OBSTRUCTION TO NAÏVE LIFTABILITY OF DG MODULES

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Abstract. The notion of naïve liftability of DG modules is introduced in [10] and [11]. The main purpose of this paper is to explicitly describe the obstruction to naïve liftability along extensions $A \to B$ of DG algebras, where B is projective as an underlying graded A-module. In particular, we give an explicit description of a DG B-module homomorphism which defines the obstruction to naïve liftability of a semifree DG B-module N as a certain cohomology class in $\mathrm{Ext}^1_B(N, N \otimes_B J)$, where J is the diagonal ideal. Our results on the obstruction class enable us to give concrete examples of DG modules that do and do not satisfy the naïve lifting property.

1. Introduction

Throughout the paper, R is a commutative ring.

Lifting and weak lifting properties have been studied in commutative ring theory by Auslander, Ding, and Solberg [1] for modules and by Yoshino [17] for complexes over certain ring extensions. These notions are tightly connected to the deformation theory of modules and have been applied in the theory of maximal Cohen-Macaulay approximations. The existing lifting and weak lifting results for modules and complexes have been generalized recently in several papers [10, 11, 13, 14, 16] to the case of differential graded (DG) modules over certain DG algebra extensions.

There exist three versions of liftability for DG modules. Let $\varphi\colon A \to B$ be a DG R-algebra homomorphism, and let N be a semifree DG B-module. We say that N is *liftable* to A if there is a semifree DG A-module M such that $N \cong M \otimes_A B$. The DG B-module N is *weakly liftable* to A (in the sense of [10]) if there are non-negative integers $a_1, \ldots, a_r$ such that the finite direct sum $N \oplus N(-a_1) \oplus \cdots \oplus N(-a_r)$ is liftable to A. Finally, N is *naïvely liftable* to A if the map $\pi_N\colon N|_A \otimes_A B \to N$ defined by $\pi_N(n \otimes b) = nb$ is a split DG B-module epimorphism. Here, $N|_A$ denotes the DG B-module N considered as a DG A-module via φ. If N is naïvely liftable to A, then it is a direct summand of the liftable DG B-module $N|_A \otimes_A B$; see Remark 3.4 for more details. In the special case where A and B are commutative rings and N is a B-module, definition of lifting (resp. weak lifting) translates to the existence of an A-module M such that $N \cong M \otimes_A B$ (resp. N is a direct summand of $M \otimes_A B$) and $\mathrm{Tor}_i^A(M, B) = 0$ for all $i > 0$. Note that by [1, p. 296] and [10, Remark 6.6] the notions of liftability that we defined above are different in general.

Auslander, Ding, and Solberg [1, Proposition 1.6] proved a sufficient condition for liftability of modules over certain ring extensions. More precisely, they showed that if R is a complete local ring and $x \in R$ is a non-zero divisor, then a finitely

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generated R/xR-module N is liftable to R if $\mathrm{Ext}^2_{R/xR}(N,N) = 0$. The converse of this result does not hold in general; see [1, p. 282]. Unlike the lifting property, detecting weak liftability does not require the vanishing of the entire Ext^2. More precisely, Auslander, Ding, and Solberg proved the following result.

Theorem 1.1 ([1, Proposition 1.5])**.** *Let R be a local ring and $x \in R$ be a non-zero divisor. A finitely generated R/xR-module N is weakly liftable to R if and only if a certain cohomology class $[\Delta_N]$ in $\mathrm{Ext}^2_{R/xR}(N,N)$ vanishes.*

We call $[\Delta_N]$ the *obstruction class* to weak liftability. Similar statements for lifting and weak lifting hold when we replace modules by bounded below complexes of finitely generated free modules; see [17]. Further generalizations have been obtained in the DG algebra setting for simple extensions of DG algebras as follows.[1]

Theorem 1.2 ([14, Theorem 3.6] and [16, Theorem 4.7])**.** *Assume $B = A\langle X\rangle$ is a simple free extension of a DG R-algebra A obtained by adjunction of a variable X to kill a cycle x in A. Let N be a semifree DG B-module. Then there exists an obstruction class $[\Delta_N]$ in $\mathrm{Ext}^{|X|+1}_B(N,N)$ such that*

(a) *if $|X|$ is odd, then $[\Delta_N] = 0$ if and only if N is weakly liftable to A and*
(b) *if $|X|$ is even and N is bounded below (i.e., $N_i = 0$ for all $i \ll 0$), then $[\Delta_N] = 0$ if and only if N is liftable to A.*

Note that if $A = R$ is a local ring, x is a non-zero divisor, and $|X| = 1$, then Theorem 1.2(a) recovers Theorem 1.1 because in this case B is the Koszul complex over the element x which is quasiisomorphic, as a DG R-algebra, to R/xR.

In general, we are interested in the lifting properties of DG modules along free extensions of DG algebras that are obtained by adjoining more than one variable to another DG algebra. Our study of lifting properties along such extensions is motivated by some of the major problems in commutative algebra including the Auslander-Reiten Conjecture. This conjecture states that a finitely generated module M over a commutative noetherian local ring R is free if $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i \geqslant 1$. As we mention in [11, §7] and [12, 1.2 and A.5], the Auslander-Reiten Conjecture has an equivalent DG version over a DG algebra $Q\langle X_i \mid i \in \mathbb{N}\rangle$ obtained by adjoining countably many variables X_i to a regular local ring Q. Therefore, developing a suitable notion of lifting along the DG algebra extension $Q \to Q\langle X_i \mid i \in \mathbb{N}\rangle$ will provide an affirmative answer to the Auslander-Reiten Conjecture.

As we see in Theorem 1.2, weak liftability along $A \to A\langle X\rangle$ depends on the parity of the degree of X. Hence, it is not a suitable version of liftability when one wishes to generalize it in an inductive process to the case where $B = A\langle X_i \mid i \in \mathbb{N}\rangle$. To avoid keeping track of the parity of the degrees of the variables in each inductive step, the notion of naïve liftability was introduced by the authors in [10, 11]. This notion is independent of the parity of the degrees of variables and at the same time, it detects (weak) liftability along simple extensions of DG algebras. Hence, naïve lifting is a suitable version of lifting in dealing with the Auslander-Reiten Conjecture. Also, Theorem 1.2 is restated as follows.

Theorem 1.3 ([10, Theorem 6.8])**.** *Assume $B = A\langle X\rangle$ is a simple free extension of a DG R-algebra A obtained by adjunction of a variable X to kill a cycle in A.*

[1] The word "simple" means that the DG algebra extension is obtained by adjoining only one variable.

Let N be a bounded below semifree DG B-module. Then there is an obstruction class $[\Delta_N]$ in $\operatorname{Ext}_B^{|X|+1}(N,N)$ such that N is naïvely liftable to A if and only if $[\Delta_N]=0$.

Moreover, the obstruction classes $[\Delta_N]$ in Theorems 1.2 and 1.3 are the same by [10, Theorem 6.8]. Later, the authors proved the following *existence* result (Theorem 1.4) along more general extensions of DG algebras using the notion of diagonal ideal that they extensively studied in the DG setting in [11]. Note that the notion of diagonal ideal is a classic notion that originates in scheme theory and has a long history. In fact, if $A\to B$ is a homomorphism of commutative rings, then the kernel of the map $B\otimes_A B\to B$ defined by $b\otimes_A b'\mapsto bb'$ is the defining ideal of the diagonal set in the Cartesian product $\operatorname{Spec} B\times_{\operatorname{Spec} A}\operatorname{Spec} B$.

Theorem 1.4 ([11, Proposition 5.3 and Theorem 5.8])**.** *Assume $A\to B$ is an extension of DG R-algebras, where B is projective as an underlying graded A-module (e.g., B is a free extension of the DG R-algebra A obtained by adjoining countably many variables to A). For a semifree DG B-module N, there is an obstruction class $[\Delta_N]$ in $\operatorname{Ext}_B^1(N,N\otimes_B J)$ such that N is naïvely liftable to A if and only if $[\Delta_N]=0$. (Here, J is the diagonal ideal; see Definition 3.2.)*

The obstruction classes $[\Delta_N]$ discussed in Theorems 1.1, 1.2, and 1.3 have been *explicitly described* in [1], [14], and [16]. More precisely, in Theorems 1.2 and 1.3, we show that $[\Delta_N]=[j_X(\partial^N)]$, where j_X denotes the j-operator with respect to the variable X and ∂^N is the differential of the DG module N. Our goal in this paper is to *explicitly describe* the obstruction to naïve liftability discussed in Theorem 1.4 as well. Our main result in this paper is the following; see Theorems 3.8 and 3.10.

