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COMPONENTWISE LINEARITY UNDER SQUARE-FREE GRÖBNER DEGENERATIONS

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ABSTRACT. Using the recent results on square-free Gröbner degenerations by Conca and Varbaro, we prove that if a homogeneous ideal I of a polynomial ring is such that its initial ideal $\mathrm{in}_<(I)$ is square-free and $\beta_0(I) = \beta_0(\mathrm{in}_<(I))$, then I is a componentwise linear ideal if and only if $\mathrm{in}_<(I)$ is a componentwise linear ideal. In particular, if furthermore one of I and $\mathrm{in}_<(I)$ is componentwise linear, then their graded Betti numbers coincide.

1. Introduction

Throughout this paper, $R = K[X_1,\dots,X_n]$ is the polynomial ring in n variables over a field K with $\deg(X_i) = 1$ for each $i = 1,\dots,n$, $\mathfrak{m} = (X_1,\dots,X_n)$ is the unique homogeneous maximal ideal of R, and I is a homogeneous ideal of R. We denote by $\beta_{i,j}(I)$ the (i, j)-th graded Betti number of I and, for each $d \in \mathbb{Z}_+$, we denote by $I_{\langle d\rangle}$ the ideal generated by all homogeneous polynomials of degree d belonging to I.

The notion of componentwise linearity was introduced by Herzog and Hibi [HH1] in 1999: We say that a homogeneous ideal $I \subseteq R$ has a *d-linear resolution* if $\beta_{i,i+j}(I) = 0$ for all i and for all $j \neq d$. We say that I is *componentwise linear* if $I_{\langle d\rangle}$ has a d-linear resolution for all $d \in \mathbb{Z}$. In particular, if I has a linear resolution, then it is componentwise linear.

In their paper published in 2020 on square-free Gröbner degenerations [CoV], Conca and Varbaro showed that if I is a homogeneous ideal of a polynomial ring and if the initial ideal $\mathrm{in}_<(I)$ is square-free with respect to some term order $<$, then the Castelnuovo-Mumford regularity of I and of $\mathrm{in}_<(I))$ coincide [CoV, Corollary 2.7]. A consequence of this result is the following: if $\mathrm{in}_<(I)$ is square-free, then I has a d-linear resolution if and only if $\mathrm{in}_<(I)$ has a d-linear resolution. On the other hand, a consequence of Macaulay's Theorem [Ma] implies that $\mathrm{in}_<(I)$ and I have the same Hilbert function (see [HH2, Corollary 6.1.5]). If both I and $\mathrm{in}_<(I)$ have d-linear resolutions, using their Hilbert series:

$$\begin{aligned}\frac{\sum_{i=0}^{\mathrm{projdim}(I)}(-1)^i\sum_{j\in\mathbb{Z}}\beta_{i,j}(I)t^j}{(1-t)^n} &= \mathrm{HS}_I(t)\\ &= \mathrm{HS}_{\mathrm{in}_<(I)}(t)\\ &= \frac{\sum_{i=0}^{\mathrm{projdim}(\mathrm{in}_<(I))}(-1)^i\sum_{j\in\mathbb{Z}}\beta_{i,j}(\mathrm{in}_<(I))t^j}{(1-t)^n}\end{aligned}$$

we have that the graded Betti numbers $\beta_{i.j}(I) = \beta_{i.j}(\mathrm{in}_<(I))$ for all $i, j \in \mathbb{Z}$. Therefore, if $\mathrm{in}_<(I)$ is square-free and I has a linear resolution, then their graded Betti numbers coincide.

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In what follows of this paper we always further assume that $<$ is a graded term order, that is, $<$ is a term order which compares the total degree. For instance, graded lexicographic order and graded reverse lexicographic order are graded term orders. Since componentwise linear ideals can be considered as a generalization of ideals with linear resolution, naturally one has some questions: if one of the ideals I and $\text{in}_<(I)$ is a componentwise linear ideal, can we obtain that, under some certain assumptions, the other one is also componentwise linear? Can we have some information about their graded Betti numbers? One part of these questions has already been answered by Caviglia and Varbaro in [CaV]. They proved that if $\text{in}_<(I)$ is a componentwise linear ideal and if $\beta_0(I) = \beta_0(\text{in}_<(I))$, then I is also a componentwise linear ideal [CaV, Theorem 5.4]. In this paper, we show that if $\text{in}_<(I)$ is square-free, then the converse of the result of Caviglia and Varbaro also holds, that is,

> Assume that $\text{in}_<(I)$ is square-free and $\beta_0(I) = \beta_0(\text{in}_<(I))$. Then I is a componentwise linear ideal if and only if $\text{in}_<(I)$ is a componentwise linear ideal. In particular, if furthermore one of I and $\text{in}_<(I)$ is componentwise linear, we have $\beta_{i,i+j}(\text{in}_<(I)) = \beta_{i,i+j}(I)$ for all i, j.

