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LOCAL COHOMOLOGY OF CERTAIN DETERMINANTAL THICKENINGS

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ABSTRACT. Let $R = \mathbb{C}[X_{ij}]$ be the ring of polynomial functions in mn variables where $m > n$. Set X to be the $m \times n$ matrix in these variables and $I := I_n(X)$ the ideal of maximal minors of X. We consider the rings R/I^t; for $t \gg 0$ the depth of R/I^t is equal to $n^2 - 1$, and we show that each local cohomology module $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ is a cyclic R-module and compute its annihilator thereby completely determining its R-module structure. We also describe the modules $\operatorname{Ext}^{mn-n^2+1}_R(R/I^t, R)$ as submodules of $H^{mn}_{\mathfrak{m}}(R)$.

In the case that X is a $n \times (n-1)$ matrix we can explicitly describe the maps $\operatorname{Ext}^n_R(R/I^t, R) \to H^n_I(R) \to H^{n(n-1)}_{\mathfrak{m}}(R)$. This is done by analysing maps between the Koszul complex of the t-powers of the maximal minors and a free resolution of R/I^t. With these maps we can give explicit descriptions of $\operatorname{Ext}^n_R(R/I^t, R)$ as submodules of the top local cohomology module $H^n_I(R)$. This description allows for an alternate more constructive proof of the description of the image of the embedding $\operatorname{Ext}^{mn-n^2+1}_R(R/I^t, R) \to H^{n(n-1)}_{\mathfrak{m}}(R)$ and the annihilators of these modules.

1. Introduction

Let I be a homogeneous ideal in a polynomial ring R. Then I defines a projective variety and one may consider its thickenings, i.e., the varieties defined by the ideals I^t. Understanding the ideals I^t is an important component of understanding the singularities of the variety defined by I. For example, they comprise the graded components of Rees algebras and also appear in the study of the functors $H^i_I(-)$. It was shown in [1] that under certain conditions the graded components of the local cohomology modules $H^i_{\mathfrak{m}}(R/I^t)$ stabilize for sufficiently large t. This recent work has brought renewed attention to thickenings and created an interest in their homological properties and invariants.

In the case that $I = I_r(X)$ and $R = \mathbb{C}[X]$ where X is matrix of indeterminates, the modules $H^i_{\mathfrak{m}}(R/I^t)$, $\operatorname{Ext}^i_R(R/I^t, R)$ and $H^i_I(R)$ have been studied extensively and successfully using representation theoretic techniques. In [2], [3] and [4] Raicu–Weyman–Witt, Raicu–Weyman and Raicu described the GL-equivariant structure of $\operatorname{Ext}^i_R(R/I^t, R)$ and $H^i_I(R)$. These results have been used by Kenkel and Li in [5] and [6] to study the asymptotic length of $H^i_{\mathfrak{m}}(R/I^t)$ and find formulas for the higher epsilon multiplicity of I. In a similar flavor, the regularity of I^t was described in [3] and [4] along with a classification of which GL-invariant ideals satisfy the property that $H^i_I(R) = \bigcup_t \operatorname{Ext}^i(R/I^t, R)$.

In this paper we focus on the case that $I \subseteq \mathbb{C}[X]$ is the ideal of maximal minors of a $m \times n$ generic matrix X, with $m > n$ and examine $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$. For sufficiently large t, $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ is the first non-vanishing local cohomology module of R/I^t and it was shown by Li that $n^2 - 1$ is the only cohomological index to yield a nonzero finite length module [6]. In Proposition 2.10, we will show that $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ is in fact a cyclic R-module. This module has also been examined in the case that X is 2×3 matrix in [7] where Kenkel explicitly describes a generator of $[H^3_{\mathfrak{m}}(R/I^t)]_0$ via the Čech complex on the variables of R.

The aforementioned results about $H^i_{\mathfrak{m}}(R/I^t)$ speak about the structure of its graded components, i.e., its structure as a graded $\mathbb{C}$-vector space. Additionally, the description of $\operatorname{Ext}^i_R(R/I^t, R)$ given in [2] is as

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a GL-representation and a priori does not speak on its structure as an R-module. In this paper we study the R-module structure of these modules and explicitly describe this structure for certain Ext and local cohomology modules.

We proceed by investigating the modules $\operatorname{Ext}^i_R(R/I^t,R)$ via the natural map $\operatorname{Ext}^i_R(R/I^t,R) \to H^i_I(R)$. In the case that I is the ideal of maximal minors of a generic matrix, the natural map is an injection [2], hence describing $\operatorname{Ext}^i_R(R/I^t,R)$ is equivalent to describing its image in $H^i_I(R)$. To understand the map $\operatorname{Ext}^i_R(R/I^t,R) \to H^i_I(R)$ we first view $H^i_I(R)$ as Čech cohomology of the maximal minors and compare this to the Koszul cohomology of the powers of maximal minors in the usual way. We then examine the natural map from $\operatorname{Ext}^i_R(R/I^t,R)$ to this Koszul cohomology. As the map from Koszul cohomology to Čech cohomology is well understood, it remains to understand the map $\operatorname{Ext}^i_R(R/I^t,R) \to H^i([d_1^t \dots d_k^t];R)$ where $d_1,\cdots,d_k$ are the maximal minors of X. Thus, to explicitly describe $\operatorname{Ext}^i_R(R/I^t,R) \to H^i([d_1^t \dots d_k^t];R)$ we need to describe a map of complexes φ_t such that

$$\begin{array}{ccccc} F_\bullet & \longrightarrow & I^t & \longrightarrow & 0 \\ \uparrow{\scriptstyle \varphi_t} & & \uparrow & & \\ K_\bullet([d_1^t \cdots d_k^t];R) & \longrightarrow & (d_1^t,\dots,d_k^t) & \longrightarrow & 0 \end{array}$$

commutes, where $F_\bullet$ is a free resolution of I^t. The utility of using this approach to study $\operatorname{Ext}^i_R(R/I^t,R)$ is in that the module structure of $H^i_I(R)$ may be quite familiar, cf. [8, Main Theorem]. For example, for $i = mn-n^2+1$, the cohomological dimension of I, it has been shown that $H^i_I(R) \cong H^{mn}_{\mathfrak{m}}(R)$ [9], [10].

In the case that X is size $n\times(n-1)$ we are able to explicitly construct a map φ_t as above; this is the content of Section 3. Using this lift, in Section 4, we give the following description of $\operatorname{Ext}^n_R(R/I^t,R)$ as a submodule of $H^n_I(R)$.

Theorem (4.1)**.** *Let X be a $n\times(n-1)$ matrix of indeterminates and $R=\mathbb{C}[X]$. Set $I=(d_1,\dots,d_n)\subseteq R$ where $d_1,\dots,d_n$ are the maximal minors of X. For a tuple $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}$ write $d^\alpha = d_1^{\alpha_1}\cdots d_n^{\alpha_n}$. Then $\operatorname{Ext}^n_R(R/I^t,R)$ embeds into $H^n_I(R)$ as the submodule generated by the classes*

$$\left\{ \frac{1}{\prod_{i=1}^n d_i} \cdot \frac{1}{d^\alpha} \right\}_{|\alpha|=t-n+1}.$$

This embedding can be realized as coming from differential operators and after identifying $H^n_I(R)$ with $H^{n(n-1)}_{\mathfrak{m}}(R)$ we obtain the following key corollary.

Corollary (4.4)**.** *In the setting of the previous theorem, for $f\in R$, let f^* denote the polynomial differential operator obtained from f by replacing x_i with ∂_i. Then for $t\geq n-1$ we have that $\operatorname{Ext}^n_R(R/I^t,R)$ embeds in $H^{n(n-1)}_{\mathfrak{m}}(R)$ as the R-submodule generated by the classes*

$$\left\{ (d^\alpha)^* \bullet \frac{1}{\underline{x}} \right\}_{|\alpha|=t-n+1},$$

where $\bullet$ denotes the application of an operator and $\frac{1}{\underline{x}} := \frac{1}{\prod x_{ij}} \in H^{n(n-1)}_{\mathfrak{m}}(R)$.

The Weyl algebra annihilator of the class $\frac{1}{\underline{x}} \in H^{n(n-1)}_{\mathfrak{m}}(R)$ is well understood, see for example [11, Exercise 17.27], and in the remainder of Section 4 we use Corollary 4.4 to compute the R-annihilator of $\operatorname{Ext}^n_R(R/I^t,R)$. By graded duality, the annihilator of $\operatorname{Ext}^n_R(R/I^t,R)$ is the annihilator of $H^{(n-1)^2-1}_{\mathfrak{m}}(R/I^t)$, hence we obtain a complete description of $H^{(n-1)^2-1}_{\mathfrak{m}}(R/I^t)$ when X is size $n\times(n-1)$ as a cyclic R-module generated in degree zero, see Proposition 2.10.

In the general case of maximal minors of an $m \times n$ matrix with $m > n+1$ the map of complexes, φ_t, and with it the structure of the Ext modules, remains mysterious. However, by analyzing the GL-structure of $H_{\mathfrak{m}}^{n^2-1}(R/I^t)$ we are able to compute its annihilator and obtain a general version of Proposition 4.4 as follows:

Theorem (4.8, 5.1,5.2)**.** *Let X be a $m \times n$ matrix of indeterminates with $m > n$ and set $I = I_n(x) \subseteq R = \mathbb{C}[X]$. If $t < n$ then $H_{\mathfrak{m}}^{n^2-1}(R/I^t) = 0$. If $t \geq n$, then we have an isomorphism of graded R-modules:*

$$H_{\mathfrak{m}}^{n^2-1}(R/I^t) \cong R/I_{\lambda},$$

where I_λ is the GL-invariant ideal associated to the partition $\lambda = (t-n+1)$, i.e., the ideal generated by $\mathrm{GL}_m \times \mathrm{GL}_n$ orbit of $x_{1,1}^{t-n+1}$, i.e., the ideal of $t-n+1$ generalized permanents of X c.f. 2.2.

Additionally in this setting,

$$\mathrm{Ext}_R^{mn-n^2+1}(R/I^t, R) \cong \sum_{|\alpha|=t-n} R \cdot (d^{\alpha})^* \frac{1}{\underline{x}} \subset H_{\mathfrak{m}}^{mn}(R),$$

where as before f^* denotes the polynomial differential operator obtained from f by replacing x_i with ∂_i and $\frac{1}{\underline{x}} := \frac{1}{\prod x_{ij}} \in H_{\mathfrak{m}}^{mn}(R)$.

2. Background

Notation 2.1. Let $R = \mathbb{C}[x_{ij}]$ be a polynomial ring and $\mathscr{D} = R[\partial_{ij}]$ be the ring of differential operators on R. Fix $f \in R$ and $\psi \in \mathscr{D}$.