Theorem 1.5. *Let $A\to B$ be an extension of DG R-algebras, where B is projective as an underlying graded A-module, and let N be a semifree DG B-module with a semifree basis $\mathcal{B}=\{e_\lambda\}_{\lambda\in\Lambda}$ with a chosen well-ordering (that respects homological degree). For $e_\lambda\in\mathcal{B}$, let $\partial^N(e_\lambda)=\sum_{\mu<\lambda}e_\mu b_{\mu\lambda}$ as a finite sum with $b_{\mu\lambda}\in B^\natural$. Then, a right DG B-module homomorphism $\Delta_N\colon N\to N\otimes_B J$ of degree -1 which defines the obstruction class $[\Delta_N]\in\operatorname{Ext}_B^1(N,N\otimes_B J)$ of naïve liftability is explicitly given by the formula*

$$\Delta_N(e_\lambda)=\sum_{\mu<\lambda}e_\mu\otimes\delta(b_{\mu\lambda})$$

where δ is the universal derivation; see Definition 3.6.

The organization of this paper is as follows. Section 2 is devoted to the terminology and basic definitions which are used in subsequent sections. In Section 3 we give an explicit description of the obstruction class whose vanishing detects naïve liftability. The proof of Theorem 1.5 is given in this section. In Section 4 we provide another description of the obstruction class that is equivalent to the one from Section 3. This description is based on the notion of "connections" that was originally defined by Connes [6, II. §2] in non-commutative differential geometry. Finally, our main result in Section 5 is Theorem 5.1 which enables us to construct concrete examples of DG modules that do and do not satisfy naïve liftability.

2. Terminology and preliminaries

The main objects considered in this paper are DG algebras and DG modules; references on these subjects include [2, 3, 8, 9]. In this section, we fix our notation and specify some terminology that will be used throughout the paper. For the unspecified notation, we refer the reader to [11].

2.1. A *strongly commutative differential graded R-algebra* (or simply *DG R-algebra*) $A = (A^\natural, d^A)$ is a non-negatively graded R-algebra $A^\natural = \bigoplus_{n\geqslant 0} A_n$ such that

(a) $d^A\colon A^\natural \to A^\natural$ is a graded R-linear map of degree -1 with $(d^A)^2 = 0$, that is, $A = (A^\natural, d^A)$ is an R-complex;
(b) for all homogeneous elements $a, b \in A^\natural$ we have $ab = (-1)^{|a||b|}ba$, and $a^2 = 0$ if the degree of a (denoted $|a|$) is odd; and
(c) d^A satisfies the *Leibniz rule*, that is, for all homogeneous elements $a, b \in A^\natural$ we have $d^A(ab) = d^A(a)b + (-1)^{|a|}ad^A(b)$.

2.2. A (right) *DG A-module* $M = (M^\natural, \partial^M)$ is a graded (right) $A^\natural$-module $M^\natural = \bigoplus_{i\in\mathbb{Z}} M_i$ which is an R-complex with a differential ∂^M that satisfies the Leibniz rule, i.e., for all homogeneous elements $a \in A^\natural$ and $m \in M^\natural$ we have $\partial^M(ma) = \partial^M(m)\, a+(-1)^{|m|}m\, d^A(a)$. A *DG submodule* of a DG A-module M is a subcomplex that is a DG A-module under the operations induced by M. A *DG ideal* of A is a right and left DG submodule of A.

The DG modules considered in this paper are right DG modules, unless otherwise stated. Note that a DG A-module M is also a left DG A-module with the left A-action $am = (-1)^{|m||a|}ma$, for all homogeneous elements $a \in A^\natural$ and $m \in M^\natural$.

A *DG A-module homomorphism* between DG A-modules M, N is a graded $A^\natural$-linear map $f\colon M^\natural \to N^\natural$ of degree 0 that *commutes with differentials*, i.e., $\partial^N f = f\partial^M$. We also set

$$^*\mathrm{Hom}_{A^\natural}(M^\natural, N^\natural) = \bigoplus_{n\in\mathbb{Z}} \text{gr-Hom}_{A^\natural}(M^\natural, N^\natural(n))$$

where $\text{gr-Hom}_{A^\natural}(M^\natural, N^\natural(n))$ is the set of graded $A^\natural$-module homomorphisms from $M^\natural$ to $N^\natural$ of degree n. We denote $\text{gr-Hom}_{A^\natural}(M^\natural, N^\natural(0))$ by $\text{gr-Hom}_{A^\natural}(M^\natural, N^\natural)$. Note that $^*\mathrm{Hom}_{A^\natural}(M^\natural, N^\natural)$ is a graded $A^\natural$-module. Defining the differential ∂^* by

$$(\partial^* f)(x) = \partial^N(f(x)) - (-1)^{|f|}f(\partial^M(x))$$

for $f \in {}^*\mathrm{Hom}_{A^\natural}(M^\natural, N^\natural)$ and $x \in M^\natural$, we see that $\left({}^*\mathrm{Hom}_{A^\natural}(M^\natural, N^\natural), \partial^*\right)$ has a DG A-module structure, which we denote by $\mathrm{Hom}_A(M, N)$.

For a DG A-module M and an integer i, the *i-th shift* of M, denoted $\Sigma^i M$, is defined by $\left(\Sigma^i M\right)_j = M_{j-i}$ with $\partial_j^{\Sigma^i M} = (-1)^i \partial_{j-i}^M$. We set $\Sigma M = \Sigma^1 M$.

A DG A-module M is *semifree* if it has a *semifree basis* $\{e_\lambda\}_{\lambda\in\Lambda}$, that is, a subset $\{e_\lambda\}_{\lambda\in\Lambda} \subseteq M^\natural$ which is a graded $A^\natural$-free basis of $M^\natural$ indexed by a well-orderd set $(\Lambda, <)$ such that $\partial^M(e_\lambda) \in \sum_{\mu<\lambda} e_\mu A^\natural$ for all e_λ. See [3, 8.2] for more details.

For an integer i and DG A-modules M, N, where M is semifree, $\mathrm{Ext}_A^i(M, N)$ is defined to be $\mathrm{H}_{-i}\left(\mathrm{Hom}_A(M, N)\right)$.

2.3. Let $\mathcal{C}(A)$ denote the abelian category of DG A-modules and DG A-module homomorphisms. Let $\mathcal{K}(A)$ be the homotopy category of DG A-modules; objects of $\mathcal{K}(A)$ are DG A-modules and morphisms are the set of homotopy equivalence classes of DG A-module homomorphisms, i.e., $\mathrm{Hom}_{\mathcal{K}(A)}(M, N) = \mathrm{Hom}_A(M, N)/\sim$, where $f \sim g$ for $f, g \in \mathrm{Hom}_A(M, N)$ if and only if there is a graded $A^\natural$-module homomorphism $h\colon M^\natural \to N^\natural$ of degree -1 such that $f - g = \partial^N h + h\partial^M$. The category $\mathcal{K}(A)$ is a triangulated category and $\mathrm{Ext}_A^i(M, N) = \mathrm{Hom}_{\mathcal{K}(A)}(M, \Sigma^i N)$.

2.4. Let $\varphi\colon A \to B$ be a DG R-algebra homomorphism, that is, φ is a graded R-algebra homomorphism of degree 0 with $d^B\varphi = \varphi d^A$. Assume that $B^\natural$ is projective as a graded $A^\natural$-module. Let B^o be the opposite DG R-algebra and $B^e = B^o \otimes_A B$ be the enveloping DG R-algebra of B over A. The algebra structure on B^e is given by

$$(b_1^o \otimes b_2)({b'}_1^o \otimes {b'}_2) = (-1)^{|{b'}_1|(|b_1|+|b_2|)}({b'}_1 b_1)^o \otimes b_2 {b'}_2 \tag{2.4.1}$$

and its differential structure is described as

$$d^{B^e}(b_1^o \otimes b_2) = d^{B^o}(b_1^o) \otimes b_2 + (-1)^{|b_1|} b_1^o \otimes d^B(b_2) \tag{2.4.2}$$

for all homogeneous elements $b_1, b_2, b_1', b_2' \in B^\natural$.

