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ON THE FIXED-POINT THEORY FOR NON-COMPACT MAPS AND SPACES. I

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Dedicated to Ky Fan

1. Introduction

One of the most famous results of topological fixed-point theory is the Lefschetz–Hopf fixed-point theorem. Many generalizations of this theorem to various classes of maps and spaces were obtained in the last decades. Compactness plays an essential rôle in these generalizations. We show here how the classical compactness conditions may be weakened to conditions which are necessary or almost necessary. Recall that a topological space has the fixed-point property if and only if any continuous map of the space into itself has a fixed point. Frequently the Lefschetz–Hopf theorem is used to prove the fixed-point property for compact spaces with an additional structure, e.g. polyhedra, quasi-complexes, **ANR**'s, semicomplexes and so on. The well-known Brouwer fixed-point theorem may be obtained in the same way.

Brouwer Fixed-Point Theorem ([27], [6]). *A contractible compact polyhedron has the fixed-point property.*

One can consider the simplest contractible non-compact polyhedra like a metric hedgehog (fan) $J(\tau)$ of spinness $\tau \geq \aleph_0$ to observe that compactness is

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not a necessary condition in the Brouwer theorem for a space to have the fixed-point property. The author has recently proved that there exist some other non-trivial Lefschetz non-compact spaces and non-compact spaces with the fixed-point property ([19]–[22]). These results give rise to the following interesting problems (we assume all polyhedra to have the metric or Whitehead topology):

(P1) Characterize the contractible polyhedra having the fixed-point property.

(P2) Characterize the polyhedra which are Lefschetz spaces.

(P3) Characterize the metric **AR**'s having the fixed-point property.

(P4) Characterize the metric **ANR**'s which are Lefschetz spaces.

In this paper we completely solve problem (P1) and give a partial solution to problems (P2)–(P4).

A *topological ray* means a space homeomorphic to the half-open interval $[0,1)$. It may be noted that a normal space having the fixed-point property does not contain a topological ray as a closed subset. We call a closed subset which is a topological ray a *closed ray*. A topological space that does not contain a closed ray is called a *rayless space*. Hence the property of being a rayless space is necessary for the fixed-point property in the class of normal spaces.

Several classical situations in which the property of being rayless is sufficient for a space to have the fixed-point property are known. The first one concerning the Brouwer–Schauder fixed-point theorem was described by V. Klee in 1955 ([11]). He proved that a rayless convex subset of a metric locally convex vector space has the fixed-point property. Moreover, K. Kuratowski ([14], [15]), V. Klee [11] and E. Connell [5] showed that a rayless contractible locally compact metric **AR** has the fixed-point property. In addition to the above results V. Klee [12], W. Holsztyński [10] and V. Okhezin [23] describe the phenomenon of closed rays in products.

Let us note that in all papers except Okhezin's papers [19]–[23] the property of being a rayless space implies compactness of the space and then the classical results on fixed points are used. On the other hand, several examples such as the product of the hedgehog with a compact contractible polyhedron or the Tikhonov cube show that absence of a closed ray in a space with an additional structure implies the fixed-point property or the property of being a Lefschetz space. Using this clear topological property we completely solve problem (P1).

Theorem. *A contractible polyhedron has the fixed-point property if and only if it is a rayless space.*

By now this theorem is the most complete generalization of the Brouwer fixed-point theorem to polyhedra. Similarly we describe some special cases of non-compact Lefschetz polyhedra. Trying to extend these results to several classes of **ANR**'s one meets some difficulties which are partially overcome by using methods of $\mathcal{U}$-domination, introducing an analogue of a rayless space and covering properties like paracompactness. Even now the technique developed in this paper permits us to extend most of the classical results to the non-compact case. Therefore a new direction in topological fixed-point theory appears which may essentially influence the development of modern topology and nonlinear analysis.

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2. Preliminaries

All spaces are assumed to be Hausdorff. All polyhedra are simplicial complexes with either the metric ($|K|_m$) or Whitehead ($|K|_{CW}$) topology. We write $|K|$ if our considerations are valid for both topologies. We do not assume the simplicial complexes to be finite or locally finite. If K is a simplicial complex then the corresponding polyhedron is denoted by $|K|$, with either the metric or Whitehead topology. The cone over a simplicial complex is defined in the standard way. We need the following fact: A contractible compact polyhedron is an absolute extensor in the class of normal spaces ([16]). Any polyhedron with the Whitehead topology is a paracompact (and hence normal) locally connected space ([16]).

For any $x \in |K|$, the *carrier* $\mathrm{si}(x)$ of x is the simplex $\sigma \in K$ of lowest dimension that contains x. $K^{(m)}$ is the m-skeleton of K. If $\mathcal{V}$ and $\mathcal{U}$ are open covers of a space X then $\mathcal{U} \prec \mathcal{V}$ means that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{U} \prec\!\ast\, V$ means that $\mathcal{U}$ is a star-refinement of $\mathcal{U}$. The polyhedron $|N(\mathcal{U})|$ is the nerve of $\mathcal{U}$. Two continuous maps $f, g : X \to Y$ are called $\mathcal{U}$-*close* if for every $x \in X$ there exists $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subseteq U$ (we write $(f, g) \prec \mathcal{U}$).

Let Y be a space and let $\mathcal{W}$ be an open cover of Y. The notion of $\mathcal{W}$-homotopy may be found in [17]. If M is a finite simplicial subcomplex of K then, generally speaking, $|M|$ is not a retract of $|K|$. On the other hand, since a cone $|CM|$ is an absolute extensor there is a retraction $\rho : |CK| \to |CM|$. For any space Y, the map $R = \rho \times I_Y$, where I_Y means the identity map on Y, is called a *canonical retraction* and is denoted by

$$R = R(M) : |CK| \times Y \to |CM| \times Y.$$

Let M be a simplicial subcomplex of K. Then a subcomplex $\widetilde{M} \subseteq K$ having the same vertices as M and such that whenever all vertices of a simplex $\sigma \in K$ are in M then $\sigma \in \widetilde{M}$, is called a *shell* of M in K. Note that a shell of a finite simplicial complex is a finite simplicial complex. In any polyhedron the union of any family of closed subpolyhedra is a closed set (see [16]). A space X is *countably compact* iff any infinite subset of X has a cluster point. The product of a countably compact space and a compact space is countably compact ([7]). A map homotopic to a constant map is called *nullhomotopic* (denoted by $f \sim 0$). Indexing sets we will often use sequences of indices; $\alpha_0 = \overline{\alpha}(0)$ is an empty sequence, which will be omitted, $\alpha_0\alpha_1 \dots \alpha_i = \alpha_1 \dots \alpha_i = \overline{\alpha}(i)$.