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2. The Main Result

Notation 2.1. For each $d \in \mathbb{Z}_+$, we denote by $I_{\leq d}$ the ideal generated by all homogeneous polynomials of I whose degree is less than or equal to d.

Lemma 2.2. *Let I be a homogeneous ideal , $<$ a graded term order and let $d \in \mathbb{Z}_+$. Then following conditions are equivalent:*

i) $\beta_0(I_{\langle d\rangle}) = \beta_0(\text{in}_<(I_{\langle d\rangle}))$,
ii) $\text{in}_<(I_{\langle d\rangle}) = \text{in}_<(I)_{\langle d\rangle}$,
iii) $\text{in}_<(I_{\leq d}) = \text{in}_<(I)_{\leq d}$.

Proof. $i \Rightarrow ii$) Since $\text{in}_<(I)_{\langle d\rangle}$ is generated by all monomials of degree d belonging to $\text{in}_<(I)$, if m is a generator of $\text{in}_<(I)_{\langle d\rangle}$, then there exists $f \in I$ such that $\text{in}_<(f) = m$ and the total degree $\deg(f) = d$. It follows that $m \in \text{in}_<(I_{\langle d\rangle})$ as well. Therefore, we have $\text{in}_<(I)_{\langle d\rangle} \subseteq \text{in}_<(I_{\langle d\rangle})$. Moreover, since a Gröbner basis of $I_{\langle d\rangle}$ generates $I_{\langle d\rangle}$, and since the 0-th Betti number of an R-module refers to the minimal number of generators of that module, if $\beta_0(I_{\langle d\rangle}) = \beta_0(\text{in}_<(I_{\langle d\rangle}))$, then $I_{\langle d\rangle}$ has a Gröbner basis (with respect to $<$) which is a minimal system of generators of $I_{\langle d\rangle}$. This implies that $\text{in}_<(I_{\langle d\rangle})$ is generated by monomials of degree d. Hence $\text{in}_<(I_{\langle d\rangle}) = \text{in}_<(I_{\langle d\rangle})_{\langle d\rangle} \subseteq \text{in}_<(I)_{\langle d\rangle} \subseteq \text{in}_<(I_{\langle d\rangle})$, that is, $\text{in}_<(I_{\langle d\rangle}) = \text{in}_<(I)_{\langle d\rangle}$.

$ii \Rightarrow i$) Since $\text{in}_<(I_{\langle d\rangle}) = \text{in}_<(I)_{\langle d\rangle}$, $\text{in}_<(I_{\langle d\rangle})$ is generated by monomials of degree d and so $\text{in}_<(I_{\langle d\rangle}) = ([\text{in}_<(I_{\langle d\rangle})]_d)$. Hence, using [BH, Proposition 1.3.1] and using the fact that $I_{\langle d\rangle}$ and $\text{in}_<(I_{\langle d\rangle})$

have the same Hilbert function [HH2, Corollary 6.1.5], we obtain

$$\begin{aligned}\beta_0(\mathrm{in}_<(I_{\langle d\rangle})) &= \beta_0([\mathrm{in}_<(I_{\langle d\rangle})]_d)\\ &= \dim_K([\mathrm{in}_<(I_{\langle d\rangle})]_d)\\ &= \mathrm{HF}_{\mathrm{in}_<(I_{\langle d\rangle})}(d)\\ &= \mathrm{HF}_{I_{\langle d\rangle}}(d)\\ &= \dim_K([I_{\langle d\rangle}]_d)\\ &= \beta_0(I_{\langle d\rangle}).\end{aligned}$$

ii $\Rightarrow$ *iii*) Repeating a discussion similar to that about $\mathrm{in}_<(I)_{\langle d\rangle} \subseteq \mathrm{in}_<(I_{\langle d\rangle})$ in the part *i* $\Rightarrow$ *ii*), we have that the inclusion $\mathrm{in}_<(I)_{\leq d} \subseteq \mathrm{in}_<(I_{\leq d})$ always holds. Now we assume that $\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle}$ and we prove $\mathrm{in}_<(I_{\leq d}) \subseteq \mathrm{in}_<(I)_{\leq d}$. If $m \in \mathrm{in}_<(I_{\leq d})$ is a monomial such that $\deg(m) = a$ with a a positive integer, then there is $f \in I_{\leq d}$ such that $\mathrm{in}_<(f) = m$ and $\deg(f) = a$. Since $I_{\leq d}$ is a homogeneous ideal, by definition all homogeneous components of f belong to $I_{\leq d}$. If f is not homogeneous, then we can replace f with its homogeneous component of degree a, namely f_a, since $\mathrm{in}_<(f_a) = \mathrm{in}_<(f) = m$. Hence we may assume that f is a homogeneous polynomial. If $a \leq d$, then $m \in \mathrm{in}_<(I)_{\leq d}$ since $m \in \mathrm{in}_<(I)$ and $\deg(m) \leq d$. Now we assume that $a > d$. Since $\mathfrak{m}I_{\langle i\rangle} \subseteq I_{\langle i+1\rangle}$, for each $i < a$ we have $(I_{\langle i\rangle})_a = (\mathfrak{m}I_{\langle i\rangle})_a \subseteq (I_{\langle i+1\rangle})_a$, and it follows that

$$(I_{\langle 1\rangle})_a \subseteq (I_{\langle 2\rangle})_a \subseteq \ldots \subseteq (I_{\langle d\rangle})_a.$$