Note that DG B^e-modules are precisely DG (B,B)-bimodules. In fact, for a DG B^e-module N, the right action of an element of $(B^e)^\natural$ on $N^\natural$ yields the two-sided $B^\natural$-module structure

$$nb = n(1^o \otimes b) \qquad \text{and} \qquad bn = (-1)^{|b||n|} n(b^o \otimes 1) \tag{2.4.3}$$

for all homogeneous elements $n \in N^\natural$ and $b \in B^\natural$. In particular, if $N = B^e$ and $n = b_1^o \otimes b_2$ for $b_1, b_2 \in B^\natural$, then by (2.4.1) we have

$$(b_1^o \otimes b_2)b = b_1^o \otimes b_2 b \qquad \text{and} \qquad b(b_1^o \otimes b_2) = (bb_1)^o \otimes b_2. \tag{2.4.4}$$

2.5. Assume that A, B are DG R-algebras such that $B^\natural$ is projective as a graded $A^\natural$-module. For a DG B^e-module N, a graded $A^\natural$-linear map $D\colon B^\natural \to N^\natural$ is called an *A-derivation of N* if the following conditions are satisfied for all homogeneous elements $b_1, b_2 \in B^\natural$:

$$D(b_1 b_2) = D(b_1)b_2 + (-1)^{|D||b_1|} b_1 D(b_2) \tag{2.5.1}$$

$$D(a) = 0 \text{ for all } a \in A.$$

Note that by (2.4.3), equality (2.5.1) is equivalent to

$$D(b_1 b_2) = D(b_1)\,(1^o \otimes b_2) + (-1)^{|b_1||b_2|} D(b_2)\,(b_1^o \otimes 1)\,. \tag{2.5.2}$$

We denote the set of A-derivations of the DG B^e-module N by $\operatorname{Der}_A(B,N)$.

3. Explicit description of the obstruction class

Our main results in this section are Theorems 3.8 and 3.10 in which we explicitly describe the obstruction class to naïve liftability of DG modules along DG algebra extensions. The notation used in this section comes from Section 2. We also make the following convention for the rest of the paper.

Convention 3.1. Throughout the paper, A, B are DG R-algebras such that $B^\natural$ is projective as a graded $A^\natural$-module and $\varphi\colon A \to B$ is a *DG R-algebra homomorphism.*

The description of the obstruction class to naïve liftability of DG modules is based on the classic notion of diagonal ideal that we define next.

Definition 3.2. Let $\pi\colon B^e \to B$ denote the map defined by $\pi(b_1^o \otimes b_2) = b_1 b_2$, for all $b_1, b_2 \in B^\natural$. Note that π is a homomorphism of DG R-algebras; see [11, 3.1]. Also, $J := \operatorname{Ker}\pi$ is a DG ideal of B^e which is called the *diagonal ideal* of φ. Moreover, the isomorphism $B^e/J \cong B$ of DG R-algebras is also an isomorphism of DG B^e-modules. Hence, there is an exact sequence

$$0 \to J \xrightarrow{\iota} B^e \xrightarrow{\pi} B \to 0 \tag{3.2.1}$$

of DG B^e-modules in which ι is the natural inclusion.

Proposition 3.3. *The short exact sequence* (3.2.1) *is a splitting sequence of DG B-modules. More precisely, the maps $\rho\colon B \to B^e$ and $\sigma\colon B^e \to J$ defined by*

$$\rho(b) = 1^o \otimes b \qquad \text{and} \qquad \sigma(b_1^o \otimes b_2) = b_1^o \otimes b_2 - 1^o \otimes b_1 b_2$$

for all $b, b_1, b_2 \in B^\natural$, are DG B-module homomorphisms that satisfy the equalities

$$\pi\rho = \mathrm{id}_B, \qquad \sigma\iota = \mathrm{id}_J, \qquad \iota\sigma + \rho\pi = \mathrm{id}_{B^e}. \tag{3.3.1}$$

Proof. For all $b_1, b_2 \in B^\natural$ we have $\pi\sigma(b_1^o \otimes b_2) = \pi\left(b_1^o \otimes b_2 - 1^o \otimes b_1 b_2\right) = b_1b_2 - b_1b_2 = 0$. Hence, $\mathrm{Im}(\sigma) \subseteq J = \mathrm{Ker}\,\pi$, that is, the map σ is well-defined.

Note that the short exact sequence (3.2.1) is a splitting sequence of graded $B^\natural$-modules and the $B^\natural$-module homomorphism ρ satisfies the equality $\pi\rho = \mathrm{id}_B$. Moreover, it follows from (2.4.2) that ρ commutes with differentials. Hence, ρ is a DG B-module homomorphism.

It is straightforward to check the equality $\iota\sigma + \rho\pi = \mathrm{id}_{B^e}$. To prove the equality $\sigma\iota = \mathrm{id}_J$, note that the elements of $J^\natural$ are finite sums of the form $\sum_{i=1}^n b_i^o \otimes b_i'$ with homogeneous elements $b_i, b_i' \in B^\natural$ such that $\sum_{i=1}^n b_i b_i' = \sum_{i=1}^n \pi(b_i^o \otimes b_i') = \pi\left(\sum_{i=1}^n b_i^o \otimes b_i'\right) = 0$. For such an element in $J^\natural$ we have

$$\sigma\iota\left(\sum_{i=1}^n b_i^o \otimes b_i'\right) = \sum_{i=1}^n (b_i^o \otimes b_i' - 1^o \otimes b_i b_i') = \sum_{i=1}^n b_i^o \otimes b_i'.$$

Also, σ commutes with the differentials. In fact, for all homogeneous elements $b_1, b_2 \in B^\natural$ we have the equalities

$$\begin{aligned}
\partial^J\sigma(b_1^o \otimes b_2) &= \partial^J(b_1^o \otimes b_2 - 1^o \otimes b_1b_2) \\
&= d^{B^o}(b_1^o) \otimes b_2 + (-1)^{|b_1|} b_1^o \otimes d^B(b_2) - 1^o \otimes d^B(b_1b_2) \\
&= d^{B^o}(b_1^o) \otimes b_2 - 1^o \otimes d^B(b_1)b_2 + (-1)^{|b_1|}\left(b_1^o \otimes d^B(b_2) - 1^o \otimes b_1 d^B(b_2)\right) \\
&= \sigma(d^{B^o}(b_1^o) \otimes b_2 + (-1)^{|b_1|} b_1^o \otimes d^B(b_2)) \\
&= \sigma d^{B^e}(b_1^o \otimes b_2).
\end{aligned}$$

This computation implies that $\sigma\colon B^e \to J$ is a DG B-module homomorphism. □

Remark 3.4. Let N be a semifree DG B-module. Applying the functor $N \otimes_B -$ to the short exact sequence (3.2.1) and using the natural isomorphism $\nu\colon N \otimes_B B \to N$ of DG B-modules defined by $\nu(n \otimes b) = nb$ for all $n \in N^\natural$ and $b \in B^\natural$, we obtain the short exact sequence at the top row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes_B J & \xrightarrow{\iota_N} & N \otimes_B B^e & \xrightarrow{\nu(\mathrm{id}_N \otimes \pi)} & N & \longrightarrow & 0 \\
 & & & & {\cong}\big\downarrow{\ell} & & \big\| & & \\
 & & & & N|_A \otimes_A B & \xrightarrow{\pi_N} & N & &
\end{array} \tag{3.4.1}$$

in which $\iota_N = \mathrm{id}_N \otimes \iota$, the DG B-module N regarded as a DG A-module via the DG R-algebra homomorphism φ is denoted by $N|_A$, the map π_N is the DG B-module epimorphism defined by $\pi_N(n \otimes b) = nb$, and the isomorphism ℓ is the composition of the isomorphisms $N \otimes_B B^e = N \otimes_B (B^o \otimes_A B) \cong (N \otimes_B B^o) \otimes_A B \cong N|_A \otimes_A B$ which is described by the formula $\ell(n \otimes (b_1^o \otimes b_2)) = nb_1 \otimes b_2$ for all $n \in N^\natural$ and $b_1, b_2 \in B^\natural$. Using the commutative diagram (3.4.1), we identify the map $\nu(\mathrm{id}_N \otimes \pi)$

with π_N, i.e., we assume $\pi_N = \nu(\mathrm{id}_N \otimes \pi)$ for the rest of the paper and hence, the top row of (3.4.1) is the short exact sequence

$$0 \to N \otimes_B J \xrightarrow{\iota_N} N \otimes_B B^e \xrightarrow{\pi_N} N \to 0 \tag{3.4.2}$$

of DG B-modules; see [11, Proposition 5.3] for more details. Note that N is naïvely liftable to A if and only if (3.4.2) splits. In particular, if N is naïvely liftable to A, then it is a direct summand of the DG B-module $N|_A \otimes_A B$, which is liftable to A.