Let X, Y be topological spaces, CX be the cone over X, and $CX \times Y$ be the topological product. Then $\pi : CX \times Y \to CX$ and $p : CX \times Y \to Y$ are obvious projections. The space $X \times Y$ may obviously be considered to be a closed subset of $CX \times Y$. Let X, Y be spaces and let $\mathcal{U}$ be an open cover of Y. We say that X *$\mathcal{U}$-dominates* Y if there are two maps $g : Y \to X$ and $f : X \to Y$ such that the composition $f \circ g : Y \to Y$ is $\mathcal{U}$-homotopic to I_Y ([17]).

We consider the (co)homology functor H with rational coefficients satisfying all the axioms of Eilenberg–Steenrod. A space X having finite type homology is called *a Lefschetz space* ([3]) if for any continuous map $f : X \to X$, $\Lambda(f) \neq 0$ implies that f has a fixed point. The *Lefschetz number* is defined as follows:

$$\Lambda(f) = \sum_{k=0}^{\infty}(-1)^k \operatorname{tr}(f_{*k}),$$

where tr denotes trace, and $f_{*k} : H_k(X) \to H_k(X)$ is the linear endomorphism induced by f in k-(co)homology. A more general construction using Leray endomorphisms may be used to define the Lefschetz number.

The classical *König's lemma* says that a tree of height ω_0 with finite levels has a path through it ([1]). $X \subset Y$ denotes $X \subseteq Y$ and $X \neq Y$. By $\bigcup\{M(\alpha) \mid \alpha \in A\}$ we denote the set-theoretic union of the family. If $A \subseteq X$ is a subset, $i : A \to X$ is the inclusion. The composition of $i : A \to X$ with $f : X \to Y$ is denoted by $f|_A : A \to Y$ and is called the restriction of f to A; we then say that f extends $f|_A$ over X. $\mathbb{N}$ is the set of natural numbers.

3. Fixed-point theorems for products with non-compact polyhedra

Let K be a simplicial complex and Y be a space. Consider the following properties which will play an important rôle in this section.

(C1) Any map $f : |K| \times Y \to |K| \times Y$ has a continuous extension

$$F : |CK| \times Y \to |K| \times Y.$$

(C2) (C1) holds and there is a countably compact space S such that

$$F(|CK| \times Y) \subseteq |K| \times S \subseteq |K| \times Y.$$

(C3) (C1) holds and for any finite subcomplex $M \subseteq K$ and any canonical retraction $R = R(M)$ the map $R \circ F : |CK| \times Y \to |CM| \times Y$ has a fixed point.

Let $M \subseteq K$ be a subcomplex. The space $|K| \setminus |M|$ is the union of its components V_α: $|K| \setminus |M| = \bigcup V_\alpha$. If q, φ are vertices such that $\varphi \in \partial V_\alpha \subseteq |M|$ and $q \in V_\alpha$ for some α, then it is easily seen that there exists a simple path Δ connecting q and φ lying in the one-dimensional skeleton $K^{(1)}$, such that $|\Delta| \setminus \{\varphi\} \subseteq V_\alpha$ ($|\Delta| \cap |M| = \{\varphi\}$). The following proposition is the main technical tool of our theory:

THEOREM 3.1. *Let $|K|$ be a polyhedron, Y be a space and let $f : |K| \times Y \to |K| \times Y$ satisfy* (C2) *and* (C3). *Then one of the following holds:*

(i) *f has a fixed point* ($\mathrm{Fix}\, f \neq \emptyset$),
(ii) *the one-dimensional skeleton of $|K|$ contains a closed ray $l \subseteq |K^{(1)}| \subseteq |K|$.*

PROOF. Suppose f has no fixed point. Condition (C1) implies that F has no fixed point either. Let us show that this implies the existence of a strongly increasing sequence of simple paths $l(\overline{\alpha}(n))$ in the one-dimensional skeleton of $|K|$, i.e.

$$l(\alpha_0) \subset l(\overline{\alpha}(1)) \subset \ldots \subset l(\overline{\alpha}(n)) \subset l(\overline{\alpha}(n+1)) \subset \ldots$$

The union

$$l = \bigcup_{n=1}^{\infty} l(\overline{\alpha}(n)) \subseteq |K^{(1)}| \subseteq |K|$$

is a closed ray in $|K|$ since the union of any family of closed subpolyhedra is a closed set. We construct the sequence of simple paths inductively. Let $e = l(\alpha_0) \in |K^{(0)}|$, where e is any vertex of the polyhedron, be the starting point of the ray. Suppose the following holds:

(1) There is a tree T_n such that for any $i = 0, \ldots, n$ the i-level of the tree contains sequences $\overline{\alpha}(i) = \alpha_1 \ldots \alpha_i$ and for any $i = 0, \ldots, n-1$ the branch order of $\overline{\alpha}(i)$ is equal to the cardinality of the non-empty finite set $A(\overline{\alpha}(i))$. We also suppose that non-empty index sets $A(\overline{\alpha}(i)) \subseteq B(\overline{\alpha}(i)) \subseteq C(\overline{\alpha}(i))$ are defined such that $B(\overline{\alpha}(i))$ are finite sets where $i = 0, \ldots, n-1$, and for any $i = 1, \ldots, n$, $\overline{\alpha}(i) \in T_n$ if and only if

$$\alpha_1 \in A(\alpha_0), \quad \alpha_2 \in A(\alpha_1), \quad \ldots, \quad \alpha_i \in A(\overline{\alpha}(i-1)).$$