Hence $f \in I_{\langle d\rangle}$ and so $m \in \mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle} \subseteq \mathrm{in}_<(I)_{\leq d}$.

iii $\Rightarrow$ *ii*) Since $I_{\langle d\rangle}$ is generated by polynomials of degree d and $<$ is a graded term order, we have that $\mathrm{in}_<(I_{\langle d\rangle})$ is generated by monomials of degree greater than or equal to d. By our assumption $\mathrm{in}_<(I_{\langle d\rangle}) \subseteq \mathrm{in}_<(I_{\leq d}) = \mathrm{in}_<(I)_{\leq d}$. It follows that $\mathrm{in}_<(I_{\langle d\rangle})$ is generated by monomials of degree d and so $\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I_{\langle d\rangle})_{\langle d\rangle} \subseteq \mathrm{in}_<(I)_{\langle d\rangle}$. □

Lemma 2.3. *Let J be an ideal of R generated by homogeneous polynomials of the same degree a. If $\beta_0(J) = \beta_0(\mathrm{in}_<(J))$, then for each $d \in \mathbb{Z}_+$ we have*

i) $\mathrm{in}_<(\mathfrak{m}^d J) = \mathfrak{m}^d \mathrm{in}_<(J)$, *and*

ii) $\beta_0(\mathfrak{m}^d J) = \beta_0(\mathrm{in}_<(\mathfrak{m}^d J))$.

Proof. Using again the facts that a Gröbner basis is a system of generators and the 0-th Betti number refers to the minimal number of generators, we have that the assumption $\beta_0(J) = \beta_0(\mathrm{in}_<(J))$ implies that J has a Gröbner basis (with respect to $<$) which is a minimal system of generators of J, we denote this system by $\{h_1,\ldots,h_r\}$. So $\mathrm{in}_<(J) = (\mathrm{in}_<(h_1),\ldots,\mathrm{in}_<(h_r))$ and $\deg(h_i) = a$ for all i.

i) It is clear that $\mathfrak{m}^d\mathrm{in}_<(J) = \mathrm{in}_<(\mathfrak{m}^d)\mathrm{in}_<(J) \subseteq \mathrm{in}_<(\mathfrak{m}^d J)$. We show that $\mathrm{in}_<(\mathfrak{m}^d J) \subseteq \mathfrak{m}^d\mathrm{in}_<(J)$. If $m \in \mathrm{in}_<(\mathfrak{m}^d J)$ is a monomial, then there exists $f \in \mathfrak{m}^d J$ such that $m = \mathrm{in}_<(f)$. Since $f \in \mathfrak{m}^d J \subseteq J$, $m = \mathrm{in}_<(f) \in (\mathrm{in}_<(h_1),\ldots,\mathrm{in}_<(h_r))$. This implies that there exists a monomial $\mu \in R$ and there exists $i \in \{1,\ldots,r\}$ such that $m = \mu\mathrm{in}_<(h_i)$. Since $<$ is graded, $\deg(m) \geq d+a$, and so $\deg(\mu) \geq d$. It follows that $m = \mu\mathrm{in}_<(h_i) \in \mathfrak{m}^d\mathrm{in}_<(J)$.

ii) Since $\mathrm{in}_<(\mathfrak{m}^d J) = \mathfrak{m}^d\mathrm{in}_<(J) = \mathfrak{m}^d(\mathrm{in}_<(h_1),\ldots,\mathrm{in}_<(h_r))$ and $\mathfrak{m}^d J$ is generated by monomials of degree $d+a$, we have $\mathrm{in}_<(\mathfrak{m}^d J) = ([\mathfrak{m}^d\mathrm{in}_<(J)]_{d+a})$. Similarly, since $\mathfrak{m}^d J$ is generated by homogeneous polynomials of degree $d+a$, we have $\mathfrak{m}^d J = ([\mathfrak{m}^d J]_{d+a})$. Therefore,

$$\begin{aligned}
\beta_0(\mathrm{in}_<(\mathfrak{m}^d J)) &= \dim_K([\mathfrak{m}^d \mathrm{in}_<(J)]_{d+a}) \\
&= \dim_K([\mathrm{in}_<(J)]_{d+a}) \\
&= \mathrm{HF}_{\mathrm{in}_<(J)}(d+a) \\
&= \mathrm{HF}_J(d+a) \\
&= \dim_K(J_{d+a}) \\
&= \dim_K([\mathfrak{m}^d J]_{d+a}) = \beta_0(\mathfrak{m}^d J).
\end{aligned}$$

