_3 s_2 \varpi_2}$ we consider the equation

$$s_3(s_2(\varpi_2)) = s_2(\varpi_2) - 2\alpha_3.$$

This shows that the lift degree of $D_{\varpi_2, s_3 s_2 \varpi_2}$ is 2. Thus, it follows that the lift degree of $f \in \mathbb{C}[G/P_K^-]$ is $2\varpi_3$. Moreover, by Corollary 4.7

$$f = \frac{\Delta_{\varpi_2, s_3 s_2 \varpi_2} \Delta_{\varpi_3, \varpi_3}^2}{\Delta_{\varpi_2, \varpi_2}}$$

and

$$\begin{aligned} D_{\varpi_3, s_3 \varpi_3} D'_1 &= D_{\varpi_2, s_3 s_2 \varpi_2} + D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3} \\ &= \frac{f}{\Delta_{\varpi_3, \varpi_3}^2} + \frac{\Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_2}}{\Delta_{\varpi_3, \varpi_3}} \\ &= \frac{f + \Delta_{\varpi_3, \varpi_3} \cdot \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_2}}{\Delta_{\varpi_3, \varpi_3}^2}. \end{aligned}$$

Hence, by Lemma 3.16 and Proposition 4.4

$$\widetilde{D_{\varpi_3, s_3 \varpi_3} D'_1} = \widetilde{D}_{\varpi_3, s_3 \varpi_3} \widetilde{D}'_1 = f + \Delta_{\varpi_3, \varpi_3} \cdot \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_2}.$$

On the other hand, the exchange relation induced by mutating at 2 is

$$D_{\varpi_2, s_3 s_2 \varpi_2} D'_2 = D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}^2 D_{\varpi_1, s_3 s_2 s_1 \varpi_1} + D_{\varpi_3, s_3 \varpi_3}^2 D_{\varpi_2, s_3 s_2 s_1 s_2 \varpi_2}.$$

Now, it is not hard to see that

$$s_3(s_2 s_1(\varpi_1)) = s_2 s_1(\varpi_1) - 2\alpha_1 \quad \text{and} \quad s_3(s_2 s_1 s_3 s_2(\varpi_2)) = s_2 s_1 s_3 s_2(\varpi_2) - 2\alpha_3$$

Therefore, the lift degree of both g and h is $2\varpi_3$ again. Hence, by Corollary 4.7, again

$$\begin{aligned} g &= \frac{\Delta_{\varpi_1, s_3 s_2 s_1 \varpi_1} \Delta_{\varpi_3, \varpi_3}^2}{\Delta_{\varpi_1, \varpi_1}} \\ h &= \frac{\Delta_{\varpi_2, s_3 s_2 s_1 s_3 s_2 \varpi_2} \Delta_{\varpi_3, \varpi_3}^2}{\Delta_{\varpi_2, \varpi_2}} \end{aligned}$$

and

$$\begin{aligned}
D_{\varpi_2, s_3 s_2 \varpi_2} D'_2 &= D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}^2 D_{\varpi_1, s_3 s_2 s_1 \varpi_1} + D_{\varpi_3, s_3 \varpi_3}^2 D_{\varpi_2, s_3 s_2 s_1 s_2 \varpi_2} \\
&= \frac{\Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}^2}{\Delta_{\varpi_3, \varpi_3}^2} \cdot \frac{g}{\Delta_{\varpi_3, \varpi_3}^2} + \frac{\Delta_{\varpi_3, s_3 \varpi_3}^2}{\Delta_{\varpi_3, \varpi_3}^2} \cdot \frac{h}{\Delta_{\varpi_3, \varpi_3}^2} \\
&= \frac{\Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}^2 \cdot g + \Delta_{\varpi_3, s_3 \varpi_3}^2 \cdot h}{\Delta_{\varpi_3, \varpi_3}^4}
\end{aligned}$$

Hence, by Lemma 3.16 and Proposition 4.4 again

$$\widetilde{D_{\varpi_2, s_3 s_2 \varpi_2} D'_2} = \widetilde{D}_{\varpi_2, s_3 s_2 \varpi_2} \widetilde{D}'_2 = \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}^2 \cdot g + \Delta_{\varpi_3, s_3 \varpi_3}^2 \cdot h.$$

The mutation relation induced by mutating at 3 lifts similarly.