Remark 3.5. Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}$. Then (3.4.2) is a splitting sequence of graded $B^\natural$-modules. More precisely, the maps $\rho_N \colon N^\natural \to (N \otimes_B B^e)^\natural$ and $\sigma_N \colon (N \otimes_B B^e)^\natural \to (N \otimes_B J)^\natural$ defined by

$$\rho_N(e_\lambda b) = e_\lambda \otimes \rho(b) \qquad \text{and} \qquad \sigma_N(e_\lambda \otimes (b_1^o \otimes b_2)) = e_\lambda \otimes \sigma(b_1^o \otimes b_2)$$

for all $e_\lambda \in \mathcal{B}$ and all homogeneous elements $b, b_1, b_2 \in B^\natural$, are right $B^\natural$-linear homomorphisms for which the following equalities hold:

$$\pi_N \rho_N = \mathrm{id}_N, \qquad \sigma_N \iota_N = \mathrm{id}_{N \otimes_B J}, \qquad \iota_N \sigma_N + \rho_N \pi_N = \mathrm{id}_{N \otimes_B B^e}. \tag{3.5.1}$$

Note that ρ_N and σ_N are not DG B-module homomorphisms in general because they are not necessarily chain maps.

Next definition plays an essential role in the rest of the paper.

Definition 3.6. Let $\delta \colon B^\natural \to J^\natural$ be the $A^\natural$-linear map defined by

$$\delta(b) = b^o \otimes 1 - 1^o \otimes b$$

for all $b \in B^\natural$. Note that $|\delta| = 0$ and $\delta(a) = 0$ for all $a \in A^\natural$. Also, it follows from (2.4.1) that δ satisfies (2.5.2). Hence, $\delta \in \mathrm{Der}_A(B, J)$. The map δ is called the *universal derivation*. Since ∂^J is induced from d^{B^e}, the universal derivation δ commutes with the differentials, i.e.,

$$\delta d^B = \partial^J \delta. \tag{3.6.1}$$

Remark 3.7. Let N be a semifree DG B-module, and let $\Delta_N \colon N^\natural \to (N \otimes_B J)^\natural$ be the right $B^\natural$-linear graded homomorphism of degree -1 defined by

$$\Delta_N := \sigma_N \partial^{N \otimes_B B^e} \rho_N.$$

It follows from chasing the diagram

$$\begin{array}{ccccc}
 & & N \otimes_B B^e & \underset{\rho_N}{\overset{\pi_N}{\rightleftarrows}} & N \\
 & & \big\downarrow {\scriptstyle \partial^{N \otimes_B B^e}} & & \big\downarrow {\scriptstyle \partial^N} \\
N \otimes_B J & \underset{\sigma_N}{\overset{\iota_N}{\rightleftarrows}} & N \otimes_B B^e & \underset{\rho_N}{\overset{\pi_N}{\rightleftarrows}} & N \\
\big\downarrow {\scriptstyle \partial^{N \otimes_B J}} & & \big\downarrow {\scriptstyle \partial^{N \otimes_B B^e}} & & \\
N \otimes_B J & \underset{\sigma_N}{\overset{\iota_N}{\rightleftarrows}} & N \otimes_B B^e & &
\end{array}$$

that $\iota_N(\partial^{N \otimes_B J} \Delta_N + \Delta_N \partial^N) = 0$. Therefore, Δ_N is a DG B-module homomorphism, since ι_N is injective. In particular, Δ_N defines a cohomology class in $\mathrm{Ext}_B^1(N, N \otimes_B J)$ which is denoted by $[\Delta_N]$.

The next theorem is one of the main results in this section that describes the structure of the obstruction class to naïve liftability of DG modules along the DG algebra extensions satisfying Convention 3.1. This result is a major part of Theorem 1.5 from the introduction.

Theorem 3.8. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. For $e_\lambda \in \mathcal{B}$, if we assume $\partial^N(e_\lambda) = \sum_{\mu<\lambda} e_\mu b_{\mu\lambda}$ as a finite sum with $b_{\mu\lambda} \in B^\natural$, then the DG B-module homomorphism $\Delta_N\colon N \to N \otimes_B J$ is explicitly described by the formula*

$$\Delta_N(e_\lambda) = \sum_{\mu<\lambda} e_\mu \otimes \delta(b_{\mu\lambda}). \tag{3.8.1}$$

Proof. For $e_\lambda \in \mathcal{B}$ we have the following equalities:

$$\begin{aligned}\Delta_N(e_\lambda) &= \sigma_N \partial^{N\otimes_B B^e}\rho_N(e_\lambda) = \sigma_N \partial^{N\otimes_B B^e}(e_\lambda \otimes (1^o \otimes 1)) \\ &= \sigma_N\left(\sum_{\mu<\lambda} e_\mu \otimes (b^o_{\lambda\mu} \otimes 1)\right) = \sum_{\mu<\lambda} e_\mu \otimes \delta(b_{\lambda\mu})\end{aligned}$$

as desired. □

Remark 3.9. Consider the notation from Theorem 3.8. Since (3.2.1) (or (3.4.2)) is a split short exact sequence of graded $B^\natural$-modules, we have the isomorphism

$$N^\natural \otimes_{B^\natural} (B^e)^\natural \cong (N^\natural \otimes_{B^\natural} J^\natural) \oplus N^\natural.$$

By Remark 3.5, the sequence (3.4.2) is equivalent to a short exact sequence

$$0 \to N \otimes_B J \xrightarrow{\begin{pmatrix}\mathrm{id}_{N\otimes_B J}\\ 0\end{pmatrix}} (N \otimes_B J) \oplus N \xrightarrow{(0 \quad \mathrm{id}_N)} N \to 0 \tag{3.9.1}$$

of DG B-modules in which the differential on $(N \otimes_B J) \oplus N$ is of the form

$$\partial := \begin{pmatrix}\partial^{N\otimes_B J} & f_N \\ 0 & \partial^N\end{pmatrix} \tag{3.9.2}$$

where $f_N\colon N \to \Sigma(N \otimes_B J)$ is a chain map, thus a cycle in $\mathrm{Hom}_B(N, \Sigma(N \otimes_B J))$. Hence, f_N represents a cohomology class $[f_N]$ in $\mathrm{Ext}^1_B(N, N \otimes_B J)$. Note also that

$$f_N = \begin{pmatrix}\mathrm{id}_{N\otimes_B J} & 0\end{pmatrix} \partial \begin{pmatrix}0 \\ \mathrm{id}_N\end{pmatrix}.$$

Therefore, $[\Delta_N] = [f_N]$ in $\mathrm{Ext}^1_B(N, N \otimes_B J)$.

The following along with Theorem 3.8 completes the proof of Theorem 1.5.

Theorem 3.10. *Consider the notation from Theorem 3.8 and Remark 3.9. The following conditions are equivalent:*

(i) $[f_N] = [\Delta_N] = 0$ *in* $\mathrm{Ext}^1_B(N, N \otimes_B J)$*;*
(ii) *N is naïvely liftable to A.*

The proof of this theorem is straightforward and perhaps can be found in the literature. However, we give the proof for the convenience of the reader.

Proof. Let $\widetilde{\partial} := \begin{pmatrix} \partial^{N\otimes_B J} & 0 \\ 0 & \partial^N \end{pmatrix}$ and recall from (3.9.2) that $\partial = \begin{pmatrix} \partial^{N\otimes_B J} & f_N \\ 0 & \partial^N \end{pmatrix}$. When necessary, we use the notations $\big((N \otimes_B J)^\natural \oplus N^\natural, \partial\big)$ and $\Big((N \otimes_B J)^\natural \oplus N^\natural, \widetilde{\partial}\Big)$ to specify the DG B-module structure on $(N \otimes_B J)^\natural \oplus N^\natural$.

Note that N is naïvely liftable to A if and only if there is an isomorphism

$$\Phi\colon ((N \otimes_B J)^\natural \oplus N^\natural, \partial) \to ((N \otimes_B J)^\natural \oplus N^\natural, \widetilde{\partial})$$

of DG B-modules such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes_B J & \xrightarrow{\binom{\mathrm{id}_{N\otimes_B J}}{0}} & \big((N \otimes_B J)^\natural \oplus N^\natural, \partial\big) & \xrightarrow{(0 \;\; \mathrm{id}_N)} & N & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \Phi} & & \Big\| & & \\
0 & \longrightarrow & N \otimes_B J & \xrightarrow{\binom{\mathrm{id}_{N\otimes_B J}}{0}} & \Big((N \otimes_B J)^\natural \oplus N^\natural, \widetilde{\partial}\Big) & \xrightarrow{(0 \;\; \mathrm{id}_N)} & N & \longrightarrow & 0
\end{array}$$

commutes. Commutativity of this diagram implies that Φ is of the form

$$\Phi = \begin{pmatrix} \mathrm{id}_{N\otimes_B J} & q \\ 0 & \mathrm{id}_N \end{pmatrix}$$

where $q\colon N^\natural \to (N \otimes_B J)^\natural$ is a graded homomorphism of degree 0.

