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FIXED LOCI IN EVEN LINKAGE CLASSES

SCOTT NOLLET

ABSTRACT. Let $\mathscr{L}$ be an even linkage class of pure codimension two subschemes of a projective space. When $\mathscr{L}$ has an integral minimal element X_0, it is known which deformation classes in $\mathscr{L}$ contain integral subschemes (varieties). When $\mathscr{L}$ does not have an integral minimal element, we use fixed loci to give necessary conditions on deformation classes in $\mathscr{L}$ to contain varieties and give examples showing sharpness. As an application, we determine all deformation classes containing integral curves in even linkage classes whose corresponding Rao module is a complete intersection module.

1. Introduction

Linkage theory was used by Halphen and Noether to classify space curves in the 1880s and has been updated with scheme-theoretic foundations over the past 50 years [20, 28]. The theory works best for linkage of codimension two subschemes of $\mathbb{P}^n$, where Rao's correspondence gives a bijection between even linkage classes and stable equivalence classes of certain reflexive sheaves [22, 30, 31]. Furthermore, each non-ACM even linkage class $\mathscr{L}$ has a minimal subscheme X_0 from which all others are obtained by sequences of basic double links followed by a cohomology preserving deformation. This was first observed by Lazarsfeld and Rao for the even linkage class of a high degree embedding of a curve in $\mathbb{P}^3$ [16], conjectured in generality by Bolondi and Milgiore [4], proved for space curves by Martin-Deschamps and Perrin [18], for locally Cohen-Macaulay codimension two subschemes in $\mathbb{P}^n$ with $n \geq 3$ by Ballico, Bolondi and Migliore [3], and finally for subschemes in $\mathbb{P}^n$ of pure codimension two [22]. Thus each even linkage class $\mathscr{L}$ of codimension two subschemes of $\mathbb{P}^n$ is stratified by irreducible locally closed subspaces $H_X \subset \mathbf{Hilb}(\mathbb{P}^n)$ consisting of constant cohomology deformations of X in $\mathscr{L}$ [26, 1.3]. A question that might have interested Halphen and Noether is the following:

Question 1.1. *Fix an even linkage class $\mathscr{L}$ of codimension two subschemes in $\mathbb{P}^n$. Which deformation classes $H_X \subset \mathscr{L}$ contain integral subschemes (varieties)?*

The even linkage class $\mathscr{L}$ of ACM subschemes has been deeply studied by many authors. Here Question 1.1 has a complete answer [25, 1.9 and 3.3], which can be described in various ways: $H_X \subset \mathscr{L}$ contains an integral subscheme if and only if the gamma character γ_X of Martin-Deschamps and Perrin [18] is positive and connected $\iff$ the numerical character of Gruson and Peskine [9] has no gaps $\iff$ the invariant $m(X)$ of Sauer [32] is at least three. Steffen [33] used Chang's filtered Bertini theorem [5] to completely determine when X is smoothable, extending work of various authors [9, 17, 32]. For $n = 3,4$ the smoothable and integral answers agree, otherwise not. For $n > 5$, the only smooth ACM varieties are the complete intersections, as predicted by Hartshorne's conjecture [11].

We therefore focus on even linkage classes $\mathscr{L}$ of non-ACM subschemes, where there is a minimal element X_0 and well-defined height h_X for each $X \in \mathscr{L}$ [3, 18, 22]. The function $\eta_X : \mathbb{Z} \to \mathbb{Z}$

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defined by $\eta_X(l) = \Delta^n(h^0\mathscr{I}_X(l) - h^0\mathscr{I}_{X_0}(l-h_X))$ is non-negative, has sum $\sum \eta_X(l) = h_X$, and uniquely determines $H_X \subset \mathscr{L}$ [26, 1.9 (b)]. For $X \in \mathscr{L}$, let $e(X) = \max\{l : H^{n-2}(\mathscr{O}_X(l)) \neq 0\}$ be the speciality of X and $s(X) \leq t(X)$ be the two lowest degrees of properly intersecting hypersurfaces containing X. Then $\eta_X(l) > 0$ for $s(X) \leq l < s(X_0) + h_X$, so the function $\theta_X(l) = \eta_X(l) - \binom{l-s(X)}{1} - \binom{l-s(X_0)-h_X}{1}$ is non-negative and can give a simple test for integrality:

Theorem 1.2. *Suppose X_0 is integral and $X \in \mathscr{L}$ is not minimal. Then H_X has an integral element if and only if*

(1) *θ_X is connected about $[s(X_0)+h_X, t(X_0)+h_X-1]$ and*
(2) *$s(X) \leq e(X_0)+n+1+h_X$*

The image of a high degree embedding of a smooth curve in $\mathbb{P}^3$ is minimal in its even linkage class [16], so Theorem 1.2 tells which curves have integral deformations. Paxia and Ragusa [27] used an earlier version of this result [21] to determine the integral curves for even linkage classes $\mathscr{L}$ of Buchsbaum curves in $\mathbb{P}^3$. Theorem 1.2 is a corollary of Theorem 5.1 in the text. The proof is mostly found in [26, 3.4], but we remove the locally Cohen-Macaulay hypothesis used in that proof.

Our main purpose here is to address the case where X_0 may not be taken integral. Condition (1) in Theorem 1.2 is necessary for $H_X \in \mathscr{L}$ to contain an integral variety and sufficient if $X_0 < Y \leq X$ with Y integral, so the problem becomes locating minimal integral elements Y. A natural obstruction to integrality occurs when all elements of H_X contain a fixed variety of codimension two, much like the base locus of a family of divisors in the Bertini theorems. One can consider the naive fixed locus

$$F(\mathscr{L}) = \bigcap_{X \text{ minimal in } \mathscr{L}} X,$$

but it is unclear how $F(\mathscr{L})$ extends to other deformation classes H_X with X not minimal. The key idea here is the definition of fixed loci F_s for each $s < s(X_0)$ which depend on how deep in an $\mathscr{N}$-type resolution for $\mathscr{I}_{X_0}$ they appear. These loci form a filtration of $F(\mathscr{L})$ which are easier to compute and determine how they proliferate to other deformation classes (Proposition 4.1). We identify an element $X_1 \in \mathscr{L}$ (Definition 5.5) as a natural candidate for minimal integral element and prove an analogous result to Theorem 1.2:

Theorem 1.3. *Let $X \in \mathscr{L}$ with $h_X > 0$.*

(1) *If X is integral, then $X \geq X_1$.*
(2) *Conversely, if X_1 is integral, then H_X contains an integral element if and only if*
  (a) *$X_1 \leq X$.*
  (b) *θ_X is connected about $[s(X_0)+h_X, t(X_0)+h_X-1]$.*

Theorem 1.3 is sharp in the sense that there are examples where X_0 cannot be taken integral, but X_1 can, in which case we get a complete description of the varieties in $\mathscr{L}$ up to deformation as well as a Lazarsfeld-Rao property for these classes. There are examples in which X cannot be a variety when $\theta_X = 0$, so we also construct $X_2 \in \mathscr{L}$ giving a result analogous to Theorem 1.3 for varieties with $\theta_X \neq 0$ (Remark 5.8).

In §2 we recall linkage theory of codimension two subschemes and prove a result reminiscent of Serre duality for non-locally Cohen-Macaulay subschemes, while §3 gives some linkage-theoretic constructions of integral varieties. In §4 we introduce the fixed loci F_s for an even linkage class $\mathscr{L}$ and in §5 we use them to prove Theorem 1.2 and Theorem 1.3, including some examples to show sharpness.

In §6 we apply our results to the even linkage class $\mathscr{L}$ of curves in $\mathbb{P}^3$ corresponding to Rao module a quotient of the homogeneous coordinate ring by a regular sequence, determining the deformation classes H_X containing an integral curve. We compare this result to work of Martin-Deschamps and Perrin who determined exactly which deformation classes contain smooth (connected) curves [19]. The simplest example where the answers differ is Hartshorne's example of an integral but non-smoothable curve [13].

We work in the context of liaison theory for pure codimension two subschemes in $\mathbb{P}^n_k$ [14, §4] without assuming the locally Cohen-Macaulay hypothesis, to allow applications to integral varieties such as cones over integral varieties and general projections of smooth varieties. We take k to be an algebraically closed. We are studying integral subschemes in even linkage classes of pure codimension two subschemes on $\mathbb{P}^n_k$, so we assume $n \geq 3$. The term *deformation* in this paper always refers to cohomology-preserving deformation through schemes in a fixed even linkage class as explained in Section 2. We thank Prabhakar Rao for useful conversations.

2. Even linkage classes

We recall linkage theory for pure codimension two subschemes in $\mathbb{P}^n$ [14, 22]. We write $X \overset{S\cap T}{\sim} Y$ when X and Y are directly linked by $S \cap T$ and $X \overset{s,t}{\sim} Y$ if $s = \deg S, t = \deg T$. Subschemes $X, Y \subset \mathbb{P}^n$ are *evenly linked* if there is an even chain of direct links between them. For example, if $X \in \mathscr{L}$ lies on a hypersurface S of degree s and we link twice via $X \overset{S\cap T}{\sim} Y \overset{S\cap T'}{\sim} X'$ with $\deg T' = \deg T + h$, then X' is obtained from X by a *double link* of height h on S and write $X \overset{s,h}{\to} X'$. The double link is ascending if $h \geq 0$ (descending otherwise) and is called a *basic double link* if $T' = T \cup H$ with $\deg H = h$, when there is an exact sequence

$$0 \to \mathscr{O}(-s-h) \to \mathscr{I}_X(-h) \oplus \mathscr{O}(-s) \to \mathscr{I}_{X'} \to 0. \tag{1}$$

Closing under transitivity gives an equivalence relation whose equivalence classes $\mathscr{L}$ are called *even linkage classes*. For $X \in \mathscr{L}$, let H_X consist of members of $\mathscr{L}$ with the same cohomology as X, in other words, $H_X = \{Y \in \mathscr{L} : h^i(\mathscr{I}_X(l)) = h^i(\mathscr{I}_Y(l)) \text{ for all } i \geq 0 \text{ and } l \in \mathbb{Z}\}$: the H_X clearly form a stratification of $\mathscr{L}$. In the best understood even linkage class $\mathscr{L}$ of Arithmetically Cohen-Macaulay (ACM) subschemes, the H_X form smooth open irreducible subsets of the Hilbert scheme [6].