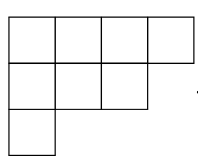
- For a $\mathscr{D}$-module M and an element $h \in M$ we write $\psi \bullet h$ for element obtained by acting on h by ψ. In particular $\psi \bullet f \in R$ is the application of ψ to f.
- We write $\psi f \in \mathscr{D}$ for the multiplication of ψ and f in $\mathscr{D}$.
- We write $f^* \in \mathscr{D}$ for $f(\partial)$, the "dual" operator to f obtained by replacing x_i by ∂_i.

Let G be a group acting on a set S. Fix $g \in G$ and $s \in S$.

- We write $g \cdot s$ for the element obtained by acting on s with g.
- We write $G \cdot s$ for the orbit of s.

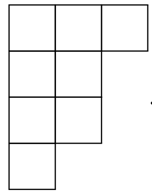
2.1. ***Dominant Weights, Partitions and Schur Functors.*** We begin by establishing some notation and recalling some useful facts about Schur functors, for a complete treatment see [12] and [13]. A vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is called a *dominant weight* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We write $\mathbb{Z}^n_{dom}$ for the set of all dominant weights in $\mathbb{Z}^n$ and write $|\lambda| = \sum_{i=1}^n \lambda_i$ for the *size* of λ. Additionally, for $c \in \mathbb{Z}$ and $0 \leq d \leq n$, we write $(c^d) \in \mathbb{Z}^n_{dom}$ for the vector with d nonzero components all equal to c. A *partition* into n parts is a dominant weight, $\lambda \in \mathbb{Z}^n_{dom}$, with $\lambda_n \geq 0$, we write $\mathscr{P}_n \subseteq \mathbb{Z}^n_{dom}$ to be the set of all such weights. An element $\lambda \in \mathscr{P}_n$ may be realized as a *Young diagram* with λ_i boxes in row i, for example the diagram associated to $(4,3,1) \in \mathscr{P}_3$ is:

(1)



.

If $m \geq n$, we can naturally identify an element of $\mathscr{P}_n$ with an element of $\mathscr{P}_m$ by adjoining zeroes, e.g., $(2,2) \in \mathscr{P}_2$ is identified with $(2,2,0,0) \in \mathscr{P}_4$. Generally we omit the trailing zeroes and would write $(2,2) \in \mathscr{P}_4$. For $\lambda \in \mathscr{P}_n$ we can consider its *transpose partition*, λ', which is the partition associated to the transpose of the Young diagram of λ. The transpose partition of $(4,3,1)$ is $(3,2,2,1)$ because the transpose of (1) is:

(2)  .

Let H be an n dimensional $\mathbb{C}$-vector space. Then to each dominant weight $\lambda \in \mathbb{Z}^n_{dom}$ we associate an irreducible representation of $\mathrm{GL}(H)$, denoted $S_\lambda H$, called a *Schur functor*. Moreover every irreducible representation of $\mathrm{GL}(H)$ can be realized in this manner. For some dominant weights, Schur functors are quite familiar: there are $\mathrm{GL}(H)$-equivariant isomorphisms:

$$S_{(1^d)}H \cong \bigwedge^d H$$

and

$$S_{(d)}H \cong \mathrm{Sym}^d H.$$

For computational purposes, frequently it is sufficient to consider $\lambda \in \mathscr{P}_n$ as we have the following $\mathrm{GL}(H)$-equivariant isomorphisms:

$$S_{\lambda+(1^n)}H \cong S_\lambda H \bigotimes \bigwedge^n H$$

and

$$S_{(\lambda_1,\dots,\lambda_n)}H \cong \mathrm{Hom}(S_{(-\lambda_n,\dots,-\lambda_1)}H, \mathbb{C}).$$

2.2. GL***-invariant ideals.*** Let $F = \mathbb{C}^m$ and $G = \mathbb{C}^n$ where $m \geq n$. Then

$$R := \mathrm{Sym}(F \otimes G) = \mathbb{C}[x_{ij}] = \mathbb{C}[X]$$

is a polynomial ring admitting a

$$\mathrm{GL} := \mathrm{GL}(F) \times \mathrm{GL}(G)$$

action as follows: for $(g_1, g_2) \in \mathrm{GL}$, $g \cdot (x_{ij}) = (z_{ij})$ where $[z_{ij}] = g_1 X g_2^{-1}$. *Cauchy's formula* describes the decomposition of R into irreducible representations [13]:

$$R = \bigoplus_{\lambda \in \mathscr{P}_n} S_\lambda F \otimes S_\lambda G \tag{3}$$

where $S_\lambda F \otimes S_\lambda G$ lives in degree $|\lambda|$.

For a number $1 \leq l \leq n$ set $\det_l := \det(x_{ij})_{1 \leq i,j \leq l}$, i.e., the $l \times l$ minor in the top left corner of X. Then for a partition, $\lambda \in \mathscr{P}_n$, with n parts let λ' be the transpose partition and define

$$\det_\lambda := \prod_{i=1}^{\lambda_1} \det_{\lambda'_i}.$$

The $\mathbb{C}$-linear span of the GL orbit of $\det_\lambda$ is equal to $S_\lambda F \otimes S_\lambda G$. We set

$$I_\lambda := (\mathrm{GL} \cdot \det_\lambda) \subseteq R,$$

the ideal generated by the GL orbit of $\det_\lambda$. This ideal is GL-invariant. We endow $\mathscr{P}_n$ with a partial ordering: for $\mu, \lambda \in \mathscr{P}_n$, we say that $\mu \geq \lambda$ if $\mu_i \geq \lambda_i$ for all i. It was shown in [14] that:

$$I_\lambda = \bigoplus_{\mu \geq \lambda} S_\mu F \otimes S_\mu G. \tag{4}$$

By taking a collection of partitions $\chi \subseteq \mathscr{P}_n$ we can form the GL-invariant ideal $I_\chi = \sum_{\lambda\in\chi} I_\lambda$. It was proven in [14] that all GL-invariant ideals may be realized in this manner for some finite subset $\chi \subseteq \mathscr{P}_n$ and so more generally GL-invariant ideals decompose as:

$$I_\chi = \bigoplus_{\substack{\mu\geq\lambda \\ \lambda\in\chi}} S_\mu F \otimes S_\mu G.$$

Example 2.2. *Let r,t be positive integers.*

(1) $I_{(1^r)} = I_r(X)$ *the ideal of $r\times r$-minors of X.*
(2) $I_{\chi_t} = I_r(X)^t$ *where* $\chi_t = \{\lambda \in \mathscr{P}_n \mid |\lambda| = rt, \lambda_1 \leq t\}$.
(2′) $I_{(t^n)} = I_n(X)^t$ *the t-th power of the ideal of maximal minors of X.*
(3) $I_{(t)}$, *the ideal of $t\times t$ generalized permanents of X, i.e., the ideal generated by the permanent of all $t\times t$ matrices of the form $[x_{\alpha_i,\beta_j}]$ where $\alpha_i \leq \alpha_{i+1}$ and $\beta_j \leq \beta_{j+1}$.*

Remark 2.3. Notice that Cauchy's Formula, (3), says that every irreducible representation of GL contained in R appears at most once. Combining this with the classification of GL-invariant ideals of [14] stated above, we have that a GL-invariant ideal is uniquely determined by its structure as a GL-representation.

2.3. ***A $\mathbb{C}$-Linear GL-Equivariant Pairing.*** Let R be as above. Let $R^* = \mathbb{C}[\partial_{ij}]$ and

$$(-)^* : R \to R^*$$

be the map induced by $x_{ij} \mapsto \partial_{ij}$. We can view R as the coordinate ring for the space of $m\times n$ complex matrices and the GL action on R described above as being induced by the GL action on this space of matrices. We now view R^* as the coordinate ring for the space of $n\times m$ matrices and hence GL acts on it as follows: for $g = (g_1, g_2) \in \mathrm{GL}$, $g\cdot x_{ij}^* = z_{ij}$ where $[z_{ij}] = (g_1^{-1})^T X^* g_2^T$.

The action of R^* on R via differentiation induces a perfect pairing

$$\langle\ ,\ \rangle : [R^*]_k \times [R]_k \to \mathbb{C}.$$

We will see below this pairing is GL-equivariant, where GL acts on R and R^* as above and fixes $\mathbb{C}$. A more general statement about differential operators acting on representations is known, see [15, Section 2.2], however we include the following proof for completeness.

Lemma 2.4. *The pairing,* $\langle\ ,\ \rangle : [R^*]_r \times [R]_k \to [R]_{k-r}$ *is* GL*-equivariant.*

By saying that the pairing is equivariant we mean that for all $\theta \in \mathrm{GL}$ and $f,g\in R$ we have that $\theta\cdot\langle f^*, g\rangle = \langle\theta\cdot f^*, \theta\cdot g\rangle$. In particular, since the action of GL on $\mathbb{C} = [R]_0$ is trivial, this means that if f and g are homogeneous of the same degree then $\langle\theta\cdot f^*, \theta\cdot g\rangle = \langle f^*, g\rangle$.

Proof. First note that we can assume that $r\leq k$ since the GL-action is degree preserving and that by linearity we may assume that f and g are monomials.

We note that the cases $k=r=0$ and $k=1$, $r=0$ are clear so to establish the base case $k=1$ we need to show $k=r=1$.

Let $\theta = (\theta_1,\theta_2) \in \mathrm{GL}$, it is sufficient to prove the statement for $\theta = (\theta_1, ID)$ and $\theta = (ID, \theta_2)$, where ID denotes the identity. The arguments in each case are analogous so we assume that $\theta = (ID,\theta_2)$.

Thus, $\theta\cdot x_{ij} = \sum_{h=1}^n x_{ih}[\theta_2^{-1}]_{hj}$ and $\theta\cdot x_{ij}^* = \sum_{h=1}^n x_{ih}^*[\theta_2^T]_{hj} = \sum_{h=1}^n x_{ih}^*[\theta_2]_{jh}$.