(i) $\Longrightarrow$ (ii) Since $[f_N] = 0$ in $\mathrm{Ext}^1_B(N, N \otimes_B J)$, there exists a graded $B^\natural$-module homomorphism $g\colon N^\natural \to (N \otimes_B J)^\natural$ of degree 0 such that $f_N = \partial^{N\otimes_B J} g - g\partial^N$. Setting $q := g$ in Φ, we see that N is naïvely liftable to A.

(ii) $\Longrightarrow$ (i) If N is naïvely liftable to A, then an isomorphism

$$\Phi = \begin{pmatrix} \mathrm{id}_{N\otimes_B J} & q \\ 0 & \mathrm{id}_N \end{pmatrix} : ((N \otimes_B J)^\natural \oplus N^\natural, \partial) \to ((N \otimes_B J)^\natural \oplus N^\natural, \widetilde{\partial})$$

of DG B-modules exists. Since Φ is a DG B-module homomorphism, it commutes with the differentials and q is a DG B-module homomorphism, that is, $q \in \mathrm{Hom}_B(N, N \otimes_B J)$. Hence, $f_N = \partial^{N\otimes_B J} q - q\partial^N$. This means that $[f_N] = 0$ in $\mathrm{Ext}^1_B(N, N \otimes_B J)$, as desired. □

Definition 3.11. Following Theorems 3.8 and 3.10, for a semifree DG B-module N, we call $[f_N] = [\Delta_N]$ the *obstruction class* to naïve liftability.

4. Another description of the obstruction class

In this section, we provide another description of the obstruction class to naïve liftability that is equivalent to the one constructed in Theorem 3.8; see Theorem 4.9 below, which is our main result in this section. The notation used in this section comes from the previous sections. Recall that we still work in the setting of Convention 3.1. We start with introducing the notion of "connections" that was originally defined by Connes [6, II. §2] in non-commutative differential geometry. See also Cuntz and Quillen [7, §8].

In the following, recall that δ is the universal derivation introduced in Definition 3.6.

Definition 4.1. Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. We define a subset $\mathrm{Diff}^\delta_A(N)$ of $\text{gr-}\mathrm{Hom}_{A^\natural}(N^\natural, (N \otimes_B J)^\natural)$ by

$$\mathrm{Diff}^\delta_A(N) = \{D\colon N^\natural \to (N \otimes_B J)^\natural \mid D(nb) = D(n)b + n \otimes \delta(b) \text{ for } n \in N^\natural, b \in B^\natural\}.$$

Each element of $\mathrm{Diff}^{\delta}_{A}(N)$ is called a *connection on N*.

Let $D^{\mathcal{B}}\colon N^{\natural} \to (N \otimes_B J)^{\natural}$ be the graded $A^{\natural}$-linear homomorphism of degree 0 satisfying $D^{\mathcal{B}}(\sum_\lambda e_\lambda b_\lambda) = \sum_\lambda e_\lambda \otimes \delta(b_\lambda)$, for all $e_\lambda \in \mathcal{B}$ and $b_\lambda \in B$. Noting that $D^{\mathcal{B}}(e_\lambda) = 0$ for all $e_\lambda \in \mathcal{B}$, for all $b \in B^{\natural}$ we have

$$D^{\mathcal{B}}(e_\lambda b) = e_\lambda \otimes \delta(b) = D^{\mathcal{B}}(e_\lambda)b + e_\lambda \otimes \delta(b).$$

Hence, for all $n \in N^{\natural}$ and $b \in B^{\natural}$ we have

$$D^{\mathcal{B}}(nb) = D^{\mathcal{B}}(n)b + n \otimes \delta(b)$$

that is, $D^{\mathcal{B}} \in \mathrm{Diff}^{\delta}_{A}(N)$. Thus, $\mathrm{Diff}^{\delta}_{A}(N)$ is non-empty.

Later, in Theorem 4.9, we will prove, under the setting of Definition 4.1, that the equality $\Delta_N(e_\lambda) = D^{\mathcal{B}}\partial^N(e_\lambda)$ holds, where Δ_N is introduced in Theorem 3.8. For this equality, one should note that $D^{\mathcal{B}}(e_\lambda) = 0$ for all $e_\lambda \in \mathcal{B}$, and hence, $\partial^{N\otimes_B J} D^{\mathcal{B}}(e_\lambda) = 0$.

Lemma 4.2. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. The following assertions hold.*

(1) *For $D, D' \in \mathrm{Diff}^{\delta}_{A}(N)$, we have $D = D'$ if and only if $D(e_\lambda) = D'(e_\lambda)$ for all $e_\lambda \in \mathcal{B}$. In particular, $D^{\mathcal{B}} = D$ if and only if $D(e_\lambda) = 0$ for all $e_\lambda \in \mathcal{B}$.*
(2) *For all $D_1, D_2 \in \mathrm{Diff}^{\delta}_{A}(N)$, the mapping $D_1 - D_2$ is $B^{\natural}$-linear.*
(3) *For all $D \in \mathrm{Diff}^{\delta}_{A}(N)$ and all $f \in \text{gr-Hom}_{B^{\natural}}(N^{\natural}, (N \otimes_B J)^{\natural})$ we have $D + f \in \mathrm{Diff}^{\delta}_{A}(N)$.*
(4) *We have the equality:*

$$\mathrm{Diff}^{\delta}_{A}(N) = D^{\mathcal{B}} + \text{gr-Hom}_{B^{\natural}}(N^{\natural}, (N \otimes_B J)^{\natural}). \tag{4.2.1}$$

Proof. (1) If $D(e_\lambda) = D'(e_\lambda)$ for all $e_\lambda \in \mathcal{B}$, then for every finite sum $\sum_\lambda e_\lambda b_\lambda \in N^{\natural}$ with $b_\lambda \in B^{\natural}$ we have the equalities $D\left(\sum_\lambda e_\lambda b_\lambda\right) = \sum_\lambda \left(D(e_\lambda)b_\lambda + e_\lambda \otimes \delta(b_\lambda)\right) = \sum_\lambda \left(D'(e_\lambda)b_\lambda + e_\lambda \otimes \delta(b_\lambda)\right) = D'\left(\sum_\lambda e_\lambda b_\lambda\right)$. Hence, the equality $D = D'$ holds.

(2) For all $n \in N^{\natural}$ and $b \in B^{\natural}$ we have the equalities $(D_1 - D_2)(nb) = D_1(n)b + n \otimes \delta(b) - D_2(n)b - n \otimes \delta(b) = (D_1 - D_2)(n)b$. Hence, $D_1 - D_2$ is $B^{\natural}$-linear.

(3) For all $n \in N^{\natural}$ and $b \in B^{\natural}$ we have $(D + f)(nb) = D(nb) + f(nb) = D(n)b + n \otimes \delta(b) + f(n)b = (D + f)(n)b + n \otimes \delta(b)$. Therefore, $D + f \in \mathrm{Diff}^{\delta}_{A}(N)$.

(4) follows immediately from statements (2) and (3). □

Notation 4.3. Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. Given a subset $\Gamma = \{\gamma_\lambda\}_{\lambda\in\Lambda}$ of $(N \otimes_B J)^{\natural}$ with $|\gamma_\lambda| = |e_\lambda|$ for all $\lambda \in \Lambda$, we define a graded $A^{\natural}$-linear homomorphism $D_\Gamma\colon N^{\natural} \to (N \otimes_B J)^{\natural}$ of degree 0 by

$$D_\Gamma\left(\sum_\lambda e_\lambda b_\lambda\right) = \sum_\lambda \left(\gamma_\lambda b_\lambda + e_\lambda \otimes \delta(b_\lambda)\right)$$

where $b_\lambda \in B^{\natural}$ and $b_\lambda \neq 0$ for only finitely many λ. Note that D_Γ is well-defined and $D_\Gamma \in \mathrm{Diff}^{\delta}_{A}(N)$ by definition.

Lemma 4.4. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. There is a one-to-one correspondence*

$$\mathrm{Diff}^{\delta}_{A}(N) \underset{g}{\overset{f}{\rightleftarrows}} \prod_{\lambda\in\Lambda} (N \otimes_B J)^{\natural}_{|e_\lambda|}$$

defined by $f(D) = \{D(e_\lambda)\}_{\lambda\in\Lambda}$ and $g(\Gamma) = D_\Gamma$.