2.1. *The Lazarsfeld-Rao property.* An even linkage class $\mathscr{L}$ of non-ACM subschemes of pure codimension two in $\mathbb{P}^n$ has an additional structure first observed in [16]. A reflexive, transitive relation on $\mathscr{L}$ is given by $X \leq Y$ if there is a sequence of height one basic double links $X \overset{s_1,1}{\to} X_1 \cdots \overset{s_n,1}{\to} X_n$ followed by a deformation to Y in H_{X_n} (i.e. a cohomology-preserving deformation through schemes in $\mathscr{L}$). The Lazarsfeld-Rao property [3, 16, 18, 22] says that $\mathscr{L}$ has a minimal element X_0 satisfying $X_0 \leq X$ for all $X \in \mathscr{L}$.

2.2. *Rao's correspondence and height.* An $\mathscr{E}$-type resolution for subscheme $X \subset \mathbb{P}^n$ is an exact sequence $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{I}_X \to 0$ where $\mathscr{F}$ is *dissocié* (a direct sum of line bundles) and $H^1_*(\mathscr{E}) = 0$. Rao's correspondence [22, 31] says that if $X \subset \mathbb{P}^n$ has pure codimension two, then the map $X \mapsto \mathscr{E}$ gives a bijection between even linkage classes $\mathscr{L}$ of pure codimension two subschemes in $\mathbb{P}^n$ and stable equivalence classes of reflexive sheaves $\mathscr{E}$ satisfying $H^1_*(\mathscr{E}) = 0$ and $\mathscr{E}xt^1(\mathscr{E}^\vee, \mathscr{O}) = 0$. An $\mathscr{N}$-type resolution is an exact sequence $0 \to \mathscr{F} \to \mathscr{N} \to \mathscr{I}_X \to 0$ where $\mathscr{F}$ is a direct sum of line bundles, $H^1_*(\mathscr{N}^\vee) = 0$ and $\mathscr{E}xt^1(\mathscr{N}, \mathscr{O}) = 0$: again, $X \mapsto \mathscr{N}$ gives a bijection between even linkage classes and stable equivalence classes of the relevant reflexive sheaves. The cone construction interchanges $\mathscr{E}$ and $\mathscr{N}$ type resolutions under direct linkage [22, 1.8 and 1.11]. Here X is locally Cohen-Macaulay if and only if $\mathscr{E}$ (or $\mathscr{N}$) is a vector bundle. The *Rao modules* of X are the graded modules $M^i(X) = H^i_*(\mathscr{I}_X), 0 < i < n-1$ over the homogeneous coordinate ring S for $\mathbb{P}^n$. A consequence of Rao's correspondence is that the Rao modules $M^i(X) = H^i_*(\mathscr{I}_X), 0 < i < n-1$ are the same modulo shift because $M^i(X) \cong H^{i+1}_*(\mathscr{E})$ and stable equivalence preserves these graded modules up to shift. The ACM even linkage class $\mathscr{L}$ corresponds to the stable equivalence class of the zero sheaf via $\mathscr{E}$ or $\mathscr{N}$ type resolution and in this case all the Rao modules $M^i(X)$ are zero. If $\mathscr{L}$ is a non-ACM even linkage class, then $\mathscr{E}$ is not stably equivalent to zero because if $\mathscr{E}$ is a reflexive sheaf with vanishing intermediate cohomology, then $\mathscr{E}$ is dissocié [1, 3.1]. Thus at least one of the Rao modules must be nonzero. Exact sequence (1) shows each height one basic double link shifts them one twist to the right, allowing us to define the *height* of $X \in \mathscr{L}$ by $h_{X_0} = 0$ and h_X is the number of twists that the higher Rao modules are twisted rightward from those of X_0.

Question 5.1 asks when the spaces H_X have an integral subscheme. The following result shows that the definition given here agrees with the definition given in [26].

Proposition 2.1. *Let $\mathscr{L}$ be an even linkage class of pure codimension two subschemes of $\mathbb{P}^n$.*

(1) *Let $X, Y \in \mathscr{L}$. The following are equivalent.*
  (a) *$h^0(\mathscr{I}_Y(l)) = h^0(\mathscr{I}_X(l))$ for all $i \geq 0$ and $l \in \mathbb{Z}$ and $h_Y = h_X$ if X is not ACM.*
  (b) *There are $\mathscr{E}$-type resolutions $0 \to \mathscr{E} \overset{\phi_X}{\to} \mathscr{F} \to \mathscr{I}_X \to 0$ and $0 \overset{\phi_Y}{\to} \mathscr{E} \to \mathscr{F} \to \mathscr{I}_Y \to 0$.*
  (c) *$Y \in H_X$.*

(2) *The stratum H_X is an irreducible locally closed subset of the Hilbert scheme.*

Proof. First we prove the equivalence.

$(a) \Rightarrow (b)$: Assuming condition (a), there is a direct sum of line bundles $\mathscr{F}$ and surjections $\mathscr{F} \to \mathscr{I}_X, \mathscr{F} \to \mathscr{I}_Y$ whose kernels $\mathscr{E}_X, \mathscr{E}_Y$ are stably isomorphic by Rao's correspondence [22, 2.4]. It follows that if $\mathscr{E}_0$ is a minimal rank element of the stable equivalence class, then there are $h_X, h_Y \in \mathbb{Z}$ and dissocié $\mathscr{Q}_X, \mathscr{Q}_Y$ with $\mathscr{E}_X \cong \mathscr{E}_0(t_X) \oplus \mathscr{Q}_X$ and $\mathscr{E}_Y \cong \mathscr{E}_0(t_Y) \oplus \mathscr{Q}_Y$ [22, 2.3]. If $\mathscr{L}$ is the ACM even linkage class, then $\mathscr{E}_0 = 0$ and $\mathscr{Q}_X \cong \mathscr{Q}_Y$ because the degrees of the twists are determined by the numbers $h^0(\mathscr{I}_X(l)) - h^0(\mathscr{F}(l))$. If $\mathscr{L}$ is a non-ACM class, then $t_X = t_Y$ because $h_X = h_Y$ so that the twists of the Rao modules agree. The twists of $\mathscr{Q}_X$ and $\mathscr{Q}_Y$ are determined by the numbers $h^0(\mathscr{I}_X(l)) - h^0(\mathscr{F}(l)) + h^0(\mathscr{E}_0(t_X + l))$ so that $\mathscr{Q}_X \cong \mathscr{Q}_Y$ and $\mathscr{E}_X \cong \mathscr{E}_Y$.

$(b) \Rightarrow (c)$: One can read off all the numbers $h^i(\mathscr{I}_Y(l))$ from the resolution combined with the equality $h^n(\mathscr{I}_Y(l)) = h^n(\mathscr{O}(l))$ for $l \in \mathbb{Z}$.

$(c) \Rightarrow (a)$: If $\mathscr{L}$ is not the ACM class, the Rao modules have the same twist so that $h_Y = h_X$.

The proof of [26, 1.3] shows that H_X is an irreducible locally closed subset of the Hilbert scheme when $\mathscr{L}$ is a non-ACM even linkage class (condition (a) defined H_X in that work). When $\mathscr{L}$ is the ACM class, the result is known from [6]. □

2.3. *Indexing the strata* H_X. Suppose $\mathscr{L}$ is a non-ACM even linkage class and let $\mathscr{N}$ be a least rank reflexive sheaf corresponding to $\mathscr{L}$ via Rao's correspondence [22, 31]. Then there is a twist $d \in \mathbb{Z}$ and a direct sum of line bundles $\mathscr{P}_0$ for which each minimal $X_0 \in \mathscr{L}$ fits into the sequence

$$0 \to \mathscr{P}_0 \overset{\varphi}{\to} \mathscr{N}_0 \to \mathscr{I}_{X_0} \to 0 \tag{2}$$

where $\mathscr{N}_0 = \mathscr{N}(d)$ [18, 22]. Using the Lazarsfeld-Rao property and repeatedly applying (1), we see that for any $X \in \mathscr{L}$, there is $Y \in H_X$ with resolution

$$0 \to \mathscr{P}_0(-h_Y) \oplus \mathscr{T}(-1) \xrightarrow{\phi} \mathscr{N}_0(-h_Y) \oplus \mathscr{T} \to \mathscr{I}_Y \to 0 \tag{3}$$

where $\mathscr{T} = \oplus \mathscr{O}(-n)^{\eta_Y(n)}$ for some non-negative $\eta_Y : \mathbb{Z} \to \mathbb{Z}$ satisfying $\sum \eta_Y(l) = h_Y$ (some summands of $\mathscr{T}$ and $\mathscr{T}(-1)$ may cancel for general maps ϕ). Taking difference functions of the spaces of global sections in exact sequence (3) shows that $\eta_X(l) = \Delta^n H^0(\mathscr{I}_X(l)) - \Delta^n H^0(\mathscr{I}_{X_0}(l-h_X))$ and η_X uniquely determines H_X because we can read off all the cohomology numbers $h^i(\mathscr{I}_X(l))$. For functions $\eta : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with finite support, we set $\sup \eta = \max\{n : \eta(n) \neq 0\}$ and $\inf \eta$ analogously. We say that η is *connected in degrees* $< d$ if $\eta(l) \neq 0$ for some $l < d$ implies $\eta(n) \neq 0$ for all $l < n < d$ with similar definitions for connectedness in degrees $\leq d, > d, \geq d$ and η is connected about an interval $[a,b]$ if η is connected in degrees $\leq b$ and connected in degrees $\geq a$. The only restriction on η_X is connectedness in degrees $< s(X_0) + h_X$ [26, 1.8] and sequence (3) shows that

$$s(X) = \min\{\inf \eta_X, s(X_0) + h_X\}. \tag{4}$$

Therefore the function $\theta_X : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $\theta_X(l) = \eta_X(l) - 1$ for $s(X) \leq l < s(X_0) + h_X$ and $\theta_X(l) = \eta_X(l)$ otherwise is non-negative.

