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ESTIMATION OF THE SECOND DERIVATIVES FOR SURFACES EVOLVING UNDER THE ACTION OF THEIR PRINCIPAL CURVATURES

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Dedicated to Louis Nirenberg

1. Introduction

In our paper [8], we have formulated some results on global unique solvability of the first initial-boundary value problem for equations of the form

$$M[u] = -\frac{u_t}{\sqrt{1+u_x^2}} + f(k(u)) = g \quad \text{in } Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1}, \tag{1}$$

$$u - \varphi = 0 \quad \text{on } \partial' Q = \partial'' Q \cup \Omega(0), \tag{2}$$

where $\partial'' Q = \partial\Omega \times [0,T]$, $\Omega(0) = \{z = (x,t) : x \in \Omega,\ t = 0\}$, and Ω is a domain in $\mathbb{R}^n$ with a smooth boundary, which only for the sake of simplicity we assumed to be bounded. In (1), (2), g and φ are smooth known functions of z, defined on $\overline{Q}$, and $k(u) = (k_1(u), \dots, k_n(u))$, where $k_i(u)(z)$ are the principal curvatures of the graph $\mathcal{T}_t$:

$$x_0 = u(x,t), \quad x \in \overline{\Omega},$$

of the sought function $u(\,\cdot\,,t) : \overline{\Omega} \to \mathbb{R}^1$ for fixed $t \in [0,T]$.

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In [8] we studied the cases

$$f(k) = f_m(k) = S_m(k)^{1/m}, \quad S_m(k) = \sum_{i_1<\ldots<i_m} k_{i_1}\ldots k_{i_m},$$

with $m = 2,\ldots,n$. Contrary to the case $m = 1$, considered by many authors, equation (1) with $f = f_m$, $m > 1$, is non-totally parabolic. Its main domain of parabolicity is the cone

$$\Gamma_m = \{k : k \in \mathbb{R}^n,\ S_l(k) > 0,\ l = 1,\ldots,m\} \subset \Pi^n_+ = \Big\{k : S_1(k) = \sum_{i=1}^n k_i > 0\Big\}$$

(for the properties of f_m on Γ_m, see [1], [4]).

Here we recall the principal result of [8]. To formulate it let us include problem (1), (2) in the family of problems

$$(3_\tau)\qquad \begin{gathered} M[u^\tau] = g^\tau \quad \text{in } Q,\\ (u^\tau - \varphi^\tau)|_{\partial''Q} = 0, \quad (u^\tau - \varphi^0)|_{\Omega(0)} = 0, \quad \tau \in [0,1], \end{gathered}$$

where $\varphi^0(x,t) = \varphi(x,0)$, $\varphi^\tau = \tau\varphi + (1-\tau)\varphi^0$, $g^\tau = \tau g + (1-\tau)g^0$ and $g^0(x,t) = f_m(k(\varphi(x,0)))$.

For $\tau = 1$, problem (3_τ) coincides with problem (1), (2) and for $\tau = 0$ it has the solution $u^0(x,t) = \varphi(x,0)$.

We call a function u^τ an *admissible solution* of (3_τ) if $u^\tau \in C^{2,1}(\overline{Q})$, u^τ satisfies (3_τ) and for any $z \in \overline{Q}$, $k(u^\tau)(z)$ belongs to Γ_m.

The following theorem holds:

Theorem 1. *Any of the problems (3_τ) with $f = f_m$, $m > 1$, $\tau \in [0,1]$, including the problem* (1), (2) *with $f = f_m$, has a unique admissible solution u^τ belonging to $H^{4+\alpha,2+\alpha/2}(\overline{Q})$ if the following conditions are satisfied:*

(a) $\partial\Omega \in \Gamma_m \cap H^{4+\alpha}$, $g \in H^{2+\alpha,1+\alpha/2}(\overline{Q})$, $\varphi \in H^{4+\alpha,2+\alpha/2}(\overline{Q})$, $k(\varphi^0)(x) \in \Gamma_m$ *for* $x \in \overline{\Omega}$, *φ and g satisfy on $\partial\Omega(0)$ the compatibility conditions up to the second order;*

(b) $$\inf_Q g \geq 0, \quad \inf_{\partial'Q} u_t + \inf_Q g \equiv \nu_1 > 0, \quad g_t \leq 0 \quad \text{in } Q,$$

(c) *there is a common minorant c for $\partial u^\tau/\partial n$, i.e.,*

$$\inf_{\tau\in[0,1]} \inf_{\partial''Q} \frac{\partial u^\tau}{\partial n} \geq c.$$

Here $H^{k+\alpha,l+\beta}(\overline{Q})$ are Hölder spaces with $\alpha,\beta \in (0,1)$. The symbol $\partial/\partial n$ in (c) is the derivative in the direction of the inner normal to $\partial\Omega$. The inclusion $\partial\Omega \in \Gamma_m$ for $m < n$ means that for $\omega(x_1,\ldots,x_{n-1})$ defining $\partial\Omega$ in local cartesian coordinates, $(k_1(\omega),\ldots,k_{n-1}(\omega)) \in \Gamma_m \subset \mathbb{R}^{n-1}$, and $\partial\Omega \in \Gamma_m$ for $m = n$ means that Ω is strictly convex[1].

