THE PERFECTION CAN BE A NON-COHERENT GCD DOMAIN

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Abstract. We show that there exists a complete local Noetherian normal domain of prime characteristic whose perfection is a non-coherent GCD domain, answering a question of Patankar in the negative concerning characterizations of F-coherent rings. This recovers and extends a result of Glaz using tight closure methods.

1. Introduction

Let R be a Noetherian ring of prime characteristic $p > 0$. The *perfection* (or *perfect closure*) of R, denoted in this note by R_{perf}, is defined as

$$R_{\mathrm{perf}} = \varinjlim \left(R \xrightarrow{F} R \xrightarrow{F} \cdots \right)$$

where $F : R \longrightarrow R$ is the Frobenius map $r \mapsto r^p$. If R is further assumed to be a domain with fraction field $K = \mathrm{Frac}(R)$, then the *absolute integral closure* of R is defined to be the integral closure of R in a choice of algebraic closure $\overline{K}$. These two objects $R_{\mathrm{perf}} \subseteq R^+$ have over the past few decades seen substantial use in subjects where one usually only a priori considers Noetherian rings, despite being highly non-Noetherian themselves. It is natural to ask which "Noetherian-like" properties these objects enjoy. Two such notions of interest in this note are *coherence* (that is, all finitely generated ideals are finitely presented) and the property of being a *GCD domain* (that is, a domain in which the intersection of two principal ideals is principal).

In the case of R^+, this can be effectively hopeless due to the fact that R^+ usually fails to be coherent regardless of characteristic [AH97; Asg17; Pat22]. For R_{perf} however, the situation is not as dire — the perfect closure of a Noetherian regular ring of prime characteristic is coherent [Shi11, Proposition 3.3(1)] and its finitely generated ideals enjoy finite primary decomposition [Rad83, Corollaire 1], essentially both because of Kunz's theorem [Kun69]. The study of Noetherian rings with coherent perfect closures was pioneered by Shimomoto in [Shi11] who called them *F-coherent*, suggesting connections to other classes of singularities defined via the Frobenius map. The results of [Shi11] for F-coherent rings R may be summarized as follows, where for simplicity we assume that R is reduced and that F_*R is a finite R-module.

(1) F-coherence is stable under localization and descends under faithfully flat maps.
(2) F-coherent rings have purely inseparable normalizations.
(3) Regular rings, purely inseparable extensions of regular rings, and purely inseparable subrings of regular rings, are all F-coherent.

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(4) If $(R, \mathfrak{m})$ is local and F-coherent, then R_{perf} is a big Cohen-Macaulay algebra.
(5) Tight closure coincides with Frobenius closure in the F-coherent setting. In particular, tight closure localizes, weak F-regularity and F-purity are equivalent, and F-injectivity and F-rationality are equivalent.

At present, the major shortcoming of this theory is the scarcity of examples. In fact, the only known examples of F-coherent rings are those which come from a purely inseparable extension or inclusion of a regular ring as in (3).

This subject has been further explored in [Asg17] using homological methods. For example, it is observed in [Asg17] (relying heavily on ideas introduced in [AH97]) that if R_{perf} is coherent then it is a GCD domain. The relationship between GCD domains and the coherence property has a peculiar history — for instance, it had been wondered for some time in the broader context of non-Noetherian ring theory whether there exists a non-coherent GCD domain. Such a ring was first constructed in [Gla01] (see Section 1 of *op. cit.* for further context) and a different construction via ultraproducts was provided in [OS03]. The question of whether this phenomenon may occur for the perfect closure of a Noetherian ring (and in particular whether R_{perf} being a GCD domain characterizes F-coherent rings) is posed in [Pat22, Remark 4.9]. The content of this note is the following:

Theorem 1.1. *(= Corollary 2.4 + Examples 2.5 and 2.6) There exists a complete local Noetherian normal domain R of prime characteristic $p > 0$ such that R_{perf} is a non-coherent GCD domain.*

To find such rings, we first observe that if R is a prime characteristic Noetherian UFD, then R_{perf} is a GCD domain (Theorem 2.1). We then appeal to (5) above to conclude that any F-pure UFD which is not Cohen–Macaulay (or even just not weakly F-regular) will satisfy this requirement. Such rings exist by work of Bertin [Ber67] and Fossum-Griffith [FG75], considering certain rings of invariants in characteristic 2 (Example 2.5). We also provide Cohen–Macaulay examples by considering certain diagonal hypersurfaces (Example 2.6).

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2. Results

Recall that a domain A (not necessarily Noetherian) is a *GCD domain* if for any $r, s \in A$, the ideal $rA \cap sA$ is principal (equivalently, the ideal $(r :_A s)$ is principal). Asgharzadeh has observed using homological methods that if R is a local Noetherian F-coherent domain, then R_{perf} is a GCD domain [Asg17, Corollary 3.7]. In this section we construct Noetherian rings R which are not F-coherent but whose perfections are GCD domains. We begin with an elementary proof that if R is a prime characteristic UFD, then R_{perf} is a GCD domain irrespective of any coherence assumptions.