Example 2.2. A double line $X_0 \subset \mathbb{P}^3$ of arithmetic genus -3 is minimal in its even linkage class $\mathscr{L}$. As will be seen in §6, X_0 has a minimal $\mathscr{N}$-type resolution $0 \to \mathscr{O} \oplus \mathscr{O}(-2) \to \mathscr{N}_0 \to \mathscr{I}_{X_0} \to 0$ for a rank three bundle $\mathscr{N}_0$ arising from a Koszul resolution; we normalize the bundle in §6 by taking $\mathscr{N}_0 = \mathscr{N}(2)$. Construct Y and Z by a sequence of basic double links $X_0 \xrightarrow{2,1} X_1 \xrightarrow{2,1} X_2 \xrightarrow{5,1} Y \xrightarrow{3,1} Z$. Applying exact sequence (1) at each step, we obtain exact sequences $0 \to \mathscr{O}(-3) \to \mathscr{O}(-2) \oplus \mathscr{I}_{X_0}(-1) \to \mathscr{I}_{X_1} \to 0$, $0 \to \mathscr{O}(-3) \to \mathscr{O}(-2) \oplus \mathscr{I}_{X_1}(-1) \to \mathscr{I}_{X_2} \to 0$, $0 \to \mathscr{O}(-6) \to \mathscr{O}(-5) \oplus \mathscr{I}_{X_2}(-1) \to \mathscr{I}_Y \to 0$ and similarly for $\mathscr{I}_Z$. Substituting each resolution into the next gives an exact sequence

$$0 \to \mathscr{T}(-1) \to \mathscr{T} \oplus \mathscr{I}_{X_0}(-4) \to \mathscr{I}_Z \to 0,$$

where $\mathscr{T} = \mathscr{O}(-6) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-3)$: here $\eta_Z(l) = 1$ for $3 \leq l \leq 6$ and 0 otherwise. In general, the invariant η_Z records the summands appearing in $\mathscr{T}$ when Z is obtained from X_0 by a sequence of height one basic double links as seen in this example. Substituting the $\mathscr{N}$-type resolution for $\mathscr{I}_{X_0}$ and cancelling like summands (possible for a general map) gives a sequence of the form

$$0 \to \mathscr{O}(-7) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-6) \to \mathscr{N}_0(-4) \oplus \mathscr{O}(-3) \to \mathscr{I}_Z \to 0. \tag{5}$$

Using the recipe above, $\theta_Z(6) = 1$ (and $\theta_Z(l) = 0$ otherwise). Here η_Z is connected in degrees < 6 and θ_Z is connected about the interval $[6,5]$: although this interval is empty, θ_Z is connected about $[6,5]$ in the sense explained. Up to deformation, we could have constructed Z as a single double link $X_0 \xrightarrow{3,4} Z$, since η_Z is agrees. Similarly, we could have constructed Y as a double link $X_0 \xrightarrow{3,3} Y$ and we could have constructed X_2 as a double line $X_0 \xrightarrow{2,2} X_2$. In particular, Y is a curve of degree 11 and genus 12 while Z is a curve of degree 14 and genus 21 [18, III, 3 (b)]. We will see in §6 that Y is smoothable in $\mathscr{L}$ while Z deforms in $\mathscr{L}$ to an integral curve that is not smooth. In fact, This is essential Hartshorne's example of an integral curve that is non-smoothable in its Hilbert scheme component [13].

We will use the following result to rule out integrality of some deformations.

Lemma 2.3. *Let $\mathscr{L}$ be an even linkage class of codimension two subschemes in $\mathbb{P}^n$ with minimal element X_0 and let $X \in \mathscr{L}$. Then*

(a) $\theta_X = 0 \iff s(X) = s(X_0)$.

(b) *If $\theta_X = 0$ and no minimal Y lies on an integral degree $s(X_0)$ hypersurface, then X does not lie on an integral surface of degree $s(X_0)$. In particular, X is not integral.*

Proof. (a) We may assume $h_X > 0$, since both statements hold for X minimal. Recalling that η_X is connected in degrees $< (X_0) + h_X$, equation (4) shows that $s(X) = \inf \eta_X = s(X_0) \iff \eta_X(l) = 1$ for $s(X_0) \le l < s(X_0) + h_X$ ($\eta_X(l) = 0$ otherwise) $\iff \theta_X = 0$ by definition of θ_X.

(b) Now suppose $\theta_X = 0$, so up to deformation X is a double link $X_0 \overset{s(X_0),h_X}{\to} X$ and for general such X we could write $X_0 \overset{s(X_0),t(X_0)}{\sim} X_0^* \overset{s(X_0),t(X_0)+h_X}{\sim} X$, where X_0^* is a minimal element for the dual even linkage class. Form the space of triples

$$D = \{(Y,S,T) : Y \in H_{X_0}, Y \subset S \cap T\}$$

where S,T are hypersurfaces of degrees $s(X_0), t(X_0)$ meeting properly. Then since H_{X_0} is irreducible and the general fibers of the projection $p : D \to H_{X_0}$ are irreducible of the same dimension, the map p is dominant and D is irreducible. Martin-Deschamps and Perrin call D a drape scheme ("schéma de drapeaux" [18, VII, §3]). There is a similar map $q : D \to H_{X_0^*}$ taking (Y,S,T) to the scheme linked to Y by $S \cap T$. Since D is irreducible, the general surface S of degree $s(X_0)$ is not integral and since q is dominant, the general surface S of degree $s(X_0)$ containing $Y^* \in H^*_{X_0}$ is also not integral. We can construct a similar drape scheme for the link $X_0^* \overset{s(X_0),t(X_0)+h_X}{\sim} X$ to see that the general $Z \in H_X$ is not contained in an integral surface of degree $s(X_0)$, hence this is true for all $Z \in H_X$, in particular X itself is not integral [26, 3.1]. □

If $X \overset{S\cap T}{\sim} Y$ via hypersurfaces S,T of degrees s,t, there is an exact sequence [14, 4.1]

$$0 \to \mathscr{I}_{S\cap T} \to \mathscr{I}_Y \to \mathscr{H}om(\mathscr{O}_X, \mathscr{O}_{S\cap T}) \to 0. \tag{6}$$

If X is locally Cohen-Macaulay, then the third sheaf is isomorphic to $\omega_X(n+1-s-t)$ and we can use Serre duality to see that $h^k(\omega_X(l)) = h^{n-2-k}(\mathscr{O}_X(-l))$. If X is not locally Cohen-Macaulay, we can still use $\mathscr{E}$ and $\mathscr{N}$ type resolutions to obtain the analogous result when $k = 0$.

Lemma 2.4. *$h^0(\mathscr{H}om(\mathscr{O}_X, \mathscr{O}_{S\cap T})(l) = h^{n-2}(\mathscr{O}_X(s+t-1-n-l))$ for all $l \in \mathbb{Z}$.*

Proof. The resolution $0 \to \mathscr{O}(-s-t) \to \mathscr{O}(-s) \oplus \mathscr{O}(-t) \to \mathscr{I}_{S\cap T} \to 0$ and an $\mathscr{E}$-type resolution $0 \to \mathscr{E} \to \mathscr{Q} \to \mathscr{I}_Y \to 0$ give rise to exact sequences on global sections. Combining with (6), $h^0(\mathscr{H}om(\mathscr{O}_X, \mathscr{O}_{S\cap T})(l)$ is equal to

$$h^0(\mathscr{Q}(l)) - h^0(\mathscr{E}(l)) - h^0(\mathscr{O}(l-s)) - h^0(\mathscr{O}(l-t)) + h^0(\mathscr{O}(l-s-t)). \tag{7}$$

The cone construction for the linkage $X \overset{S\cap T}{\sim} Y$ gives

$$0 \to \mathscr{Q}^\vee(-s-t) \to \mathscr{E}^\vee(-s-t) \oplus \mathscr{O}(-s) \oplus \mathscr{O}(-t) \to \mathscr{I}_X \to 0.$$

Twisting by $s+t-1-n-l$ and taking the long exact cohomology sequence shows that

$$h^{n-2}(\mathscr{O}_X(s+t-1-n-l)) = h^{n-1}(\mathscr{I}_X(s+t-1-n-l)) = h^n(\mathscr{Q}^\vee(-1-n-l))$$

$$-h^n(E^\vee(-1-n-l)) - h^n(\mathscr{O}(t-1-n-l)) - h^n(\mathscr{O}(s-1-n-l)) + h^n(\mathscr{I}_X(s+t-1-n-l))$$

Comparing with (7), the first, third and fourth terms agree by Serre duality. Since $\mathscr{E}$ is reflexive, we have $H^0(\mathscr{E}(l)) = \mathrm{Hom}(\mathscr{O}, \mathscr{E}(l)) \cong \mathrm{Hom}(\mathscr{E}^\vee(-l), \mathscr{O}) \cong H^n(\mathscr{E}^\vee(-l-1-n))'$ by [12, III, 7.1 (c)], so that the second terms agree. The last terms agree because $h^n(\mathscr{I}_X(l)) = h^n(\mathscr{O}_X(l))$. Combining, we see that both sides are equal. □

3. Construction of integral subschemes

We construct integral subschemes with linkage. Proposition 3.4 fully proves [26, 3.5] as it removes the locally Cohen-Macaulay hypothesis. We consider subschemes of pure codimension two in $\mathbb{P}^n$ with $n \geq 3$. Since there are no embedded primes, to show integrality it suffices to show integrality away from a set of codimension at least three.

Lemma 3.1. *Let $Y \subset T \subset \mathbb{P}^n$ be a generic Cartier divisor on an integral hypersurface. Suppose $\mathscr{I}_{X,T}(s)$ is globally generated away from B,* codim $B \geq 3$, *and $H^0(\mathscr{I}_{X,T}(s-1)) \neq 0$. Then the general link $Y \sim X$ by $S \cap T$ with* $\deg S = s$ *yields an integral subscheme X.*

Proof. Since $\mathscr{I}_{Y,T}$ is generically a line bundle along Y and $H^0(\mathscr{I}_{Y,T}(s))$ is globally generated away from B of codimension ≥ 3, the divisor given by a general section $g \in H^0(\mathscr{I}_{Y,T}(s))$ is generically equal to Y along Y. Lifting g via the surjection $H^0(\mathscr{I}_Y(s)) \to H^0(\mathscr{I}_{Y,T}(s))$ gives a hypersurface $S \subset \mathbb{P}^n$ with $S \cap T = Y \cup X$ and $Y \cap X$ a proper intersection. Since there exists $0 \neq h \in H^0(\mathscr{I}_{Y,T}(s-1))$, the sections $h \cdot l$ with $l \in H^0(\mathscr{O}_T(1))$ separate points and tangent vectors away from the zero set H of h and the corresponding map $T - H \to \mathbb{P}^N$ is unramified and has image of dimension equal to $\dim T \geq 2$. Therefore by Bertini's theorem [15, 6.10] (valid for unramified maps over arbitrary algebraically closed fields), the general hyperplane section is geometrically irreducible and reduced, so $(S \cap T) - H$ is integral for general S. Since $\mathscr{I}_{Y,T}(s)$ is globally generated away from B, $S \cap T$ contains no irreducible components of $H \cap T$, hence $(S \cap T) - Y$ is integral and so is $X = \overline{(S \cap T) - Y}$. □

Example 3.2. The conclusion can fail when $H^0(\mathscr{I}_{X,T}(s-1)) = 0$: take $T \subset \mathbb{P}^3$ to be a smooth quadric surface, Y a union of two skew lines and $s = 2$.

