Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 3, 1994, 235–256

A PRIORI ESTIMATES FOR THE GRADIENT OF THE SOLUTION TO THE SYSTEM OF VISCOELASTICITY IN SEVERAL DIMENSIONS

Piotr Rybka

Dedicated to Jean Leray

1. Introduction

The aim of this note is to study a priori estimates for the gradient of strong solutions u to the system of viscoelasticity

$$(1)\qquad u_{tt} = \operatorname{div}(\sigma(\nabla u) + \nabla u_t), \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary; $u : \Omega \to \mathbb{R}^n$, $n > 1$. The main technical assumption is Lipschitz continuity of the nonlinearity and that σ is close to a linear mapping for large arguments. We consider data such that ∇u_0 is in the space of functions of bounded mean oscillation (BMO) or u_0, u_1 are spherically symmetric and ∇u_0 is essentially bounded. For the definition of BMO we refer the reader to Section 2 or to the original paper [13].

One can hope for such estimates because of a hidden parabolic structure of (1). Its parabolic structure may be made transparent after a diagonalization procedure (cf. [14]–[16]): the introduction of new variables P, Q,

$$\operatorname{div} P = u_t, \qquad Q = \nabla u - P,$$

This research has been done with partial support of KBN, grant number 2 P301 017 05. I would like to thank my colleagues Nick Firoozye, Stefan Müller and Vladimir Šverak for stimulating discussions. Last but not least, I would like to thank Universität Bonn for hospitality; part of this research has been done there.

where P and Q are gradients, leads to a new system

$$P_t = \nabla \operatorname{div} P + \pi\sigma(P+Q), \qquad Q_t = -\pi\sigma(P+Q),$$

where π is an integral operator acting by removal of the divergence-free part of a matrix (vector) field (see Section 2 below).

For u_0 in $W^{1,\infty}$ we study spherically symmetric solutions and we show that $\|\nabla u\|_\infty$ stays bounded in time only if u_0 has some additional regularity. Otherwise, the apparent parabolic structure does not prevent instantaneous formation of singularities at the origin. This suggests that the space of functions of bounded mean oscillation would be the right substitute for L^∞. This seems the right choice since it is well known that π considered on the whole space maps L^∞ into BMO (see [10]). Indeed, we show that for $\nabla u_0 \in \mathrm{BMO}$ the gradient of the solution u is bounded in BMO. Moreover, no symmetry of data is required.

Our result, which is valid for vector-valued u, is in contrast with the known results which mostly concern the case of real-valued u. For one-dimensional problem (1),

$$u_{tt} = (\sigma(u_x) + u_{xt})_x \text{ in } (0,1), \qquad u(t,0) = 0, \quad (\sigma(u_x) + u_{xt})(t,1) = 0,$$

Pego [14] shows that $\|u_x\|_\infty$ stays finite over time, where u is a strong solution (see also [5] for related problems). The approach of [14] and [5] is essentially one-dimensional and cannot be generalized directly to higher dimensions. In a recent paper Grippenberg [12] considers a larger class of hyperbolic-parabolic Volterra equations of which the above equation is a special case. He establishes L^∞ a priori estimates for weak solutions of equations from this class. Engler [7] constructs spherically symmetric mild solutions to (1) provided u is real-valued. Along the way he establishes L^∞ a priori estimates for gradients using a method similar to ours. It turns out that strong dissipation may not prevent formation of singularities at the origin for spherically symmetric vector-valued solutions. Thus our result for ∇u_0 in BMO is in a sense optimal.

The motivation for this work stems from attempts to model phase transitions in solids (see [3], [4] and references therein). If we assume that there exists a smooth function $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that $\partial W(F)/\partial F = \sigma(F)$, then the stored energy function W has several local minima, and we cannot count on convexity nor ellipticity of W. The consequence of this fact is that the system of elasticity

$$u_{tt} = \operatorname{div} \sigma(\nabla u)$$

is ill-posed. The equation (1) is a regularization of the above system, and may be used as a model for dynamics of the process (see e.g. [14]–[16]).

The existence of vector-valued solutions u was obtained under the rather technical assumption that σ is globally Lipschitz continuous (cf. [15], [16]). This assumption puts severe restrictions on the growth of the nonlinearity for large arguments. Despite this we cannot guarantee that the solution stays in $W^{1,\infty}$ if u_0 is in this space. The negative result of this paper suggests that $W^{1,\infty}$ may not be the proper space of admissible data for dynamics of diffusionless phase transitions.

The paper is organized as follows. In the second section we revisit existence theory for (1) in order to make it work for u_0 such that $\nabla u_0 \in \mathrm{BMO}(\Omega)$. We do that using the variables P, Q introduced in [15], generalizing the one-dimensional transformation of Pego [14]. In the third section we present the a priori estimates. We deduce them from the solvability of

$$Q_t = -\pi\sigma(P+Q)$$

on various function spaces under suitable conditions on σ.

Throughout the paper we shall use the following notation: the space of n by n matrices is denoted by $M(n \times n)$ or just $\mathbb{R}^{n\times n}$; for a matrix $\xi \in M(n\times n)$ we define its norm $|\xi|$ by

$$|\xi| := \sqrt{\langle \xi, \xi\rangle} = (\operatorname{tr} \xi\xi^T)^{1/2};$$

B_ρ is the ball in $\mathbb{R}^n$ centered at the origin with radius ρ, $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$ and we use the shorthands $\|\cdot\|_{k,q}$ (resp. $\|\cdot\|_q$) for $\|\cdot\|_{W^{k,q}}$ (resp. $\|\cdot\|_{L^q}$), and $\|\cdot\|_* = \|\cdot\|_{\mathrm{BMO}}$.

2. Existence revisited

In this section we present concisely the existence results for the system

$$u_{tt} = \operatorname{div}(\sigma(\nabla u) + \nabla u_t) \quad \text{in } \Omega, \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \tag{2}$$

with either Neumann boundary conditions

$$(\sigma(\nabla u) + \nabla u_t)\cdot \nu = 0 \qquad \text{on } \partial\Omega, \tag{3}$$

where ν is the outer normal to the boundary of Ω, or Dirichlet boundary data

$$u = 0. \tag{4}$$

We assume that $u : \Omega \to \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary.

We use new variables (see [14], [15])

$$\operatorname{div} P = u_t, \qquad Q = \nabla u - P,$$

leading to diagonalization of (2). We assume that P and Q are gradients and P satisfies

$$P\cdot\nu=0 \qquad \text{on } \partial\Omega \tag{5}$$

in the case of Neumann data (3) or

$$\operatorname{div} P=0 \qquad \text{on } \partial\Omega \tag{6}$$

in the case of Dirichlet data. In the new variables equation (2) becomes

$$P_t=\pi\sigma(P+Q)+\nabla\operatorname{div} P, \qquad Q_t=-\pi\sigma(P+Q), \tag{7}$$

where $\pi=\pi^N$ or $\pi=\pi^D$, depending on boundary conditions. Here $\pi^N v$ and $\pi^D v$, $v\in L^p(\Omega,\mathbb{R}^{n\times n})$, are continuous projections onto subspaces of gradients satisfying some additional boundary conditions. Precisely, $\nabla\phi=\pi^N v$, where $\phi\in W^{1,p}(\Omega;\mathbb{R}^n)$, $1<p<\infty$, if ϕ is a weak solution of

$$\Delta\phi=\operatorname{div} v \quad \text{in } \Omega, \qquad (\nabla\phi-v)\cdot\nu=0 \quad \text{on } \partial\Omega; \tag{8_N}$$

and $\nabla\phi=\pi^D v$, where $\phi\in W_0^{1,p}(\Omega;\mathbb{R}^n)$, if ϕ is a unique weak solution to

$$\Delta\phi=\operatorname{div} v \qquad \text{in } \Omega. \tag{8_D}$$

We refer the interested reader to [15, Appendix] and [9] for more details. In order to simplify the notation, from now on we shall drop the superscripts D and N. This should not lead to confusion since the form of boundary conditions is mostly immaterial.