Since $k=1$ we set $f = x_{ab}$, $g = x_{cd}$. Then $\langle f^*, g\rangle$ is 1 if $(a,b) = (c,d)$ and 0 otherwise. Now consider,

$$\begin{aligned}
\langle\theta\cdot f^*,\theta\cdot g\rangle &= \Big(\sum_{h=1}^{n} x^*_{ah}[\theta_2]_{bh}\Big)\bullet\Big(\sum_{h=1}^{n} x_{ch}[\theta_2^{-1}]_{hd}\Big)\\
&= \sum_{k=1}^{n}\sum_{l=1}^{n}[\theta_2]_{bh}[\theta_2^{-1}]_{ld}(x^*_{ah}\bullet x_{cl})\\
&= \begin{cases} 0 & a\neq c\\ \sum_{h=1}^{n}[\theta_2]_{bh}[\theta_2^{-1}]_{hd} & \text{else}\end{cases}\\
&= \begin{cases} 0 & a\neq c\\ ID_{b,d} & \text{else}\end{cases}\\
&= \begin{cases} 1 & (a,b)=(c,d)\\ 0 & \text{else}\end{cases}.
\end{aligned}$$

Now assuming $k>1$ we may write $f=x_{ab}f'$ and $g=x_{cd}g'$, then :

$$(\theta\cdot f^*)\bullet(\theta\cdot g) = (\theta\cdot f'^*)\bullet(\theta\cdot x^*_{ab}\bullet(\theta\cdot g')(\theta\cdot x_{cd})) \tag{5}$$
$$= (\theta\cdot f'^*)\bullet((\theta\cdot x_{cd})(\theta\cdot x^*_{ab}\bullet(\theta\cdot g'))+(\theta\cdot g')(\theta\cdot x^*_{ab}\bullet(\theta\cdot x_{cd}))) \tag{6}$$
$$= (\theta\cdot f'^*)\bullet((\theta\cdot x_{cd})(\theta\cdot(x^*_{ab}\bullet g'))+(\theta\cdot g')(\theta\cdot(x^*_{ab}\bullet x_{cd})) \tag{7}$$
$$= (\theta\cdot f'^*)\bullet(\theta\cdot(x_{cd}(x^*_{ab}\bullet g'))+\theta\cdot(g'\cdot(x^*_{ab}\bullet x_{cd}))) \tag{8}$$
$$= (\theta\cdot f'^*)\bullet(\theta\cdot(x_{cd}(x^*_{ab}\bullet g')+g'\cdot(x^*_{ab}\bullet x_{cd}))) \tag{9}$$
$$= \theta\cdot(f'^*\bullet((x_{cd}(x^*_{ab}\bullet g')+g'\cdot(x^*_{ab}\bullet x_{cd})))) \tag{10}$$
$$= \theta\cdot(f'^*\bullet(x^*_{ab}\bullet(x_{cd}g'))) \tag{11}$$
$$= \theta\cdot(f^*\bullet g), \tag{12}$$

where (5) to (6) is by the product rule since $\theta\cdot x^*_{ab}$ is a linear operator. From (6) to (7) is by induction. From (7) to (8) and (8) to (9) are by factoring. From (9) to (10) is by induction hypothesis. Finally (10) to (11) is again by the product rule.

□

Now given an GL-invariant subspace of $[R]_k$, e.g., a graded component of a GL-invariant ideal, we can use this pairing to analyze which polynomial operators annihilate that subspace.

Lemma 2.5. *Let $N\subseteq[R]_k$ be a* GL*-invariant subspace and suppose $f\in[R]_j$ such that $f^*\bullet N=0$. Then the* GL *orbit of f^* also annihilates N.*

Proof. Let $h\in N$ and $\theta\in\mathrm{GL}$. Since N is GL-equivariant $\theta^{-1}\cdot h\in N$ and by assumption $f^*\bullet\theta^{-1}\cdot h=0$. Thus by Lemma 2.4,

$$0=f^*\bullet\theta^{-1}\cdot h=\theta\cdot(f^*\bullet\theta^{-1}\cdot h)=(\theta\cdot f^*)\bullet(\theta\cdot(\theta^{-1}\cdot h))=(\theta\cdot f^*)\bullet h$$

□

2.4. GL***-equivariant description of certain*** *Ext* ***modules.*** Let R,F,G and GL be as defined above in Section 2.2. Let $I=I_{(1^n)}$ be the ideal generated by the maximal minors of X. In [2] the authors gave a GL-equivariant description of $H^i_I(R)$ as a direct sum of irreducible GL-representations. They also prove a number of results about the modules $\mathrm{Ext}^i_R(R/I^t,R)$, we recall two of these results below.

Theorem 2.6. [2, Theorem 4.3] *Let $m > n$ and $t \geq n$. If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ we write*

$$\lambda(s) = (\lambda_1, \dots, \lambda_{n-s}, \underbrace{-s, \dots, -s}_{m-n}, \lambda_{n-s+1} + (m-n), \dots, \lambda_n + (m-n)) \in \mathbb{Z}^m.$$

Writing $W(r;s)$ for the set of dominant weights $\lambda \in \mathbb{Z}^n_{dom}$ with $|\lambda| = r$ such that $\lambda(s)$ is also dominant. We have

$$[\mathrm{Ext}_R^{s(m-n)}(I^t, R)]_r = \bigoplus_{\substack{\lambda \in W(r;s) \\ \lambda_n \geq -t-(m-n)}} S_{\lambda(s)} F \otimes S_\lambda G.$$

An analogous description of $\mathrm{Ext}^i_R(J, R)$ was computed in [4] for any GL-invariant ideal J not just powers of maximal minors.

We will use Theorem 2.6 in Section 5 in conjunction with graded duality to compute the GL-structure of $H^{mn-n^2+1}_{\mathfrak{m}}(R/I^t)$. To make use of this description it will be useful to understand how the modules $\mathrm{Ext}^i_R(R/I^t, R)$ sit inside $H^i_I(R)$.

Theorem 2.7. [2, Section 4] *Let $i \in \mathbb{Z}_{\geq 0}$, for all $t \geq 1$, the induced maps $\mathrm{Ext}^i_R(R/I^t, R) \to \mathrm{Ext}^i_R(R/I^{t+1}, R)$ are injective.*

This immediately gives us the following:

Corollary 2.8.

$$H^i_I(R) = \bigcup_{t \geq 0} \mathrm{Ext}^i_R(R/I^t, R).$$

More generally, the pairs of GL-invariant ideals I_{χ_1} and I_{χ_2} for which

$$\mathrm{Ext}^i_R(R/I_{\chi_1}, R) \to \mathrm{Ext}^i_R(R/I_{\chi_2}, R)$$

is injective is classified in [3] and [4]. In particular, Theorem 2.7 and Corollary 2.8 fail for ideals of non-maximal minors.

2.5. ***Other facts on local cohomology as related to determinantal ideals.*** Let R, I be as in Section 2.4 and let $\mathscr{D} = R[\partial_{ij}]$ be the Weyl algebra. The action of differentiation makes R a left $\mathscr{D}$-module and formal application of the quotient rule then gives $R_a = R[\frac{1}{a}]$ a $\mathscr{D}$-modules structure for all $a \in R$. Thus for any ideal $J = (a_1, \dots, a_k)$, the Čech complex, $\check{C}^\bullet(a_1, \dots, a_k; R)$, is a complex of $\mathscr{D}$-modules, hence $H^i_J(R) \cong H^i(\check{C}^\bullet(a_1, \dots, a_k; R))$ carries the structure of a $\mathscr{D}$-modules.

Theorem 2.9. [9, Theorem 1.2][10, Theorem 1.2, Theorem 3.1] *There exists a degree preserving isomorphism of $\mathscr{D}$-modules:*

$$H^{mn-r^2+1}_{I_r(X)}(R) \cong H^{mn}_{\mathfrak{m}}(R),$$

in particular,

$$H^{mn-n^2+1}_I(R) \cong H^{mn}_{\mathfrak{m}}(R).$$

As these are cyclic $\mathscr{D}$-modules, in order to describe an isomorphism as above, we just need to choose a socle generator of $H^{mn-n^2+1}_I(R)$ and $H^{mn}_{\mathfrak{m}}(R)$. Such a map will be constructed in section 4.1 in the case where X is size $n \times (n-1)$.

In light of Corollary 2.8, Theorem 2.9 and local duality gives us an avenue to examine $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$. Some implications of this result to the asymptotic structure of the graded components of $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ has been remarked on in [1] and [7]. We obtain the following result on the structure of $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ as an R-module.

Proposition 2.10. *$H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ is a cyclic R-module generated in degree 0. In other words, there exists $J \subseteq R$ such that $H^{n^2-1}_{\mathfrak{m}}(R/I^t) \cong R/J$.*

Proof. First to show $H_m^{(n-1)^2-1}(R/I^t)$ is cyclic, by graded duality it is sufficient to show that $\operatorname{Ext}_R^{mn-n^2+1}(R/I^t,R)$ is finite length and has socle dimension at most 1. By Theorem 2.7 and Theorem 2.9 we have that $\operatorname{Ext}_R^{mn-n^2+1}(R/I^t,R) \hookrightarrow H^{mn}_{\mathfrak{m}}(R)$. Thus $\operatorname{Ext}_R^{mn-n^2+1}(R/I^t,R)$ is a finitely generated submodule of an Artinian module hence has finite length. Moreover, since $H^{mn}_{\mathfrak{m}}(R)$ has socle dimension 1 we have that $\operatorname{Ext}_R^{mn-n^2+1}(R/I^t,R)$ has socle dimension at most 1. That $H_m^{(n-1)^2-1}(R/I^t)$ is generated in dimension 0 follows by graded duality since $\operatorname{Soc}(\operatorname{Ext}_R^{mn-n^2+1}(R/I^t,R)) = \operatorname{Soc}(H^{mn}_{\mathfrak{m}}(R))$ is generated in degree $-mn$. □

3. A Map Between Complexes

Let A be a set, we write $\#A$ for the cardinality of A. Let X be a $n \times (n-1)$ matrix of indeterminates. For $A \subseteq \{1,\dots,n\}$ and $H \subseteq \{1,\dots,n-1\}$ with $\#A = \#H$, we write $X_{A,H}$ for the determinant of the submatrix of X coming with rows in A and columns in H. We will make use of the Hilbert-Burch theorem so it is convenient for this section to use signed minors: set $\Delta_i = (-1)^i X_{\{i\}^c,\emptyset^c}$, that is to say $(-1)^i$ times the maximal minor of X obtained by deleting the ith row.

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[X]$, set $I = I_n(X) = (\Delta_1,\dots,\Delta_n) \subseteq R$ the ideal of maximal minors of X. In this case the *Rees algebra* of I,

$$\mathscr{R}(I) := \bigoplus_{i \geq 0} I^i t^i \subseteq R[t],$$

is linear type [16], i.e.,

$$\mathscr{R}(I) \cong S/(F_1,\dots,F_{n-1})$$

where $S = R[T_1,\dots,T_n]$ and $F_j = \sum_{i=1}^n x_{ij}T_i$. Moreover $\mathscr{R}(I)$ is a complete intersection so the Koszul complex of $[F_1 \cdots F_{n-1}] : S^{n-1} \to S$ is a resolution. In this section we will compare the linear strands of this Koszul complex to the Koszul complexes of $[\Delta_1^t \cdots \Delta_n^t] : R^n \to R$.

More precisely, we will describe maps of complexes, φ_r, for each r making the following diagram commute.

$$\begin{array}{ccccc} [K_\bullet([F_1\cdots F_{n-1}];S)]_t & \longrightarrow & [\mathscr{R}(I)]_t & \longrightarrow & 0 \\ \varphi_t \uparrow & & \uparrow & & \\ K_{\geq 1}([\Delta_1^t \dots \Delta_n^t];R)[1] & \longrightarrow & (\Delta_1^t,\dots,\Delta_n^t) & \longrightarrow & 0. \end{array} \tag{13}$$

Where $[K_\bullet([F_1\cdots F_{n-1}];S)]_t$ denotes the t-th linear strand of $K_\bullet([F_1\cdots F_{n-1}];S)$ and $K_{\geq 1}([\Delta_1^t \dots \Delta_n^t];R)[1]$ denotes the Koszul complex with shifted homological degree and $K_0([\Delta_1^t \cdots \Delta_n^t];R)$ removed. For more on linear strands, we refer the reader to [17, Chapter 7].