Corollary 3.3. *Let $Y \subset \mathbb{P}^n$ be a generic local complete intersection of codimension two. If $s \leq t$, $\varphi : \oplus_{l \leq s} \mathscr{O}(-l)^{q(l)} \oplus \mathscr{O}(-t) \to \mathscr{I}_Y$ is a map with* codim Supp Coker $\varphi \geq 3$ *and $H^0(\mathscr{I}_Y(s-1)) \neq 0$, then the general link $Y \overset{s,t}{\sim} X$ is integral.*

Proof. Since $\mathscr{I}_Y(t)$ is globally generated away from a set of codimension ≥ 3 and is generically generated by two sections, an argument like that of Lemma 3.1 shows the general hypersurface T of degree t containing Y is integral and Y is generically Cartier on T. For the general map $\mathscr{O}(-t) \to \mathscr{I}_Y$ defining T, a lift of the map to $\oplus_{l \leq s} \mathscr{O}(-l)^{q(l)} \oplus \mathscr{O}(-t)$ splits and we get an induced map $\oplus_{l \leq s} \mathscr{O}(-l)^{q(l)} \to \mathscr{I}_{Y,T}$ whose cokernel is supported on a set of codimension ≥ 2. Now apply Lemma 3.1 to find a linking hypersurface S. □

The following result removes the locally Cohen-Macaulay assumption from [26, 3.5].

Proposition 3.4. *Let $X \in \mathscr{L}$ be integral and $0 < h$. Let $d \leq e(X) + n + 1 + h$ be an integer with $d = s(X)$ or $d \geq t(X)$. Then the general scheme X' obtained from X by a double link of type (d, h) is integral.*

Proof. From [26, 3.1] we know that a general hypersurface T of degree d containing X is integral and X is generically Cartier on T. Since X is a generic local complete intersection, a general hypersurface

S of degree $s \gg 0$ containing X meets T properly and links X geometrically to Y and we have the exact sequence (6). Twisting by $d+h$ and taking sections gives

$$0 \to H^0(\mathscr{O}_T(h)) \xrightarrow{\cdot f} H^0(\mathscr{I}_{Y,T}(d+h)) \to H^0(\mathscr{H}om(\mathscr{O}_X, \mathscr{O}_{S\cap T})(d+h)) \to 0,$$

where f is the equation of S on T. The hypothesis on d and Lemma 2.4 show that the rightmost group is nonzero, so the linear system in the middle cuts out a scheme Y' with $Y \subset Y' \subset Y \cup X = T \cap S$ and the second inclusion is proper because not every section of $H^0(\mathscr{I}_{Y,T}(d+h))$ is a multiple of f, so $\operatorname{codim}(Y'-Y) \geq 3$. Apply Lemma 3.1 to see that the general link $Y \overset{T\cap S'}{\sim} X'$ with $\deg S' = s+h$ is integral. □

4. Fixed loci of an even linkage class

Fix an even linkage class $\mathscr{L}$ of codimension two subschemes in $\mathbb{P}^n$ with X_0 minimal. We introduce fixed loci $F_s \subset F(\mathscr{L})$ for $s < s(X_0)$ that are amenable to calculation. We prove some elementary properties and use these fixed loci to give obstructions to integrality of subschemes in $\mathscr{L}$. We also recover the result that if $s(X) > e(X)+n+1$, then X is the unique minimal element in $\mathscr{L}$.

Start with the $\mathscr{N}$-type resolution (2) for X_0, write $\mathscr{P}_0 = \oplus \mathscr{O}(-n)^{p(n)}$ and for $s \in \mathbb{Z}$, split $\mathscr{P}_0$ into lower and upper summands as $\mathscr{P}_0^{\leq s} = \oplus_{n \leq s} \mathscr{O}(-n)^{p(n)}$ and $\mathscr{P}_0^{>s} = \oplus_{n>s} \mathscr{O}(-n)^{p(n)}$. The quotients $\mathscr{Q}_s = \mathscr{N}_0/\mathscr{P}_0^{\leq s}$ are independent of the choice of φ for $s < s(X_0)$ because $H^0(\mathscr{I}_{X_0}(s)) = 0$, so we unambiguously define

$$F_s = \operatorname{Sing} \mathscr{Q}_s \tag{8}$$

where $\operatorname{Sing} \mathscr{F} = \{x : \dim_{k(x)} \mathscr{F}_x \otimes k(x) > \operatorname{rank} \mathscr{F}\}$ is the singular scheme of the sheaf $\mathscr{F}$.

Proposition 4.1. *Fix $s < s(X_0)$.*

(a) *If $X \in \mathscr{L}$ is minimal, then $F_s \subset X$.*
(b) *If $Y \in \mathscr{L}$ and $s + h_Y < s(Y)$, then $F_s \subset Y$.*
(c) *If $s_1 < s_2 < s(X_0)$, then $F_{s_1} \subset F_{s_2}$.*

Proof. Suppose $X \in H_{X_0}$ corresponds to a map φ in (2). The snake lemma gives an exact sequence $0 \to \mathscr{P}_0^{\leq s} \to \mathscr{Q}_s \to \mathscr{I}_X \to 0$. Localizing at $x \in \operatorname{Sing} \mathscr{Q}_s$ and tensoring with $k(x)$ gives the right exact $\mathscr{P}_0^{\leq s} \otimes k(x) \to \mathscr{Q}_s \otimes k(x) \to \mathscr{I}_X \otimes k(x) \to 0$ which shows that $\dim(\mathscr{I}_X) \otimes k(x) > \operatorname{rank} \mathscr{I}_X = 1$, so $x \in X$. We conclude that $\operatorname{Sing} \mathscr{Q}_s \subset \operatorname{Sing} \mathscr{I}_X = X$. This proves part (a).

Consider $Y \in \mathscr{L}$ as in (b). First assume Y is obtained from X_0 by a sequence of height one basic double links, so Y has resolution (3). The hypothesis $s(Y) > s + h_Y$ assures that the summands $\mathscr{O}(-a)$ of $\mathscr{T}$ satisfy $a > s + h_Y$, hence the composite map $\mathscr{P}_0^{<s}(-h_Y) \subset \mathscr{P}_0(-h_Y) \to \mathscr{T}$ is zero, giving a snake diagram

$$\begin{array}{ccccccccc}
 & & & & & & \mathscr{P}_0^{>s}(-h_Y) \oplus \mathscr{T}(-1) & & \\
 & & & & & & \downarrow & & \\
0 & \to & \mathscr{P}_0^{\leq s}(-h_Y) & \to & \mathscr{N}_0(-h_Y) \oplus \mathscr{T} & \to & \mathscr{Q}_s(-h_Y) \oplus \mathscr{T} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{P}_0(-h_Y) \oplus \mathscr{T}(-1) & \to & \mathscr{N}_0(-h_Y) \oplus \mathscr{T} & \to & \mathscr{I}_Y & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathscr{P}_0^{>s}(-h_Y) \oplus \mathscr{T}(-1) & & & & & &
\end{array}$$

and we conclude as in part (a) that $F_s \subset Y$. Since general $Z \in H_Y$ has resolution (3), we see that $F_s \subset Z$ for general $Z \in H_Y$. Containment of the closed set F_s is a closed property of the Hilbert scheme, so $F_s \subset Y$ holds for all such Y.

For part (c), write $\mathscr{P}_0^{\leq s_1} = \oplus_{n \leq s_1} \mathscr{O}(-n)^{p(n)}$, $\mathscr{P}_0^{\leq s_2} = \oplus_{n \leq s_2} \mathscr{O}(-n)^{p(n)}$ and $\mathscr{R} = \oplus_{s_1 < n \leq s_2} \mathscr{O}(-n)^{p(n)}$ to obtain the exact sequence $0 \to \mathscr{R} \to \mathscr{Q}^{s_1} \to \mathscr{Q}^{s_2} \to 0$. Then apply another snake diagram argument as in part (a). $\square$

Thus for $s = s(X_0)$ we obtain closed sets $\dots F_{s-2} \subset F_{s-1} \subset F(\mathscr{L})$. The last inclusion can be strict (compare Lemma 6.1 and Remark 6.2 (b)). These fixed loci give a necessary condition on integral elements.

Corollary 4.2. *Let $f = \inf\{s \in \mathbb{Z} : \operatorname{codim} F_s = 2$ or $s = s(X_0)\}$. If $Y \in \mathscr{L}$ is non-minimal and integral, then $s(Y) \leq f + h_Y$.*

Proof. If there exists $s < s(X_0)$ with $\operatorname{codim} F_s = 2$, then $f < s(X_0)$ and if $s(Y) > f + h_Y$, then $F_f \subset Y$ by Proposition 4.1 and $\deg F_f \leq \deg X_0 < \deg Y$, so Y is cannot be integral. Otherwise $f = s(X_0)$ and $s(Y) \leq f + h_Y$ for all Y by (4). $\square$

Consequently we obtain a familiar condition for $\mathscr{L}$ to have a unique minimal element.