The following existence result is valid.

EXISTENCE THEOREM. *Suppose σ is Lipschitz continuous, $2\le p<\infty$, P_0 and Q_0 are initial data for (7) such that $P_0\in\pi W^{1,p}(\Omega;\mathbb{R}^{n\times n})$, $Q_0\in\pi L^p(\Omega,\mathbb{R}^{n\times n})$ where the region Ω is smooth, and either*

(a) *$\pi=\pi^N$ and $P_0\cdot\nu=0$;*

or

(b) *$\pi=\pi^D$.*

Then there is a unique solution $(P(t),Q(t))$ of (7) such that

(a) *for $\pi=\pi^N$, $P(t)$ satisfies the boundary condition $P(t)\cdot\nu=0$*

or

(b) *for $\pi=\pi^D$, $P(t)$ satisfies $\operatorname{div} P(t)=0$ on $\partial\Omega$, for $t>0$;*

moreover,

$$P \in C([0,\infty); \pi W^{1,p}) \cap C^1((0,\infty); \pi L^p) \cap C((0,\infty); \pi W^{2,p}),$$
$$Q \in C^1([0,\infty); \pi L^p).$$

PROOF. If $p = 2$, this result was proved in the present form for both boundary conditions in [15]. Swart and Holmes [16] sharpened the result for Dirichlet boundary conditions to accommodate $p \neq 2$. Actually, their argument can be extended to boundary data (3) to yield existence in the present form. □

This theorem provides a unique solution to the original problem (2), (3) or (2), (4) satisfying

$$u_t \in C([0,\infty); L^p(\Omega, \mathbb{R}^n)), \qquad \nabla u \in C([0,\infty); L^p(\Omega, \mathbb{R}^{n\times n})),$$
$$\nabla u_t + \pi\sigma(\nabla u) \in C((0,\infty); \pi W^{1,p}(\Omega; \mathbb{R}^{n\times n})).$$

This result was proved for $p = 2$ and either of the boundary conditions (3) or (4) in [15]. Swart and Holmes [16] sharpened this result to accommodate the case $p > 2$ for Dirichlet boundary condition (4). It turns out that their argument goes through only with minor changes for Neumann condition (3). However, we shall content ourselves with an immediate corollary to the Existence Theorem:

COROLLARY. *If σ is Lipschitz continuous, $u_0 \in W^{1,p}(\Omega; \mathbb{R}^n)$, $u_1 \in L^p(\Omega, \mathbb{R}^n)$, then the unique solution of* (2) *and* (3) (*or* (2) *and* (4)) *satisfies*

$$u \in C([0,\infty); W^{1,p}(\Omega; \mathbb{R}^n)), \qquad u_t \in C([0,\infty); L^p(\Omega, \mathbb{R}^n)).$$

For the reader's convenience we recall the definition of the function space BMO(Ω) introduced by John-Nirenberg [13] (see also [1]):

$$\mathrm{BMO}(\Omega) = \Bigg\{ f \in L^1(\Omega, \mathbb{R}^n) : \\ \|f\|_* := \sup_{x_0\in\Omega, r>0} \left\{ r^{-n} \int_{Q(x_0,r)\cap\Omega} |f - f_{Q(x_0,r)\cap\Omega}|\, dx \right\} < \infty \Bigg\},$$

where f_A denotes the mean value of f over the set A,

$$f_A = \frac{1}{|A|} \int_A f(x)\, dx,$$

and $Q(x,r)$ is the cube centered at x with edges of length r parallel to the axes.

We shall prove existence of solutions for (2) with data such that $\nabla u_0 \in$ BMO(Ω) only for Dirichlet boundary conditions. For these boundary conditions we have a priori estimates for the gradients of solutions of elliptic systems (see [1]).

PROPOSITION 1. *If we add to assumptions of the Existence Theorem, the conditions* $Q_0 \in \mathrm{BMO}(\Omega)$ *and* $p > n$, *then* $Q \in C^1([0,\infty);\mathrm{BMO}(\Omega))$.

PROOF. Since $\mathrm{BMO}(\Omega) \subset \bigcap_{q=1}^{\infty} L^q(\Omega)$, by the Existence Theorem we have $Q \in C^1([0,\infty);L^p)$. In particular, Q solves

$$Q_t = -\pi\sigma(P+Q).$$

We shall see that this equation is well-posed in BMO, and hence the uniqueness of solutions will follow. We note that $P(t) \in W^{1,p}(\Omega;\mathbb{R}^{n\times n})$ with $p > n$, so $P(t)$ is in BMO. By Lipschitz continuity of σ the composition $\phi \mapsto \sigma(\phi)$ maps BMO into itself. It is sufficient to check that (a) the composition $\phi \mapsto \sigma(\phi)$ is a Lipschitz mapping, (b) the projection π is continuous.

We begin with (a). We note that if $f \in \mathrm{BMO}(\Omega)$ then there is a positive number c such that

$$\|f\|_p \le cp\|f\|_* + \frac{1}{|\Omega|}\|f\|_1 \qquad \forall p \ge 2.$$

This can be deduced easily from Sections 1 and 2 of Chapter VI in [11].

Let $A > 0$ be the infimum of numbers c such that the above inequality holds for all f in $\mathrm{BMO}(\Omega)$ and all $p \ge 2$. Thus, if for $f \in \mathrm{BMO}(\Omega)$ we have

$$\|f\|_p \le \alpha p + \frac{1}{|\Omega|}\|f\|_1,$$

then $\|f\|_* \le \alpha/A$. Therefore, since $\sigma(\phi), \sigma(\psi) \in \mathrm{BMO}$ for $\phi, \psi \in \mathrm{BMO}$ and

$$\|\sigma(\phi) - \sigma(\psi)\|_p \le L\|\phi-\psi\|_p \le 2LAp\|\phi-\psi\|_* + \frac{L}{|\Omega|}\|\phi-\psi\|_1,$$

where L is the Lipschitz constant of σ, we obtain

$$\|\sigma(\phi) - \sigma(\psi)\|_* \le 2L\|\phi-\psi\|_*.$$

The validity of (b) for $\Omega = \mathbb{R}^n$ follows from the fundamental paper of Fefferman and Stein [10]. For bounded domains the continuity of π follows from a priori estimates in BMO for the gradients of solutions of elliptic systems applied to (8_D) (see e.g. [1]).

From (a) and (b) we easily conclude that $Q \in C^1([0,\infty);\mathrm{BMO}(\Omega))$. □

By the definition of P and Q we immediately obtain

COROLLARY. *In addition to the assumptions of the Existence Theorem, suppose that* $\nabla u_0 \in \mathrm{BMO}(\Omega)$. *Then* $\nabla u \in C([0,\infty);\mathrm{BMO}(\Omega))$.

3. A priori estimates

We shall examine two cases:

1) BMO estimates, no symmetry of data assumed;
2) L^∞ estimates for spherically symmetric solutions.