(2) There is a connected finite simplicial complex M_n such that $e \in M_n \subseteq K$. The space $|K| \setminus |\widetilde{M_n}|$ is the union of its components. Moreover, we have

$$\begin{aligned}|K| \setminus |\widetilde{M_n}| = &\bigcup_{j=1}^{n} \bigcup \{V(\overline{\alpha}(j)) \mid \overline{\alpha}(j) \in T_n, \alpha_j \in C(\overline{\alpha}(j-1)) \setminus B(\overline{\alpha}(j-1))\} \\ &\cup \bigcup \{V(\overline{\alpha}(n+1)) \mid \overline{\alpha}(n) \in T_n, \alpha_{n+1} \in C(\overline{\alpha}(n))\},\end{aligned}$$

where $C(\overline{\alpha}(n))$ is a non-empty set of indices. There is a similar representation for $|CK| \setminus |C\widetilde{M_n}|$ in which we use W instead of V for components. The boundary of each component V is contained in the polyhedron $|\widetilde{M}_n|$ and

$$\emptyset \neq V(\overline{\alpha}(n+1)) \subset V(\overline{\alpha}(n)).$$

(3) To any $\alpha_{n+1} \in C(\overline{\alpha}(n))$ we assign a vertex

$$\varphi_n(\overline{\alpha}(n+1)) \in \partial V(\overline{\alpha}(n+1)) \subseteq |\widetilde{M}_n|.$$

(4) There is a canonical retraction

$$R_n : |CK| \times Y \to |C\widetilde{M}_n| \times Y, \quad R_n = \rho_n \times I_Y, \quad \rho_n : |CK| \to |C\widetilde{M}_n|,$$

such that for any $z \in \operatorname{Fix} R_n \circ F$,

$$\pi \circ F(z) \in \bigcup \{W(\overline{\alpha}(n+1)) \mid \overline{\alpha}(n) \in T_n, \alpha_{n+1} \in C(\overline{\alpha}(n))\}.$$

(5) For any $i = 0, \dots, n$ there are simple paths $l(\overline{\alpha}(i))$, $\overline{\alpha}(i) \in T_n$, such that

$$l(\alpha_0) \subset l(\overline{\alpha}(1)) \subset \dots \subset l(\overline{\alpha}(n-1)) \subset l(\overline{\alpha}(n)).$$

We define $B(\overline{\alpha}(n)) \subseteq C(\overline{\alpha}(n))$ as follows:

$$B(\overline{\alpha}(n)) = \{\alpha_{n+1} \in C(\overline{\alpha}(n)) \mid \text{there exists } z_n \in \operatorname{Fix} R_n \circ F \text{ such that } F(z_n) \in W(\overline{\alpha}(n+1)) \times S\}.$$

Condition (4) implies that $B(\overline{\alpha}(n))$ is a non-empty set. Suppose $B(\overline{\alpha}(n))$ is infinite. Then there is a countable set $\widehat{C} \subseteq B(\overline{\alpha}(n))$ of pairwise distinct indices such that for any $\alpha^k_{n+1} \in \widehat{C}$ there is $z^k_n \in \operatorname{Fix} R_n \circ F \subseteq |C\widetilde{M}_n| \times S$ and $F(z^k_n) \in W(\overline{\alpha}(n)\alpha^k_{n+1}) \times S$. Since $|C\widetilde{M}_n| \times S$ is a countably compact space, the infinite set $\{z^k_n \mid \alpha^k_{n+1} \in \widehat{C}\}$ has a cluster point $z_n \in |CK| \times Y$. The set $\operatorname{Fix} R_n \circ F$ is closed, hence $z_n \in \operatorname{Fix} R_n \circ F$. Now $\{W(\overline{\alpha}(n)\alpha^k_{n+1}) \times Y \mid \alpha^k_{n+1} \in \widehat{C}\}$ is a disjoint family of open sets, so $F(z_n) \in |C\widetilde{M}_n| \times S$ and thus $z_n = R_n \circ F(z_n) = F(z_n)$. But we have supposed that F has no fixed point. This contradiction shows that $B(\overline{\alpha}(n))$ is a non-empty finite set.

For any $\alpha_{n+1} \in B(\overline{\alpha}(n))$ we let

$$P(\overline{\alpha}(n+1)) = V(\overline{\alpha}(n+1)) \cup |\widetilde{M}_n|.$$

Define

$$A(\overline{\alpha}(n)) = \{\alpha_{n+1} \in B(\overline{\alpha}(n)) \mid P(\overline{\alpha}(n+1)) \text{ is a non-compact space}\}.$$

We shall show that $A(\overline{\alpha}(n))$ is non-empty. Suppose not. Then $P(\overline{\alpha}(n+1))$ is compact for any $\alpha_{n+1} \in B(\overline{\alpha}(n))$. We construct an auxiliary canonical retraction corresponding to a compact polyhedron Q, where

$$Q = \bigcup \{P(\overline{\alpha}(n+1)) \mid \overline{\alpha}(n) \in T_n,\ \alpha_{n+1} \in B(\overline{\alpha}(n))\}.$$

Consider a closed set $\Psi \subseteq |CK|$, where

$$\Psi = |\widetilde{M}_n| \cup \bigcup \{W(\overline{\alpha}(n+1)) \mid \overline{\alpha}(n) \in T_n,\ \alpha_{n+1} \in C(\overline{\alpha}(n)) \setminus B(\overline{\alpha}(n))\}$$

and a continuous mapping ${}'\rho_n : \Psi \cup |CQ| \to |CQ|$ defined as follows:

$$ {}'\rho_n = \begin{cases} z & \text{if } z \in |CQ|, \\ \rho_n(z) & \text{if } z \in \Psi. \end{cases} $$

Since $|CQ|$ is an absolute extensor, there is a continuous extension of ${}'\rho_n$ over the cone $|CK|$. This extension $\widetilde{\rho}_n : |CK| \to |CQ|$ will be an auxiliary retraction and we let $\widetilde{R}_n = \widetilde{R}_n(Q) = \widetilde{\rho}_n \times I_Y$. This $\widetilde{R}_n$ will be the desired auxiliary canonical retraction. Condition (C3) implies the existence of a fixed point $z_n = \widetilde{R}_n \circ F(z_n)$ and $z_n \in |CQ| \times S$. Since $\widetilde{R}_n \circ F(z_n) \neq F(z_n)$ there exists an index $\alpha_{n+1} \in C(\overline{\alpha}(n)) \setminus B(\overline{\alpha}(n))$ such that $F(z_n) \in W(\overline{\alpha}(n+1)) \times S$. Therefore $\widetilde{R}_n \circ F(z_n) = R_n \circ F(z_n) = z_n$, i.e. $z_n \in |C\widetilde{M}_n| \times S$. This contradicts the definition of $B(\overline{\alpha}(n))$, so that finally $A(\overline{\alpha}(n))$ is a non-empty finite set.