Corollary 4.3. *If $s(X_0) > e(X_0) + n + 1$, then X_0 is the unique minimal element of $\mathscr{L}$.*

Proof. There is a direct link between X_0 and minimal X_0^* for the dual linkage class by hypersurfaces S, T of degrees $s = s(X_0), t = t(X_0)$ [22, 3.30]. The linkage sequence

$$0 \to \mathscr{I}_{S \cap T} \to \mathscr{I}_{X_0^*} \to \mathscr{H}om(\mathscr{O}_{X_0}, \mathscr{O}_{S \cap T}) \to 0$$

shows that Y has an $\mathscr{E}$-type resolution of the form

$$0 \to \mathscr{F} \to \mathscr{O}(-s) \oplus \mathscr{O}(-t) \oplus \mathscr{R} \to \mathscr{I}_Y \to 0, \ \ \mathscr{R} = \bigoplus_{n>t} \mathscr{O}(-n)^{r(n)}$$

because $H^0(\mathscr{H}om(\mathscr{O}_{X_0}, \mathscr{O}_{S \cap T})(l)) = 0$ for $l \leq t$ due to Lemma 2.4 and $s > e(X_0) + n + 1$. The cone construction gives $\mathscr{N}$-type resolution

$$0 \to \mathscr{P}_0 \to \mathscr{N} \to \mathscr{I}_{X_0} \to 0$$

with $\mathscr{N} = \mathscr{F}^\vee(-s-t)$ and $\mathscr{P}_0 = \mathscr{R}^\vee(-s-t) = \oplus_{n>t} \mathscr{O}(n-s-t)^{r(n)} = \oplus_{m<s} \mathscr{O}(-m)^{r(s+t-m)}$ so that $\mathscr{P}_0^{<s} = \mathscr{P}_0$ and $\operatorname{Sing} \mathscr{Q}_s = X_0$ is the unique minimal element in $\mathscr{L}$. $\square$

Example 4.4. Let $\mathscr{L}$ be the even linkage class of two skew lines $X_0 \subset \mathbb{P}^3$. Then $s(X_0) = 2$ and sequence (2) takes the form

$$0 \to \mathscr{O}(-2)^2 \to \Omega_{\mathbb{P}^3} \to \mathscr{I}_{X_0} \to 0$$

so that $\mathscr{Q}_s = \Omega_{\mathbb{P}^3}$ and $\operatorname{Sing} \mathscr{Q}_s = \emptyset$, consistent with the fact that the family of pairs of skew lines has no fixed points.

Example 4.5. If $X_0 \subset \mathbb{P}^4$ is a union of two planes meeting at a point p, then X_0 is not locally Cohen-Macaulay at p and $\operatorname{Sing} \mathscr{N} = \{p\}$. We can take $s < s(X_0) = 2$ small enough that $\mathscr{P}^{\leq s} = 0$ to conclude that $F_s = \{p\}$ and p is contained in every surface $Y \in \mathscr{L}(X_0)$.

5. Integral subschemes in an even linkage class

Fix an even linkage class $\mathscr{L}$ of codimension two subschemes in $\mathbb{P}^n$ with X_0 minimal. We give a complete answer to Question 1.1 if X_0 may be taken integral; otherwise we use Corollary 4.2 and results from [26] to give necessary conditions which we show to be sharp by example. First we correct a statement from [26].

Theorem 5.1. *Let $X,Y \in \mathscr{L}$ such that X is integral and $X < Y$. Then H_Y contains an integral element if and only if*

(a) *θ_Y is connected about $[s(X_0)+h_Y, t(X_0)+h_Y-1]$.*
(b) *$s(Y) \le e(X_0)+n+1+h_Y$.*

Proof. This is [26, 3.6] with two fixes. Firstly $X<Y$ replaces $X \le Y$. The logic gap came from applying [26, 3.4] which assumes Y is not minimal. Secondly Proposition 3.4 replaces [26, 3.5] in the proof to remove the locally Cohen-Macaulay assumption. □

Theorem 5.1 completely answers Question 1.1 when X_0 is integral (see Theorem 1.2). For example, if $X_0 \subset \mathbb{P}^3$ is the image of a general high degree embedding of a smooth connected curve, then $s(X_0) > e(X_0)+4$ [16, 3.1] and X_0 is the unique minimal element in its even linkage class $\mathscr{L}$ by Corollary 4.3. Therefore Theorem 1.2 describes all the cohomology-preserving deformation classes in $\mathscr{L}$ containing an integral curve.

Example 5.2. A general rational quintic curve $X_0 \subset \mathbb{P}^3$ has maximal rank by [2], so $s(X_0)=3$ and $e(X_0)=-1$, therefore X_0 is minimal in its even linkage class [16, 1.8]. It is not unique because automorphisms of $\mathbb{P}^3$ taking the line $L=Z(x,y)$ to itself fix the Rao module $M=S/(x,y,z^2,zw,w^2)^*(-1)$. These automorphisms show that $F(\mathscr{L})=\emptyset$. Theorem 1.2 says that a curve $Y \in \mathscr{L}$ deforms with constant cohomology to an integral curve if and only if θ_Y is connected about $[3+h_Y, 2+h_Y]$ and $s(Y) \le 3+h_Y$, but the latter condition holds for all $Y \in \mathscr{L}$ by (4). Since $\mathscr{I}_{X_0}$ is 4-regular, this a sharp instance of the theorem of Gruson, Lazarsfeld and Peskine [8], which says the ideal sheaf $\mathscr{I}_C$ of an integral curve C of degree d is at most $(d-1)$ regular with equality if and only if C is rational with a $(d-1)$-secant line. The secant line can be seen here because restricting the exact sequence $0 \to \mathscr{I}_{X_0} \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{X_0} \to 0$ to $L=Z(x,y)$ shows that L is a 4-secant line to X_0.

When $\mathscr{L}$ does not have a minimal integral element, we reduce Question 1.1 to finding minimal integral elements with the following.

Proposition 5.3. *Let $X \in \mathscr{L}$ be a non-minimal integral element and suppose $X \le Y$. Then the following are equivalent.*

(a) *The deformation space H_Y has an integral element.*
(b) *θ_Y is connected about $[s(X_0)+h_Y, t(X_0)+h_Y-1]$.*

Proof. If $Y=X$, then (a) is true because X is integral and [26, Theorem 3.4 (a)] says (b) is true, so assume $X<Y$. Then the function $\eta_{X,Y}(l)=\Delta^n h^0((\mathscr{I}_Y(l))-h^0(\mathscr{I}_X(h_Y-h_Z+l)))$ is non-negative [26, Proposition 1.12 (c)], so by the relative version of (4), the minimal hypersurface degree is $s(Y)=\min\{s(X)+h_Y-h_X, \inf \eta_{X,Y}\} \le s(X)+h_Y-h_X$. Now $s(X) \le e(X_0)+n+1+h_X$ follows from [26, Theorem 3.4 (b)] because X is integral and non-minimal. Thus $s(Y) \le e(X_0)+n+1+h_Y$, so condition (b) of Theorem 5.1 holds for all $Y \ge X$ and the equivalence follows. □

In view of Proposition 5.3, the set of all deformation classes containing an integral element is determined by the minimal integral elements. Since we have seen no counterexample, we ask the following natural question:

Question 5.4. Up to cohomology preserving deformation, are there at most finitely many minimal integral elements in $\mathscr{L}$?

Propositions 5.3 (b) and Corollary 4.2 suggest two places to look for minimal integral candidates when X_0 cannot be chosen integral.

Definition 5.5. Let $\mathscr{L}$ be an even linkage class of codimension two subschemes in $\mathbb{P}^n$ with X_0 minimal and set $f = \min\{s : \operatorname{codim} F_s = 2 \text{ or } s = s(X_0)\}$ as in Corollary 4.2. The *first candidate for non-minimal integral element* is the scheme X_1 is defined up to deformation in $\mathscr{L}$ by

$$X_0 \overset{s(X_0),s(X_0)-f}{\longrightarrow} X_1. \tag{9}$$

Remark 5.6. We make two remarks.

(a) X_1 is minimal if and only if $s(X_0) - f = 0$, i.e. $f = s(X_0)$.

(b) A minimal element X_0 is linked to a minimal element X_0^* in the dual linkage class by hypersurfaces of degrees $s(X_0)$ and $t(X_0)$ [22, 3.30], so we can define X_1 by direct link

$$X_0^* \overset{s(X_0),t(X_0)+s(X_0)-f}{\sim} X_1. \tag{10}$$

Definition 5.5 is justified by Theorem 1.3 from the introduction, which we now prove:

Proof. If $X \in \mathscr{L}$ is non-minimal and integral, then $s(X) \le f + h_Y$ by Corollary 4.2. From (4) and connectedness of η_X in degrees $< s(X_0) + h_X$, we see that $\eta_X(l) \ge 1$ for $f + h_X \le l < s(X_0) + h_X$. On the other hand, from (1) and (9) we compute that $\eta_{X_1}(l) = 1$ for $f + h_{X_1} \le l < s(X_0) + h_{X_1}$ and $\eta_{X_1}(l) = 0$ otherwise, so $\eta_{X_1}(l - h_{X_1}) \le \eta_X(l - h_X)$ for all $l \in \mathbb{Z}$ and therefore we have the inequality $X_1 \le X$ by [26, 1.12 (c)], proving (1).

If X may be taken integral, then (a) and (b) follow from Proposition 5.3 and part (1). Conversely if X_1 integral and conditions (a) and (b) hold, we need to show X can be taken integral. We may assume $X_1 < X$. If X_1 is non-minimal, then Proposition 5.3 shows that H_X contains an integral element. If X_1 is minimal, then by Remark 5.6(a) we have $f = s(X_0)$ and $s(X_0) \le e(X_0) + n + 1$ by Corollary 4.3 via the contrapositive. It follows that $s(X) \le e(X_0) + n + 1 + h_X$ and we can apply Theorem 5.1 to finish. □

Example 5.7. We give an example where X_1 is not minimal, but may be taken integral in Theorem 1.3. Let X_0 be the general rational quintic from Example 5.2 and let Y_0 be a minimal curve in the dual even linkage class $\mathscr{L}^*$, linked to X_0 by two cubic surfaces. The Rao module $M(Y_0) \cong S/(x,y,z^2,zw,w^2)$ has minimal resolution of form

$$0 \to S(-5)^2 \to S(-4)^7 \to S(-2) \oplus S(-3)^8 \xrightarrow{\sigma} S(-1)^2 \oplus S(-2)^3 \xrightarrow{\pi} S \to M \to 0.$$

If $\tilde{\sigma}$ is the sheafification of σ, then $\mathscr{N}_0 = \operatorname{Ker} \tilde{\sigma}$ is the indecomposable rank four bundle which corresponds to $\mathscr{L}^*$ by Rao's correspondence [31]. From the construction of minimal curves due to Martin-Deschamps and Perrin [18, IV] there is an exact sequence

$$0 \to \oplus(-2) \oplus \mathscr{O}(-3)^2 \to \mathscr{N}_0 \to \mathscr{I}_{Y_0} \to 0$$

for any minimal $Y_0 \in \mathscr{L}$. The map π is given by (x, y, z^2, zw, w^2) and σ is given by

$$\sigma = \begin{pmatrix} y & z^2 & zw & w^2 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & z^2 & zw & w^2 & 0 & 0 \\ 0 & -x & 0 & 0 & -y & 0 & 0 & w & 0 \\ 0 & 0 & -x & 0 & 0 & -y & 0 & -z & w^2 \\ 0 & 0 & 0 & -x & 0 & 0 & -y & 0 & -z \end{pmatrix}.$$

Looking at the first column, $\tilde{\sigma}|_{\mathscr{O}(-2)}$ drops rank along $L = Z(x, y)$, so $F_2 = L$. Here $s(X_0) = t(X_0) = 3$ and $f = 2$ in Definition 5.5, so the first candidate for non-minimal integral element is $X_1 \overset{3,4}{\sim} X_0$ via (12). Using the cone construction and the $\mathscr{N}$-type resolution for Y_0 above, we obtain an $\mathscr{E}$-type resolution for X_0 of the form

$$0 \to \mathscr{E}_0 \to \mathscr{O}(-3)^4 \oplus \mathscr{O}(-4) \to \mathscr{I}_{X_0} \to 0.$$

Now X_0 is a generically Cartier divisor on a cubic surface S [26, 3.1] and $\mathscr{I}_{X_0,S}(4)$ is globally generated, so by Lemma 3.1 we may take X_1 integral. By Theorem 1.3 the integral curves in $\mathscr{L}^*$ are cohomology preserving deformations of $Y \geq X_1$ with θ_Y connected about $[3 + h_Y, 2 + h_Y]$.