First we need to establish some notation and prove a small lemma that will be helpful later.

Notation 3.1.

(1) Let $A = \{a_1,\dots,a_r\} \subseteq \mathbb{Z}_{\geq 1}$ with $a_1 < a_2 < \dots < a_r$. Then set

$$e_A := e_{a_1} \wedge e_{a_2} \wedge \dots \wedge e_{a_r}$$

and

$$\Delta_A := \prod_{a \in A} \Delta_a.$$

(2) Let A, B be ordered sets of integers then

$$\rho(A,B) := \begin{cases} 0 & \text{if } A \cap B \neq \emptyset \\ (-1)^{\nu(A,B)} & \text{else} \end{cases}$$

where $\nu(A,B) = \#\{(a,b) \subseteq A \times B \mid a > b\}$.

(3) Let $A \subseteq \{1,\dots,n\}$ and $H \subseteq \{1,\dots,n-1\}$ with $\#A = r$ and $\#H = r-2$. Then set

$$Y_{A,H,i} := \det \begin{bmatrix} r_{i,H^c} \\ Z \end{bmatrix},$$

where r_{i,H^c} is the entries of the i-th row of X with columns in H^c and Z is the submatrix of X with rows in A^c and columns in H^c.

(4) For A an ordered sets of integers set $(-1)^A := (-1)^{\sum_{a\in A} a}$.

The function $\rho(-,-)$ describes sign appearing in higher order determinantal expansions, see [18, Chapter 3, Section 8] for a complete treatment. Additionally, the Koszul differential can be written using $\rho(-,-)$: the differential of a Koszul complex $K_\bullet([f_1 \dots f_l]; S)$ is given by $e_A \mapsto \rho(\alpha, A\setminus\{\alpha\}) f_\alpha e_{A\setminus\{\alpha\}}$.

Lemma 3.2.

(1)

$$Y_{A,H,i} = \sum_{\alpha\in H^c} \rho(\{\alpha\}, H^c \setminus \{\alpha\}) x_{i\alpha} X_{A^c (H\cup\alpha)^c}.$$

(2) *Let $A \subseteq \{1,\dots,m\}$ and $\alpha \in A$. Then,*

$$\rho(\{\alpha\}, A\setminus\alpha)\rho(\{\alpha\}, A^c) = (-1)^{\alpha-1}.$$

Proof.

(1) This is an expansion of the determinant along the first row.

(2) $\rho(\{\alpha\}, A\setminus\alpha)\rho(\{\alpha\}, A^c) = (-1)^{\nu(\{\alpha\},A\setminus\alpha)+\nu(\{\alpha\},A^c)}$. Now,

$$\nu(\{\alpha\}, A\setminus\alpha) + \nu(\{\alpha\}, A^c) = \#\{b \in A\setminus\{\alpha\} \mid \alpha > b\} + \#\{b \in A^c \mid \alpha > b\}.$$

Since $A \setminus \{\alpha\}$ and A^c are disjoint we have that

$$\nu(\{\alpha\}, A\setminus\alpha) + \nu(\{\alpha\}, A^c) = \#\{b \in \{1\dots m\}\setminus\{\alpha\} \mid \alpha > b\} = \alpha - 1.$$

□

Our strategy will be to consider each commutative square of Diagram (13) and induct on t. Theorem 3.3 and Corollary 3.5 will constitute the base case of this induction with Theorem 3.3 addressing the first t for which a every module in a square is non-zero.

Theorem 3.3. *Consider the following diagram for $n \geq r \geq 2$.*

$$\begin{array}{ccc} \left[\bigwedge^{r-1} S^{n-1}\right]_0 & \xrightarrow{\ \delta\ } & \left[\bigwedge^{r-2} S^{n-1}\right]_1 \\ {\scriptstyle \varphi^{r-1}_{r-1}}\big\uparrow & & {\scriptstyle \varphi^{r-2}_{r-1}}\big\uparrow \\ \bigwedge^{r} R^n & \xrightarrow{\ \partial\ } & \bigwedge^{r-1} R^n, \end{array}$$

where δ is the map on the $(r-1)$-st linear strand of $K_\bullet([F_1 \dots F_{n-1}]; S)$ and the bottom map, ∂, is the differential of $K([\Delta_1^{r-1} \dots \Delta_n^{r-1}]; R)$.

Let $(f_j)_{j=1}^{n-1}$ denote the standard S-basis of S^{n-1} and $(e_i)_{i=1}^{n}$ denote the standard R-basis for R^n. Define the vertical maps as follows: let $A, B \subseteq \{1,\dots,n\}$ with $\#A = r$ and $\#B = r-1$, set

$$\varphi_{r-1}^{r-1}(e_A) := (-1)^{r-1}\Delta_A^{r-2} \sum_{\substack{K\subseteq\{1\dots n-1\}\\ \#K=r-1}} (-1)^{A+K} X_{A^cK^c} f_K$$

$$\begin{aligned}\varphi_{r-1}^{r-2}(e_B) &:= (-1)^r\Delta_B^{r-3} \sum_{\substack{K\subseteq\{1\dots n-1\}\\ \#K=r-2}} (-1)^{B+K} X_{B^cK^c} f_K \sum_{b\in B} \Delta_B \frac{T_b}{\Delta_b}\\ &= (-1)^r\Delta_B^{r-2} \sum_{b\in B}\frac{T_b}{\Delta_b} \sum_{\substack{K\subseteq\{1\dots n-1\}\\ \#K=r-2}} (-1)^{B+K} X_{B^cK^c} f_K\end{aligned}$$

Then the diagram above commutes. (Note that for $r=2$, φ_{r-1}^{r-2} is the map that takes $e_i \to T_i$).

Before we begin the proof we first give an example.

Example 3.4. *Suppose $n=3$ and $t=2$ then Diagram (13) becomes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\bigwedge^2 S^2]_0 & \longrightarrow & [\bigwedge^1 S^2]_1 & \longrightarrow & [\bigwedge^0 S^2]_2 & \longrightarrow & I^2 & \longrightarrow & 0\\ & & \varphi_2^2\uparrow & & \varphi_2^1\uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & \bigwedge^3 R^3 & \longrightarrow & \bigwedge^3 R^2 & \longrightarrow & \bigwedge^1 R^3 & \longrightarrow & (\Delta_1^2,\Delta_2^2,\Delta_3^2) & \longrightarrow & 0. \end{array}$$

Then, Theorem 3.3 says the left square commutes for,

$$\begin{aligned}\varphi_2^2(e_1\wedge e_2\wedge e_3) &= (-1)^2\Delta_1\Delta_2\Delta_3((-1)^{(6+3)}f_1\wedge f_2)\\ &= -\Delta_1\Delta_2\Delta_3(f_1\wedge f_2)\end{aligned}$$

and

$$\begin{aligned}\varphi_2^1(e_a\wedge e_b) &= (-1)^{3-2}\Delta_a\Delta_b\left(\frac{T_a}{\Delta_a}+\frac{T_b}{\Delta_b}\right)((-1)^{a+b+1}X_{\{a,b\}^c,2}f_1 + (-1)^{a+b+2}X_{\{a,b\}^c,1}f_2)\\ &= -(\Delta_bT_a+\Delta_aT_b)((-1)^{a+b+1}X_{\{a,b\}^c,2}f_1 + (-1)^{a+b+2}X_{\{a,b\}^c,1}f_2)\\ &= \begin{cases} -(\Delta_2T_1+\Delta_1T_2)(x_{3,2}f_1 - x_{3,1}f_2) & (a,b)=(1,2),\\ -(\Delta_3T_1+\Delta_1T_3)(-x_{2,2}f_1 + x_{2,1}f_2) & (a,b)=(1,3),\\ -(\Delta_3T_2+\Delta_2T_3)(x_{1,2}f_1 - x_{1,1}f_2) & (a,b)=(2,3).\end{cases}\end{aligned}$$

To check that the square does indeed commute amounts to repeated application of the relation $\sum_{i=1}^n x_{i,\alpha}\Delta_i$. This relation should most relevantly be thought of as a determinantal expansion of a matrix with a repeated column and lies at the heart of the computations in the remainder of this sections.

One may notice that the only way to possibly complete this diagram with a map $\varphi_2^0 : \bigwedge^1 R^3 \to [\bigwedge^0 S^2]_2$ and have any hope that it commutes is to set $\varphi_2^0(e_i) = T_i^2$. Later, in Theorem 3.7 we will see that this is the correct choice to make the diagram commute, along with how to construct the maps for other t.

Proof of Theorem 3.3. To show this diagram commutes we simply compute the two compositions of maps. Fix $A\subseteq\{1,\dots,n\}$. Then,

$$\begin{aligned}
\delta(\varphi^{r-1}_{r-1}(e_A)) &= \delta((-1)^{r-1}\Delta_A^{r-2}\sum_{\substack{K\subseteq\{1\ldots n-1\}\\ \#K=r-1}}(-1)^{A+K}X_{A^cK^c}f_K)\\
&= (-1)^{r-1}\Delta_A^{r-2}\sum_{\substack{K\subseteq\{1\ldots n-1\}\\ \#K=r-1}}(-1)^{A+K}X_{A^cK^c}\delta(f_K)\\
&= (-1)^{r-1}\Delta_A^{r-2}\sum_{\substack{K\subseteq\{1\ldots n-1\}\\ \#K=r-1}}(-1)^{A+K}X_{A^cK^c}(\sum_{k\in K}\rho(\{k\},K\setminus\{k\})F_kf_{K\setminus\{k\}})\\
&= (-1)^{r-1}\Delta_A^{r-2}\sum_{\substack{K\subseteq\{1\ldots n-1\}\\ \#K=r-1}}(-1)^{A+K}X_{A^cK^c}(\sum_{k\in K}\rho(\{k\},K\setminus\{k\})(\sum_{i=1}^{n}x_{ik}T_i)f_{K\setminus\{k\}})\\
&= (-1)^{r-1}\Delta_A^{r-2}\sum_{i=1}^{n}T_i\sum_{\substack{K\subseteq\{1\ldots n-1\}\\ \#K=r-1}}\sum_{k\in K}(-1)^{A+K}X_{A^cK^c}\rho(\{k\},K\setminus\{k\})x_{ik}f_{K\setminus\{k\}}\\
&= (-1)^{r-1}\Delta_A^{r-2}\sum_{i=1}^{n}T_i\sum_{\substack{H\subseteq\{1\ldots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{\alpha\in H^c}(-1)^{\alpha}\rho(\{\alpha\},H)x_{i\alpha}X_{A^c(H\cup\{\alpha\})^c}
\end{aligned}$$

Now by Lemma 3.2 (2) we know that $\rho(\{\alpha\},H^c\setminus\alpha)\rho(\{\alpha\},H)=(-1)^{\alpha-1}$. Hence, $(-1)^{\alpha}\rho(\{\alpha\},H)=(-1)\rho(\{\alpha\},H^c\setminus\alpha)$. So the above is

$$\begin{aligned}
&= (-1)^{r-1}\Delta_A^{r-2}\sum_{i=1}^{n}T_i\sum_{\substack{H\subseteq\{1\ldots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{\alpha\in H^c}(-1)\rho(\{\alpha\},H^c\setminus\alpha)x_{i\alpha}X_{A^c(H\cup\{\alpha\})^c}\\
&= (-1)^{r}\Delta_A^{r-2}\sum_{i=1}^{n}T_i\sum_{\substack{H\subseteq\{1\ldots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\alpha)x_{i\alpha}X_{A^c(H\cup\{\alpha\})^c}
\end{aligned}$$

Now applying Lemma 3.2 (1) we get

$$\begin{aligned}
&= (-1)^{r}\Delta_A^{r-2}\sum_{i=1}^{n}T_i\sum_{\substack{H\subseteq\{1\ldots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_HY_{A,H,i}\\
&= (-1)^{r}\Delta_A^{r-2}\sum_{\substack{H\subseteq\{1\ldots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{i=1}^{n}Y_{A,H,i}T_i\\
&= (-1)^{r}\Delta_A^{r-2}\sum_{\substack{H\subseteq\{1\ldots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{i\in A}Y_{A,H,i}T_i.
\end{aligned}$$

Here the last equality follows from the fact that $Y_{A,H,i}=0$ if $i\notin A$.