Our method of solving these problems is loosely related to the idea of "linearization at infinity" due to Evans and Chipot [8]. That is, we assume that for large ξ the term $\sigma(\xi)$ is linear up to a bounded perturbation, $\sigma(\xi) = \Lambda\xi + E(\xi)$. We now treat equation (7_2) as

$$Q_t = -\pi\Lambda Q - (\pi\Lambda P + \pi E(P+Q))$$

in an appropriate function space $X(\Omega)$. The variation of constants formula yields

$$Q(t) = \exp(-\pi\Lambda t)Q_0 - \int_0^t \exp(-(t-s)\pi\Lambda)\pi(\Lambda P(s) + E(P(s)+Q(s)))\,ds. \tag{9}$$

This formula will be justified if $\pi\Lambda$ is a continuous mapping on $X(\Omega)$ and $P(t)$ is continuous with values in $X(\Omega)$. Subsequently we deduce the desired estimates from (9) and additional properties of σ, Λ. From now on we assume that

$$|\sigma(\xi) - \Lambda\xi| \leq M < \infty, \qquad \forall \xi \in M(n\times n), \tag{10}$$

where $\Lambda \in M(n\times n)$ and Λ is symmetric, positive definite, i.e.

$$\Lambda \geq \lambda I > 0. \tag{11}$$

We can now state our main results.

THEOREM 2. *Suppose that the assumptions of the Existence Theorem are satisfied and there is a smooth function $W : M(n\times n) \to \mathbb{R}$ such that $\sigma(\xi) = DW(\xi)$. In addition, assume that $p > n$ and there is a positive definite matrix Λ such that* (10) *and* (11) *are satisfied. Then, if $\nabla u_0 \in \mathrm{BMO}(\Omega)$, the solution u of* (2), (4) *satisfies*

$$\|\nabla u(t)\|_* \leq \|\nabla u_0\|_* + K,$$

where K depends only on M, λ and u_1.

THEOREM 3. *Suppose that the assumptions of the previous theorem hold. Assume that σ is isotropic, i.e.*

$$\sigma(RFR^T) = R\sigma(F)R^T, \qquad \forall R \in SO(n),$$

for all $F \in M(n \times n)$*;* u_0 *and* u_1 *are spherically symmetric (i.e.* $u_0 = a_0(r)x/r$ *and* $u_1 = a_1(r)x/r$ *where* $r = |x|$*), and* $u_0 \in W^{1,\infty}(B_\rho; \mathbb{R}^n)$ *is such that the integral*

$$J(u_0) := \int_0^\rho \left| \frac{a_0(r)/r - a_0'(r)}{r} \right| dr \tag{12}$$

is finite. Then the solution u *of* (2) *with either of the boundary conditions* (3) *or* (4) *satisfies*

$$\|\nabla u(t)\|_\infty \le (\|\nabla u_0\|_\infty + J(u_0) + \kappa + Mt)e^{nKt}$$

where the constant K *depends only on the nonlinearity* σ *and* κ *depends only on* p*,* ρ*, the dimension* n*, and the initial velocity* u_1*. Moreover,*

$$J(u(t)) \le (\|\nabla u_0\|_\infty + J(u_0) + \kappa + Mt)e^{nKt}$$

for all times $t > 0$*.*

The condition $J(u_0) < \infty$ is essential. Its violations may lead to instantaneous formation of singularities of u at the origin. We stress that this is possible only because u is vector-valued.

PROPOSITION 4. *There is an isotropic* σ *satisfying the assumptions of the Existence Theorem and* $u_0 \in W^{1,\infty}(B_\rho; \mathbb{R}^n)$ *with* $J(u_0) = \infty$*,* $u_1 \in L^p(\Omega, \mathbb{R}^n)$*,* $p > n$*, such that for no positive* C*,* T *does one have*

$$\|\nabla u(t)\|_\infty < C$$

for $0 < t < T$*; moreover, a singularity forms at the origin, i.e.* $|\nabla u|(x,t) \to \infty$ *as* $x \to 0$*.*

REMARK 1. The linear mapping $\Phi_\Lambda : M(n \times n) \to M(n \times n)$, $\Phi_\Lambda(\xi) = \Lambda \circ \xi$, where $\Lambda \in M(n \times n)$, is a rather special example of a linear transformation of $M(n \times n)$ but it has the property that $\Lambda \nabla \phi = \nabla(\Lambda \phi)$ for $\phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ or

$$\pi \Lambda \nabla \phi = \Lambda \nabla \phi. \tag{13}$$

Moreover, it is the only map with this property.

PROPOSITION 5. *If* $L : M(n \times n) \to M(n \times n)$ *is linear and has property* (13) *for all* $\phi \in W^{1,1}(\Omega; \mathbb{R}^n)$ *then there exists a matrix* $\Lambda \in M(n \times n)$ *such that* $L = \Phi_\Lambda$*.*

PROOF. It is sufficient to consider smooth, compactly supported functions ϕ. If necessary, we can shift the origin so that $\operatorname{supp}\phi \subset B_r$, $r > 0$.

The property (13) means that there exists ψ such that

$$L\nabla\phi = \nabla\psi \tag{14}$$

and $\nabla\psi$ vanishes outside the support of ϕ, so we may assume that $\operatorname{supp}\phi \subset B_r$. We can take the Fourier transform of both sides of (14) to obtain

$$(L\nabla\phi)^\wedge = L(i\widehat{\phi}\otimes\zeta) = i\widehat{\psi}\otimes\zeta, \qquad \zeta\in\mathbb{R}^n,$$

or

$$\ell_{ijkl}\widehat{\phi}_k\zeta_l = \widehat{\psi}_i\zeta_j.$$

We have assumed that

$$(L\xi)_{ij} = \ell_{ijkl}\xi_{kl}$$

(the summation convention is in force).

Fix b. By the Plancherel-Pólya Theorem we may find a smooth function ϕ such that $\operatorname{supp}\phi\subset B_r$ and $b = \widehat{\phi}(\zeta)$. Thus, we may vary ζ and we conclude that

$$\ell_{ijkl}\widehat{\phi}_k = \widehat{\psi}_i\delta_j^l.$$

Thus there is a matrix $A = \{a_{ij}\}$ such that $\ell_{ijkl} = a_{ik}\delta_j^l$, namely $a_{ik} = \ell_{ijkl}\delta_j^k$. This yields the desired result. □

The common ingredients of the proofs of Theorems 2 and 3 are: boundedness of $\exp(-\pi\Lambda t)$ in $X(\Omega)$ and L^∞ bounds for P. We begin with the first fact. By Remark 1 we have $\exp(-t\pi\Lambda) = \exp(-t\Lambda)$. Positivity of Λ yields

$$\|\exp(-t\pi\Lambda)x\|_X \le e^{-\lambda t}\|x\|_X \tag{15}$$

for $X(\Omega) = \mathrm{BMO}(\Omega)$ or $X(\Omega) = L^q(\Omega,\mathbb{R}^{n\times n})$, $1 < q \le \infty$. Now, the triangle inequality applied to (9) yields

$$\begin{aligned}\|Q(t)\|_X &\le \|Q_0\|_X \\ &\quad + \frac{1}{\lambda}\sup_{s>0}(\|\pi\sigma(P(s)+Q(s)) - \Lambda(P(s)+Q(s))\|_X + \|\Lambda P\|_X).\end{aligned} \tag{16}$$

Now we shall show

LEMMA 6. *Suppose that the assumptions of the Existence Theorem and the conditions* (10) *and* (11) *hold. If* $u_0\in W^{1,p}(\Omega;\mathbb{R}^n)$ *and* $u_1\in L^p(\Omega,\mathbb{R}^n)$ *where* $p>n$, *then*

$$\sup_{0<t<\infty}\|P(t)\|_\infty < \infty \quad \textit{and} \quad \sup_{0<t<\infty}\|P(t)\|_{1,p} < \infty. \tag{17}$$

PROOF. We shall rely on Theorem 4.6 and Corollary 4.7 in [15] guaranteeing for either the boundary condition (3) or (4) that

$$\|P\|_{2,2} \to 0 \qquad \text{as } t \to \infty, \tag{18}$$

provided the condition (B) in [15] is satisfied, i.e. provided there are positive numbers c, C, D and a real number d such that

$$d + c|\xi|^2 \le W(\xi) \le D + C|\xi|^2$$

for all $\xi \in M(n \times n)$.