Remark 5.8. In the next section we will see many examples in which $\theta_X = 0 \implies X$ is not integral for the reason shown in Lemma 2.3. In this case we define a *second candidate for minimal integral element* X_2 defined up to deformation in $\mathscr{L}$ by

$$X_0 \overset{t(X_0), t(X_0) - f}{\longrightarrow} X_2 \tag{11}$$

or equivalently through direct linkage by

$$X_0^* \overset{t(X_0), t(X_0) + s(X_0) - f}{\sim} X_2. \tag{12}$$

With this definition we can use the methods above to prove a version of Theorem 1.3 for subschemes $X \in \mathscr{L}$ with $\theta_X \neq 0$:

(1) If X is integral, then $X \geq X_2$.
(2) Conversely, if X_2 is integral, then H_X contains an integral element if and only if
   (a) $X_2 \leq X$.
   (b) θ_X is connected about $[s(X_0) + h_X, t(X_0) + h_X - 1]$.

This will be useful for many examples in the following section.

Example 5.9. Let $\mathscr{L}_k$ be the even linkage class with minimal element X_0 consisting of $k > 2$ skew lines on a smooth quadric $Q \subset \mathbb{P}^3$, so X_0 has type $(k, 0) \in \mathbb{Z} \oplus \mathbb{Z} \cong \operatorname{Pic} Q$. The $\mathscr{N}$-type resolution for X_0 has the form $0 \to \mathscr{O}(-2)^k \to \mathscr{N} \to \mathscr{I}_{X_0} \to 0$ with $\mathscr{N}$ a rank $k+1$ bundle. The dual minimal curve X_0^* is a divisor of type $(0, k)$ on Q and we have $s(X_0) = 2, t(X_0) = k, f = 2$ so that $X_0^* \overset{2,k}{\sim} X_1$ and $X_0^* \overset{k,k}{\sim} X_2$. Here $X_1 = X_0$ cannot be taken integral, but the curves $Y > X_1$ with $\theta_Y = 0$ have type $(k + l, l)$ with $l \geq 1$ on Q and can be taken smooth. Since $\mathscr{I}_{X_0^*}(k)$ is globally generated, X_2 can be taken to be a smooth connected curve of degree $k^2 - k$. Thus H_Y contains an integral element if and only if $X_1 < Y$ and θ_Y is connected about $[2 + h_Y, k - 1 + h_Y]$.

6. The even linkage class of a complete intersection module

We study the even linkage class $\mathscr{L}$ of curves in $\mathbb{P}^3$ corresponding to Rao module

$$M = S/(f_1, f_2, f_3, f_4), n_i = \deg f_i, n_1 \leq n_2 \leq n_3 \leq n_4, f_i \text{ a regular sequence in } S, \tag{13}$$

where S is the homogeneous coordinate ring of $\mathbb{P}^n$. We compute the fixed loci F_s and determine which deformation classes $H_X \subset \mathscr{L}$ contain integral curves. Comparing with Martin-Deschamps and Perrin's results on smooth connected curves [19, V, 2.6], we find that typically when the fixed loci are non-empty, many integral curves are not smoothable within $\mathscr{L}$. We assume char $k = 0$, as it affects the answers both for smooth connected curves and for integral curves (Remark 6.11).

6.1. ***Minimal curves and fixed loci.*** The Koszul resolution for M has form

$$0 \to S(-\nu) \to \bigoplus S(-\nu + n_i) \stackrel{\sigma_3}{\to} \bigoplus_{i \neq j} S(-n_i - n_j) \stackrel{\sigma_2}{\to} \bigoplus S(-n_i) \stackrel{\sigma_1}{\to} S \to M \to 0 \tag{14}$$

where $\nu = \sum n_i$, $\sigma_1 = (f_1, f_2, f_3, f_4)$ and σ_2 is given by the Koszul relations

$$\sigma_2 = \begin{pmatrix} f_2 & f_3 & f_4 & 0 & 0 & 0 \\ -f_1 & 0 & 0 & f_3 & f_4 & 0 \\ 0 & -f_1 & 0 & -f_2 & 0 & f_4 \\ 0 & 0 & -f_1 & 0 & -f_2 & -f_3 \end{pmatrix}.$$

Martin-Deschamps and Perrin [18, IV, 6.7] prove that if $N = \operatorname{Ker} \sigma_1$ and $\mathscr{N} = \tilde{N}$, then a minimal curve X_0 for the even linkage class $\mathscr{L}(M)$ has $\mathscr{N}$-type resolution

$$0 \to \mathscr{O}(-n_1 - n_2) \oplus \mathscr{O}(-\mu) \stackrel{\varphi}{\to} \mathscr{N} \to \mathscr{I}_{X_0}(h_0) \to 0 \tag{15}$$

and $\mathscr{E}$-type resolution (note $\mathscr{E} = \mathscr{N}^\vee(-\nu)$ by self-duality of the Koszul complex)

$$0 \to \mathscr{E} \to \mathscr{O}(-n_1 - n_3) \oplus \mathscr{O}(-\mu') \oplus \mathscr{O}(-n_2 - n_4) \oplus \mathscr{O}(-n_3 - n_4) \to \mathscr{I}_{X_0}(h_0) \to 0 \tag{16}$$

where $\mu = \sup\{n_1 + n_4, n_2 + n_3\}$, $\mu' = \inf\{n_1 + n_4, n_2 + n_3\}$ and $h_0 = \mu - n_3 - n_4$.

They also construct explicit minimal curves X_0 [18, IV, 6.8]: general polynomials f, g of degrees $\mu - n_1 - n_4, \mu - n_2 - n_3$ give rise to minimal $X_0 \in \mathscr{L}$ with ideal

$$I_{X_0} = (f_1 f_2, g f_2^2, f f_1^2, f f_1 f_4 - g f_2 f_3). \tag{17}$$

Since $\deg f = 0$ or $\deg g = 0$, one of f or g is a nonzero constant, leading to three possibilities. Let D be the locally Cohen-Macaulay double structure on $F = V(f_1, f_2)$ contained in the surface $V(f f_1 f_4 - g f_2 f_3)$. Then

(1) If $\deg f = \deg g = 0$, then $X_0 = D$.
(2) if $\deg f = 0$ and $\deg g > 0$, then $X_0 = D \cup V(g, f_1)$.
(3) if $\deg g = 0$ and $\deg f > 0$, then $X_0 = D \cup V(f, f_2)$.

From this one reads off numerical invariants [18, IV, 6.7] for X_0:

$$\deg X_0 = \mu(n_1 + n_2) - n_1 n_3 - n_2 n_4, \ \ s(X_0) = n_1 + n_3 + h_0, \ \ t(X_0) = n_2 + n_4 + h_0. \tag{18}$$

Lemma 6.1. *Let $f = \inf\{s : \operatorname{codim} F_s = 2$ or $s = s(X_0)\}$ as in Prop. 4.2.*

(a) *If $n_2 = n_3$, then $f = s(X_0)$ and F_s is empty for all $s < s(X_0)$.*
(b) *If $n_2 < n_3$, then $f = n_1 + n_2 + h_0$ and $F_f = V(f_1, f_2)$*

Proof. We can see from equations (18) that $s(X_0) = n_1 + n_3 + h_0$ and resolution (15) shows that $\mathscr{P}_0 = \mathscr{O}(-n_1 - n_2 - h_0) \oplus \mathscr{O}(-\mu - h_0)$. If $n_2 = n_3$, then $\mathscr{P}_0^{\leq s} = 0$ for $s < s(X_0)$, so that $\mathscr{Q}_s = \mathscr{N}_0$ and F_s is empty. If $n_2 < n_3$, then $\mathscr{P}_0^{\leq s} = \mathscr{O}(-n_1 - n_2 - h_0)$ for $s = n_1 + n_2 + h_0$ and the map to $\mathscr{N}(-h_0)$ is given by the first column of matrix σ_2. Since the sequence $0 \to \mathscr{N} \to \oplus \mathscr{O}(-n_i) \to \mathscr{O} \to 0$ is locally split, the map $\mathscr{P}_0^{\leq s} \to \mathscr{N}$ drops rank at x iff the composite map $\mathscr{P}_0^{\leq s} \to \oplus \mathscr{O}(-n_i)$ drops rank, which occurs precisely for $x \in V(f_1, f_2)$ in view of the first column of the matrix σ_2. □

Remark 6.2. Taking intersections over the explicit examples above and combining with Lemma 6.1, one can compute the full fixed locus $F(\mathscr{L})$:

(1) If $n_2 = n_3 = n_4$, then $F(\mathscr{L}) = \emptyset$.
(2) If $n_2 = n_3 < n_4$, then $F(\mathscr{L}) = V(f_1, f_2, f_3)$.
(3) If $n_2 < n_3$, then $F(\mathscr{L}) = V(f_1, f_2)$.

In particular, none of the F_s from Lemma 6.1 are equal to $F(\mathscr{L})$ when $n_2 = n_3 < n_4$.

To find the deformation classes of integral curves, we need more information about X_0.