Now for the other composition,

$$\begin{aligned}
\varphi_{r-1}^{r-2}(\partial(e_A)) &= \varphi_{r-1}^{r-2}\Big(\sum_{\beta\in A}\rho(\{\beta\},A\setminus\{\beta\})\Delta_\beta^{r-1}e_{A\setminus\{\beta\}}\Big)\\
&= \sum_{\beta\in A}\rho(\{\beta\},A\setminus\{\beta\})\Delta_\beta^{r-1}\varphi_{r-1}^{r-2}(e_{A\setminus\{\beta\}})\\
&= \sum_{\beta\in A}\rho(\{\beta\},A\setminus\{\beta\})\Delta_\beta^{r-1}(-1)^r\Delta_{A\setminus\{\beta\}}^{r-2}\sum_{b\in A\setminus\{\beta\}}\frac{T_b}{\Delta_b}\sum_{\substack{H\subseteq\{1\dots n-1\}\\ \#H=r-2}}(-1)^{A\setminus\{\beta\}+H}X_{(A\setminus\{\beta\})^cH^c}f_H\\
&= (-1)^r\sum_{\substack{H\subseteq\{1\dots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{\beta\in A}\sum_{b\in A\setminus\{\beta\}}(-1)^\beta\frac{T_b}{\Delta_b}\rho(\{\beta\},A\setminus\{\beta\})\Delta_A^{r-2}\Delta_\beta X_{(A\setminus\{\beta\})^cH^c}\\
&= (-1)^r\Delta_A^{r-2}\sum_{\substack{H\subseteq\{1\dots n-1\}\\ \#H=r-2}}(-1)^{A+H}f_H\sum_{i\in A}\frac{T_i}{\Delta_i}\sum_{\gamma\in A\setminus\{i\}}(-1)^\gamma\rho(\{\gamma\},A\setminus\{\gamma\})\Delta_\gamma X_{(A\setminus\{\gamma\})^cH^c}
\end{aligned}$$

We now write

$$X_{(A\setminus\{\gamma\})^cH^c}=\rho(\{\gamma\},A^c\setminus\{\gamma\})Y_{A,H,\gamma}=\rho(\{\gamma\},A^c\setminus\{\gamma\})\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c}.$$

Hence by Lemma 3.2 (2) we have

$$\begin{aligned}
\rho(\{\gamma\},A\setminus\{\gamma\})X_{(A\setminus\{\gamma\})^cH^c} &= \rho(\{\gamma\},A\setminus\{\gamma\})\rho(\{\gamma\},A^c\setminus\{\gamma\})\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c}\\
&= (-1)^{\gamma-1}\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c}.
\end{aligned}$$

So, returning to the original expression,

$$\begin{aligned}
\varphi_{r-1}^{r-2}(\partial(e_A)) &= \cdots\sum_{i\in A}\frac{T_i}{\Delta_i}\sum_{\gamma\in A\setminus\{i\}}(-1)^\gamma\Delta_\gamma(-1)^{\gamma-1}\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c}\\
&= \cdots(-1)\sum_{i\in A}\frac{T_i}{\Delta_i}\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})X_{A^c(H\cup\alpha)^c}\sum_{\gamma\in A\setminus\{i\}}\Delta_\gamma x_{\gamma\alpha}
\end{aligned}$$

Using the fact that $\sum_{i=1}^n\Delta_i x_{i\alpha}=0$ we get that $\sum_{\gamma\in A\setminus\{i\}}\Delta_\gamma x_{\gamma\alpha}=-\sum_{\gamma\in A^c\cup\{i\}}\Delta_\gamma x_{\gamma\alpha}$. Therefore the previous line becomes

$$\begin{aligned}
&= \cdots(-1)\sum_{i\in A}\frac{T_i}{\Delta_i}\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})X_{A^c(H\cup\alpha)^c}\Big((-1)\sum_{\gamma\in A^c\cup\{i\}}\Delta_\gamma x_{\gamma\alpha}\Big)\\
&= \cdots\sum_{i\in A}\frac{T_i}{\Delta_i}\sum_{\gamma\in A^c\cup\{i\}}\Delta_\gamma\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c}
\end{aligned}$$

Using Lemma 3.2 (1), we see that $\sum_{\alpha\in H^c}\rho(\{\alpha\},H^c\setminus\{\alpha\})x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c}=Y_{A,H,\gamma}$. Thus, we have

$$=\cdots\sum_{i\in A}\frac{T_i}{\Delta_i}\sum_{\gamma\in A^c\cup\{i\}}\Delta_\gamma Y_{A,H,\gamma}.$$

Finally, using that $Y_{A,H,\gamma} = 0$ for $\gamma \in A^c$, the expression simplifies to

$$
\begin{aligned}
&= \cdots \sum_{i \in A} \frac{T_i}{\Delta_i} \Delta_i Y_{A,H,i} \\
&= (-1)^r \Delta_A^{r-2} \sum_{\substack{H \subseteq \{1 \ldots n-1\} \\ \#H = r-2}} (-1)^{A+H} f_H \sum_{i \in A} \frac{T_i}{\Delta_i} \Delta_i Y_{A,H,i} \\
&= (-1)^r \Delta_A^{r-2} \sum_{\substack{H \subseteq \{1 \ldots n-1\} \\ \#H = r-2}} (-1)^{A+H} f_H \sum_{i \in A} Y_{A,H,i} T_i.
\end{aligned}
$$

We have shown that

$$\varphi_{r-1}^{r-2}(\partial(e_A)) = (-1)^r \Delta_A^{r-2} \sum_{\substack{H \subseteq \{1 \ldots n-1\} \\ \#H = r-2}} (-1)^{A+H} f_H \sum_{i \in A} Y_{A,H,i} T_i$$

and

$$\delta(\varphi_{r-1}^{r-1}(e_A)) = (-1)^r \Delta_A^{r-2} \sum_{\substack{H \subseteq \{1 \ldots n-1\} \\ \#H = r-2}} (-1)^{A+H} f_H \sum_{i \in A} Y_{A,H,i} T_i$$

so the commutativity of the diagram is proven.

□

Corollary 3.5. *Suppose φ_{r-1}^{r-1} and φ_{r-1}^{r-2} are the maps defined in Theorem 3.3. Consider the following two squares of Diagram (13) where $t = r-1$.*

$$
\begin{array}{ccccc}
0 = \left[\bigwedge^r S^{n-1}\right]_{-1} & \longrightarrow & \left[\bigwedge^{r-1} S^{n-1}\right]_0 & \xrightarrow{\delta} & \left[\bigwedge^{r-2} S^{n-1}\right]_1 \\
\uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \varphi_{r-1}^{r-1}} & & \uparrow{\scriptstyle \varphi_{r-1}^{r-2}} \\
\bigwedge^{r+1} R^n & \xrightarrow{\partial_{r-1}^{r+1}} & \bigwedge^r R^n & \xrightarrow{\partial_{r-1}^{r}} & \bigwedge^{r-1} R^n
\end{array}
$$

This diagram commutes.

Proof. This follows from the injectivity of δ, since the the top row is the tail of a resolution of I^{r-1}, and Theorem 3.3: We have that $\operatorname{im}(\varphi_{r-1}^{r-1} \circ \partial_{r-1}^{r+1}) \subseteq \ker \delta = 0$. □

Notation 3.6. Let $s_l(y_1, \ldots, y_d)$ be the complete homogeneous symmetric function of degree l in $y_1, \ldots, y_d$. For $A = \{a_1, \ldots, a_d\} \subseteq \{1, \ldots, n\}$ define $h_l(A) = s_l(\frac{T_{a_1}}{\Delta_{a_1}}, \ldots, \frac{T_{a_d}}{\Delta_{a_d}})$.

Theorem 3.7. *For $r > 1$ let φ_{r-1}^{r-1} be the maps defined in Theorem 3.3 and let $\varphi_0^0 : \bigwedge^1 R^n \to [\bigwedge^0 S^{n-1}]_0$ be the map $\varphi_0^0(e_a) = 1$. Then for all $t, r \geq 1$ define functions $\varphi_t^{r-1} : \bigwedge^r R^n \to [\bigwedge^{r-1} S^{n-1}]_{t-r+1}$ as follows:*

$$\varphi_t^{r-1}(e_A) := \begin{cases} 0 & t < r-1, \\ \varphi_{r-1}^{r-1}(e_A) & t = r-1, \\ \varphi_{r-1}^{r-1}(e_A)(\Delta_A^{t-r+1} h_{t-r+1}(A)) & t > r-1. \end{cases}$$

(Note that this definition of φ_{r-1}^{r-2} agrees with Theorem 3.3). Then

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \left[\bigwedge^{r-1} S^{n-1}\right]_{t-r+1} & \xrightarrow{\delta^{r-1}} & \cdots & \longrightarrow & \left[\bigwedge^{1} S^{n-1}\right]_{t-1} & \xrightarrow{\delta^{1}} & [S]_t & \xrightarrow{\delta^{0}} & I^t \\
 & & \varphi_t^{r-1}\Big\uparrow & & & & \varphi_t^{1}\Big\uparrow & & \varphi_t^{0}\Big\uparrow & & \Big\uparrow \\
\cdots & \longrightarrow & \bigwedge^{r} R^n & \xrightarrow{\partial_t^{r}} & \cdots & \longrightarrow & \bigwedge^{2} R^n & \xrightarrow{\partial_t^{2}} & \bigwedge^{1} R^n & \xrightarrow{\partial_t^{1}} & (\Delta_1^t,\dots,\Delta_n^t)
\end{array}$$

commutes, where the rightmost vertical map is the natural inclusion, the top row is the t-th strand of $K_\bullet([F_1 \dots F_{n-1}];S)$ and the bottom row is a truncation of $K_\bullet([\Delta_1^t \dots \Delta_n^t];R)$.