□

Theorem 2.4. *If I is a componentwise linear ideal, $\mathrm{in}_<(I)$ is a square-free ideal and $\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle}$ for all $d \in \mathbb{Z}_+$, then $\mathrm{in}_<(I)$ is a componentwise linear ideal. Moreover, we have $\beta_{i,i+j}(\mathrm{in}_<(I)) = \beta_{i,i+j}(I)$ for all i, j.*

Proof. We denote by $h = \mathrm{reg}(I) := \max\{j \mid \beta_{i,i+j}(I) = 0 \text{ for some } i\}$ the Castelnuovo-Mumford regularity of I. Since $\mathrm{in}_<(I)$ is a square-free ideal, we have $\mathrm{reg}(I) = \mathrm{reg}(\mathrm{in}_<(I))$ by [CoV, Corollary 2.7]. It follows that $\beta_{0,i}(\mathrm{in}_<(I)) = 0$ for all $i > h$. Furthermore, since I is componentwise linear, $\beta_{0,h}(\mathrm{in}_<(I)) \geq \beta_{0,h}(I) > 0$. Hence h is the highest degree of a generator in a minimal system of generators of $\mathrm{in}_<(I)$. We show that $\mathrm{in}_<(I)$ is componentwise linear by induction on h.

$h = 1$: $\mathrm{in}_<(I)$ has 1-linear resolution and so it is componentwise linear.

$h > 1$: Since $\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle}$ for all $d \in \mathbb{Z}_+$, $\mathrm{in}_<(I_{\leq d}) = \mathrm{in}_<(I)_{\leq d}$ for all $d \in \mathbb{Z}_+$ by Lemma 2.2 *ii* ⇒ *iii*). For each $d \in \mathbb{Z}_+$, since

$$(I_{\leq h-1})_{\leq d} = \begin{cases} I_{\leq d} & \text{if } d \leq h-1, \\ I_{\leq h-1} & \text{if } d > h-1, \end{cases}$$

we have

$$\begin{aligned}
\mathrm{in}_<((I_{\leq h-1})_{\leq d}) &= \begin{cases} \mathrm{in}_<(I_{\leq d}) & \text{if } d \leq h-1, \\ \mathrm{in}_<(I_{\leq h-1}) & \text{if } d > h-1 \end{cases} \\
&= \begin{cases} \mathrm{in}_<(I)_{\leq d} & \text{if } d \leq h-1, \\ \mathrm{in}_<(I)_{\leq h-1} & \text{if } d > h-1 \end{cases} \\
&= (\mathrm{in}_<(I)_{\leq h-1})_{\leq d} \\
&= \mathrm{in}_<(I_{\leq h-1})_{\leq d}.
\end{aligned}$$

Using again Lemma 2.2 *iii* ⇒ *ii*) we have $\mathrm{in}_<((I_{\leq h-1})_{\langle d\rangle}) = \mathrm{in}_<(I_{\leq h-1})_{\langle d\rangle}$ for all $d \in \mathbb{Z}_+$. Moreover, since I is componentwise linear, $I_{\leq h-1}$ is componentwise linear and $\mathrm{reg}(I_{\leq h-1}) = h-1$. Since $\mathrm{in}_<(I)$ is a square-free ideal, $\mathrm{in}_<(I_{\leq h-1}) = \mathrm{in}_<(I)_{\leq h-1}$ is also a square-free ideal and $\mathrm{reg}(\mathrm{in}_<(I_{\leq h-1})) = h-1$ by [CoV, Corollary 2.7]. Hence $\mathrm{in}_<(I_{\leq h-1})$ is componentwise linear by inductive hypothesis. It follows that

$$\mathrm{in}_<(I_{\langle h-1\rangle}) = \mathrm{in}_<((I_{\leq h-1})_{\langle h-1\rangle}) = \mathrm{in}_<(I_{\leq h-1})_{\langle h-1\rangle}$$

has $(h-1)$-linear resolution and so $\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle} = \mathfrak{m}\,\mathrm{in}_<(I_{\langle h-1\rangle})$ has h-linear resolution by [HH2, Lemma 8.2.10].