Proposition 6.3. *Let $\mathscr{L} = \mathscr{L}(M)$ with general minimal curve X_0.*

(a) *If $n_1 = n_2 < n_3 = n_4$, then X_0 is a double structure on $V(f_1, f_2)$ having ideal of the form $(f_1^2, f_1 f_2, f_2^2, f_1 F - f_2 G)$, with F, G general linear combinations of f_3, f_4.*
(b) *X_0 lies on an integral surface of degree $s(X_0) \iff n_1 = n_2 = n_3 = n_4$.*

Proof. Part (a). Suppose $n_1 = n_2 < n_3 = n_4$. The map $\varphi : \mathscr{O}(-n_1 - n_2) \oplus \mathscr{O}(-\mu) = \mathscr{P}_0 \to \mathscr{N}$ drops rank where the composite map $\varphi : \mathscr{O}(-n_1 - n_2) \oplus \mathscr{O}(-\mu) \to \oplus \mathscr{O}(-n_i - n_j)$ drops rank because $\mathscr{N}$ is a subbundle of $\oplus \mathscr{O}(-n_i - n_j)$. When $n_1 = n_2 < n_3 = n_4$, the summand $\mathscr{O}(-n_1 - n_2)$ only has nonzero maps to the first column. The summand $\mathscr{O}(-\mu)$ only has nonzero maps to the first five columns and we can take the coefficient to the first column to be zero without changing the dependency locus. If the remaining coefficients are a, b, c, d for the middle 4 columns, this locus is given by the 2×2 minors of the matrix

$$\begin{pmatrix} f_2 & af_3 + bf_4 \\ -f_1 & cf_3 + df_4 \\ 0 & -af_1 - cf_2 \\ 0 & -bf_1 - df_2 \end{pmatrix}$$

leading to the ideal $I_{X_0} = (f_1^2, f_1 f_2, f_2^2, f_2(cf_3 + df_4) + f_1(af_3 + bf_4))$ if $ad - bc \neq 0$. The generator degrees agree with the degrees of the minimal generators from (16), so this is the total ideal, which gives a double structure as stated.

Part (b). From Resolution (16), we see that the lowest three degrees of generators for I_{X_0} are $s(X_0) = n_1 + n_3 + h_0 \leq \mu' + h_0 \leq t(X_0) = n_2 + n_4 + h_0$, where $\mu' = \min\{n_2 + n_3, n_1 + n_4\}$. If X_0 lies on an integral surface S of degree $s(X_0)$, then $\mu' + h_0 = n_2 + n_4 + h_0$, since otherwise S would meet a surface of degree $\mu' + h_0 < t(X_0)$ properly, contradicting the definition of $t(X_0)$. Therefore $\mu' = n_2 + n_4$, hence $n_1 = n_2$ and $n_3 = n_4$. Part (a) shows that if $n_1 = n_2 < n_3 = n_4$, then the general surface of degree $s(X_0)$ has equation $af_1^2 + bf_1 f_2 + cf_2^2$ and is not integral, so if X_0 lies on an integral surface of degree $s(X_0)$, then all the n_i must be equal.

Conversely suppose the n_i are equal. Then by [19, V, 2.3] the curve X_0 may be taken smooth (after replacing f_i be general linear combinations of themselves, X_0 may be taken a disjoint union $V(f_1, f_3) \cup V(f_2, f_4)$ of smooth curves [18, IV, 6.8]) and I_{X_0} is generated by four equations of degree

$n_1+n_3+h_0=s(X_0)$, hence $\mathscr{I}_{X_0}(s(X_0))$ is generated by global sections. Since char $k=0$, this implies that the general surface of degree $s(X_0)$ containing X_0 is smooth [25, 2.7], hence integral. □

Remark 6.4. Letting the f_i vary, we obtain an irreducible family of curves. In case $n_1=n_2=1$, it is the family of double lines of fixed negative genus and the closure is an irreducible component of the Hilbert scheme [23, 1.6]. We don't know if the closures of these families always form irreducible components of the Hilbert scheme.

6.2. ***Integral curves.*** We determine the deformation classes $H_X \subset \mathscr{L}$ containing integral curves. The case $n_1 \neq n_2$ or $n_3 \neq n_4$ is easy, since X_0 lies on no integral surface of degree $s(X_0)$ by Prop. 6.3, hence no curve X with $\theta_X=0$ is integral by Lemma 2.3 (b). On the other hand, the second candidate for minimal integral curve X_2 may be taken integral, so Remark 5.8 determines all integral curves. When $n_1=n_2=a\leq b=n_3=n_4$, we must adjust the argument because the general X_2 is not connected. In this situation we define curves C_1 and C_2 as the height one double links

$$X_2 \overset{a+b,1}{\to} C_1, \quad X_2 \overset{a+b+1,1}{\to} C_2. \tag{19}$$

When $n_1=n_2\leq n_3=n_4$ and $V(f_1,f_2)$ is superficial, Martin-Deschamps and Perrin prove that C_1 and C_2 can be taken smooth and connected [19, 2.9], but below we will show they are integral whether $V(f_1,f_2)$ is superficial or not. We conclude the following:

Theorem 6.5. *Let $X\in\mathscr{L}$. Then H_X contains an integral curve if and only if*

(a) *$X_2\leq X$ (resp. $C_1\leq X$ or $C_2\leq X$ if $n_1=n_2=a\leq b=n_3=n_4$).*
(b) *θ_X is connected about $[s(X_0)+h_X, t(X_0)+h_X-1]$.*

Proof. Assuming H_X contains an integral curve, the connectedness condition for θ_X holds by Proposition 5.3 and Remark 5.8 show that $X_2\leq X$ or $X_1\leq X$ and $\theta_X=0$. If $\theta_X=0$ and X is integral, then X lies on an integral surface of degree $s(X_0)$ [26, 3.1], but this is only possible if all the n_i are equal by Lemma 2.3 and Proposition 6.3, in which case $X_1=X_2$, so we conclude that $X_2\leq X$. In case $n_1=n_2, n_3=n_4$, the curve X_2 satisfies $h^1(\mathscr{I}_{X_2})=1$, so X_2 cannot be integral and since $\theta_X\neq 0$ by the argument above, condition (b) implies $\theta_X(2a+h_X)>0$ (or in case the n_i are equal, $\eta(2a+h_X)>0$: in either case $C_1\leq X$) or $\theta_X(2a+h_X-1)>0$ (so that $C_2\leq X$).

To prove the converse, it suffices in view of Proposition 5.3 to show that X_2 may be taken integral if $n_1\neq n_2$ or $n_3\neq n_4$ and that C_1 and C_2 may be taken integral if $n_1=n_2=a<b=n_3=n_4$.

First suppose $n_1\neq n_2$ or $n_3\neq n_4$. Looking at ideal (17) we see that X_0 is a generic local complete intersection, since away from $V(f_1,f_2)$ it is a complete intersection or empty and at points along $V(f_1,f_2)$ where $fgf_4\neq 0$ the ideal (17) is locally generated by gf_2^2 and $ff_1f_4-gf_2f_3$. The $\mathscr{E}$-type resolution (16) gives a surjection

$$S(-n_1-n_3-h_0)\oplus S(-\mu'-h_0)\oplus S(-n_2-n_4-h_0)\oplus S(-n_3-n_4)\to I_{X_0}\to 0$$

where $\mu'=\inf\{n_1+n_4, n_2+n_3\}$. The degrees of the generators satisfy

$$n_1+n_2+h_0\leq \mu'+h_0\leq n_2+n_4+h_0\leq n_3+n_4+h_0$$

and one of the first two inequalities is strict because $n_1<n_2$ or $n_3<n_4$. From (18) and Proposition 6.1 (b) we have $t(X_0)=n_2+n_4+h_0$ and $s(X_0)+t(X_0)-f=n_3+n_4+h_0$, so X_2 may be taken integral by Proposition 3.3.

Now suppose $n_1 = n_2 = a < b = n_3 = n_4$. Then $s(X_0) = t(X_0) = 2a$ so that X_2 can be taken as a basic double link $X_0 \overset{2a,b-a}{\to} X_2$ and we will show that C_1 and C_2 may be taken integral. Up to deformation there are direct links $C_1 \overset{2a+1,a+b}{\sim} D \overset{2a,a+b}{\sim} X_2 \overset{2a,a+b}{\sim} X_0$ and if we think of these double links taking place on a surface of degree $a+b$, we may take $X_0 \overset{2a+1,a+b}{\sim} C_1$. Since X_0 is a generic local complete intersection curve, $h^0(\mathscr{I}_{X_0}(2a)) \neq 0$ and $\mathscr{I}_{X_0}(a+b)$ is globally generated by (16), we can apply Corollary 3.3 to take C_1 integral.

Similarly $C_2 \overset{2a+1,a+b+1}{\sim} D \overset{2a,a+b+1}{\sim} X_2 \overset{2a,a+b}{\sim} X_0$ up to deformation and $X_0 \overset{2a,1}{\to} D$. Since X_0 is a generic local complete intersection, then so is the typical basic double link D and since $\mathscr{I}_{X_0}(a+b)$ is globally generated, so is $\mathscr{I}_D(a+b+1)$ by resolution (1). Furthermore $h^0(\mathscr{I}_D(2a)) \neq 0$, so by Corollary 3.3 we may take C_2 integral.

□

6.3. *Comparison to smooth connected curves.* Martin-Deschamps and Perrin completely described the deformation classes in $\mathscr{L}$ containing a smooth curve [19, V, 2.6]. Their methods are based on vector bundle constructions, so their solution is stated in terms of free resolutions. We translate Theorem 6.5 into the same language to compare the two results. In particular, there are many examples of integral curves that cannot be smoothed in their even linkage classes. The simplest example was discovered by Hartshorne [13], an integral curve which cannot be smoothed within the Hilbert scheme.