Proof. Let $D \in \mathrm{Diff}^{\delta}_{A}(N)$ and $\Gamma = \{\gamma_\lambda\}_{\lambda\in\Lambda} \subset (N \otimes_B J)^\natural$ such that $|\gamma_\lambda| = |e_\lambda|$ for $\lambda \in \Lambda$. We have $(gf)(D)(e_\mu) = D_{\{D(e_\lambda)\}}(e_\mu) = D(e_\mu)$ for all $e_\mu \in \mathcal{B}$. It follows from Lemma 4.2(1) that $(gf)(D) = D$. On the other hand, $(fg)(\Gamma) = \{D_\Gamma(e_\lambda)\}_{\lambda\in\Lambda} = \{\gamma_\lambda\}_{\lambda\in\Lambda} = \Gamma$. □

Notation 4.5. Let N be a semifree DG B-module. For every $D \in \mathrm{Diff}^{\delta}_{A}(N)$, let

$$\alpha(D) := D\partial^N - \partial^{N\otimes_B J} D.$$

Note that $\alpha(D)\colon N^\natural \to (N \otimes_B J)^\natural$ is a graded $A^\natural$-linear map of degree -1.

In the following, recall that $\mathcal{K}(B)$ denotes the homotopy category defined in 2.3.

Lemma 4.6. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$, and let $D \in \mathrm{Diff}^{\delta}_{A}(N)$. Then the map $\alpha(D)\colon N \to \Sigma(N \otimes_B J)$ is a DG B-module homomorphism. Hence, there is a mapping*

$$\mathrm{Diff}^{\delta}_{A}(N) \to \mathrm{Hom}_{\mathcal{K}(B)}(N, \Sigma(N \otimes_B J))$$

defined by $D \mapsto [\alpha(D)]$. Moreover, for all $D_1, D_2 \in \mathrm{Diff}^{\delta}_{A}(N)$, the equality

$$[\alpha(D_1)] = [\alpha(D_2)]$$

holds in $\mathrm{Hom}_{\mathcal{K}(B)}(N, \Sigma(N \otimes_B J))$.

Proof. It is straightforward to see that $\partial^{\Sigma(N\otimes_B J)}\alpha(D) - \alpha(D)\partial^N = 0$. Hence, $\alpha(D)$ is a chain map. On the other hand, for all $n \in N^\natural$ and $b \in B^\natural$ we have the equalities

$$\begin{aligned}
\alpha(D)(nb) =& (D\partial^N - \partial^{N\otimes_B J} D)(nb) \\
=& D(\partial^N(n)b + (-1)^{|n|} n d^B(b)) - \partial^{N\otimes_B J}(D(n)b + n \otimes \delta(b)) \\
=& D(\partial^N(n))b + \partial^N(n) \otimes \delta(b) + (-1)^{|n|} D(n) d^B(b) + (-1)^{|n|} n \otimes \delta(d^B(b)) \\
&- \partial^{N\otimes_B J}(D(n))b - (-1)^{|n|} D(n) d^B(b) - \partial^N(n) \otimes \delta(b) - (-1)^{|n|} n \otimes \partial^J(\delta(b)) \\
=& (D\partial^N - \partial^{N\otimes_B J} D)(n)b \\
=& \alpha(D)(n)b
\end{aligned}$$

where the fourth equality uses (3.6.1). Hence, $\alpha(D)$ is $B^\natural$-linear and we conclude that $\alpha(D) \in \mathrm{Hom}_B(N, \Sigma(N \otimes_B J))$. Note that $\alpha(D)$ defines an element $[\alpha(D)]$ in $\mathrm{Hom}_{\mathcal{K}(B)}(N, \Sigma(N \otimes_B J))$.

For the last assertion, assume D_1, $D_2 \in \mathrm{Diff}^{\delta}_{A}(N)$. Let $f = D_1 - D_2$ and note that f is $B^\natural$-linear by Lemma 4.2(2). Then we have the equalities

$$\alpha(D_1) - \alpha(D_2) = (D_1\partial^N - \partial^{N\otimes_B J} D_1) - (D_2\partial^N - \partial^{N\otimes_B J} D_2) = \partial^{\Sigma N\otimes_B J} f + f\partial^N.$$

Hence, $[\alpha(D_1)] = [\alpha(D_2)]$ in $\mathrm{Hom}_{\mathcal{K}(B)}(N, \Sigma(N \otimes_B J))$, as desired. □

Definition 4.7. Using the notation from Lemma 4.6, it follows that the class $[\alpha(D)]$ in $\mathrm{Hom}_{\mathcal{K}(B)}(N, \Sigma(N \otimes_B J))$ does not depend on the choice of $D \in \mathrm{Diff}^{\delta}_{A}(N)$. We call $[\alpha(D)]$ the Atiyah class of N.

Remark 4.8. There are notions of Atiyah class in various areas; see, for instance, [4, 5, 15]. The above definition of Atiyah class is a DG version of the one for modules over a non-commutative algebra; we refer the reader to [5, 15] for more details on this notion.

Theorem 4.9. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. The following equality holds:*

$$\alpha(D^{\mathcal{B}}) = \Delta_N. \tag{4.9.1}$$

Proof. For a basis element e_λ of $\mathcal{B}$, write $\partial^N(e_\lambda) = \sum_{\mu<\lambda} e_\mu b_{\mu\lambda}$, which is a finite sum with $b_{\mu\lambda} \in B^\natural$. Then we have the equalities

$$\alpha(D^{\mathcal{B}})(e_\lambda) = (D^{\mathcal{B}}\partial^N - \partial^{N\otimes_B J}D^{\mathcal{B}})(e_\lambda) = D^{\mathcal{B}}\left(\partial^N(e_\lambda)\right) = \sum_{\mu<\lambda} e_\mu \otimes \delta(b_{\mu\lambda}) = \Delta_N(e_\lambda)$$

in which the last equality follows from (3.8.1). Now, the equality (4.9.1) follows from the fact that $\alpha(D^{\mathcal{B}})$ and Δ_N are both $B^\natural$-linear. □

Moreover, we can prove the following result.

Proposition 4.10. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. The following statements are equivalent.*

(i) *N is naïvely liftable to A.*
(ii) *$[\alpha(D)] = 0$ holds in $\operatorname{Ext}^1_B(N, N\otimes_B J)$ for all $D \in \operatorname{Diff}^\delta_A(N)$.*
(iii) *$[\alpha(D)] = 0$ holds in $\operatorname{Ext}^1_B(N, N\otimes_B J)$ for some $D \in \operatorname{Diff}^\delta_A(N)$.*
(iv) *$\alpha(D) = 0$ holds in $\operatorname{Hom}_B(N, \Sigma(N\otimes_B J))$ for some $D \in \operatorname{Diff}^\delta_A(N)$.*
(v) *$\alpha(D_\Gamma) = 0$ holds in $\operatorname{Hom}_B(N, \Sigma(N\otimes_B J))$ for some subset $\Gamma = \{\gamma_\lambda\}_{\lambda\in\Lambda}$ of $(N\otimes_B J)^\natural$ with $|\gamma_\lambda| = |e_\lambda|$ for all $\lambda\in\Lambda$.*

Proof. (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (i) follow from Theorem 3.10 and Theorem 4.9.

(iii) $\Longrightarrow$ (ii) follows from Lemma 4.6.

(iv) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (iv) Let $D \in \operatorname{Diff}^\delta_A(N)$ such that $[\alpha(D)] = 0$ in $\operatorname{Ext}^1_B(N, N\otimes_B J)$. There is a graded $B^\natural$-module homomorphism $h\colon N^\natural \to (N\otimes_B J)^\natural$ of degree 0 such that $\alpha(D) = \partial^{\Sigma N\otimes_B J}h + h\partial^N$. Note that $D - h \in \operatorname{Diff}^\delta_A(N)$ by Lemma 4.2(3). By definition we also have $\alpha(D) = D\partial^N - \partial^{N\otimes_B J}D$. Therefore, $\alpha(D-h) = (D-h)\partial^N - \partial^{N\otimes_B J}(D-h) = \alpha(D) - \alpha(D) = 0$.

(iv)$\Longleftrightarrow$(v) is clear from Lemma 4.4. □

5. Some concrete examples

In this section, we will construct concrete examples of DG modules that do and do not satisfy naïve liftability; see Examples 5.6 and 5.7. The main tool to construct such examples is the following theorem, which is our main result in this section. (Again, our notation in this section comes from the previous sections and we still work in the setting of Convention 3.1.)