Now we consider the following two short exact sequences

$$0 \longrightarrow \mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle} \longrightarrow \mathrm{in}_<(I)_{\langle h\rangle} \longrightarrow M_h \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{in}_<(I)_{\leq h-1} \longrightarrow \mathrm{in}_<(I) \longrightarrow M_h \longrightarrow 0,$$

where $M_h = \mathrm{in}_<(I)_{\langle h\rangle}/\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle}$. For each i,j they yield the following long exact sequences

$$\begin{aligned} \ldots \longrightarrow\ & \mathrm{Ext}_R^{i-1}(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle},K)_{i+j} \longrightarrow \mathrm{Ext}_R^i(M_h,K)_{i+j} \longrightarrow \\ & \mathrm{Ext}_R^i(\mathrm{in}_<(I)_{\langle h\rangle},K)_{i+j} \longrightarrow \mathrm{Ext}_R^i(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle},K)_{i+j} \longrightarrow \ldots \end{aligned}$$

and

$$\begin{aligned} \ldots \longrightarrow\ & \mathrm{Ext}_R^{i-1}(\mathrm{in}_<(I)_{\leq h-1},K)_{i+j} \longrightarrow \mathrm{Ext}_R^i(M_h,K)_{i+j} \longrightarrow \\ & \mathrm{Ext}_R^i(\mathrm{in}_<(I),K)_{i+j} \longrightarrow \mathrm{Ext}_R^i(\mathrm{in}_<(I)_{\leq h-1},K)_{i+j} \longrightarrow \ldots. \end{aligned}$$

Since $\beta_{i-1,i+j}(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle}) = \beta_{i,i+j}(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle}) = 0$ for each $j > h$,

$$\mathrm{Ext}_R^{i-1}(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle},K)_{i+j} = \mathrm{Ext}_R^i(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle h-1\rangle},K)_{i+j} = 0$$

for each $j > h$. This implies

$$\mathrm{Ext}_R^i(M_h,K)_{i+j} \cong \mathrm{Ext}_R^i(\mathrm{in}_<(I)_{\langle h\rangle},K)_{i+j}$$

for each i and for each $j > h$ by the first long exact sequence. Since $\mathrm{reg}(\mathrm{in}_<(I)_{\leq h-1}) = \mathrm{reg}(\mathrm{in}_<(I_{\leq h-1})) = h-1$, we have

$$\dim_K(\mathrm{Ext}_R^{i-1}(\mathrm{in}_<(I)_{\leq h-1},K)_{i+j}) = \beta_{i-1,i+j}(\mathrm{in}_<(I_{\leq h-1})) = 0$$

and

$$\dim_K(\mathrm{Ext}_R^{i}(\mathrm{in}_<(I)_{\leq h-1},K)_{i+j}) = \beta_{i,i+j}(\mathrm{in}_<(I_{\leq h-1})) = 0$$

for all i and for all $j \geq h$, and so

$$\mathrm{Ext}_R^i(M_h,K)_{i+j} \cong \mathrm{Ext}_R^i(\mathrm{in}_<(I),K)_{i+j}$$

for all i and for all $j \geq h$ by the second long exact sequence. Therefore, for all i and for all $j > h$,

$$\mathrm{Ext}_R^i(\mathrm{in}_<(I)_{\langle h\rangle},K)_{i+j} \cong \mathrm{Ext}_R^i(\mathrm{in}_<(I),K)_{i+j}.$$

We have

$$\begin{aligned} \beta_{i,i+j}(\mathrm{in}_<(I)_{\langle h\rangle}) &= \dim_K(\mathrm{Ext}_R^i(\mathrm{in}_<(I)_{\langle h\rangle},K)_{i+j}) \\ &= \dim_K(\mathrm{Ext}_R^i(\mathrm{in}_<(I),K)_{i+j}) \\ &= \beta_{i,i+j}(\mathrm{in}_<(I)) \\ &= 0 \end{aligned}$$

for all i and for all $j > h = \mathrm{reg}(\mathrm{in}_<(I))$. Since $\mathrm{in}_<(I)_{\langle h\rangle}$ is generated by generators of degree h, $\beta_{0,j}(\mathrm{in}_<(I)_{\langle h\rangle}) = 0$ for $j < h$, and so $\beta_{i,i+j}(\mathrm{in}_<(I)_{\langle h\rangle}) = 0$ for $j < h$. It follows that $\beta_{i,i+j}(\mathrm{in}_<(I)_{\langle h\rangle}) = 0$ for all $j \neq h$.

By inductive hypothesis and by Lemma 2.2, $\mathrm{in}_<(I)_{\leq h-1}$ is componentwise linear. Hence for each $d \leq h-1$ and for each $j \neq d$,

$$\beta_{i,i+j}(\mathrm{in}_<(I)_{\langle d\rangle}) = \beta_{i,i+j}((\mathrm{in}_<(I)_{\leq h-1})_{\langle d\rangle}) = 0.$$

Therefore, $\beta_{i,i+j}(\mathrm{in}_<(I)_{\langle d\rangle}) = 0$ for each d and for each $j \neq d$, that is, $\mathrm{in}_<(I)$ is a componentwise linear ideal.