Again, before proving this theorem lets return to Example 3.4.

Example 3.8. *Suppose $n = 3$ and $t = 2$ then Diagram (13) becomes:*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \left[\bigwedge^{2} S^{2}\right]_{0} & \longrightarrow & \left[\bigwedge^{1} S^{2}\right]_{1} & \longrightarrow & \left[\bigwedge^{0} S^{2}\right]_{2} & \longrightarrow & I^2 & \longrightarrow & 0 \\
 & & \varphi_2^{2}\Big\uparrow & & \varphi_2^{1}\Big\uparrow & & \varphi_2^{0}\Big\uparrow & & \Big\uparrow & & \\
0 & \longrightarrow & \bigwedge^{3} R^3 & \longrightarrow & \bigwedge^{3} R^2 & \longrightarrow & \bigwedge^{1} R^3 & \longrightarrow & (\Delta_1^2,\Delta_2^2,\Delta_3^2) & \longrightarrow & 0.
\end{array}$$

We saw in Example 3.4 that

$$\begin{aligned}
\varphi_2^2(e_1 \wedge e_2 \wedge e_3) &= (-1)^2\Delta_1\Delta_2\Delta_3((-1)^{(6+3)} f_1 \wedge f_2) \\
&= -\Delta_1\Delta_2\Delta_3(f_1 \wedge f_2).
\end{aligned}$$

Now using Theorem 3.7 we compute φ_2^0 and φ_2^1: $\varphi_0^0(e_a) = 1$ and $h_2(\{a\}) = \frac{T_a^2}{\Delta_a^2}$, so

$$\varphi_2^0(e_a) = (1)(\Delta_a^2)\left(\frac{T_a^2}{\Delta_a^2}\right) = T_a^2.$$

Note that this agrees with the observation we made in Example 3.4 of what this map should be. For φ_2^1 we first need to compute φ_1^1:

$$\begin{aligned}
\varphi_1^1(e_a \wedge e_b) &= (-1)^{2-1}(\Delta_a\Delta_b)^{2-2}((-1)^{a+b+1}X_{\{a,b\}^c,1}f_1 + (-1)^{a+b+1}X_{\{a,b\}^c,1}f_2) \\
&= \begin{cases} -(x_{3,2}f_1 - x_{3,1}f_2) & (a,b) = (1,2) \\ -(-x_{2,2}f_1 + x_{2,1}f_2) & (a,b) = (1,3) \\ -(x_{1,2}f_1 - x_{1,1}f_2) & (a,b) = (2,3) \end{cases}.
\end{aligned}$$

Now $h_1(\{a,b\}) = \frac{T_a}{\Delta_a} + \frac{T_b}{\Delta_b}$ and we have,

$$\begin{aligned}
\varphi_2^1(e_a \wedge e_b) &= \varphi_1^1(e_a \wedge e_b)(\Delta_a\Delta_b)\left(\frac{T_a}{\Delta_a} + \frac{T_b}{\Delta_b}\right) \\
&= \varphi_1^1(e_a \wedge e_b)(\Delta_b T_a + \Delta_a T_b) \\
&= \begin{cases} -(\Delta_2 T_1 + \Delta_1 T_2)(x_{3,2}f_1 - x_{3,1}f_2) & (a,b) = (1,2) \\ -(\Delta_3 T_1 + \Delta_1 T_3)(-x_{2,2}f_1 + x_{2,1}f_2) & (a,b) = (1,3) \\ -(\Delta_3 T_2 + \Delta_2 T_3)(x_{1,2}f_1 - x_{1,1}f_2) & (a,b) = (2,3) \end{cases},
\end{aligned}$$

which agrees with the computation in Example 3.4.

Proof of Theorem 3.7. The commutativity of the rightmost square is immediate so we are done once we show that for all $r \geq 2$ the following square commutes:

$$\begin{array}{ccc} [\bigwedge^{r-1} S^{n-1}]_{t-r+1} & \xrightarrow{\delta^{r-1}} & [\bigwedge^{r-2} S^{n-1}]_{t-r+2} \\ \varphi_t^{r-1}\uparrow & & \varphi_t^{r-2}\uparrow \\ \bigwedge^r R^n & \xrightarrow{\partial_t^r} & \bigwedge^{r-1} R^n \end{array}.$$

If $t < r-2$ the top row vanishes and both vertical maps are zero, so commutativity is clear. The case that $t = r-2$ is addressed by Corollary 3.5. Finally, the case where $t = r-1$ is handled by Theorem 3.3. So it is left to check the cases $t > r-1$.

Before computing the two compositions we note the following key identity. Let $A \subseteq \{1,\dots,n\}$ with $\#A = r$. Using Corollary 3.5 we have that:

$$\begin{aligned} 0 &= \varphi_{r-2}^{r-2}(\partial_{r-2}^r)(e_A) \\ &= \varphi_{r-2}^{r-2}\Big(\sum_{\alpha \in A} \rho(\{\alpha\}, A\setminus\{\alpha\})\Delta_\alpha^{r-2} e_{A\setminus\alpha}\Big) \\ &= \sum_{\alpha\in A} \rho(\{\alpha\}, A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\alpha}). \end{aligned} \tag{14}$$

Now we are ready to show that the square commutes. Let $A \subseteq \{1,\dots,n\}$ with $\#A = r$ and set $l = t-r+1$. Then,

$$\begin{aligned} \delta^{r-1}(\varphi_t^{r-1}(e_A)) &= \delta^{r-1}(\varphi_{r-1}^{r-1}(e_A)(\Delta_A^l h_l(A))) \\ &= \Delta_A^l h_l(A)\delta^{r-1}(\varphi_{r-1}^{r-1}(e_A)), \end{aligned}$$

where the second equality follows from the S-linearity of δ. Now using Theorem 3.3 we have,

$$\begin{aligned} &= \Delta_A^l h_l(A)\varphi_{r-1}^{r-2}(\partial_{r-1}^r(e_A)) \\ &= \Delta_A^l h_l(A)\varphi_{r-1}^{r-2}\Big(\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-1}e_{A\setminus\{\alpha\}}\Big) \\ &= \Delta_A^l h_l(A)\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-1}\varphi_{r-1}^{r-2}(e_{A\setminus\{\alpha\}}) \\ &= \Delta_A^l h_l(A)\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-1}\Big(\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\frac{\Delta_A}{\Delta_\alpha}h_1(A\setminus\{\alpha\})\Big) \\ &= \Delta_A^{l+1}h_l(A)\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})h_1(A\setminus\{\alpha\}) \\ &= \Delta_A^{l+1}\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\sum_{\beta\in A\setminus\{\alpha\}}\frac{T_\beta}{\Delta_\beta}h_l(A). \end{aligned}$$

Now apply the fact that for $\beta \in A$, $h_{l+1}(A) = \frac{T_\beta}{\Delta_\beta}h_l(A) + h_{l+1}(A\setminus\{\beta\})$ to see that the above is

$$\begin{aligned} &= \Delta_A^{l+1}\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\sum_{\beta\in A\setminus\{\alpha\}}(h_{l+1}(A) - h_{l+1}(A\setminus\{\beta\})) \\ &= \Delta_A^{l+1}\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\sum_{\beta\in A\setminus\{\alpha\}}h_{l+1}(A) \\ &\quad - \Delta_A^{l+1}\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\sum_{\beta\in A\setminus\{\alpha\}}h_{l+1}(A\setminus\{\beta\})) \end{aligned}$$

But $\sum_{\beta\in A\setminus\{\alpha\}} h_{l+1}(A) = (\#A-1)h_{l+1}(A) = (r-1)h_{l+1}(A)$. So by identity (14),

$$\begin{aligned}0 &= \Delta_A^{l+1}\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\sum_{\beta\in A\setminus\{\alpha\}} h_{l+1}(A)\\ &= (r-1)\Delta_A^{l+1}h_{l+1}(A)\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}}).\end{aligned}$$

Hence continuing we have that

$$\begin{aligned}\delta^{r-1}(\varphi_t^{r-1}(e_A)) &= -\Delta_A^{l+1}\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\sum_{\beta\in A\setminus\{\alpha\}} h_{l+1}(A\setminus\{\beta\}))\\ &= -\Delta_A^{l+1}\sum_{\alpha\in A}h_{l+1}(A\setminus\{\alpha\})\sum_{\beta\in A\setminus\{\alpha\}}\rho(\{\beta\},A\setminus\{\beta\})\Delta_\beta^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\beta\}}).\end{aligned}$$

By identity (14), we see that $0=\sum_{\beta\in A\setminus\{\alpha\}}\rho(\{\beta\},A\setminus\{\beta\})\Delta_\beta^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\beta\}})+\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})$. So the above expression is equal to

$$\begin{aligned}&= -\Delta_A^{l+1}\sum_{\alpha\in A}h_{l+1}(A\setminus\{\alpha\})(-\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}}))\\ &= \sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\Delta_A^{l+1}h_{l+1}(A\setminus\{\alpha\})\\ &= \sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2+l+1}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\left(\frac{\Delta_A}{\Delta_\alpha}\right)^{l+1}h_{l+1}(A\setminus\{\alpha\})\\ &= \sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{r-2+l+1}\varphi_{r-2}^{r-2}(e_{A\setminus\{\alpha\}})\Delta_{A\setminus\alpha}^{l+1}h_{l+1}(A\setminus\{\alpha\})\\ &= \sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^{t}\varphi_t^{r-2}(e_{A\setminus\{\alpha\}})\\ &= \varphi_t^{r-2}(\sum_{\alpha\in A}\rho(\{\alpha\},A\setminus\{\alpha\})\Delta_\alpha^t e_{A\setminus\{\alpha\}})\\ &= \varphi_t^{r-2}(\partial_t^r(e_A)).\end{aligned}$$

□

The results of this section are highly specialized to the case that X is size $n\times(n-1)$, in all other cases the Rees algebra of the ideal of maximal minors is substantially less nice and it is much more difficult to access a resolution of I^t, cf. [19]. However, this is the only specialized aspect of this argument. Due to the elementary computational nature of the proof, Theorem 3.7 holds for any grade 2 perfect ideal of linear type with mild assumptions assumptions on the ambient ring.

4. The $n\times(n-1)$ Case

For this section let X be a $n\times(n-1)$ matrix of indeterminates, $R=\mathbb{C}[X]$ and $I=I_{n-1}(X)$. We write d_i for the determinant of the matrix obtained by deleting the i-th row of X. As noted in Section 2.2 $R\cong \mathrm{Sym}(F\otimes G)$ where $F=\mathbb{C}^n$, $G=\mathbb{C}^{n-1}$ and $\mathrm{GL}=\mathrm{GL}_n\times\mathrm{GL}_{n-1}$ acts on R.