In order to check the validity of this condition we note that

$$\begin{aligned} W(\xi) - W(0) &= \int_0^{|\xi|} \frac{d}{dt} W\left(t\frac{\xi}{|\xi|}\right) dt \\ &= \int_0^{|\xi|} \left\langle \sigma\left(t\frac{\xi}{|\xi|}\right), \frac{\xi}{|\xi|}\right\rangle dt \\ &= \int_0^{|\xi|} \left\langle \sigma\left(t\frac{\xi}{|\xi|}\right) - \Lambda t\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}\right\rangle dt + \int_0^{|\xi|} \left\langle \Lambda t\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}\right\rangle dt. \end{aligned}$$

By (10), (11) and the Young inequality, we obtain

$$\frac{\lambda}{4}|\xi|^2 - \frac{4M^2}{\lambda} + W(0) \le W(\xi) \le W(0) + \frac{1}{2}M^2 + \frac{1}{2}(1 + \|\Lambda\|)|\xi|^2,$$

as desired.

So, if $n = 2$ or $n = 3$, then by (18) and the embedding $W^{2,2}(\Omega) \subset C^0(\Omega)$ the lemma follows. A little more work has to be done in order to prove (17) in general. We only sketch the argument. Set $k^* = kn/(n-k)$ for $k < n$. We note that $k^* > k + 1/n$ if $k > 1$. We put $k_0 = 2$, $k_{l+1} = (k_l^*)^* - 1/(nl)$. This sequence is increasing, and for some l_0, $k_{l_0} > n$, otherwise the sequence would have a limit g satisfying $g = gn/(n-2g)$, which is impossible.

We shall show that $\|P\|_{2,k_l} \to 0$ implies $\|P\|_{2,k_{l+1}} \to 0$. Thus after a finite number of steps we obtain $\|P\|_{2,\min\{k_{l_0},p\}} \to 0$ and (17) holds. Indeed, if $\|P\|_{2,k_l} \to 0$ then, by the embedding theorems, $\|P\|_{1,k_l^*} \to 0$ and $\|P\|_{(k_l^*)^*} \to 0$, and consequently $\sup_{t>0} \|P\|_{(k_l^*)^*} \le M_l$. Since π is continuous on $L^q(\Omega, \mathbb{R}^{n\times n})$ the inequality (16) implies that $\|Q\|_{(k_l^*)^*}$ is bounded, because $\|\pi\sigma(P+Q)\|_{(k_l^*)^*}$ is. Since we know [15, Proposition 4.5] that $\|\pi\sigma(P+Q)\|_2 \to 0$, the Hölder inequality implies that $\|\pi\sigma(P+Q)\|_{k_{l+1}} \to 0$. By Lemma 3.3 of [16], which is valid for either of the boundary conditions (5) or (6), and by an estimate for time derivatives of solutions of parabolic equations (Lemma A.3 of [14]), we deduce that

$$\|P_t\|_{1,k_{l+1}} \to 0, \qquad \|Q_t\|_{k_{l+1}} \to 0,$$

so $\|(P+Q)_t\|_{k_{l+1}} \to 0$ and $\|P\|_{2,k_{l+1}} \to 0$. The inequalities (17) follow. □

PROOF OF THEOREM 2. Having established (17) our result is a straightforward application of (16) since

$$\|\pi(\sigma(P+Q) - \Lambda(P+Q))\|_* \leq M\|\pi\|_*,$$

and finally by (17),

$$\|\nabla u(t)\|_* \leq \|\nabla u_0\|_* + \frac{1}{\lambda}\|\pi\|_*(M + \|\Lambda\| \sup_{s\geq 0}\|P(s)\|_\infty) + \sup_{s\geq 0}\|P(s)\|_\infty < \infty$$

for $\nabla u = Q + P$. □

It remains to consider the case of spherically symmetric data in $W^{1,\infty}(\Omega;\mathbb{R}^n)$,

$$u_0(x) = a_0(r)x/r, \qquad u_1(x) = a_1(r)x/r,$$

where $r = |x|$. We note that if $y(x) = a(r)x/r$ belongs to $W^{1,p}(B_\rho;\mathbb{R}^n)$ then

$$\nabla y(x) = \frac{a(r)}{r}\mathrm{Id} + \frac{x\otimes x}{r^3}[ra'(r) - a(r)].$$

One can check that ∇y has the following symmetry:

$$R\nabla y(x) = \nabla y(Rx)R$$

for all $R \in SO(n)$. We shall call a tensor field $A : B_\rho \to M(n\times n)$ *spherically symmetric* if

$$RA(x)R^T = A(Rx) \qquad \forall R \in SO(n). \tag{19}$$

We note that if σ is isotropic, then $\sigma(\nabla y)$ is a spherically symmetric matrix field for $y(x)$ spherically symmetric. Moreover, a spherically symmetric matrix field can be easily characterized. Namely, we have

PROPOSITION 7. *If $A : B_\rho \to M(n\times n)$ satisfies* (19) *then*

$$A(x) = \alpha(r)\mathrm{Id} + \beta(r)\frac{x\otimes x}{r^2}$$

for some functions α and β.

PROOF. Fix $x = re$, $r > 0$, e being a unit vector. Consider all $R \in SO(n)$ such that $Re = e$. If we choose an orthonormal basis $e_1, \ldots, e_n$ such that $e_1 = e$, then

$$R = \begin{pmatrix} 1 & 0 \\ 0 & R' \end{pmatrix}$$

and A takes the form

$$A(re) = \begin{pmatrix} \mu & 0 \\ 0 & A' \end{pmatrix}.$$

For, if

$$A(re) = \begin{pmatrix} \mu & w^T \\ v & A' \end{pmatrix},$$

and $RA(re)R^T = A(Rre) = A(re)$ then

$$\begin{pmatrix} \mu & w^T \\ v & A' \end{pmatrix} = \begin{pmatrix} \mu & w^T R'^T \\ R'v & R'A'R'^T \end{pmatrix},$$

that is, for all R', one has $R'v = v$ and $R'w = w$. Therefore $v = w = 0$.

We now observe that A' has only one eigenvalue. If v and w are unit eigenvectors orthogonal to e corresponding to α and β, then there is R' such that $R'^T v = w$ and

$$\alpha v = A'(re)v = R'A'(R^T re)R'^T v = R'A'(re)w = \mu R'w = \beta v.$$

Hence $\alpha = \beta$.