The initial exchange matrix of the cluster algebra $\mathbb{C}[N_K]$ given in Definition 5.1 is

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 4 \\ 3 \\ 5 \\ 6 \end{matrix}.$$

Using the calculations above and Theorem 5.4, we get that $\mathbb{C}[G/P_K^-]$ contains the cluster algebra $\widehat{\mathcal{A}}$ whose initial extended cluster variables are the ones in the list above and whose initial exchange matrix is

$$\widehat{B} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ \hline -1 & 0 & 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 4 \\ 3 \\ 5 \\ 6 \\ j \in J \end{matrix}.$$

Remark 6.3. Note that our goal once we find the additional rows of \widehat{B} is to make the relations homogeneous. Observing the previous examples, we can easily see that if $J = \{i_1\}$, the last row of the matrix \widehat{B} is obtained by making the summation of the degrees of the positive monomial and the negative monomial equal in each column. Hence, the entries of the last row are obtained according to that.

7. GENERAL THEOREMS

In the previous section, we saw how one uses the results of this paper to obtain explicit examples of type A and type B . In this section, we generalize these examples to any 2-step parabolic subgroup of type A and to any maximal parabolic subgroup of type B . Other general examples can be obtained similarly to the combination of the results of this section and the previous one.

Example 7.1 (2-step parabolic subgroups of type A). Let G be of type A_n , that is, $G = SL_{n+1}$. Take $J = \{j_1, j_2\}$ with $j_1 < j_2$ and $K = \{1, \dots, n\} \setminus \{j_1, j_2\}$. The longest word can be determined once we know the positions of j_1 and j_2 . Indeed, the desired expression of the longest word starts with the longest word of A_{j_1-1} indexed by $1, \dots, j_1 - 1$, followed by the longest word of $A_{j_2-j_1-1}$ indexed by $j_1 + 1, \dots, j_2 - 1$ and then followed by the longest word of A_{n-j_2} indexed by $j_2 + 1, \dots, n$, followed by the completion of that longest word by a subword generating N_K . More concretely, let

$$\begin{aligned} u_1 &= s_1 s_2 \dots s_{j_1-1} s_1 s_2 \dots s_{j_1-2} \dots s_1 s_2 s_1 \\ u_2 &= s_{j_1+1} s_{j_1+2} \dots s_{j_2-1} s_{j_1+1} s_{j_1+2} \dots s_{j_2-2} \dots s_{j_1+1} s_{j_1+2} s_{j_1+1} \\ u_3 &= s_{j_2+1} s_{j_2+2} \dots s_{n-1} s_{j_2+1} s_{j_2+2} \dots s_{n-2} \dots s_{j_2+1} s_{j_2+2} s_{j_2+1} \end{aligned}$$

Then, there exists a subword u_4 in which the longest word w_0 can be expressed as $w_0 = u_1 u_2 u_3 u_4$. One way to construct u_4 is to take $s_1 \dots s_n$, together with a reduced word consisting of $j_2 - 1$ reduced subwords each of length $n - j_2$, followed by a reduced word consisting $j_2 - j_1$ reduced subwords each of length j_1 . In particular, we may choose u_4 to be

$$\begin{aligned} u_4 &= s_1 \dots s_n u_5 u_6, \\ u_5 &= s_1 \dots s_{n-j_2} s_2 \dots s_{n+1-j_2} s_3 \dots s_{n+2-j_2} \dots s_{j_2-1} \dots s_{n-2}, \\ u_6 &= s_1 \dots s_{j_1} s_2 \dots s_{j_1+1} s_3 \dots s_{j_1+2} \dots s_{j_2-j_1} \dots s_{j_2-1}. \end{aligned}$$

In fact, this u_4 generates N_K and it is of length $n + (n - j_2)(j_2 - 1) + j_1(j_2 - j_1)$.