Theorem 5.1. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$. Write $\partial^N(e_\lambda) = \sum_{\mu<\lambda} e_\mu b_{\mu\lambda}$ as a finite sum with $b_{\mu\lambda} \in B^\natural$. Then the following assertions are equivalent.*

(i) *N is naïvely liftable to A.*
(ii) *There is a subset $\Gamma = \{\gamma_\lambda\}_{\lambda\in\Lambda}$ of $(N\otimes_B J)^\natural$ with $|\gamma_\lambda| = |e_\lambda|$ for all $\lambda\in\Lambda$ such that $\partial^{N\otimes_B J}(\gamma_\lambda) = \sum_{\mu<\lambda}(\gamma_\mu b_{\mu\lambda} + e_\mu\otimes\delta(b_{\mu\lambda}))$.*

Proof. Let $\Gamma = \{\gamma_\lambda\}_{\lambda\in\Lambda}$ be a subset of $(N\otimes_B J)^\natural$ with $|\gamma_\lambda| = |e_\lambda|$ for $\lambda\in\Lambda$. Recall from Notation 4.3 that the $A^\natural$-linear homomorphism $D_\Gamma \in \operatorname{Diff}^\delta_A(N)$ is defined by the formula $D_\Gamma\left(\sum_\lambda e_\lambda b_\lambda\right) = \sum_\lambda \gamma_\lambda b_\lambda + e_\lambda \otimes \delta(b_\lambda)$. Now, for all $\lambda \in \Lambda$ we have

$$\begin{aligned}\alpha(D_\Gamma)(e_\lambda) &= D_\Gamma(\partial^N(e_\lambda)) - \partial^{N\otimes_B J}(D_\Gamma(e_\lambda)) \\ &= D_\Gamma\left(\sum_{\mu<\lambda} e_\mu b_{\mu\lambda}\right) - \partial^{N\otimes_B J}(\gamma_\lambda) \\ &= \sum_{\mu<\lambda}(\gamma_\mu b_{\mu\lambda} + e_\mu \otimes \delta(b_{\mu\lambda})) - \partial^{N\otimes_B J}(\gamma_\lambda).\end{aligned} \tag{5.1.1}$$

If (i) holds, then by the equivalence (i)$\Longleftrightarrow$(v) in Proposition 4.10 there exists a subset $\Gamma = \{\gamma_\lambda\}_{\lambda\in\Lambda}$ of $(N\otimes_B J)^\natural$ with $|\gamma_\lambda| = |e_\lambda|$ for all $\lambda \in \Lambda$ such that $\alpha(D_\Gamma) = 0$. Therefore, for all $\lambda \in \Lambda$ we have $\alpha(D_\Gamma)(e_\lambda) = 0$. Now, (ii) follows from (5.1.1).

For the converse, if (ii) holds, then by (5.1.1) we have $\alpha(D_\Gamma)(e_\lambda) = 0$ for all $\lambda \in \Lambda$. Hence, $\alpha(D_\Gamma) = 0$, and (i) follows again from the equivalence (i)$\Longleftrightarrow$(v) in Proposition 4.10. □

Remark 5.2. If the elements γ_λ of a set $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ satisfy Theorem 5.1(ii), then they can be described by induction. Let ϖ denote the minimum element of Λ and assume $\varpi < \lambda$. Consider the element $\xi_\lambda = \sum_{\mu<\lambda}(\gamma_\mu b_{\mu\lambda} + e_\mu \otimes \delta(b_{\mu\lambda}))$ of $(N \otimes_B J)^\natural$ which is constructed using the inductive step and note that it is a cycle in the DG B-module $N \otimes_B J$, by Proposition 5.3 below. This ξ_λ defines an element $[\xi_\lambda]$ in $\mathrm{H}_{|e_\lambda|}(N\otimes_B J)$. By assumption, we obtain $\gamma_\lambda \in (N\otimes_B J)^\natural$ satisfying $\partial^{N\otimes_B J}(\gamma_\lambda) = \xi_\lambda$, that is, $[\xi_\lambda] = 0$ in $\mathrm{H}_{|e_\lambda|}(N\otimes_B J)$.

Proposition 5.3. *Let N be a semifree DG B-module with a semifree basis $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$, and let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ be a subset of $(N\otimes_B J)^\natural$ with $|\gamma_\lambda| = |e_\lambda|$ for all $\lambda \in \Lambda$. Then, $\sum_{\mu<\lambda}(\gamma_\mu b_{\mu\lambda} + e_\mu \otimes \delta(b_{\mu\lambda}))$ is a cycle in $N\otimes_B J$ for $\lambda \in \Lambda$.*

Proof. We proceed by induction on λ. The assertion is trivial for $\lambda = \varpi$, where ϖ denotes the minimum element of Λ. Now, assume $\lambda > \varpi$. Use the general protocol that $\partial^N(e_i) = \sum_{j<i} e_j b_{ji}$ is a finite sum with all $b_{ji} \in B^\natural$. Then we have

$$\begin{aligned}&\partial^{N\otimes_B J}\left(\sum_{\mu<\lambda}(\gamma_\mu b_{\mu\lambda} + e_\mu\otimes\delta(b_{\mu\lambda}))\right) \\ &= \sum_{\mu<\lambda}\Big(\partial^{N\otimes_B J}(\gamma_\mu)b_{\mu\lambda} + (-1)^{|\gamma_\mu|}\gamma_\mu d^B(b_{\mu\lambda}) + \partial^N(e_\mu)\otimes\delta(b_{\mu\lambda}) + (-1)^{|e_\mu|}e_\mu\otimes\partial^J\delta(b_{\mu\lambda})\Big) \\ &= \sum_{\mu<\lambda}\left(\sum_{\nu<\mu}(\gamma_\nu b_{\nu\mu} + e_\nu\otimes\delta(b_{\nu\mu}))\right)b_{\mu\lambda} + \sum_{\mu<\lambda}(-1)^{|\gamma_\mu|}\gamma_\mu d^B(b_{\mu\lambda}) \\ &\qquad + \sum_{\mu<\lambda}\sum_{\nu<\mu}(e_\nu b_{\nu\mu}\otimes\delta(b_{\mu\lambda})) + \sum_{\mu<\lambda}\Big((-1)^{|e_\mu|}e_\mu\otimes\partial^J\delta(b_{\mu\lambda})\Big) \\ &= \sum_{\nu<\lambda}\gamma_\nu\left(\sum_{\nu<\mu<\lambda} b_{\nu\mu}b_{\mu\lambda} + (-1)^{|\gamma_\nu|}d^B(b_{\nu\lambda})\right) + \sum_{\nu<\lambda}e_\nu\otimes\left(\sum_{\nu<\mu<\lambda}\delta(b_{\nu\mu}b_{\mu\lambda}) + (-1)^{|e_\nu|}\delta(d^B(b_{\nu\lambda}))\right) \\ &= 0\end{aligned}$$

where the second equality follows from the inductive hypothesis and the third equality follows from (3.6.1). To see the last equality, note that

$$\begin{aligned} 0=(\partial^N)^2(e_\lambda)=\partial^N\left(\sum_{\mu<\lambda}e_\mu b_{\mu\lambda}\right) &= \sum_{\mu<\lambda}\left(\partial^N(e_\mu)b_{\mu\lambda}+(-1)^{|e_\mu|}e_\mu d^B b_{\mu\lambda}\right)\\ &= \sum_{\mu<\lambda}\left(\sum_{\nu<\mu}e_\nu b_{\nu\mu}b_{\mu\lambda}+(-1)^{|e_\mu|}e_\mu d^B b_{\mu\lambda}\right)\\ &= \sum_{\nu<\lambda}e_\nu\left(\sum_{\nu<\mu<\lambda}b_{\nu\mu}b_{\mu\lambda}+(-1)^{|e_\nu|}d^B b_{\nu\lambda}\right).\end{aligned}$$

Since $\mathcal{B}$ is a semifree basis for N, for $\nu<\lambda$, we conclude that

$$\sum_{\nu<\mu<\lambda}b_{\nu\mu}b_{\mu\lambda}+(-1)^{|e_\nu|}d^B b_{\nu\lambda}=0$$

and hence, the last equality holds. □

The following result is an application of Theorem 5.1 and will be used to construct Examples 5.6 and 5.7 below.