Moreover, A' is symmetric. If we take arbitrary vectors v, w orthogonal to e, then there is R' such that $R'v = w$ and we have

$$\langle A'v, w\rangle = \langle A'v, R'v\rangle = \langle R'^T A'R'R'^T v, v\rangle = \langle A'w, v\rangle.$$

Thus there exists an orthogonal transformation R_1 such that

$$R_1 A'(re)R_1^T = \alpha(r)\mathrm{Id}$$

for some α. Hence $A'(re) = \alpha(r)\mathrm{Id}$. Finally,

$$A(x) = \beta(r)\frac{x}{r} \otimes \frac{x}{r} + \alpha(r)\mathrm{Id},$$

where $\beta(r) = \mu(r) - \alpha(r)$. □

We note that if $v(x) = \alpha(r)\mathrm{Id} + \beta(r)x \otimes x/r^2$ then $\operatorname{div} v = (\alpha'(r) + \beta'(r) + (n-1)\beta(r)/r)x/r$, so that $\operatorname{div} v$ is also spherically symmetric. Thus, πv also enjoys this property and it is then legitimate to restrict π to spherically symmetric tensor fields. We count on great simplifications. Indeed, if we set

$$L^p_{\mathrm{rad}}(B_\rho, \mathbb{R}^{n\times n}) = \{f \in L^p(B_\rho, \mathbb{R}^{n\times n}) \,:\, f \text{ is spherically symmetric}\}$$

then we have

LEMMA 8. *For $v = \alpha(r)\mathrm{Id} + \beta(r)x \otimes x/r^2$, where α, β are in $L^\infty(0,\rho)$, we have $\pi^N v = \nabla y$ where $y(x) = a(r)x/r$ and*

$$\begin{aligned}
a(r) &= r^{1-n}\int_0^r \Big(\alpha(s) + \frac{1}{n}\beta(s)\Big)s^{n-1}\,ds \\
&\quad + r(n-1)\Big(\frac{1}{\rho^n}\int_0^\rho \Big(\alpha(s)+\frac{1}{n}\beta(s)\Big)s^{n-1}\,ds + \frac{1}{n}\int_\rho^r \frac{\beta(s)}{s}\,ds\Big), \\
a'(r) &= (1-n)r^{-n}\int_0^r \Big(\alpha(s) + \frac{1}{n}\beta(s)\Big)s^{n-1}\,ds \\
&\quad + (n-1)\Big(\frac{1}{\rho^n}\int_0^\rho \Big(\alpha(s)+\frac{1}{n}\beta(s)\Big)s^{n-1}\,ds + \frac{1}{n}\int_\rho^r \frac{\beta(s)}{s}\,ds\Big) \\
&\quad + \alpha(r) + \beta(r),
\end{aligned}$$

i.e. $\pi : L^\infty_{\mathrm{rad}}(B_\rho, \mathbb{R}^{n\times n}) \to L^q_{\mathrm{rad}}(B_\rho, \mathbb{R}^{n\times n})$ is continuous for $q < \infty$.

REMARK 2. It is well known that $\pi : L^\infty(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)$ (cf. [10]), so it is not surprising that these formulas show that finite L^∞ norm of v does not imply that πv will stay bounded. This is why the condition $J(u_0) < \infty$ has been introduced.

REMARK 3. Similar formulas are valid for Dirichlet boundary conditions; however, we shall not present here the calculations which are analogous to those below.

PROOF OF LEMMA 8. First assume that v is smooth. We may write the equation (8_N) for y using the spherical symmetry of data as

$$\begin{aligned}
&a''(r) + \frac{n-1}{r}a'(r) - \frac{n-1}{r^2}a(r) = \alpha'(r) + \beta'(r) + \frac{n-1}{r}\beta(r), \\
&a'(\rho) = \alpha(\rho) + \beta(\rho).
\end{aligned} \tag{20}$$

We rewrite (20) as a first order ODE,

$$X' = AX + G' + H, \tag{21}$$

where $X^T = (X_0, X_1) = (a, a')$,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{n-1}{r^2} & -\frac{n-1}{r} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ \alpha+\beta \end{pmatrix}, \qquad H = \begin{pmatrix} 0 \\ \frac{n-1}{r}\beta \end{pmatrix}. \tag{22}$$

We deal here with a special example of a singular ODE of the second kind. We need quite precise information about the solution, so we prefer to solve the above equation directly, instead of invoking the general theory of singular ODE.

Easy calculations yield that the matrix

$$\Phi(r) = \begin{pmatrix} r^{1-n} & r \\ (1-n)r^{-n} & 1 \end{pmatrix} \tag{23}$$

is the fundamental matrix of the system such that $\Phi(\rho) \neq \mathrm{Id}$. The variation of constants formula allows us to write solutions of (21) as

$$X(r) = \Phi(r)\Phi^{-1}(\rho)D + \Phi(r)\int_\rho^r \Phi^{-1}(s)(G'(s) + H(s))\,ds$$

where D is to be determined according to our needs. After an integration by parts, this equation takes the form

$$\begin{aligned} X(r) &= \Phi(r)\Phi^{-1}(\rho)D \\ &\quad + \Phi(r)\int_\rho^r \Phi^{-1}(s)(A(s)G(s) + H(s))\,ds + G(r) - \Phi(r)\Phi^{-1}(\rho)G(\rho) \end{aligned}$$

where we used $(\Phi^{-1})' = -\Phi^{-1}\Phi'\Phi^{-1} = -\Phi^{-1}A$. The solution has to satisfy the boundary condition (20_2) and to be continuous at $r = 0$, for it belongs to $W^{1,p}(B_\rho;\mathbb{R}^n)$, $p > n$. If we take

$$D^T = \Big(-n\rho^{1-n}\int_\rho^0 \Big(\alpha + \frac{1}{n}\beta\Big)s^{n-1}\,ds, \alpha(\rho) + \beta(\rho)\Big)$$

then the above requirements are satisfied for essentially bounded α and β.

We can combine the results of calculations into formulas for $a(r)$ and $a'(r)$:

$$\begin{aligned} a(r) &= r^{1-n}\int_0^r \Big(\alpha(s) + \frac{1}{n}\beta(s)\Big)s^{n-1}\,ds \\ &\quad + r(n-1)\Big(\frac{1}{\rho^n}\int_0^\rho \Big(\alpha(s) + \frac{1}{n}\beta(s)\Big)s^{n-1}\,ds + \frac{1}{n}\int_\rho^r \frac{\beta(s)}{s}\,ds\Big), \\ a'(r) &= (1-n)r^{-n}\int_0^r \Big(\alpha(s) + \frac{1}{n}\beta(s)\Big)s^{n-1}\,ds \\ &\quad + (n-1)\Big(\frac{1}{\rho^n}\int_0^\rho \Big(\alpha(s) + \frac{1}{n}\beta(s)\Big)s^{n-1}\,ds + \frac{1}{n}\int_\rho^r \frac{\beta(s)}{s}\,ds\Big) \\ &\quad + \alpha(r) + \beta(r). \end{aligned}$$

We can now drop the smoothness assumption on data. The above formulas make sense for arbitrary α, β in L^∞, and an easy argument shows that $y(x) = a(r)x/r$ with $a(r)$ as above is a weak solution to (8_N), i.e.

$$\int_{B_\rho} (\nabla y - v)\cdot\nabla\phi\,dx = 0$$

for all $\phi \in C^\infty(\overline{B}_\rho; \mathbb{R}^n)$. □

PROOF OF THEOREM 3. It is not difficult to check that the solution will be spherically symmetric for spherically symmetric data, provided that σ is isotropic.