Furthermore, by Lemma 2.2 *ii* $\Rightarrow$ *i*) and Lemma 2.3 *i*) we get $\mathrm{in}_<(\mathfrak{m}I_{\langle d\rangle}) = \mathfrak{m}\,\mathrm{in}_<(I_{\langle d\rangle}) = \mathfrak{m}\,\mathrm{in}_<(I)_{\langle d\rangle}$ for each d. If both I and $\mathrm{in}_<(I)$ are componentwise linear, then both $I_{\langle d\rangle}$ and $\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle}$ have d-linear resolutions for each d. It follows that both $\mathfrak{m}I_{\langle d\rangle}$ and $\mathrm{in}_<(\mathfrak{m}I_{\langle d\rangle})$ have $(d+1)$-linear resolutions for all d. Using their Hilbert series we obtain $\beta_{i,i+j}(I_{\langle d\rangle}) = \beta_{i,i+j}(\mathrm{in}_<(I_{\langle d\rangle}))$ and $\beta_{i,i+j}(\mathfrak{m}I_{\langle d\rangle}) = \beta_{i,i+j}(\mathrm{in}_<(\mathfrak{m}I_{\langle d\rangle}))$ for all i, j. Therefore, by [HH1, Proposition 1.3] we have that

$$\begin{aligned}\beta_{i,i+j}(\mathrm{in}_<(I)) &= \beta_i(\mathrm{in}_<(I)_{\langle j\rangle}) - \beta_i(\mathfrak{m}\,\mathrm{in}_<(I)_{\langle j-1\rangle})\\ &= \beta_i(\mathrm{in}_<(I_{\langle j\rangle})) - \beta_i(\mathrm{in}_<(\mathfrak{m}I_{\langle j-1\rangle}))\\ &= \beta_i(I_{\langle j\rangle}) - \beta_i(\mathfrak{m}I_{\langle j-1\rangle})\\ &= \beta_{i,i+j}(I)\end{aligned}$$

for each i, j. □

Notice that the first part of the proof of [CaV, Theorem 5.4] showed that if $\beta_0(I) = \beta_0(\mathrm{in}_<(I))$, then for each $d \in \mathbb{Z}_+$ the initial ideal $\mathrm{in}_<(I_{\langle d\rangle})$ is generated in degree d, and so $\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle}$ for all $d \in \mathbb{Z}_+$. Therefore, the above theorem has the following consequence:

Corollary 2.5. *Let I be a homogeneous ideal, and let $<$ be a graded term order such that $\mathrm{in}_<(I)$ is square-free and $\beta_0(I) = \beta_0(\mathrm{in}_<(I))$. Then I is a componentwise linear ideal if and only if $\mathrm{in}_<(I)$ is a componentwise linear ideal. In particular, if furthermore one of I and $\mathrm{in}_<(I)$ is componentwise linear, we have $\beta_{i,i+j}(\mathrm{in}_<(I)) = \beta_{i,i+j}(I)$ for all i, j.*

Proof. It follows by Theorem 2.4 and [CaV, Theorem 5.4]. □

Now we make a short discussion on the necessity of the assumptions of Theorem 2.4. First notice that, by the second part of Theorem 2.4 we obtain that whenever both I and $\mathrm{in}_<(I)$ are componentwise linear and I is minimally generated by a Gröbner basis (with respect to $<$), the graded Betti numbers of I and of $\mathrm{in}_<(I)$ coincide, and it follows that they have the same Castelnuovo-Mumford regularity. However, if $\mathrm{in}_<(I)$ is not square-free, as indicated in [CoV], it can happen that $\mathrm{reg}(I) < \mathrm{reg}(\mathrm{in}_<(I))$. In such instances, even if I is componentwise linear and $\beta_0(I) = \beta_0(\mathrm{in}_<(I))$, $\mathrm{in}_<(I)$ cannot be a componentwise linear ideal. Thus, the assumption "$\mathrm{in}_<(I)$ is a square-free ideal" is a necessary condition for our result. A counterexample is provided as follows.

Example 2.6. Let $X = \begin{pmatrix} a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}$ be a symmetric matrix and let $<$ be the graded reverse lexicographic order on $K[a,b,c,d,e,f]$ induced by $a > b > c > d > e > f$. The ideal I generated by the

2 Feb 2024 07:45:35 PST
230330-HongmiaoYu Version 4 - Submitted to J. Comm. Alg.

2-minors of X is

$$I = (-b^2+ad, -bc+ae, -cd+be, -c^2+af, -ce+bf, -e^2+df).$$

Using Macaulay2 [M2] we compute the Betti table of R/I:

	0	1	2	3
0	1	0	0	0
1	0	6	8	3

Notice that I has 2-linear resolution so it is a componentwise linear ideal. Using again Macaulay2 [M2] we obtain that the initial ideal of I with respect to $<$

$$\mathrm{in}_<(I) = (e^2, ce, cd, c^2, bc, b^2)$$

is not a square-free ideal. And according to the Betti table of $R/\mathrm{in}_<(I)$:

	0	1	2	3	4
0	1	0	0	0	0
1	0	6	8	4	1
2	0	0	1	1	0

we have that $\mathrm{in}_<(I)$ is not a componentwise linear ideal even if $\beta_0(I) = \beta_0(\mathrm{in}_<(I)) = 6$.