4.1. ***The Cyclic Local Cohomology Module.*** By Proposition 2.10, we have that $H_{\mathfrak{m}}^{(n-1)^2-1}(R/I^t) = H_{\mathfrak{m}}^{n^2-2n}(R/I^t)$ is a cyclic R-module. Define J_t to be the ideal such that

$$H_{\mathfrak{m}}^{n^2-2n}(R/I^t)\cong R/J_t.$$

We will utilize the lift constructed in Section 3 to describe the modules $\mathrm{Ext}_R^n(R/I^t,R)$ as submodules of $H_I^n(R)$. After constructing an isomorphism of D-modules $H_I^n(R)\to H_m^{n(n-1)}(R)$ we obtain a description of

$\operatorname{Ext}_R^n(R/I^t, R)$ as a submodule of $H_m^{n(n-1)}(R)$ which we can use to directly compute $\operatorname{ann}_R \operatorname{Ext}_R^n(R/I^t, R) = J_t$.

4.2. *Description of* $\operatorname{Ext}_R^n(R/I^t, R)$. Let $\Delta_i = (-1)^i d_i$ and $S = \operatorname{Sym}(R^{n-1})$. Then we have the following commutative diagram:

$$(15)\qquad \begin{array}{ccccc} R\left[\frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n}\right] & \longrightarrow & H^n(\check{C}^\bullet(\Delta_1, \ldots, \Delta_n; R)) \cong H_I^n(R) & \longrightarrow & 0 \\ {\scriptstyle f \mapsto \frac{f}{(\prod \Delta_i)^t}}\uparrow & & \uparrow & & \\ R & \longrightarrow & H^n(K^\bullet(\Delta_1^t, \ldots, \Delta_n^t; R)) & \longrightarrow & 0 \\ & & {\scriptstyle \psi_t}\uparrow & & \\ & & \operatorname{Ext}_R^n(R/I^t, R) & & \end{array}$$

By Corollary 2.8, the composition of vertical maps on the right is injective. Moreover ψ_t is induced by the map $\varphi_t^{n-1} : R \cong \bigwedge^n R^n \to [\bigwedge^{n-1}(S)^{n-1}]_{t-n} \cong [S]_{t-n+1}$ described in Theorem 3.3 and Theorem 3.7. This map is zero for $t < n-1$. For $t = n-1$, we have φ_{n-1}^{n-1}, and hence ψ_{n-1} is multiplication by the constant:

$$(-1)^{n-1}(\prod_{i=1}^n \Delta_i)^{n-2}.$$

Thus, the image of $\operatorname{Ext}_R^n(R/I^{n-1}, R)$ is generated by $\frac{(\prod_{i=1}^n \Delta_i)^{n-2}}{(\prod_{i=1}^n \Delta_i)^{n-1}} = \frac{1}{\prod_{i=1}^n \Delta_i}$ in $H_I^n(R)$. For $t \geq n$ we see that for $|\alpha| = t - n + 1$,

$$\psi_t(T^\alpha) = (-1)^{n-1}(\prod_{i=1}^n \Delta_i)^{n-2}(\prod_{i=1}^n \Delta_i)^{t-n+1}\frac{1}{\Delta^\alpha} = (-1)^{n-1}(\prod_{i=1}^n \Delta_i)^{t-1}\frac{1}{\Delta^\alpha}.$$

Since d_i and Δ_i agree up to sign the above discussion proves the following:

Theorem 4.1. *Under the embedding* $\operatorname{Ext}_R^n(R/I^t, R) \hookrightarrow H_I^n(R)$ *of Diagram* (15)*,* $\operatorname{Ext}_R^n(R/I^t, R)$ *is the submodule generated by*

$$\left\{ \frac{1}{\prod_{i=1}^n d_i} \cdot \frac{1}{d^\alpha} \right\}_{|\alpha| = t-n+1}.$$

Recall that $H_I^n(R)$ is a cyclic $\mathscr{D}$-module. The following result allows us to describe the images of the modules $\operatorname{Ext}_R^n(R/I^t, R)$ in $H_I^n(R)$ in a manner related to the $\mathscr{D}$-module structure of $H_I^n(R)$.

Proposition 4.2. [15, Remark 3.8] [20] *Let* $\underline{s} = (s_1, \ldots, s_n)$ *and* $s = \sum s_i$*. For each i, we have*

$$d_i^* \bullet (d_i \cdot d^{\underline{s}}) = (s_i + 1)(s+2)(s+3)\cdots(s+n)d^{\underline{s}}$$

This proposition immediately gives us the following.

Proposition 4.3. *Under the embedding induced by Diagram* (15)*, for* $t \geq n-1$*, we have*

$$\operatorname{Ext}_R^n(R/I^t, R) = \sum_{|\alpha| = t-n+1} R \cdot (d^\alpha)^* \bullet \frac{1}{\prod_{i=1}^n d_i}.$$

By Theorem 2.9 the $\mathscr{D}$-modules $H_I^n(R)$ and $H_{\mathrm{m}}^{n(n-1)}(R)$ are isomorphic cyclic $\mathscr{D}$-modules. To describe a $\mathscr{D}$-isomorphism between them it is sufficient to choose a socle generator of $H_I^n(R)$ and of $H_{\mathrm{m}}^{n(n-1)}(R)$. Choose

$$\frac{1}{\prod_{i=1}^n d_i} \in \operatorname{Soc}(H_I^n(R))$$

and

$$\frac{1}{\underline{x}} := \frac{1}{\prod_{ij} x_{ij}} \in \operatorname{Soc}(H^{n(n-1)}_{\mathfrak{m}}(R)).$$

We observe the image of $\operatorname{Ext}^n_R(R/I^t, R)$ in $H^{n(n-1)}_{\mathfrak{m}}(R)$ under the map induced by $\frac{1}{\prod_{i=1}^n d_i} \mapsto \frac{1}{\underline{x}}$.

Proposition 4.4. *For $t \geq n-1$, we have*

$$\operatorname{Ext}^n_R(R/I^t, R) \cong \sum_{|\alpha|=t-n+1} R \cdot (d^\alpha)^* \bullet \frac{1}{\underline{x}},$$

where we write $\frac{1}{\underline{x}}$ for the class in $H^{n(n-1)}_{\mathfrak{m}}(R)$.

Example 4.5. *Let $n = t = 3$. Then,*

$$\operatorname{Ext}^3_R(R/I^3, R) \cong \sum_{i=1}^{3} R \cdot (d_i)^* \bullet \frac{1}{x_{1,1}x_{1,2}x_{2,1}x_{2,2}x_{3,1}x_{3,2}}.$$

Thus, $\operatorname{Ext}^3_R(R/I^3, R) \subseteq H^{n(n-1)}_{\mathfrak{m}}(R)$ is generated as an R-module by

$$\frac{1}{\underline{x}}\left(\frac{1}{x_{2,1}x_{3,2}} - \frac{1}{x_{2,2}x_{3,1}}\right),$$

$$\frac{1}{\underline{x}}\left(\frac{1}{x_{1,1}x_{3,2}} - \frac{1}{x_{1,2}x_{3,1}}\right),$$

and

$$\frac{1}{\underline{x}}\left(\frac{1}{x_{1,1}x_{2,2}} - \frac{1}{x_{1,2}x_{2,1}}\right).$$

Using this description $\operatorname{Ext}^n_R(R/I^t, R)$, we can utilize the $\mathscr{D}$-module structure of $H^{n(n-1)}_{\mathfrak{m}}(R)$ to describe the annihilator of $\operatorname{Ext}^n_R(R/I^t, R)$. Recall from Section 2.3 that $R^* = \mathbb{C}[\partial_{ij}]$ and for a polynomial $f \in R$ we write $f^* = f(\partial_{ij}) \in R^*$. For an element $f \in R$ we can form the R^* module generated by f, where R^* acts by differentiation.

Proposition 4.6. *Let $t \geq n-1$. Then*

$$(\operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R))^* = \operatorname{ann}_{R^*} \sum_{|\alpha|=t-n+1} R^* \cdot d^\alpha.$$

Proof. Let $\zeta = \frac{1}{\underline{x}} \in H^{n(n-1)}_{\mathfrak{m}}(R)$ then $H^{n(n-1)}_{\mathfrak{m}}(R) = \mathscr{D} \cdot \zeta$ and $\operatorname{ann}_{\mathscr{D}} \zeta = \mathscr{D} \cdot \mathfrak{m}$. Now $f \in \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$ if and only if for all $|\alpha| = t-n+1$ we have $f d^{\alpha *} \bullet \zeta = 0$. Now $f d^{\alpha *} \bullet \zeta = 0$ if and only if $f d^{\alpha *} \in \mathscr{D} \cdot \mathfrak{m}$. So, applying the Fourier automorphism which sends $x_{ij} \mapsto \partial_{ij}$,$\partial_{ij} \mapsto -x_{ij}$, we have that $f d^{\alpha *} \in \mathscr{D} \cdot \mathfrak{m}$ if and only if $f^* d^\alpha \in \mathscr{D} \cdot (\mathfrak{m}^*)$ if and only if $f^* \bullet d^\alpha = 0$. Hence $f \in \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$ if and only if $f^* \in \operatorname{ann}_{R^*} \sum_{|\alpha|=t-n+1} R^* \cdot d^\alpha$. □

4.3. *The annihilator of* $\operatorname{Ext}^n_R(R/I^t, R)$. Recall from Section 2.3 that for all $k \geq 0$ there exists a GL-equivariant pairing $\langle\ ,\ \rangle : [R^*]_k \times [R]_k \to \mathbb{C}$ induced by differentiation.

Proposition 4.7. *Let $k \geq 1$, $\lambda = (k+1)$ and $N = [I^k]_{(n-1)k} = \sum_{|\alpha|=k} \mathbb{C} \cdot d^\alpha$. Then for all f in the* GL*-orbit of* $\det_\lambda$, $f^* \bullet N = 0$.

Proof. $\det_\lambda = x_{1,1}^{t+1}$ so for all $|\alpha| = t$ we have that $(\det_\lambda)^* \bullet d^\alpha = 0$. The claim then follows from Lemma 2.5. □

We are now ready to prove Theorem 5.1 in the $n \times (n-1)$ case.

Theorem 4.8. *If $t \le n-2$ then $J_t = R$. If $t \ge n-1$ then*

$$\operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R) = J_t = I_{(t-n+2)}.$$

Proof. In the case that $t \le n-2$ we have that $\operatorname{projdim}_R(R/I^t) < n$ so clearly

$$\operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R) = R.$$

For $t \ge n-1$, first, we claim that

$$I_{(t-n+2)} \subseteq \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R).$$

Let $f \in I_{(t-n+2)}$, then by Proposition 4.7, $f^* \bullet d^\alpha = 0$ for all $|\alpha| = t-n+1$. Thus $f^* \in \operatorname{ann}_{R^*} \sum_{|\alpha|=t-n+1} R^* \cdot d^\alpha$, so by Proposition 4.6, $f \in \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$.