Let us explore further the consequences of isotropy of σ (cf. [2], §§3, 4, and [6], §1.3),

$$R\sigma(F)R^T = \sigma(RFR^T) \qquad \forall R \in SO(n).$$

If F is the gradient of a spherically symmetric deformation, then the Rivlin-Ericksen Theorem implies that σ has the form

$$\sigma(F) = \gamma_0 \mathrm{Id} + \gamma_1 B$$

where $B = FF^T$ and γ_i, $i = 0, 1$, depend only on the invariants of $B^{1/2}$. On the other hand, we assumed that σ is a derivative, $\sigma(F) = DW(F)$, and W is isotropic, i.e. $W(F) = \Phi(\lambda_1, \dots, \lambda_n)$ where Φ is a symmetric function and λ_i are eigenvalues of $(FF^T)^{1/2}$; and then

$$\frac{\partial W}{\partial F}(F) = \Phi_2 \mathrm{Id} + (\Phi_1 - \Phi_2) x \otimes x / r^2,$$

where

$$\Phi_i = \frac{\partial \Phi}{\partial \lambda_i}(\lambda_1, \dots, \lambda_n).$$

Thus the assumption (10) reads $|\Phi_i - \mu v_i| <$ const for some $\mu > 0$ or $\Lambda = \mu \mathrm{Id}$.

In the sequel we shall write

$$F = \nabla(\phi(r)x/r) = (\phi(r)/r)\mathrm{Id} + (\phi'(r) - \phi(r)/r) x \otimes x r^{-2},$$
$$P = \nabla(\eta(r)x/r), \qquad Q = \nabla(\theta(r)x/r).$$

Thus

$$\sigma(F) = \alpha \mathrm{Id} + \beta x \otimes x r^{-2},$$

where

$$\alpha(\lambda_1, \lambda_2) = \gamma_0(\lambda_1, \lambda_2) + \lambda_1^2 \gamma_1(\lambda_1, \lambda_2),$$
$$\beta(\lambda_1, \lambda_2) = (\lambda_2 - \lambda_1)(\lambda_2 + \lambda_1)\gamma_1(\lambda_1, \lambda_2),$$

and

$$\lambda_1 = \phi(r)/r, \qquad \lambda_2 = \phi'(r).$$

As before, we consider (7_2) in an appropriate function space. For this purpose we introduce the space

$$\pi L^\infty_{J,\mathrm{rad}}(B_\rho;\mathbb{R}^{n\times n}) = \{\nabla f \,:\, f \text{ is radially symmetric}, \\ f\in W^{1,\infty}(B_\rho;\mathbb{R}^n),\ J(f)<\infty\}.$$

The functional J defined by (12) is in fact a seminorm. The space $\pi L^\infty_{J,\mathrm{rad}}(B_\rho;\mathbb{R}^{n\times n})$ equipped with the norm $\|\cdot\|_{J,\infty} = \|\cdot\|_\infty + J(\cdot)$ becomes a Banach space. We shall show that if Ξ is in $\pi L^\infty_{J,\mathrm{rad}}(B_\rho;\mathbb{R}^{n\times n})$ then $\pi\sigma(\Xi)$ is in this space too. Moreover, this map is locally Lipschitz continuous. We also need

$$J(P) \le C(n,p)\rho^{1-n/p}\|P\|_{1,p}. \tag{24}$$

Hence the existence and uniqueness of

$$Q_t = -\pi\sigma(P+Q)$$

in $\pi L^\infty_{J,\mathrm{rad}}(B_\rho;\mathbb{R}^{n\times n})$ follow.

We first check (24). We note that P is continuous since $p>n$ and from the form of $P = (\eta(r)/r)\mathrm{Id} + (\eta'(r)-\eta(r)/r)x\otimes xr^{-2}$ it follows that $\eta'(r)-\eta(r)/r$ goes to 0 as $r\to 0$. One can check, in a manner similar to that of the proof of Lemma 4.1 in [2], that

$$\int_0^\rho |\eta''(r)|^p r^{n-1}\,dr \le \|P\|_{1,p}.$$

Next we calculate

$$\begin{aligned} J(\eta) &= \int_0^\rho |\eta'(r)-\eta(r)/r|r^{-1}\,dr \\ &= \int_0^\rho \left|\int_0^r \frac{d}{ds}(\eta'(s)-\eta(s)/s)\,ds\right| r^{-2}\,dr = \int_0^\rho r^{-2}\int_0^r |\eta''(s)|s\,ds \\ &\le \left(\frac{2p-n}{p-1}\right)^{p/(p-1)} \rho^{1-n/p}\int_0^\rho |\eta''(r)|^p r^{n-1}\,dr, \end{aligned}$$

and (24) follows.

We now investigate $\pi\sigma(F)$. By the representation $\pi\sigma(F) = \nabla y$, $y(x) = a(r)x/r$, we must show that $a(r)/r$ and $a'(r)$ are in L^∞. Our estimates are as follows:

$$\begin{aligned} \|a(r)/r\|_\infty \le \frac{1}{n}\left\|\alpha+\frac{1}{n}\beta\right\|_\infty + \frac{n-1}{n}\left\|\alpha+\frac{1}{n}\beta\right\|_\infty \\ + \frac{n-1}{n}J(\phi)\|(\phi(r)/r+\phi'(r))\gamma_1\|_\infty, \end{aligned}$$

and similar calculations are also valid for $a'(r)$. Since we assumed that α and β are Lipschitz continuous, and by the structural assumptions (10) and (11), we deduce that

$$|\alpha| \leq (|\lambda_1| + |\lambda_2|)\mu + M, \qquad |\beta| \leq (|\lambda_1| + |\lambda_2|)\mu + M,$$

for some positive L, m where $\lambda_1 = \phi(r)/r$ and $\lambda_2 = \phi'(r)$. In this way we obtain

$$\|\nabla y\|_\infty \leq (\|\nabla\phi\|_\infty + J(\phi))(\mu + M) + M.$$

Now, we estimate $J(y)$. We see that

$$\frac{a' - a/r}{r} = \frac{1}{r}\left((\alpha+\beta) - \frac{n}{r^n}\int_0^r \left(\alpha + \frac{\beta}{n}\right)s^{n-1}ds\right)$$

and

$$\begin{aligned}\int_0^\rho \left|\frac{a' - a/r}{r}\right| dr \leq & \int_0^\rho \frac{n}{r^{n+1}} \int_0^r |\alpha(s) - \alpha(r)| s^{n-1}\, ds \\ & + \int_0^\rho \frac{1}{r^{n+1}} \int_0^r |\beta(s)| s^{n-1}\, ds + \int_0^\rho \frac{1}{r}\beta(r)\, dr \\ = & I_1 + I_2 + I_3.\end{aligned}$$

We immediately obtain

$$I_3 \leq \|(\phi(r)/r + \phi'(r))\gamma_1\|_\infty \int_0^\rho \left|\frac{\phi(r)/r - \phi'(r)}{r}\right| dr \leq (\mu + M)J(\phi),$$

and

$$I_2 \leq \|(\phi(r)/r + \phi'(r))\gamma_1\|_\infty \int_0^\rho \frac{1}{r^{n+1}} \int_0^r |\phi(s)/s - \phi'(s)| s^{n-1}\, ds.$$

An integration by parts yields

$$\begin{aligned} I_2 \leq & (\mu + M)\int_0^\rho \frac{(\phi(r)/r - \phi'(r))r^{n-1}}{r^n}\, dr \\ & + (\mu + M)\Big(\rho^{-n}\int_0^\rho |\phi(r)/r - \phi'(r)| r^{n-1}\, dr \\ & + \lim_{\varepsilon\to 0} \varepsilon^{-n} \int_0^\varepsilon |\phi(r)/r - \phi'(r)| r^{n-1}\, dr\Big) \\ \leq & 2(\mu + M)\left(\int_0^\rho \left|\frac{\phi(r)/r - \phi'(r)}{r}\right| dr + \|\phi' - \phi/r\|_\infty\right).\end{aligned}$$