Theorem 2.1. *Let R be a Noetherian UFD of prime characteristic $p > 0$. Then R_{perf} is a GCD domain.*

Proof. Let $a, b \in R_{\mathrm{perf}}$. We will show that $(a :_{R_{\mathrm{perf}}} b)$ is a principal ideal. There exists $N_1 \gg 0$ such that $a^{p^{N_1}}, b^{p^{N_1}} \in R$. Let

$$a^{p^{N_1}} = u\pi_1^{s_1} \cdots \pi_m^{s_m}$$
$$b^{p^{N_1}} = v\pi_1^{t_1} \cdots \pi_m^{t_m}$$

be prime factorizations of $a^{p^{N_1}}$ and $b^{p^{N_1}}$ where $u, v \in R$ are units, $\pi_i \in R$ are distinct irreducible elements, and $s_i, t_i \geq 0$. Note that

$$\left(a^{p^{N_1}} :_R b^{p^{N_1}}\right) = \left(\frac{\operatorname{lcm}\left(a^{p^{N_1}}, b^{p^{N_1}}\right)}{b^{p^{N_1}}}\right) = (c)$$

where $c = \pi_1^{\max\{0, s_1 - t_1\}} \cdots \pi_m^{\max\{0, s_m - t_m\}}$. We claim that $(a :_{R_{\mathrm{perf}}} b) = (c^{1/p^{N_1}})R_{\mathrm{perf}}$, where the $\supseteq$ direction is clear. Let $\eta \in (a :_{R_{\mathrm{perf}}} b)$ and write $\eta b = \xi a$ for some $\xi \in R_{\mathrm{perf}}$. Choose $N_2 \gg 0$ such that $\eta^{p^{N_1+N_2}}, \xi^{p^{N_1+N_2}} \in R$. We see that

$$\eta^{p^{N_1+N_2}} \in \left(a^{p^{N_1+N_2}} :_R b^{p^{N_1+N_2}}\right) = \left(\pi_1^{\max\{0, p^{N_2}(s_1 - t_1)\}} \cdots \pi_m^{\max\{0, p^{N_2}(s_m - t_m)\}}\right) = \left(c^{p^{N_2}}\right). \tag{2.1}$$

Write $\eta^{p^{N_1+N_2}} = rc^{p^{N_2}}$ for some $r \in R$. Taking $p^{N_1+N_2}$-th roots, we have the equation $\eta = r^{1/p^{N_1+N_2}} c^{1/p^{N_1}}$ (in R_{perf}), so $\eta \in (c^{1/p^{N_1}})R_{\mathrm{perf}}$ as desired. □

Remark 2.2. If R is not a UFD then the equality $(f :_R g)^{[p^e]} = (f^{p^e} :_R g^{p^e})$ in Equation (2.1) need not hold; in fact, such behavior characterizes local prime characteristic UFDs by [Zha09, Theorem 3.5].

We briefly recall for the reader the notions of F-purity and weak F-regularity. Let R be a Noetherian ring of prime characteristic $p > 0$. R is said to be *F-pure* if the Frobenius map $F : R \to F_*R$ is a pure R-module homomorphism. If F_*R is a finite R-module or if R is a complete local ring, then R is F-pure if and only if $R \to F_*R$ splits as a map of R-modules. R is said to be *cyclically F-pure* if $I = I^F$ for all ideals $I \subseteq R$, where $I^F := IR_{\mathrm{perf}} \cap R$ denotes the *Frobenius closure of I*. F-purity always implies cyclic F-purity, and the converse holds, for example, for rings which are excellent and reduced [Hoc77]. Finally, R is said to be *weakly F-regular* if $I = I^*$ for all ideals $I \subseteq R$, where

$$I^* = \left\{ r \in R \mid \text{there exists } c \in R \setminus \bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p} \text{ such that } cr^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0 \right\}$$

denotes the *tight closure of I*. We recall the following theorem of Shimomoto since it is a key ingredient in our construction.

Theorem 2.3. *[Shi11, Proposition 3.12 & Corollary 3.15] Let R be a reduced cyclically F-pure local ring of prime characteristic $p > 0$ whose perfection R_{perf} is coherent. Then R is weakly F-regular. If R is further assumed to be excellent, then R is Cohen–Macaulay.*

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The proof of Theorem 2.3 uses a valuation argument to show that coherence of R_{perf} implies the equality $I^F = I^*$ for all ideals $I \subseteq R$, and weakly F-regular excellent rings are well-known to be Cohen–Macaulay. Despite having a simple proof, Theorem 2.1 has the following unexpected consequence when combined with Theorem 2.3.