Now for the other inclusions we note that $\operatorname{Ext}^n_R(R/I^t, R)$ is GL-equivariant hence $\operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$ is a GL-invariant ideal. As was noted in Subsection 2.2, [14] proved that every GL-invariant ideal is of the form $I_\chi = \sum_{\lambda \in \chi} I_\lambda$ for some finite collection of incomparable partitions χ.

Suppose for the sake of contradiction that $I_{(t-n+2)} \subsetneq \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$ and set $I_\chi = \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$ where χ is a collection of incomparable partitions. Thus there exists a partition $\mu \in \chi$ such that either $(t-n+2) > \mu$ or $(t-n+2)$ is incomparable to μ. In either case we have that $((t-n+1)^{n-1}) \ge \mu$, hence $I_{((t-n+1)^{n-1})} \subseteq \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R)$. In particular this implies that

$$\det{}_{((t-n+1)^{n-1})} = d_n^{t-n+1} \in \operatorname{ann}_R \operatorname{Ext}^n_R(R/I^t, R).$$

However, this is a contradiction because by Theorem 4.1 we have that $\frac{1}{\prod_{i=1}^n d_i} \frac{1}{d_n^{t-n+1}} \in \operatorname{Ext}^n_R(R/I^t, R)$ but

$$d_n^{t-n+1} \cdot \left(\frac{1}{\prod_{i=1}^n d_i} \frac{1}{d_n^{t-n+1}} \right) = \frac{1}{\prod_{i=1}^n d_i} \neq 0.$$

□

In the next section we will generalize the results of Section 4 to maximal minors of arbitrary size matrices using graded duality and results from [2]. The cost of this increased generality is that no longer have explicit isomorphisms. For X an arbitrary size $m \times n$ matrix, when one computes local cohomology with the the Čech complex on the maximal minors of X, writing down a description of a socle generator or even a non-zero element of $H^{mn-n^2+1}_{I_n(X)}(R)$ becomes non-trivial. As a consequense of this describing an explicit isomorphism $H^{mn-n^2+1}_{I_n(X)}(R) \cong H^{mn}_{\mathfrak{m}}(R)$ is challenging.

5. The General Case

We return to the setting of Section 2.2: Let $F = \mathbb{C}^m$ and $G = \mathbb{C}^n$ where $m \ge n$. Then

$$R := \operatorname{Sym}(F \otimes G) = \mathbb{C}[\partial_{ij}] = \mathbb{C}[X] \text{ and } \mathrm{GL} := \mathrm{GL}(F) \times \mathrm{GL}(G).$$

Fix I to be the ideal of $n \times n$ minors of X.

Theorem 5.1. *Let R, I be as above and set $\mathfrak{m}$ to be the homogeneous maximal ideal. Then*

$$H^{n^2-1}_{\mathfrak{m}}(R/I^t) \cong R/J_t,$$

where $J_t = R$ for $t < n$, and for $t \ge n$, $J_t = I_{(t-n+1)}$, i.e., the ideal generated by the GL *orbit of x_{11}^{t-n+1}.*

Proof. By graded duality we have the following isomorphism:

$$H^{n^2-1}_{\mathfrak{m}}(R/I^t) \cong \operatorname{Hom}_R(\operatorname{Ext}^{mn-n^2+1}_R(R/I^t, R), H^{mn}_{\mathfrak{m}}(R)).$$

The GL structure of $H^{mn}_{\mathfrak{m}}(R)$ is given by,

$$H^{mn}_{\mathfrak{m}}(R) = \bigoplus_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ \lambda_1 \le -m}} S_{\lambda(n)} F \otimes S_\lambda G,$$

where $S_{\lambda(n)} F \otimes S_\lambda G$ lives in degree $|\lambda|$. We begin describing the GL structure of $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ by first analyzing a single graded component.

$$\begin{aligned}
[H^{n^2-1}_{\mathfrak{m}}(R/I^t)]_r &= [\mathrm{Hom}_R(\mathrm{Ext}^{mn-n^2+1}_R(R/I^t, R), H^{mn}_{\mathfrak{m}}(R))]_r \\
&= \mathrm{Hom}_{\mathbb{C}}([\mathrm{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-mn-r}, [H^{mn}_{\mathfrak{m}}(R)]_{-mn}) \\
&= \mathrm{Hom}_{\mathbb{C}}([\mathrm{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-mn-r}, (\bigwedge^m F)^{-n} \otimes (\bigwedge^n G)^{-m}) \\
&= \mathrm{Hom}_{\mathbb{C}}([\mathrm{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-mn-r}, \mathbb{C}) \otimes (\bigwedge^m F)^{-n} \otimes (\bigwedge^n G)^{-m}.
\end{aligned}$$

Now by Theorem 2.6 we have that

$$\mathrm{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-mn-r} = \bigoplus_{\lambda \in A(r)} S_{\lambda(n)} F \otimes S_\lambda G,$$

where

$$A(r) = \{\lambda \in \mathbb{Z}^n | \sum_{i=1}^n \lambda_i = -mn - r \text{ and } -m \ge \lambda_1 \ge \cdots \ge \lambda_n \ge -t-(m-n)\}.$$

Dualizing into $\mathbb{C}$ we get that

$$\mathrm{Hom}_{\mathbb{C}}([\mathrm{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-mn-r}, \mathbb{C}) = \bigoplus_{\lambda \in B(r)} S_{\lambda+(-m^n)+(n^m)} F \otimes S_\lambda G,$$

where

$$B(r) = \{\lambda \in \mathbb{Z}^n | \sum_{i=1}^n \lambda_i = mn + r \text{ and } t + (m-n) \ge \lambda_1 \ge \cdots \ge \lambda_n \ge m\}.$$

With this we can now describe the decomposition of $H^{n^2-1}_{\mathfrak{m}}(R/I^t)$ into irreducible GL-representations:

$$\begin{aligned}
[H^{n^2-1}_{\mathfrak{m}}(R/I^t)]_r &= \bigoplus_{\lambda \in B(r)} S_{\lambda+(-m^n)+(n^m)} F \otimes S_\lambda G \otimes (\bigwedge^m F)^{-n} \otimes (\bigwedge^n G)^{-m} \\
&= \bigoplus_{\lambda \in B(r)} S_{\lambda+(-m^n)+(n^m)+(-n^m)} F \otimes S_{\lambda+(-m^n)} G \\
&= \bigoplus_{\lambda \in B(r)} S_{\lambda+(-m^n)} F \otimes S_{\lambda+(-m^n)} G \\
&= \bigoplus_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ t-n \ge \lambda_1 \\ \lambda_n \ge 0 \\ |\lambda| = r}} S_\lambda F \otimes S_\lambda G.
\end{aligned}$$

Thus by Cauchy's formula (3) we see that J_t as a GL-representation is a direct sum of terms $S_\mu F \otimes S_\mu G$ not present in the above direct sum. Hence by Remark 2.3 and Formula (4) we have that

$$J_t = \bigoplus_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ \lambda_1 \geq t-n+1 \\ \lambda_n \geq 0}} S_\lambda F \otimes S_\lambda G = I_{(t-n+1)}.$$

□

In a similar manner we can also obtain a general version of Proposition 4.3. Note however that unlike in Section 4.1 the isomorphism here is abstract. To the author's knowledge there is no known description of the socle element of $H_I^{mn-t^2+1}(R)$ as a class in Čech cohomology, this precludes the constructions of an explicit isomorphism as in Section 4.1.

Theorem 5.2. *Let R and I be as above, then*

$$\operatorname{Ext}_R^{mn-n^2+1}(R/I^t, R) \cong \sum_{|\alpha|=t-n} R \cdot (d^\alpha)^* \frac{1}{\underline{x}},$$

where as before f^* denotes the polynomial differential operator obtained from f by replacing x_i with ∂_i..

Proof. Note that under graded duality the $\mathbb{C}$-vector space generated by $\frac{1}{x^\alpha}\frac{1}{\underline{x}} = (x^\alpha)^* \bullet \frac{1}{\underline{x}} \subseteq H_{\mathfrak{m}}^{mn}(R)$ corresponds to the $\mathbb{C}$-vector space generated by $x^\alpha \subseteq R \cong \operatorname{Hom}_R(H_{\mathfrak{m}}^{mn}(R), H_{\mathfrak{m}}^{mn}(R))$. Thus, $\operatorname{Ext}_R^{mn-n^2+1}(R/I^t, R)$ is generated by elements $f^* \bullet \frac{1}{\underline{x}}$ where $f + I_{(t-n+1)} \in \operatorname{Soc} H_{\mathfrak{m}}^{n^2-1}(R/I^t)$.

From 5.1 we have that $H_{\mathfrak{m}}^{n^2-1}(R/I^t) \simeq R/I_{(t-n+1)}$. Since the socle is equivariant by [14] we have that

$$\operatorname{Soc} R/I_{(t-n+1)} = \bigoplus_{\lambda \in C} S_\lambda F \otimes S_\lambda G,$$

where

$$C = \{\lambda \in \mathscr{P}_n | (t-n+1) \not\leq \lambda \text{ and } (t-n+1) \leq \tau \text{ for all } \lambda \geq \tau \in \mathscr{P}_n\}.$$

On one hand $(t-n+1) \not\leq \lambda$ if and only if $\lambda_1 < t-n+1$, on the other hand $(t-n+1) \leq \tau$ for all $\lambda < \tau$ if and only if $\lambda_n \geq t-n$. So we conclude that $C = \{(t-n^n)\}$ and $\operatorname{Soc} R/I_{(t-n+1)} = I^{t-n} + I_{(t-n+1)}$. Thus

$$\operatorname{Ext}_R^{mn-n^2+1}(R/I^t, R) = \sum_{|\alpha|=t-n} R \cdot (d^\alpha)^* \bullet \frac{1}{\underline{x}}.$$

□

Comments on Characteristic $p > 0$. The description of these local cohomology modules in characteristic $p > 0$ is almost completely unknown. While the results of Section 3 are not dependent on characteristic, the approach used for the $n \times (n-1)$ case fails completely. Since I is Cohen-Macaulay of height $(m-n+1)$, we have that $H_I^{mn-n^2+1}(R) = 0$ so extracting information from the maps $\operatorname{Ext}_R^{mn-n^2+1}(R/I^t, R) \to H_I^{mn-n^2+1}(R)$ is challenging.

Computer computations in Macaulay2 [21] show that in prime characteristic the modules $H_{\mathfrak{m}}^{n^2-1}(R/I^t)$ are not always cyclic and may have generators in multiple degrees. In [7] it was shown that the degree 0 component of $H_{\mathfrak{m}}^{n^2-1}(R/I^t)$ can have arbitrarily large vector space dimension, suggesting these modules may have arbitrarily many generators.

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