Each curve $C \in \mathscr{L}$ has an $\mathscr{N}$-type resolution of the form

$$0 \to \mathscr{P} \overset{u}{\to} \mathscr{N} \oplus \mathscr{S} \to \mathscr{I}_C(h) \to 0 \qquad (20)$$

with $\mathscr{N}$ is as in (15) so that $h_C = h - h_0$, $\mathscr{P} = \oplus \mathscr{O}(-n)^{p(n)}$ and $\mathscr{S} = \oplus \mathscr{O}(-n)^{s(n)}$. Taking u general, we cancel off redundant summands to assume $p(n)s(n) = 0$ for all n. Set

$$a = \begin{cases} n_2 + n_4 & \text{if } V(f_1, f_2) \text{ is superficial} \\ n_3 + n_4 & \text{otherwise} \end{cases} \qquad (21)$$

and define q_l by $\oplus \mathscr{O}(-n)^{q_l(n)} = \mathscr{O}(\mu) \oplus \mathscr{O}(-a)$. With this preamble, Martin-Deschamps and Perrin describe all classes in $\mathscr{L}$ having a smooth curve C [19, V, 2.6]:

Theorem 6.6. *Let $C \in \mathscr{L}$ with notations above with $c = \sup \mathscr{P}$.*

1. *If C is smooth and connected, then*
   - (a) *$p^\sharp(n) \leq s^\sharp(n) + q_l^\sharp(n)$ for all $n \in \mathbb{Z}$.*
   - (b) *If $n < n_1 + n_2$, then $p^\sharp(n) \leq \sup(s^\sharp(n) - 2, 0)$.*
   - (c) *If $n_1 + n_2 \leq n < n_1 + n_3$, then $p^\sharp(n) \leq \sup(s^\sharp(n) - 1, 0)$ except in the* **paradoxal case**: *if $n_2 < n_3, p^\sharp(n_1+n_2) = s^\sharp(n_1+n_2) = 1, p(n_1+n_2) = 1$ (hence there is $m < n_1 + n_2$ with $s(m) = 1$), $V(f_1, f_2)$ is superficial and $c \geq n_3 + n_4$.*
   - (d) *If $a \leq n < c$, then $p^\sharp(n) \leq s^\sharp(n) + 1$*
2. *Conversely, if* (1) *holds, then there is a sequence* (20) *with C a smooth curve. Furthermore, C is connected unless $n_1 = n_2, n_3 = n_4$, $\mathscr{S} = 0$ and $\mathscr{P} = \mathscr{O}(-\mu)^2$.*

This result is remarkably compactly presented, perhaps difficult to grasp at a glance. We translate Theorem 6.5 to the same notation for comparison. The curve X_2 is obtained from X_0 as a double link $X_0 \overset{t(X_0), t(X_0)-f}{\to} X_2$. Combining (15) and (1) we find an $\mathscr{N}$-type resolution for X_1 of the form

$$0 \to \mathscr{O}(-\mu) \oplus \mathscr{O}(-t + h_0) \to \mathscr{N} \to \mathscr{I}_{X_2}(h_0 + t - f) \to 0 \qquad (22)$$

with $t = t(X_0) = n_2 + n_4 + h_0$ and $f = n_1 + n_2 + h_0$, which we used to cancel the summands $\mathscr{O}(-n_1 - n_2)$ on the left and $\mathscr{O}(-f - h_0)$ in the middle, so we define q by $\oplus\mathscr{O}(-n)^{q(n)} = \mathscr{O}(-\mu) \oplus \mathscr{O}(-n_2 - n_4)$. Similarly, in view of (15) we define p_0 by $\mathscr{P}_0 = \oplus\mathscr{O}(-n)^{p_0(n)} = \mathscr{O}(-n_1 - n_2) \oplus \mathscr{O}(-\mu)$. For a curve $C \in \mathscr{L}$, we compare sequences (15) and (20) to arrive at the key connection

$$\eta_C(n+h) = s^\sharp(n) - p^\sharp(n) + p_0^\sharp(n) \tag{23}$$

where $h_C = h - h_0$ is the height of C. In particular, $\eta_{X_1}(l + h_0 + h_{X_2}) = -q^\sharp(l) + p_0^\sharp(l) = 1$ for $n_1 + n_2 \le l < n_2 + n_4$ and 0 otherwise.

Corollary 6.7. *Let $C \in \mathscr{L}$. Then H_C contains an integral curve if and only if*

(a) *$p^\sharp(n) \le s^\sharp(n) + q^\sharp(n)$ for all $n \in \mathbb{Z}$; if $n_1 = n_2 = a \le b = n_3 = n_4$, then the inequality is strict for $n = 2a$ or $n = 2a - 1$.*
(b) *If $n < n_1 + n_2$, then $p^\sharp(n) \le \sup(s^\sharp(n) - 2, 0)$.*
(c) *If $n_1 + n_2 \le n < n_1 + n_3$, then $p^\sharp(n) \le \sup(s^\sharp(n) - 1, 0)$ except when $p^\sharp(n) = s^\sharp(n) = 1$ and $p(n_1 + n_2) = 1$.*
(d) *If $n_2 + n_4 \le n < c$, then $p^\sharp(n) \le s^\sharp(n) + 1$*

Proof. Condition (a) says that $\eta_C(n + h_C) \ge \eta_{X_1}(n + h_{X_2})$ for all $n \in \mathbb{Z}$, which is equivalent to $X_2 \le C$ [26, 1.8 and 1.11 (a)]. When $n_1 = n_2 = a \le b = n_3 = n_4$ the strict inequality gives $C_1 \le C$ or $C_2 \le C$. Therefore condition (1) of Corollary 6.7 and condition (a) of Theorem 6.5 are equivalent. Here we have $\theta_C(n+h) = \eta_C(n+h) \ge \eta_{X_1}(n + h_{X_1}) = 1$ for

$$s(X_0) + h_C = n_1 + n_3 + h_0 + h_C \le n + h < n_2 + n_4 + h_0 + h_C = t(X_0) + h_C,$$

so Condition (b) of Theorem 6.5, connectedness of θ_C about $[s(X_0) + h_C, t(X_0) + h_C - 1]$, is equivalent to (A) connectedness of θ_C in degrees $< s(X_0) + h_C$ and (B) connectedness of θ_C in degrees $\ge t(X_0) + h_C$. We show these are equivalent to (b)-(d) above.

Condition (B) is equivalent to condition (d), which says that $\theta_C(l) = \eta_C(l)$ is positive until is becomes zero for $l \ge t(X_0) + h_C$. Bearing in mind that $p_0(n) = 0$ for $n < n_1 + n_2$ and $p_0(n) = 1$ for $n_1 + n_2 \le n < n_1 + n_3$, conditions (b) and (c) are equivalent to saying that for $l < n_1 + n_2 + h_0 + h_C$, $\eta_C(l)$ is non-decreasing until it possibly reaches a value ≥ 2, after which it remains ≥ 2: in view of the definition of θ_C, this is equivalent to connectedness of θ_C in degrees $< n_1 + n_3 + h_0 + h_C = s(X_0) + h_C$. The special case in condition (c) handles the scenario where $s^\sharp(n)$ increases to 1 at some point $n < n_1 + n_2$ and $p^\sharp(n)$ increases to 1 at $n = n_1 + n_2$. □

It follows that when the fixed locus is empty, every integral curve is smoothable:

Corollary 6.8. *Assume $n_2 = n_3$ and let $C \in \mathscr{L}$. Then H_C contains a smooth connected curve $\iff$ H_C contains an integral curve.*

Proof. $(1) \Rightarrow (2)$ holds because every smooth connected curve is integral. Conversely, if H_C contains an integral curve, then condition (c) in Theorem 6.6 holds vacuously and since $n_2 + n_4 \le a$, so do conditions (a), (b) and (d). □

Example 6.9. Taking $n_1 = n_1 = 1 < b = n_3 = n_4$, the minimal curve X_0 is a double line of genus $-b$ as in Remark 6.4. When $b = 2$, all integral curves are smoothable because in comparing Theorem 6.6 and Corollary 6.7 we have $n_2 + n_4 = a, q = q_l$ so conditions (a), (b), (d) line up almost exactly, meanwhile condition turns out to be the same because it need only be checked for $n = n_1 + n_2$, when they read the same. However when $b = 3$, there are integral curves not smoothable in $\mathscr{L}$, such as the curve Z from Example 2.2. Using $\mathscr{N}_0 = \mathscr{N}(2)$, Sequence (5) becomes

$$0 \to \mathscr{O}(-2) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-5) \to \mathscr{N} \oplus \mathscr{O}(-1) \to \mathscr{I}_Z(2) \to 0$$

which lines up with Sequence (20), so we can compute the functions $s^\sharp, p^\sharp, q_l^\sharp$ and $q^\sharp$. By Lemma 6.1 we have $f = 0$ and the fixed locus is a line, which is superficial, so $a = 4$ and $q = q_l$, leading to

n	0	1	2	3	4	5	6
$p^\sharp$	0	0	1	1	2	3	3
$s^\sharp$	0	1	1	1	1	1	1
$p_0^\sharp$	0	0	1	1	2	2	2
$q_l^\sharp$	0	0	0	0	2	2	2
$q^\sharp$	0	0	0	0	2	2	2

Looking at the values of $s^\sharp, p^\sharp, p_0^\sharp$ and formula (23), we see consistency with the calculation of η_Z from Example 2.2. It's easy to check that all conditions of Corollary 6.7 hold, so Z deforms to an integral curve. When we look at Theorem 6.6 we see that conditions (a), (b) and (d) hold, but (c) fails because $5 = c < 6 = n_3 + n_4$, so Z is not smoothable in $\mathscr{L}$. The curve Y from Example 2.2 is smoothable, it's the curve C_2 from the proof of Theorem 6.5, it also appears in [19, V, 2.9]. One way to see that Z is not smoothable is to show that the cubic surface S used for a double link $Y \stackrel{3,1}{\to} Z$ necessarily contains a double line, specifically $L^{(2)} \subset S$, and this forces Z to have some nodes along L [21, 8.2.4]. Hartshorne showed more strongly that Z is not only unsmoothable in $\mathscr{L}$, but in the entire Hilbert scheme [13].

Question 6.10. For an even linkage class $\mathscr{L}$ of curves in $\mathbb{P}^3$ with empty fixed loci F_s, is every integral curve smoothable in $\mathscr{L}$?

Example 6.11. Theorems 6.5 and 6.6 fail when char $k = p > 0$. Take $f_i = x_{i-1}^p$ for $1 \leq i \leq 4$ so that $M = S/(x_0^p, x_1^p, x_2^p, x_3^p)$. Here $\mathscr{P}_0 = \mathscr{O}(-2p)^2$ and the general map $\varphi : \mathscr{P}_0 \to \mathscr{N}$ is given by taking a linear combination of two columns of matrix σ_2. Since $(x+y)^p = x^p + y^p$, the 2×2 minors of the resulting 4×2 matrix have the form $L_1^p L_2^p - L_3^p L_4^p$ for linear forms L_i and the total ideal I_{X_0} is generated by such. The partial derivative criterion shows that these surfaces are generically non-reduced, hence not integral. In particular, the curve $C_0 = X_0$ is non-reduced, while Theorem 6.6 says it is smooth and disconnected when char $k = 0$. Furthermore, the double link $X_0 \stackrel{2p,1}{\to} C_1$ lies on only non-reduced surfaces of minimal degree, hence C_1 is not integral and Theorem 6.5 fails.

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TEXAS CHRISTIAN UNIVERSITY, DEPARTMENT OF MATHEMATICS, FORT WORTH, TX 76129
Email address: s.nollet@tcu.edu