Finally, we estimate I_1. Since σ is Lipschitz continuous, from the definition of α we obtain

$$I_1 \leq L \int_0^\rho \frac{n}{r^{n+1}} \int_0^r (|\phi(s)/s - \phi(r)/r| + |\phi'(s) - \phi'(r)|)s^{n-1}\, ds$$

where L is the Lipschitz constant of σ, and by the triangle inequality we have

$$I_1 \leq L \int_0^\rho \frac{n}{r^{n+1}} \int_0^r (|\phi(s)/s - \phi(r)/r| + |\phi(s)/s - \phi'(s)| + |\phi(r)/r - \phi'(r)|)s^{n-1}\, ds = L(I_{11} + I_{12} + I_{13}).$$

The integrals I_{12} and I_{13} are easy to estimate:

$$I_{13} \leq \int_0^\rho \left| \frac{\phi(r)/r - \phi'(r)}{r} \right| dr,$$

and after one integration by parts I_{12} may be estimated as

$$I_{12} \leq \left(\int_0^\rho \left| \frac{\phi(r)/r - \phi'(r)}{r} \right| dr + \|\phi(r)/r\|_\infty + \|\phi'\|_\infty \right).$$

In order to estimate I_{11} we again integrate by parts to obtain

$$\begin{aligned} I_{11} &\leq \int_0^\rho r^{-n} \int_0^r |(\phi(r)/r)'| s^{n-1}\, ds + 2\|\phi(r)/r\|_\infty \\ &\leq n\left(\int_0^\rho \left| \frac{\phi(r)/r - \phi'(r)}{r} \right| dr + \|\phi(r)/r\|_\infty \right). \end{aligned}$$

We can thus combine our estimates into one:

$$\|\nabla y\|_{J,\infty} \leq (\mu + M + L)n(J(\phi) + \|\nabla\phi\|_\infty) \leq n(\mu + M + L)\|\nabla\phi\|_{J,\infty}. \tag{25}$$

Our proof of Lipschitz continuity of $\pi\sigma$ besides utilizing the above estimates uses a simple fact:

CLAIM. *If $g : \mathbb{R} \to \mathbb{R}$ is C^1 and $x^2 g(x)$ is Lipschitz continuous, then $xg(x)$ is also Lipschitz continuous.*

For, by Lipschitz continuity we have $|x^2 g(x)| \leq \lambda|x| + m$, where λ is the Lipschitz constant, so $|xg(x)| \leq \lambda + m$. On the other hand,

$$\lambda \geq |(x^2 g(x))'| = |x^2 g'(x) + 2xg(x)|,$$

so $|x^2 g'(x)|$ stays bounded by $3\lambda + 2m$. Therefore,

$$|(xg(x))'| = |xg'(x) + g(x)| \leq 4\lambda + 3m$$

and our claim follows.

Let us estimate $\|a_1(r)/r - a_2(r)/r\|_\infty$. Set $F_i = \nabla(\phi_i(r)x/r)$, $i = 1, 2$. Then

$$\begin{aligned}\|a_1(r)/r - a_2(r)/r\|_\infty \leq{}& L\|F_1 - F_2\|_\infty \\ &+ (\mu + M)\int_r^\rho |(\phi_1' - \phi_1/s) - (\phi_2' - \phi_2/s)|s^{-1}\,ds \\ &+ \int_r^\rho |\phi_1/s - \phi_1'|s^{-1}|[(\phi_1' + \phi_1/s)\gamma_1(\phi_1) \\ &- (\phi_2' + \phi_2/s)\gamma_1(\phi_2)]|\,ds.\end{aligned}$$

The first two terms require no further work, the third is estimated by the above Claim, and so

$$\begin{aligned}\|a_1(r)/r - a_2(r)/r\|_\infty \leq{}& L\|\nabla\phi_1 - \nabla\phi_2\|_\infty \\ &+ (\mu + M)J(\phi_1 - \phi_2) + \|F_1 - F_2\|_\infty J(\phi_1)(4L + 3M).\end{aligned}$$

In order to estimate $J(y_1 - y_2)$ we note that

$$\frac{\alpha(r)}{r} - \frac{n}{r^{n+1}}\int_0^r \alpha(s)s^{n-1}\,ds = \frac{d}{dr}\left(r^{-n}\int_0^r \alpha(s)s^{n-1}\,ds\right) \quad \text{a.e.,}$$

so

$$\int_0^\rho \left|\frac{d}{dr}\left(r^{-n}\int_0^r \alpha(s)s^{n-1}\right)\right| ds = \left(\int_{E+} - \int_{E^-}\right)\frac{d}{dr}\left(r^{-n}\int_0^r \alpha(s)s^{n-1}\right) ds = I.$$

By continuity of $\int_D f\,dx$ with respect to the integration set, for a given ε there is $\delta > 0$ so that if the measure $|E|$ is less than δ then $|\int_E f\,dx| < \varepsilon$. The sets E^+, E^- can be approximated by open ones, i.e. there are a_i, b_i, c_i, d_i, $i = 1, 2, \ldots$, such that $|\bigcup_i(a_i, b_i) \setminus E^+| < \delta/2$, $|\bigcup_i(c_i, d_i) \setminus E^-| < \delta/2$. Then

$$I \leq \left(r^{-n}\int_0^r \alpha(s)s^{n-1}\,ds\right)\Big|_{r=a_i}^{r=b_i} - \left(r^{-n}\int_0^r \alpha(s)s^{n-1}\,ds\right)\Big|_{r=c_i}^{r=d_i} + \varepsilon$$

and

$$I \leq \frac{4}{n}\|\alpha\|_\infty + \varepsilon.$$

Since ε was arbitrary we obtain $I \leq \frac{4}{n}\|\alpha\|_\infty$. Thus, this estimate combined with the estimates for the integrals I_2, I_3 yields

$$\begin{aligned}J(y_1 - y_2) \leq{}& L\|F_1 - F_2\|_\infty \\ &+ n(\mu + M)(J(\phi_1 - \phi_2) + \|F_1 - F_2\|_\infty) + J(\phi_1)\|F_1 - F_2\|_\infty \\ &+ (\mu + M)J(\phi_1 - \phi_2) + J(\phi_1)\|F_1 - F_2\|_\infty \\ \leq{}& \|F_1 - F_2\|_\infty(L + n(\mu + M) + 2J(\phi_1)) \\ &+ J(\phi_1 - \phi_2)(n + 1)(\mu + M).\end{aligned}$$

This completes the proof of local Lipschitz continuity of $\pi\sigma$ on $\pi L^\infty_{J,\mathrm{rad}}(B_\rho;\mathbb{R}^{n\times n})$.