But the necessity of assumption "$\mathrm{in}_<(I_{\langle d\rangle}) = \mathrm{in}_<(I)_{\langle d\rangle}$ for all $d \in \mathbb{Z}_+$" is still an open question. Using Lemma 2.2, this question can be reduced to the following one:

Question. *If I is a componentwise linear ideal such that $\mathrm{in}_<(I)$ is square-free, can we obtain that $\mathrm{in}_<(I_{\leq h-1})$ is also a square-free ideal with $h = \mathrm{reg}(I)$?*

If the answer to the above question is true, then the necessity of this assumption can be denied.

3. Some Applications

The notion of "N-fiber-full up to h modules", introduced and studied in [Yu], aims to generalize certain statements on square-free Gröbner degenerations by Conca and Varbaro [CoV]. In this section, we discuss consequences of our work in Section 2 specifically in the context of these modules. In particular, we improve a crucial result in [Yu] and show some applications to it.

In what follows, we suppose furthermore that $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ is a weight vector and N is a finitely generated $R[t]$-module such that it is a graded $K[t]$-module and it is flat over $K[t]$.

Let us recall that for each $f \in R$ there exists a unique (finite) subset of the set of monomials of R, denoted by $\mathrm{Supp}(f)$, such that

$$f = \sum_{\mu \in \mathrm{Supp}(f)} a_\mu \mu \quad \text{with} \quad a_\mu \in K \setminus \{0\}.$$

If $\mu = X^u = X_1^{u_1} \cdot \dots \cdot X_n^{u_n}$ with $u = (u_1, \dots, u_n)$ is a monomial of R, then we set $w(\mu) = w_1u_1 + \dots + w_nu_n$. If $f = \sum_{\mu \in \mathrm{Supp}(f)} a_\mu \mu \in R$, $f \neq 0$, we set

$$w(f) = \max\{w(\mu) : \mu \in \mathrm{Supp}(f)\},$$

$$\text{init}_w(f) = \sum_{\substack{\mu \in \text{Supp}(f) \\ w(\mu)=w(f)}} a_\mu \mu,$$

and we call

$$\text{hom}_w(f) = \sum_{\mu \in \text{Supp}(f)} a_\mu \mu t^{w(f)-w(\mu)} \in R[t]$$

the *w-homogenization* of f.

Given an ideal $J \subseteq R$, $\text{in}_w(J)$ denotes the ideal of R generated by $\text{init}_w(f)$ with $f \in J$, and $\text{hom}_w(J)$ denotes the ideal of $R[t]$ generated by $\text{hom}_w(f)$ with $f \in J$.

Moreover, recall that $S = R[t]/\text{hom}_w(J)$ is N-fiber-full up to an integer h as an $R[t]$-module in the sense of [Yu, Definition 1.1] if, for any $m \in \mathbb{Z}_+$, the natural projection $S/t^mS \longrightarrow S/tS$ induces injective maps $\text{Ext}^i_{R[t]}(S/tS, N) \longrightarrow \text{Ext}^i_{R[t]}(S/t^mS, N)$ for all $i \leq h$. One result related to this notion is the following:

> If J is a homogeneous ideal and if $S = R[t]/\text{hom}_w(J)$ is N-fiber full up to h as an $R[t]$-module, then
>
> $$\dim_K(\text{Ext}^i_R(R/J, N/tN)_j) = \dim_K(\text{Ext}^i_R(R/\text{in}_w(J), N/tN)_j)$$
>
> for all $i \leq h-2$ and for all $j \in \mathbb{Z}$. (see [Yu, Corollary 3.2])

Actually the converse of the above result also holds and we have the following one:

Proposition 3.1. *Let $J \subseteq R$ be an ideal. Then*

i) $S = R[t]/\text{hom}_w(J)$ is N-fiber-full up to h as an $R[t]$-module if and only if $\text{Ext}^i_{R[t]}(S, N)$ is a flat $K[t]$-module for $i \leq h-1$.

ii) If furthermore J is homogeneous, then S is N-fiber-full up to h as an $R[t]$-module if and only if

$$\dim_K(\text{Ext}^i_R(R/J, N/tN)_j) = \dim_K(\text{Ext}^i_R(R/\text{in}_w(J), N/tN)_j)$$

for all $i \leq h-2$ and for all $j \in \mathbb{Z}$.