Corollary 5.4. *Let N be a semifree DG B-module with a semifree basis consisting of only two elements $\{e,e'\}$ with $\partial^N(e')=eb$ for $b\in B^\natural$. Then the following hold:*

(a) *If $\delta(b)$ is a boundary of J, then N is naïvely liftable to A.*
(b) *Conversely, if $A_0=B_0$ and N is naïvely liftable to A, then $\delta(b)$ is a boundary of J.*

Proof. (a) By assumption, there is $c\in J^\natural$ such that $\delta(b)=\partial^J(c)$. Let $\gamma=0$ be in $(N\otimes_B J)_{|e|}$ and $\gamma'=e\otimes c$, where $\gamma'\in(N\otimes_B J)_{|e'|}$. Note that $|\partial^N(e)|=|e|-1$. Hence, $\partial^N(e)=0$ and we have

$$\partial^{N\otimes_B J}(\gamma')=\partial^N(e)\otimes c+e\otimes\partial^J(c)=e\otimes\delta(b)=\gamma b+e\otimes\delta(b).$$

This means that $\{\gamma,\gamma'\}$ satisfies condition (ii) in Theorem 5.1. Hence, N is naïvely liftable to A.

(b) The assumption $B_0=A_0$ implies $J_0=0$. Then, we have $(N\otimes_B J)_n=\bigoplus_{i>0}N_{n-i}\otimes J_i$. In particular, $(N\otimes_B J)_{|e|}=0$. Since N is naïvely liftable to A, it follows from Theorem 5.1 that there are elements $\gamma=0$ in $(N\otimes_B J)_{|e|}$ and $\gamma'\in(N\otimes_B J)_{|e'|}$ such that $\partial^{N\otimes_B J}(\gamma')=\gamma b+e\otimes\delta(b)=e\otimes\delta(b)$. We know that $\gamma'=e\otimes c$, for some $c\in J$. Therefore, $\partial^{N\otimes_B J}(\gamma')=\partial^{N\otimes_B J}(e\otimes c)=e\otimes\partial^J(c)$. Hence, $\delta(b)=\partial^J(c)$ and this means that $\delta(b)$ is a boundary of J. □

Corollary 5.5. *Consider the assumption of Corollary 5.4. If $\mathrm{H}_{|e'|-|e|-1}(J)=0$, then N is naïvely liftable to A.*

Proof. Note that $0=(\partial^N)^2(e')=\partial^N(eb)=(-1)^{|e|}ed^B(b)$. This implies that $d^B(b)=0$. It follows from (3.6.1) that $\partial^J\delta(b)=\delta d^B(b)=0$, Hence, $\delta(b)$ is always a cycle in J. By our assumption, $\delta(b)$ is a boundary of J as well. Now, it follows from Corollary 5.4(a) that N is naïvely liftable to A. □

Next, we construct a DG module that satisfies naïve liftability. For the notation and more details about free extensions of DG algebras see [11].

Example 5.6. Assume x, y are non-zero elements of R such that $xR \cap yR = (0)$ and $x^2 = 0$. Let $B = R\langle X, Y \mid dX = x, dY = Xy\rangle$ be a free extension of the DG R-algebra R with $|X| = 1$ and $|Y| = 2$. Let N be the semifree DG B-module with $N^\natural = eB^\natural \oplus e'B^\natural$ and with the differential defined by $\partial^N(e) = 0$ and $\partial^N(e') = eYXy$. By (3.6.1) we have $\delta(YXy) = \delta(d^B(Y^{(2)})) = \partial^J\delta(Y^{(2)})$. Now it follows from Corollary 5.4(a) that N is naïvely liftable to R.

In the next example we construct a DG module that does not satisfy naïve liftability.

Example 5.7. Consider the setting of Example 5.6. Let M be the semifree DG B-module with $M^\natural = uB^\natural \oplus u'B^\natural$ and with the differential defined by $\partial^M(u) = 0$ and $\partial^M(u') = uYXx$, where $|u| = 0$.

Claim: M is not naïvely liftable to R.

To prove the claim, set

$$\begin{aligned} T &= B\langle X', Y' \mid dX' = 0, dY' = X'y\rangle \\ &= R\langle X, X', Y, Y' \mid dX = x, dX' = 0, dY = Xy, dY' = X'y\rangle \end{aligned}$$

where $|X'| = |X| = 1$ and $|Y'| = |Y| = 2$. Note that

$$T^\natural = \bigoplus_{n\geqslant 0} \left(Y'^{(n-1)}X'B^\natural \oplus Y'^{(n)}B^\natural\right)$$

with the convention $Y^{(-1)} = 0$. The map $f\colon B^e \to T$ defined by $f(1 \otimes X) = X$, $f(X \otimes 1) = X' + X$, $f(1 \otimes Y^{(n)}) = Y^{(n)}$, $f(Y^{(n)} \otimes 1) = (Y' + Y)^{(n)}$ is a DG R-algebra isomorphism. In particular, if we consider the DG ideal J' of T, where $J'^\natural = \bigoplus_{n>0} \left(Y'^{(n-1)}X'B^\natural \oplus Y'^{(n)}B^\natural\right)$ and $\partial^{J'}$ is induced by d^T, then we obtain an isomorphism $f|_J\colon J \xrightarrow{\cong} J'$ which induces a mapping $\mathrm{H}(f|_J)\colon \mathrm{H}(J) \to \mathrm{H}(J')$ on homology. Note that $f(\delta(YXx)) = X'Yx + Y'Xx + X'Y'x$. Since $\partial^{J'}(-X'XY) = X'Yx$ and $\partial^{J'}(-X'Y'X) = X'Y'x$, we have the following equalities:

$$\mathrm{H}(f|_J)([\delta(YXx)]) = [X'Yx + Y'Xx + X'Y'x] = [Y'Xx].$$

We show that $[Y'Xx] \neq 0$ in $\mathrm{H}_3(J')$. To see this, note that for a general element $\chi = X'XYa_1 + Y'Ya_2 + Y'^{(2)}a_3 + X'Y'Xa_4 \in J'_4 = X'XYR \oplus Y'YR \oplus Y'^{(2)}R \oplus X'Y'XR$ with $a_i \in R$ we have

$$\partial^{J'}(\chi) = X'Y(-xa_1 + ya_2) + Y'Xya_2 + X'Y'ya_3 - X'Y'xa_4.$$

If $[Y'Xx] = 0$, then there is $b \in R$ such that $yb = x$. It then follows from the assumption $xR \cap yR = (0)$ that $x = 0$. This contradicts the fact that x is nonzero. Hence, $[Y'Xx] \neq 0$, that is, $\delta(YXx)$ is not boundary in J. It then follows from Corollary 5.4(b) that M is not naïvely liftable to R, as desired.

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References

1. M. Auslander, S. Ding, and Ø. Solberg, *Liftings and weak liftings of modules*, J. Algebra **156** (1993), 273–317.
2. L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118.
3. L. L. Avramov, H.-B. Foxby, and S. Halperin, *Differential graded homological algebra*, in preparation.
4. R. Buchweitz and H. Flenner, *A semiregularity map for modules and applications to deformations*, Compositio Math. **137** (2003), no. 2, 135–210.
5. N. Carqueville and D. Murfet, *Adjunctions and defects in Landau-Ginzburg models*, Adv. Math. **289** (2016), 480–566.
6. A. Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.
7. J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. 8 (1995), no. 2, 251–289.
8. Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.
9. Tor H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Paper in Pure and Applied Mathematics, No. 20 (1969), Queen's University, Kingston, Ontario, Canada.
10. S. Nasseh, M. Ono, and Y. Yoshino, *The theory of j-operators with application to (weak) liftings of DG modules*, J. Algebra, **605** (2022), 199–225.
11. S. Nasseh, M. Ono, and Y. Yoshino, *Naïve liftings of DG modules*, Math. Z., **301** (2022), no. 1, 1191–1210.
12. S. Nasseh, M. Ono, and Y. Yoshino, *Diagonal tensor algebra and naïve liftings*, preprint 2023, `arXiv:2309.05293`.
13. S. Nasseh and S. Sather-Wagstaff, *Liftings and Quasi-Liftings of DG modules*, J. Algebra, **373** (2013), 162–182.
14. S. Nasseh and Y. Yoshino, *Weak liftings of DG modules*, J. Algebra, **502** (2018), 233–248.
15. P. Nuss, *Noncommutative descent and non-abelian cohomology* (English summary), K-Theory, **12** (1997), no. 1, 23–74.
16. M. Ono and Y. Yoshino, *A lifting problem for DG modules*, J. Algebra **566** (2021), 342–360.
17. Y. Yoshino, *The theory of L-complexes and weak liftings of complexes*, J. Algebra **188** (1997), no. 1, 144–183.

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