The information we obtain from the differential equation for Q is the following: $Q(t)$ is continuous with values in $\pi L^\infty_{J,\mathrm{rad}}(B_\rho;\mathbb{R}^{n\times n})$ and the integral

$$Q(t) = Q_0 - \int_0^t \pi\sigma(Q+P)(s)ds$$

is well defined. Recalling that $\nabla u = Q + P$ we obtain

$$\|\nabla u(t)\|_{J,\infty} \le \|\nabla u_0\|_{J,\infty} + \|P(t)\|_{J,\infty} + \int_0^t \|\pi\sigma(P+Q)(s)\|_{J,\infty}\, ds.$$

By (25) we obtain

$$\begin{aligned}\|\nabla u(t)\|_{J,\infty} \le\ & \|\nabla u_0\|_{J,\infty} + C(n,p)\rho^{1-n/p}\|P(t)\|_{1,p} + \|P(t)\|_\infty \\ & + \int_0^t (n(\mu+M+L)\|P+Q\|_{J,\infty} + M)\, ds,\end{aligned}$$

and the Gronwall inequality yields

$$\|\nabla u(t)\|_{J,\infty} \le (\|\nabla u_0\|_{J,\infty} + C(n,p)\rho^{1-n/p}\|P(t)\|_{1,p} + \|P(t)\|_\infty + Mt)e^{n(\mu+M+L)t}.$$

Finally, inequalities (17) yield

$$\begin{aligned}&\|\nabla u(t)\|_{J,\infty} \le (\|\nabla u_0\|_{J,\infty} \\ &\quad + \sup_{0<t<\infty} (C(n,p)\rho^{1-n/p}\|P(t)\|_{1,p} + \|P(t)\|_\infty) + Mt)e^{n(\mu+M+L)t} < \infty.\end{aligned}$$

□

PROOF OF PROPOSITION 4. Set

$$\sigma(F) = -|\lambda_n - \lambda_1|\mathrm{Id} + 2|\lambda_n - \lambda_1| x \otimes x r^{-2},$$

where $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge 0$ are the eigenvalues of $(FF^T)^{1/2}$. For radially symmetric deformations the above formula reduces to

$$\sigma(F) = -|\phi'(r) - \phi(r)/r|\mathrm{Id} + 2|\phi'(r) - \phi(r)/r| x \otimes x r^{-2}.$$

Certainly $\sigma(F)$ is isotropic and Lipschitz continuous.

For initial data we choose $u_1 = 0$ and $u_0(x) = \theta_0(r)x/r$, where θ_0 is continuous, $\theta_0(0) = 0$ and

$$\theta_0'(r) = \begin{cases} \lambda & \text{if } a_{i+1} < r < (a_i + a_{i+1})/2, \\ \mu & \text{if } (a_i + a_{i+1})/2 < r < a_i, \end{cases}$$

where $0 < \lambda < \mu$ and $\{a_i\}$ is a strictly decreasing sequence of positive numbers such that $a_i \leq M a_{i+1}$. Then one can check that $\mu - \lambda \geq |\theta_0'(r) - \theta_0(r)/r| \geq (\mu - \lambda)/(M+1) = \delta$, and obviously $J(u_0) = \infty$.

Suppose that for σ defined as above and any spherically symmetric $u_0 \in W^{1,\infty}(B_\rho; \mathbb{R}^n)$ there are positive C, T such that

$$\|\nabla u(t)\|_\infty \leq C \qquad \text{for } 0 \leq t < T.$$

By Lemma 8 the equation

$$Q_t = -\pi\sigma(P+Q)$$

for our choice of σ takes the form

$$\begin{aligned}
&(\theta/r)_t = T_1(\eta, \eta', \theta, \theta') + 2\frac{n-1}{n}\int_\rho^r |(\eta(s)+\theta(s))' - (\eta(s)+\theta(s))s^{-1}|s^{-1}\,ds,\\
&(\theta' - \theta/r)_t = T_2(\eta, \eta', \theta, \theta'),
\end{aligned}$$

where T_1, T_2 are bounded maps from L^∞ into itself. Moreover, the right-hand sides are Riemann integrable in L^p, so we can write

$$\begin{aligned}
(\theta/r)(t) - (\theta/r)(s) &= \int_s^t \Big(T_1(\eta, \eta', \theta, \theta')\\
&\quad + 2\frac{n-1}{n}\int_\rho^r |(\eta(s)+\theta(s))' - (\eta(s)+\theta(s))s^{-1}|s^{-1}\,ds\Big)\,d\tau, \qquad (26)\\
(\theta' - \theta/r)(t) - (\theta' - \theta/r)(s) &= \int_s^t T_2(\eta, \eta', \theta, \theta')\,d\tau.
\end{aligned}$$

By assumption, the left-hand side is essentially bounded in x for $0 \leq t < T$, so the same is true for the right-hand side. Moreover, if T is sufficiently small we have $\|T_2\|T < \delta/2$. Thus we have

$$\|(\theta' - \theta/r)(t) - (\theta_0' - \theta_0/r)\|_\infty \leq \delta/2, \qquad t < T,$$

and

$$|(\theta' - \theta/r)(t)| \geq \delta/2 \qquad \text{a.e. } x.$$

Hence $J(\theta(t)) = \infty$ for $0 \leq t < T$. Since $J(u(t)) \geq J(\theta(t)) - J(\eta(t)) = \infty$, equation (26_1) allows us to conclude that $\theta(r,t)/r$ remains unbounded and $\nabla u(x,t)$ has a singularity at the origin for all $0 \leq t < T$. $\square$

REFERENCES

[1] P. ACQUISTAPACE, *On BMO regularity for linear elliptic systems*, Ann. Mat. Pura Appl. **161** (1992), 231–269.

[2] J. M. BALL, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Philos. Trans. Roy. Soc. London Ser. A **306** (1982), 557–611.

[3] J. M. BALL AND R. D. JAMES, *Fine phase mixtures as minimizers of energy*, Arch. Rational Mech. Anal. **100** (1987), 13–52.

[4] ———, *Proposed experimental tests of a theory of fine microstructure and the two-well problem*, Philos. Trans. Roy. Soc. London Ser. A **338** (1992), 389–450.

[5] J. M. BALL, R. D. JAMES, PH. HOLMES, R. L. PEGO AND P. SWART, *On the dynamics of fine structure*, J. Nonlinear Sci. **1** (1991), 17–70.

[6] P. G. CIARLET, *Lectures on Three-Dimensional Elasticity*, Springer, Berlin, 1983.

[7] H. ENGLER, *Existence of radially symmetric solutions of strongly damped wave equation*, Nonlinear Semigroups, Partial Differential Equations and Attractors (T. L. Gill and W. W. Zachary, eds.), Springer, Berlin, 1987.

[8] L. C. EVANS AND M. CHIPOT, *Linearization at infinity and Lipschitz estimates for certain problems in the calculus of variations*, Proc. Roy. Soc. Edinburgh Sect. A **102** (1986), 291–303.

[9] D. FUJIWARA AND H. MORIMOTO, *An L_r-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo Sect. I **2** (1977), 685–700.

[10] C. FEFFERMAN AND E. M. STEIN, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[11] J. B. GARNETT, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[12] G. GRIPPENBERG, *Weak solutions of hyperbolic-parabolic Volterra equations*, Trans. Amer. Math. Soc. (to appear).

[13] F. JOHN AND L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 175–188.

[14] R. L. PEGO, *Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability*, Arch. Rational Mech. Anal. **97** (1987), 353–394.

[15] P. RYBKA, *Dynamical modelling of phase transitions by means of viscoelasticity in many dimensions*, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), 101–138.

[16] P. SWART AND PH. HOLMES, *Energy minimization and the formation of microstructure in dynamic anti-plane shear*, Arch. Rational Mech. Anal. **121** (1992), 37–85.

Manuscript received January 4, 1994

PIOTR RYBKA
Institute of Applied Mathematics and Mechanics
Warsaw University
Banacha 2, 02-097 Warsaw, POLAND