Corollary 2.4. Let R be a local Noetherian F-pure UFD of prime characteristic $p > 0$. Then R_{perf} is a GCD domain which is not coherent in either of the following two cases:

(1) R is not weakly F-regular;
(2) R is excellent and not Cohen–Macaulay.

Proof. R_{perf} is a GCD domain by Theorem 2.1. Since R is F-pure, it is reduced and cyclically F-pure. The fact that R_{perf} is not coherent in the two cases above now follows from the contrapositives of both statements in Theorem 2.3, as desired. □

We demonstrate that rings satisfying the above hypotheses exist:

Example 2.5. Let $\Bbbk$ be a field of characteristic 2 and let $B := \Bbbk[x_0, x_1, x_2, x_3]$. Consider the action of the group $G := \mathbb{Z}/4\mathbb{Z}$ on B induced by the $\Bbbk$-algebra automorphism $\sigma(x_i) = x_{i+1}$ (where the indices are viewed $\mod 4$). Let $\mathfrak{m} = (x_0, x_1, x_2, x_3)$ be the homogeneous maximal ideal of B, and let $A = B^G$ be the invariant subring. It is shown in [Ber67] that A is a UFD which is not Cohen–Macaulay. Moreover, A is F-pure by [Gla95, Proposition 2.4(a)]. Let $\mathfrak{n} = \mathfrak{m} \cap A$, and denote $R = \widehat{A_{\mathfrak{n}}}$. R is F-pure by [MP21, Exercise 11 & Corollary 2.3], and it is shown in [FG75] that R is a UFD. It follows that R_{perf} is a non-coherent GCD domain by Corollary 2.4 (2). □

We can also use (1) instead of (2) in Corollary 2.4 to produce Cohen–Macaulay examples exhibiting the conclusion of Theorem 1.1.

Example 2.6. Let $p \equiv 1 \mod 5$ and consider the degree 5 diagonal hypersurface

$$A = \mathbb{F}_p[x, y, z, u, v]/(x^5 + y^5 + z^5 + u^5 + v^5)$$

with homogeneous maximal ideal $\mathfrak{m}$. By the Jacobian criterion, $\mathfrak{m}$ defines the singular locus of A, hence A and its $\mathfrak{m}$-adic completion $R := \widehat{A}$ are UFDs by Grothendieck's parafactoriality theorem [SGA2, XI Corollaire 3.10 and XI Théorème 3.13(ii)] (see also [CL94]). By the assumption on the characteristic, the coefficient of $(xyzuv)^{p-1}$ in the monomial expansion of $(x^5 + y^5 + z^5 + u^5 + v^5)^{p-1}$ is nonzero, hence A and R are both F-pure by Fedder's criterion [Fed83]. However, it is well-known that A and R are not weakly F-regular. Indeed, the ideal (y, z, u, v) is not tightly closed in either ring as one checks that $x^4 \in (y, z, u, v)^*$ (see also [MP21, Exercise 17] and [Hun96, Example 1.6.3]). We then apply Corollary 2.4 (1) to conclude that R_{perf} is a non-coherent GCD domain.

We remark that this reasoning applies more generally to any F-pure local hypersurface of dimension at least four which is not weakly F-regular and which has an isolated singularity. □

We may also deduce non-explicitly that there are plenty of the eponymous rings by combining Corollary 2.4 with a well-known result of Heitmann, as suggested to the author by B. Olberding:

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Corollary 2.7. Let $(T, \mathfrak{m})$ be a complete local domain with depth $T \geq 2$ which is F-pure but not weakly F-regular. Then there exists an excellent local ring $(A, \mathfrak{m} \cap A)$ such that A_{perf} is a non-coherent GCD domain and such that $\widehat{A} \cong T$.

Proof. We use [Hei93, Theorem 8] to construct an excellent local UFD A such that $\widehat{A} \cong T$. Since T is not weakly F-regular, the excellence of A together with [HH94, Corollary 7.28] implies that A is not weakly F-regular. Since A is F-pure, we may apply Corollary 2.4 (1) to conclude that A_{perf} is a non-coherent GCD domain. □

We conclude with the following remarks.

Remark 2.8.
(1) Corollary 2.4 (2) does not provide additional content to Corollary 2.7 because all rings under consideration are excellent, and excellent weakly F-regular rings are Cohen–Macaulay by [HH94, Proposition 4.2(c)].
(2) In view of Theorem 2.1 we may ask whether the property of R_{perf} being a GCD domain characterizes those rings R sharing a perfect closure with a prime characteristic UFD.
(3) Corollary 2.7 suggests a potential approach to showing that the completion of a local F-coherent ring need not be F-coherent – the existence of such a ring is unknown to the author.

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