For any $\alpha_{n+1} \in A(\alpha(n))$ choose an arbitrary point $z_n \in \operatorname{Fix} R_n \circ F$ such that $F(z_n) \in W(\overline{\alpha}(n+1)) \times S$ (see (4)). Since $\pi \circ F(z_n) \notin |\widetilde{M}_n|$ there are vertices of the simplex $|\mathrm{si}(\pi \circ F(z_n))|$ which are not in $|\widetilde{M}_n|$. This follows from the definition of the shell $\widetilde{M}_n$.

For any $\alpha_{n+1} \in A(\overline{\alpha}(n))$ we choose a vertex $q_n(\overline{\alpha}(n+1))$ such that

$$ q_n(\overline{\alpha}(n+1)) \notin |\widetilde{M}_n|, \qquad q_n(\overline{\alpha}(n+1)) \in |(\mathrm{si}(\pi \circ F(z_n)))^{(0)}|. $$

Using the remark before the theorem we find that there is a simple path

$$ |\Delta^{n+1}(\overline{\alpha}(n+1))| \subseteq |K^{(1)}| $$

connecting the points $\varphi_n(\overline{\alpha}(n+1))$ and $q_n(\overline{\alpha}(n+1))$ such that

$$ |\Delta^{n+1}(\overline{\alpha}(n+1))| \setminus \{\varphi_n(\overline{\alpha}(n+1))\} \subseteq V(\overline{\alpha}(n+1)). $$

Let us construct auxiliary connected finite simplicial complexes:

$$ \begin{aligned} M_{n+1}(\overline{\alpha}(n+1)) &= \widetilde{M}_n \cup \Delta^{n+1}(\overline{\alpha}(n+1)) \cup \mathrm{si}(\pi \circ F(z_n(\overline{\alpha}(n+1)))), \\ M_{n+1} &= \bigcup \{\widetilde{M}_{n+1}(\overline{\alpha}(n+1)) \mid \alpha_1 \in A(\alpha_0), \ldots, \ \alpha_{n+1} \in A(\overline{\alpha}(n))\} \\ &\quad \cup \bigcup \{P(\overline{\alpha}(n+1)) \mid \overline{\alpha}(n) \in T_n, \alpha_{n+1} \in B(\overline{\alpha}(n)) \setminus A(\overline{\alpha}(n))\}. \end{aligned} $$

The space $|K| \setminus |\widetilde{M}_{n+1}|$ is the union of its components and we have the following representation:

$$ \begin{aligned} |K| \setminus |\widetilde{M}_{n+1}| = \bigcup_{j=1}^{n+1} & \bigcup \{V(\overline{\alpha}(j)) \mid \overline{\alpha}(j) \in T_n, \alpha_j \in C(\overline{\alpha}(j-1)) \setminus B(\overline{\alpha}(j-1))\} \\ & \cup \bigcup \{V(\overline{\alpha}(n+2)) \mid \overline{\alpha}(n) \in T_n, \alpha_{n+1} \in A(\overline{\alpha}(n)), \alpha_{n+2} \in C(\overline{\alpha}(n+1))\}, \end{aligned} $$

where $C(\overline{\alpha}(n+1))$ is a non-empty index set. Replacing V by W we obtain a similar representation for $|CK| \setminus |C\widetilde{M}_{n+1}|$. Each component has its boundary in $|\widetilde{M}_{n+1}|$ and

$$ \emptyset \neq V(\overline{\alpha}(n+2)) \subset V(\overline{\alpha}(n+1)). $$

Choose any vertex $x \in V(\overline{\alpha}(n+2))$ and consider a simple path in

$$V(\overline{\alpha}(n+1)) \cap |K^{(1)}|$$

which connects the points x and $q_n(\overline{\alpha}(n+1)) \notin V(\overline{\alpha}(n+2))$. This simple path contains a boundary point of the component $V(\overline{\alpha}(n+2))$. Therefore

$$\partial V(\overline{\alpha}(n+2)) \cap V(\overline{\alpha}(n+1)) \neq \emptyset.$$

Choose arbitrary vertices

$$\varphi_{n+1}(\overline{\alpha}(n+2)) \in \partial V(\overline{\alpha}(n+2)) \cap V(\overline{\alpha}(n+1)), \qquad \varphi_{n+1}(\overline{\alpha}(n+2)) \notin |\widetilde{M}_n|$$

and consider the following simple path in $|K^{(1)}|$:

$$[\varphi_n(\overline{\alpha}(n+1)), \varphi_{n+1}(\overline{\alpha}(n+2))] \subseteq |\Delta^{n+1}(\overline{\alpha}(n+1))| \subseteq |\widetilde{M}_{n+1}|.$$

Since $|\Delta^{n+1}(\overline{\alpha}(n+1))| \cap |\widetilde{M}_n| = \{\varphi_n(\overline{\alpha}(n+1))\}$ and $l(\overline{\alpha}(n+1)) \subseteq |\widetilde{M}_n|$, it follows that

$$l(\overline{\alpha}(n+2)) = l(\overline{\alpha}(n+1)) \cup [\varphi_n(\overline{\alpha}(n+1)), \varphi_{n+1}(\overline{\alpha}(n+2))]$$

is a simple path connecting the points e and φ_{n+1} such that

$$l(\overline{\alpha}(n+1)) \subset l(\overline{\alpha}(n+2)) \qquad (\text{since } \varphi_{n+1}(\overline{\alpha}(n+2)) \notin |\widetilde{M}_n|).$$

We now construct a tree $T_{n+1} \supset T_n$. The i-level of T_{n+1} $(i \leq n)$ is the i-level of T_n, and the $(n+1)$-level consists of the sequences $\overline{\alpha}(n+1)$, where $\alpha_{n+1} \in A(\overline{\alpha}(n)), \dots, \alpha_1 \in A(\alpha_0)$. The branch order of $\overline{\alpha}(n)$ is finite and equal to the cardinality of $A(\overline{\alpha}(n))$.