Obviously, one now can use the fact that the generalized minor is a flag minor in type A and use the Plucker relation. Then, follow the procedure of Example 10.3 of [5] to get the general picture of how the cluster algebra $\widehat{\mathcal{A}}$ is constructed generally from a 2-step parabolic subgroup, that is, a parabolic subgroup P_K^- with $|J| = 2$. Let $M := n + (n - j_2)(j_2 - 1)$ and

$$\begin{aligned} A &:= \{n - 1, n\} \\ &\cup \left\{ a \mid a \in [n + 1, M] \text{ and } i_a \geq j_2 - 1 \right\} \\ &\cup \left\{ \begin{array}{l} M + 1, \quad M + j_1 + 1, \quad M + 2j_1 + 1, \quad \dots, \quad M + (j_2 - j_1 - 1)j_1 + 1, \\ M + (j_2 - j_1 - 1)j_1 + 2, \quad M + (j_2 - j_1 - 1)j_1 + 3, \quad \dots, \quad M + (j_2 - j_1)j_1 \end{array} \right\}. \end{aligned}$$

It is not hard to see that $s(i_k) = \infty$ if and only if $k \in A$. Thus, by Definition 5.1, these k 's are exactly the indexes of the frozen variables of the desired initial seed of $\mathcal{A} = \mathbb{C}[N_K]$, while the other indexes are the ones of the mutable variables. More concretely, the frozen variables are

$$D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}, \quad (k \in A),$$

while the mutable variables are

$$D_{\varpi_{i_l}, w_{\leq l} \varpi_{i_l}}, \quad (l \notin A).$$

Also, by Definition 5.1, the exchange matrix of this seed is

$$B_{jk} = \begin{cases} 1, & \text{if } j = p(k), \\ -1, & \text{if } j = s(k), \\ a_{i_j i_k}, & \text{if } j < k < s(j) < s(k), \\ -a_{i_j i_k}, & \text{if } k < j < s(k) < s(j), \\ 0, & \text{otherwise;} \end{cases}$$

Now, one can rewrite the indexes of the flag minors and find the use the Plucker relation to homogenize the lift of each exchange relation.

Example 7.2 (Maximal parabolic subgroups of type B). Let G be a semisimple algebraic group of type B_n . Let $J = \{j\}$ and $K = \{1, \dots, n\} \setminus \{j\}$. The description of the longest word w_0 here depends on the position of j . We generalize Example 6.2 by taking $j = n$. Here w_0 is constructed by taking the longest word u of A_{n-1} indexed by $1, \dots, n-1$, completed by some subword generating N_k , for instance,

$$v = s_n s_{n-1} \dots s_1 s_n s_{n-1} \dots s_2 s_n s_{n-1} \dots s_3 s_n s_{n-1} s_n.$$

Therefore, in this case, we have $w_0 = uv$. Let

$$A = \left\{ n, 2n-1, 3n-3, 4n-5, \dots, \frac{n(n+1)}{2} - 1, \frac{n(n+1)}{2} \right\}.$$

Note that $s(i_k) = \infty$ if and only if $k \in A$. Thus, the frozen variables are those indexed by such k 's. Same as before, the frozen variables are

$$D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}, \quad (k \in A),$$

while the mutable variables are

$$D_{\varpi_{i_l}, w_{\leq l} \varpi_{i_l}}, \quad (l \notin A).$$

Again, Definition 5.1, gives the recipe of the exchange matrix B .

Now, the exchange matrix \widehat{B} of the lift of this seed to the cluster algebra $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ is B extended by the row indexed by $j = n$, whose entries are obtained from Theorem 5.4. The frozen variables are $\Delta_{\varpi_n, \varpi_n}$ together with

$$\widetilde{D}_{\varpi_{i_k}, w \leq k \varpi_{i_k}} = \frac{\Delta_{\varpi_{i_k}, w \leq k \varpi_{i_k}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_k}}{\Delta_{\varpi_{i_k}, \varpi_{i_k}}},$$

where $k \in A$. On the other hand, the mutable ones are

$$\widetilde{D}_{\varpi_{i_l}, w \leq l \varpi_{i_l}} = \frac{\Delta_{\varpi_{i_l}, w \leq l \varpi_{i_l}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_l}}{\Delta_{\varpi_{i_l}, \varpi_{i_l}}},$$

where $l \notin A$.

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