In particular,

iii) if $N = R[t]$, then S is $R[t]$-fiber-full up to h as an $R[t]$-module if and only if

$$\dim_K(H^i_{\mathfrak{m}}(R/J)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\text{in}_w(J))_j)$$

for all $i \geq n-h+2$ and for all $j \in \mathbb{Z}$.

iv) If $N = K[t]$, then S is $K[t]$-fiber-full up to h as an $R[t]$-module if and only if

$$\beta_{i,j}(R/J) = \beta_{i,j}(R/\text{in}_w(J))$$

for all $i \leq h-2$ and for all $j \in \mathbb{Z}$.

Proof. The part *i*) is a direct consequence of [Yu, Theorem 2.6]. For the part *ii*) we only have to notice that the converse of some steps of the proof of [Yu, Corollary 3.2] are also true. More precisely, for the same reason discussed in [Yu, Corollary 3.2]: if $R = K[X_1, \dots, X_n]$ is equipped with the graded structure $\deg(X_i) = g_i$, where $g_1, \dots, g_n$ are positive integers, we provide a bi-graded structure on $R[t]$ by putting $\deg(X_i) = (g_i, w_i)$ and $\deg(t) = (0, 1)$. Since $S = R[t]/\text{hom}_w(J)$ and $\text{Ext}^i_{R[t]}(S, N)$ are finitely generated bi-graded $R[t]$-modules, using the structure theorem for finitely generated modules over a principal ideal domain, we have that for each $i,j \in \mathbb{Z}$,

$$\bigoplus_{l\in\mathbb{Z}} \operatorname{Ext}^i_{R[t]}(S,N)_{(j,l)} \cong K[t]^{a_{i,j}} \oplus \big(\bigoplus_{k\in\mathbb{Z}_+} (K[t]/(t^k))^{b_{i,j,k}}\big)$$

for some natural numbers $a_{i,j}$ and $b_{i,j,k}$. And for every $i,j\in\mathbb{Z}$,

$$\dim_K(\operatorname{Ext}^i_R(R/J,N/tN)_j) = a_{i,j},$$

$$\dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(J),N/tN)_j) = a_{i,j} + b_{i,j} + b_{i+1,j},$$

where $b_{i,j} = \sum_{k\in\mathbb{Z}_+} b_{i,j,k}$.

Therefore, S is N-fiber-full up to h if and only if $\operatorname{Ext}^i_{R[t]}(S,N)$ is a flat $K[t]$-module for all $i \le h-1$, if and only if $b_{i,j,k} = 0$ for all $i \le h-1$ and for all j,k, if and only if $b_{i,j} = b_{i+1,j} = 0$ for all $i \le h-2$ and for all j, if and only if $\dim_K(\operatorname{Ext}^i_R(R/J,N/tN)_j) = a_{i,j} = \dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(J),N/tN)_j)$ for all $i \le h-2$ and for all j.

In particular, if $N = R[t]$, then $N/tN \cong R$ and the part *iii*) is obtained by using the local duality theorem for graded modules (see [BH, Theorem 3.6.19]).

If $N = K[t]$, then $N/tN \cong K$. By [BH, Proposition 1.3.1] we have $\dim_K(\operatorname{Ext}^i_R(R/J,K)_j) = \beta_{i,j}(R/J)$ and $\dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(J),K)_j) = \beta_{i,j}(R/\operatorname{in}_w(J))$ for all i,j. □

Now considering the following fact:

> Given an ideal $J \subseteq R$ and given a monomial order $<$ on R, there exists a suitable weight vector w such that $\operatorname{in}_w(J) = \operatorname{in}_<(J)$. (see [HH2, Theorem 3.1.2] or [Va, Proposition 3.4])

we show that the above proposition and Corollary 2.5 imply the following result:

Proposition 3.2. *Let I be a homogeneous ideal and let $<$ be a graded term order. Assume that one of the following two conditions holds*

i) *I is a componentwise linear ideal and $\operatorname{in}_<(I)$ is square-free,*

ii) *$\operatorname{in}_<(I)$ is a componentwise linear ideal.*

Then $S = R[t]/\operatorname{hom}_w(I)$ is $K[t]$-fiber-full up to 3 if and only if S is $K[t]$-fiber-full up to h for all $h \in \mathbb{Z}$, where w is a weight vector such that $\operatorname{in}_w(I) = \operatorname{in}_<(I)$.

Proof. One implication is trivial. On the other hand, if S is $K[t]$-fiber-full up to 3, then $\beta_{1,j}(R/I) = \beta_{1,j}(\operatorname{in}_<(I))$ for all $j \in \mathbb{Z}$ by Proposition 3.1 *iv*), and it follows that $\beta_0(I) = \beta_0(\operatorname{in}_<(I))$. If one of the two conditions of our assumption holds, we obtain that $\beta_{i,i+j}(I) = \beta_{i,i+j}(\operatorname{in}_<(I))$ for all i,j by Corollary 2.5. This implies that S is $K[t]$-fiber-full up to h for all $h \in \mathbb{Z}$ by using again Proposition 3.1 *iv*). □

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