Now we shall define a canonical retraction $R_{n+1} : |CK| \times Y \to |C\widetilde{M}_{n+1}| \times Y$, $R_{n+1} = \rho_{n+1} \times I_Y$, $\rho_{n+1} : |CK| \to |C\widetilde{M}_{n+1}|$. Consider the closed set

$$\Phi = |C\widetilde{M}_n| \cup \bigcup_{j=1}^{n+1} \bigcup \{W(\overline{\alpha}(j)) \mid \alpha_1 \in A(\alpha_0), \dots, \alpha_{j-1} \in A(\overline{\alpha}(j-2)),$$
$$\alpha_j \in C(\overline{\alpha}(j-1)) \setminus B(\overline{\alpha}(j-1))\}$$

and the continuous mapping $\rho'_{n+1} : \Phi \cup |C\widetilde{M}_{n+1}| \to |C\widetilde{M}_{n+1}|$ defined as follows:

$$\rho'_{n+1}(z) = \begin{cases} z & \text{if } z \in |C\widetilde{M}_{n+1}|, \\ \rho_n(z) & \text{if } z \in \Phi. \end{cases}$$

Since $|C\widetilde{M}_{n+1}|$ is an absolute extensor, there is a continuous extension of ρ'_{n+1} to the cone $|CK|$. This extension $\rho_{n+1} : |CK| \to |C\widetilde{M}_{n+1}|$ is the desired retraction, $R_{n+1} = \rho_{n+1} \times I_Y$ is the canonical retraction. If $z \in \operatorname{Fix} R_{n+1} \circ F \neq \emptyset$ and

$$\pi \circ F(z) \notin \bigcup \{W(\overline{\alpha}(n+2)) \mid \overline{\alpha}(n+1) \in T_{n+1}, \alpha_{n+2} \in C(\overline{\alpha}(n+1))\},$$

then $\pi \circ F(z) \in \Phi \setminus |C\widetilde{M}_{n+1}|$; but then $R_{n+1} \circ F(z) = R_n \circ F(z) = z$, i.e. $z \in \operatorname{Fix} R_n \circ F$ and

$$\pi \circ F(z) \notin \bigcup \{W(\overline{\alpha}(n+1)) \mid \overline{\alpha}(n) \in T_n, \alpha_{n+1} \in C(\overline{\alpha}(n))\},$$

contradicting condition (4).

Consider the tree $T = \bigcup_{n=0}^{\infty} T_n$. The height of T is equal to ω_0 and every level of T is finite by the way the T_n's were constructed. Using König's lemma we see that there exists an infinite simple path $\alpha_0\alpha_1 \ldots \alpha_n\alpha_{n+1} \ldots$ This path immediately gives an increasing sequence of simple paths

$$l(\alpha_0) \subset l(\alpha_1) \subset \ldots \subset l(\overline{\alpha}(n)) \subset l(\overline{\alpha}(n+1)) \subset \ldots$$

The union of these paths,

$$l = \bigcup_{n=1}^{\infty} l(\overline{\alpha}(n)) \subseteq |K^{(1)}| \subseteq |K|,$$

is a closed ray in the one-dimensional skeleton. Roughly speaking, the map $f : |K| \times Y \to |K| \times Y$ is a "nonlinear translation along the product $l \times S$". This concludes the proof. □

We now consider some important situations in which (C1), (C2), (C3) hold automatically. When $Y = \{*\}$ is a singleton we may identify $|CK| \times Y$ and $|K| \times Y$ with the cone $|CK|$ and polyhedron $|K|$ respectively. In this case (C1) holds for any nullhomotopic map $f : |K| \to |K|$. (C2) obviously holds. The Brouwer fixed-point theorem implies (C3). So we have the following theorem:

Theorem 3.2. *Let $f : |K| \to |K|$ be a continuous map. Then one of the following holds:*

(i) *f has a fixed point* ($\operatorname{Fix} f \neq \emptyset$),
(ii) *f is not nullhomotopic* ($f \not\simeq 0$),
(iii) *the polyhedron $|K|$ contains a closed ray.* □

Using Theorem 3.2 we can obtain several interesting propositions the first of which may be considered to be a weakened Lefschetz–Hopf fixed-point theorem for non-compact polyhedra.

Definition 3.3. A space is called a *rayless space* if it does not contain a closed ray.

Theorem 3.4. *Let $f : |K| \to |K|$ be a nullhomotopic map of a rayless polyhedron. Then f has a fixed point.* □

The following theorem completely solves problem (P1) mentioned above.

THEOREM 3.5. *A contractible polyhedron has the fixed-point property if and only if it is a rayless space.* □

This theorem is the most complete version of the Brouwer fixed-point theorem for polyhedra; we hope it will be useful in topology and nonlinear analysis.

The next situation concerns contractible polyhedra. If $|K|$ is a contractible polyhedron then there is a retraction $\rho : |CK| \to |K|$. Consider the retraction $R = \rho \times I_Y : |CK| \times Y \to |K| \times Y$. Such a retraction exists in view of condition (C1). Suppose that Y is a countably compact space and whenever $|M| \subseteq |K|$ is a compact polyhedron then $|CM| \times Y$ is a Lefschetz space. Consider the following retractions:

$$\rho' : |CK| \to |K| \subseteq |CK|, \qquad \rho'' : |CK| \to |CM| \subseteq |CK|,$$
$$R' = \rho' \times I_Y : |CK| \times Y \to |K| \times Y \subseteq |CK| \times Y,$$
$$R'' = \rho'' \times I_Y : |CK| \times Y \to |CM| \times Y.$$

One can easily see that the maps $f \circ R$, $f \circ R'$, and $R'' \circ f \circ R'$ are homotopic. Then we can obviously define the Lefschetz number by the formula

$$\Lambda(f) = \Lambda(R'' \circ f \circ R' | |CM| \times Y).$$

Note that a normal Lefschetz space is rayless. Otherwise there exists a null-homotopic map having no fixed point and the Lefschetz number of the map is equal to 1. So we have the following analogue of the Lefschetz–Hopf fixed-point theorem.

THEOREM 3.6. *Let $|K|$ be a contractible polyhedron, and Y be a countably compact space such that for any compact polyhedron $|M| \subseteq |K|$ the product $|CM| \times Y$ is a Lefschetz space. Then for any continuous map $f : |K| \times Y \to |K| \times Y$ such that $\Lambda(f) \neq 0$ one of the following holds:*

(i) *f has a fixed point* ($\mathrm{Fix}\, f \neq \emptyset$),
(ii) *$|K|$ contains a closed ray.*

PROOF. Indeed, if $\Lambda(f) \neq 0$ then for any two retractions

$$\rho' : |CK| \to |K| \subseteq |CK|, \qquad \rho'' : |CK| \to |CM| \subseteq |CK|$$

we have

$$\Lambda(f) = \Lambda(R'' \circ f \circ R' | |CM| \times Y) = \Lambda(R'' \circ F | |CM| \times Y) \neq 0.$$

Since $|CM| \times Y$ is a Lefschetz space the map $R \circ F$ has a fixed point, i.e. (C3) holds. Then Theorem 3.1 implies (i) or (ii). □

The following theorem describes a special class of Lefschetz spaces.

Theorem 3.7. *Let $|K|$ be a contractible polyhedron, and Y be a countably compact space such that for any compact polyhedron $|M| \subseteq |K|$ the product $|CM| \times Y$ is a Lefschetz space. Then $|K| \times Y$ is a Lefschetz space if and only if $|K| \times Y$ is rayless.* □

Corollary 3.8. *Let $|K|$ be a contractible rayless polyhedron and Y belong to one of the following classes:*

(i) *weak semicomplexes of Browder,*
(ii) *compact Q-simplicial spaces of finite type* ([13]),
(iii) *finite unions of convex compact subsets of locally convex topological vector spaces,*
(iv) *compact metric* **ANR***'s,*
(v) *compact metric approximate* **ANR***'s* ([9]),
(vi) *compact* **ANR** *spaces.*

Then $|K| \times Y$ is a Lefschetz space. □

The proof of Corollary 3.8 may be obtained by observing that for the classes (i)–(vi) the product of a space from one of these classes and the cone over a compact polyhedron also belongs to the class and is a compact Lefschetz space.

Theorem 3.7, Corollary 3.8 and Theorem 3.4 give a partial solution to problems (P2)–(P4). We now formulate several results on fixed points for non-compact spaces.

Theorem 3.9. *Let $|K|$ be a contractible polyhedron and Y belong to one of the following classes:*

(i) *compact metric* **AR***'s,*
(ii) *compact convex subsets of locally convex topological vector spaces,*
(iii) *I^τ, the Tikhonov cube of weight τ,*
(iv) *compact polyhedra Y such that $Y \times I$ has the fixed-point property,*
(v) *compact nested spaces* ([18]–[20]).

Proof. For (i)–(iii) the proof follows immediately from Corollary 3.8. For case (iv) the author [24] showed that any polyhedron homotopically equivalent to Y has the fixed-point property, i.e. for any compact polyhedron $|M| \subseteq |K|$ the space $|CM| \times Y$ has the fixed-point property. We complete the proof using Theorem 3.6. For the proof of (v) see [19], [20]. □

Let us conclude this section by showing the naturality of conditions (C1), (C2), (C3).

(C1) may be considered to be an analogue of nullhomotopy.

Condition (C2) is essential for the above theorems since for a non-countably compact arcwise connected space Y and a connected non-compact polyhedron

$|K|$ (even rayless) such that $|K| \times Y$ is a normal space there is a nullhomotopic map $f : |K| \times Y \to |K| \times Y$ without a fixed point. W. Holsztyński [10] and V. Okhezin [23] showed that such products contain closed rays. Other theorems on the necessity of countable compactness of factors were proved by the author in [23]. Taking into account the results of this section we have the following statement:

Theorem 3.10. *Let $|K|_m$ and $|L|_m$ be contractible polyhedra. Then the product $|K|_m \times |L|_m$ has the fixed-point property if and only if one factor is compact and the other is a rayless polyhedron.* □

Condition (C3) may be effectively verified using the classical methods of fixed-point theory.

4. Fixed-point theorems for non-compact metric ANR's

It is well known that the classical Lefschetz–Hopf fixed-point theorem may be extended to several classes of compact maps and spaces. It is of great interest to extend the above results to classes of non-compact **ANR**'s, i.e. to solve problems (P2), (P3), (P4). In this section we make such an extension using methods of $\mathcal{U}$-dominating **ANR**'s by polyhedra and introducing some approximative analogues of a rayless space.

First consider the following definitions which will play a central rôle in our discourse.

Definition 4.1. An open cover $\mathcal{U}$ of a topological space is called *rayless* if and only if the nerve of $\mathcal{U}$ is a rayless polyhedron.

Definition 4.2. A topological space is called *approximatively rayless* if any open cover of it has a rayless refinement.

I do not know if the property of X being approximatively rayless is necessary for X to have the fixed-point property. On the other hand, in the following typical situation any rayless space is approximatively rayless. In [8] V. Filippov constructed an example of a complete metric locally arcwise connected space with an open cover having no locally finite open refinement consisting of arcwise connected sets. The following definition is natural here.

Definition 4.3. A topological space is called *finely paracompact* if any open cover has a locally finite open refinement consisting of arcwise connected sets.

It may be worth observing that a finely paracompact space is a paracompact locally arcwise connected space. Any locally compact or σ-compact or totally paracompact locally arcwise connected space is finely paracompact. Below we describe some more classes of finely paracompact spaces. The following proposition shows the importance of the notion of fine paracompactness in fixed-point theory.

Proposition 4.4. *A finely paracompact rayless space is approximatively rayless.*

Proof. Assume the contrary, i.e. there is an open cover $\mathcal{U}$ of a finely paracompact rayless space such that the nerve of any open refinement of $\mathcal{U}$ contains a closed ray. Consider a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of arcwise connected sets. Then the polyhedron $|N(\mathcal{V})^{(1)}|$ contains a closed ray. There is a sequence $\{V_n \mid n \in \mathbb{N}\}$ of pairwise distinct elements of $\mathcal{V}$ such that $V_n \cap V_{n+1} \neq \emptyset, n \in \mathbb{N}$. Choose an arbitrary point $x_n \in V_n \cap V_{n+1}$ and consider the family $\mathcal{F} = \{[x_n, x_{n+1}] \subseteq V_{n+1} \mid n \in \mathbb{N}\}$ of arcs. Since $\mathcal{V}$ is locally finite, $\mathcal{F}$ is a locally finite family of compact arcs. Hence $L = \bigcup\{[x_n, x_{n+1}] \mid n \in \mathbb{N}\}$ is a connected, locally compact, locally arcwise connected metric non-compact space and L is a closed subset of the space. It follows ([5]) that there is a topological ray $l \subseteq L$ such that l is closed in L, thus l is a closed ray in the space. This contradiction concludes the proof. $\square$

The following theorem is well known and is one of the most important in the theory of **ANR**'s.

Theorem 4.5 ([17]). *Let X be a metric* **ANR**. *Then for every open cover $\mathcal{U}$ of X there exists a polyhedron P* (*with the Whitehead topology*) *such that P $\mathcal{U}$-dominates X.* $\square$

Arguing similarly to the proof of this theorem we prove the following theorem.

Theorem 4.6. *Let X be an approximatively rayless metric* **ANR**. *Then for every open cover $\mathcal{U}$ of X there exists a rayless polyhedron P* (*with the Whitehead topology*) *such that P $\mathcal{U}$-dominates X.*

Proof. Let $\mathcal{U}$ be an arbitrary open cover of X. It is well known (see [17]) that there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for every space Y, any two $\mathcal{V}$-close maps $f, g : Y \to X$ are $\mathcal{U}$-homotopic. As in the classical theory ([17]), $\mathcal{V}$ has an open refinement $\mathcal{W}$ such that for every simplicial complex $\mathcal{T}$ and any subcomplex $\mathcal{S} \subseteq \mathcal{T}$ containing all the vertices of $\mathcal{T}$, and for every partial realization $f : |\mathcal{S}| \to X$ of $\mathcal{T}$ in X relative to $(\mathcal{S}, \mathcal{W})$ there exists a full realization $g : |\mathcal{T}| \to X$ of $\mathcal{T}$ in X relative to $\mathcal{V}$ having the following properties:

(i) g extends f and
(ii) for every $\sigma \in \mathcal{S}$ and for every $W \in \mathcal{W}$ with $f(|\sigma| \cap |\mathcal{S}|) \subseteq W$ there exists $V \in \mathcal{V}$ such that $g(|\sigma|) \cup W \subseteq V$.

Note that we do not assume the local finiteness of $|\mathcal{T}|$. Since X is a metric and hence paracompact space there is a star-refinement $\mathcal{B}$ of $\mathcal{W}$. Since X is an approximatively rayless space there exists a rayless refinement $\mathcal{A}$ of $\mathcal{B}$. Finally, we have a rayless star-refinement $\mathcal{A}$ of $\mathcal{W}$. Using paracompactness again, consider a locally finite refinement $\mathcal{F}$ of $\mathcal{A}$ and a continuous k-mapping $k : X \to |N(\mathcal{F})|$ (the canonical projection). Let $p : |N(\mathcal{F})| \to |N(\mathcal{A})|$ be the natural simplicial continuous mapping (such a map exists since $\mathcal{F} \prec \mathcal{A}$). Then $\pi = p \circ k : X \to |N(\mathcal{A})|$ is a continuous mapping. Consider the rayless polyhedron $P = |N(\mathcal{A})|$. For every vertex $x(A) \in N(\mathcal{A})^{(0)}$ choose an arbitrary point $f(x(A)) \in A$. This defines a continuous function $f : |N(\mathcal{A})^{(0)}| \to X$. $\mathcal{A}$ being a star-refinement of $\mathcal{W}$ ($\mathcal{A} \prec* \mathcal{W}$) implies that f is a partial realization of $N(\mathcal{A})$ in X relative to $(N(\mathcal{A})^{(0)}, \mathcal{W})$ (see [17]) (i.e. for any simplex $\sigma \in N(\mathcal{A})$ there exists $W \in \mathcal{W}$ such that $f(|\sigma| \cap |N(\mathcal{A})^{(0)}|) \subseteq W$).

Let $g : |N(\mathcal{A})| \to X$ satisfy conditions (i) and (ii). We shall show that I_X and $g \circ \pi$ are $\mathcal{V}$-close mappings. Take an arbitrary $x \in X$. Since $\mathcal{F}$ is a locally finite open cover, there exist finitely many elements of $\mathcal{F}$ that contain x, say $F_0, F_1, \ldots, F_n$. Then $|\sigma'| = |\{x(F_0), \ldots, x(F_n)\}|$ is a simplex in $|N(\mathcal{F})|$ and $k(x) \in |\sigma'|$. For every $i = 0, \ldots, n$ there is an element A_i of $\mathcal{A}$ such that $F_i \subseteq A_i$. Let $p(x(F_i)) = x(A_i)$. Then the image of $|\sigma'|$ under p is a simplex $|\sigma| = |\{x(A_0), \ldots, x(A_n)\}|$ in $|N(\mathcal{A})|$. Indeed, $\bigcap_{i=0}^n F_i \neq \emptyset$ implies that $\bigcap_{i=0}^n A_i \neq \emptyset$, i.e. $|\sigma|$ is a simplex in $|N(\mathcal{A})|, |\sigma| = p(|\sigma'|)$. Therefore $p \circ k(x) \in |\sigma|$, $x \in \bigcup_{i=0}^n A_i$. There exists $W \in \mathcal{W}$ such that $x \in \bigcup_{i=0}^n A_i \subseteq W$. Since $f(x(A_i)) \in A_i$ for any $i = 0, \ldots, n$, we have

$$f(|\sigma| \cap |N(\mathcal{A})^{(0)}|) \subseteq W.$$

Condition (ii) implies that there is $V \in \mathcal{V}$ such that $g(|\sigma|) \cup W \subseteq V$. Since $p \circ k(x) \in |\sigma|$ we have $g \circ p \circ k(x) \in g(|\sigma|) \subseteq V$, i.e. $g \circ p \circ k(x) \in V$. Since $W \subseteq V$ it follows that $x \in V$. We conclude that $\{x, g \circ p \circ k(x)\} \subseteq V$. By the special choice of $\mathcal{V}$, we now conclude that the $\mathcal{V}$-close maps I_X and $g \circ \pi$ are $\mathcal{U}$-homotopic. Hence P $\mathcal{U}$-dominates X. □

The following is the main theorem in this section.

THEOREM 4.7. *Let X be a metric* **ANR**. *If $f : X \to X$ is a continuous map then one of the following holds:*

(i) *f has a fixed point* ($\mathrm{Fix}\, f \neq \emptyset$),
(ii) *f is not nullhomotopic* ($f \not\simeq 0$),
(iii) *X is not an approximatively rayless space.*

In the proof of this theorem one observation from approximative fixed-point theory is used (see [4]).

Proof. It is sufficient to prove that X being approximatively rayless and $f : X \to X$ being nullhomotopic imply that f has a fixed point. Take any open cover $\mathcal{U}$, and let $\mathcal{V}$ be a star-refinement of $\mathcal{U}$. By Theorem 4.6 there are a rayless polyhedron P and maps $g : P \to X$ and $\pi : X \to P$ such that $(g \circ \pi, I_X) \prec \mathcal{V}$, and $g \circ \pi$ and I_X are $\mathcal{U}$-homotopic. Then $\psi = \pi \circ f \circ g : P \to P$ is a nullhomotopic map of the rayless polyhedron P into itself. By Theorem 3.4 there exists a fixed point $y \in P$, $y = \psi(y) = \pi \circ f \circ g(y)$. Consider the point $x = g(y)$, $y = \pi \circ f(x)$. We have $x = g(y) = g \circ \pi \circ f(x)$. Now $(g \circ \pi, I_X) \prec \mathcal{V}$ implies that there is $V \in \mathcal{V}$ such that $\{x, f(x)\} \subseteq V$. We conclude that

($*$) *for any open cover $\mathcal{U}$ of X there is $x \in X$ such that there exists $U \in \mathcal{U}$ such that $\{x, f(x)\} \subseteq U$.*

Assume that for every $x \in X$, $x \neq f(x)$. Since X is a Hausdorff space, for every $x \in X$ there are open neighborhoods U_x and $V_{f(x)}$ of x and $f(x)$ respectively such that $U_x \cap V_{f(x)} = \emptyset$. Since f is continuous, for any $x \in X$ there is a neighborhood W_x of x, $W_x \subseteq U_x$, such that $f(W_x) \subseteq V_{f(x)}$. For any $U \in \{W_x\}$ and any $x \in X$, $x \in U$ implies that $f(x) \notin U$. This contradicts ($*$). Therefore f has a fixed point. □

Now we formulate a theorem giving a partial answer to problem (P3).

Theorem 4.8. *An approximatively rayless metric* **AR** *has the fixed-point property.* □

This easily follows from Theorem 4.7 and the fact that any **AR** is a contractible space.

The property of being an approximatively rayless space is extremely close to being a necessary condition for the fixed-point property. Using Proposition 4.4 we obtain the following characterization of the fixed-point property in several classes of non-compact **AR**'s.

Theorem 4.9. *A finely paracompact metric* **AR** *has the fixed-point property if and only if it is a rayless space.* □

Theorem 4.10. *A σ-compact metric* **AR** *has the fixed-point property if and only if it is a rayless space.* □

Theorem 4.11. *A totally paracompact metric* **AR** *has the fixed-point property if and only if it is a rayless space.* □

In conclusion we note that a complete solution of problem (P3) is not known.

5. Fixed-point theorem in locally convex spaces

In this section we use the notion of a topologically convex set to obtain a theorem on fixed points for contractible spaces which are not necessarily **AR**. Let us recall the definition of a convex set.

Definition 5.1 ([16]). A space X is called *equi-locally convex* (ELCX) if there exists an open neighborhood U of the diagonal Δ in $X \times X$, a continuous mapping $\varphi : U \times I \to X$ and an open cover $\mathcal{V} = \{V_\alpha\}$ of X such that:

(i) for any $t \in I$ and $x \in X$, $\varphi(x, x, t) = x$,
(ii) for any $x, y \in X$, $\varphi(x, y, 0) = x$, $\varphi(x, y, 1) = y$,
(iii) for any $V_\alpha \in \mathcal{V}$, $V_\alpha \times V_\alpha \subseteq U$,
(iv) for any $V_\alpha \in \mathcal{V}$, $\varphi(V_\alpha \times V_\alpha \times I) \subseteq V_\alpha$.

A subset $W \subseteq X$ is called *convex* if $W \times W \subseteq U$ and $\varphi(W \times W \times I) \subseteq W$, and the cover $\mathcal{V}$ is called a *convex open cover* of X.

Definition 5.2. An ELCX space is called *coparacompact* if any open cover has a locally finite convex refinement.

Note that any convex set is arcwise connected so we have the following proposition.

Proposition 5.3. *Any coparacompact space is finely paracompact.* □

Using Proposition 4.4 we obtain the following proposition.

Proposition 5.4. *A coparacompact rayless space is approximatively rayless.* □

The main result of this section is the following theorem.

Theorem 5.5. *A contractible coparacompact space has the fixed-point property if and only if it is a rayless space.*

Proof. Let X be a contractible coparacompact rayless space. Then X is approximatively rayless. Let $\mathcal{U}$ be an arbitrary open cover of X. Then there exists an open locally finite convex refinement $\mathcal{V}$ of $\mathcal{U}$. Observe that $|N(\mathcal{V})| = P$ is a rayless polyhedron. In ([16], p. 134) it is proved that the canonical projection $k : X \to |N(\mathcal{V})|$ is a homotopy equivalence. Therefore X is approximatively dominated by a family of contractible rayless polyhedra which have the fixed-point property. On the other hand, the fixed-point property is an approximatively dominated invariant (see [4]) in the class of paracompact spaces. So X has the fixed-point property. □

Remark. Since a cone is a particular case of the cylinder of a continuous mapping into a compact polyhedron, we may consider polyhedra homotopically equivalent to compact ones. Using the results from [2] we can construct an analogue of our canonical retraction. Then arguing similarly to the proof of Theorem 3.1 we obtain the following theorem:

Theorem. *Let $|K|$ be a polyhedron homotopically equivalent to a compact one. Then $|K|$ is a Lefschetz space if and only if it is a rayless space.*

The proof will be published in the second part of the paper.

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