[1] The first part of the hypothesis b) in Theorem 1 of [8] can be eliminated for $T < \infty$.

The core of the proof of Theorem 1 consists in getting a priori estimates for the norms of u^τ in the spaces $H^{2+\beta,1+\beta/2}(\overline{Q})$ with a $\beta > 0$. The conditions under which this was done for problem (1), (2) are satisfied for any of the problems (3_τ), $\tau \in [0,1]$, and majorants could be chosen to be independent of τ. This is why in what follows we speak only about problem (1), (2) and its admissible solutions.

In this paper we estimate $\sup_Q |u_{xx}|$, supposing that the estimates

$$\sup_Q |u_x| \le M_1 \quad \text{and} \quad \sup_{\partial' Q} |u_{xx}| \le M_2 \tag{4}$$

are known. We will do that for a class of symmetric functions f (this means that f is invariant with respect to transpositions of the arguments $k_1, \dots, k_n$) which are defined on a domain $\mathcal{D} \subset \mathbb{R}^n$ containing the values of $k(u)(z)$, $z \in \overline{Q}$, for the solution $u(z)$ under consideration. One of the conditions imposed on f is its ellipticity on $\mathcal{D}$, i.e.,

$$f^i(k) \equiv \frac{\partial f(k)}{\partial k_i} > 0, \quad i = 1, \dots, n,\ k \in \mathcal{D}. \tag{5}$$

Other conditions on f will be formulated in Sec. 3. All of them are satisfied for $f = f_m$, $m = 2, \dots, n$.

It is known (see [3]) that any real-valued smooth symmetric function f can be represented as a smooth function F of symmetric matrices which is invariant under the transformations $A \to BAB^*$ with any orthogonal matrix B. Let us write this in the form

$$f(\lambda(A)) = F(A), \tag{6}$$

where $\lambda_i(A)$, $i = 1, \dots, n$, are the eigenvalues of A. The properties (5) guarantee the inequalities

$$\frac{\partial F(A)}{\partial A_{ij}} \xi_i \xi_j > 0, \tag{7}$$

where A_{ij} are the elements of A, and ξ is any vector from $\mathbb{R}^n$ with $|\xi| = 1$. Here and later, a double repeated index implies summation from 1 up to n.

The principal curvatures $k_i(u)$ of the surface $\mathcal{T}_t$ for $x \in \overline{\Omega}$ are the eigenvalues of the matrix

$$\frac{1}{\gamma(u)} g(u_x)^{-1/2} u_{xx} g(u_x)^{-1/2} \equiv \frac{1}{\gamma(u)} u_{(xx)}, \tag{8}$$

calculated at the point $z = (x,t) \in \overline{Q}$. Here and in the sequel we use the notations: $\gamma(u) = \sqrt{1 + u_x^2}$, u_{xx} is the Hessian of u with the elements $u_{ij} = u_{x_i x_j}$;

$g(u_x)$ is the metric tensor of the surface $\mathcal{T}_t$, its elements are $g_{ij}(u_x) = \delta_i^j + u_i u_j$, where $u_i = u_{x_i}$. From this and (6) we have

$$f(k(u)) = F\left(\frac{1}{\gamma(u)} u_{(xx)}\right) \equiv \mathcal{F}(u_x, u_{xx}). \tag{9}$$

The property (7) implies the inequalities

$$\frac{\partial \mathcal{F}(u_x, u_{xx})}{\partial u_{ij}} \xi_i \xi_j > 0 \quad \text{for all } \xi \text{ with } |\xi| = 1. \tag{10}$$

2. Estimation of u_t

First, we estimate $\sup_Q |u_t|$ for the equations

$$-\frac{u_t}{1+u_x^2} + \mathcal{F}(u_x, u_{xx}) = g(x,t) \tag{2.1}$$

with an arbitrary smooth function $\mathcal{F}$ satisfying only the inequalities (10) on the solution u. Let us differentiate (2.1) with respect to t. The result can be represented as a linear equation for $w = u_t$:

$$-a w_t + a_{ij} w_{ij} + b_i w_i = g_t. \tag{2.2}$$

Its coefficients are bounded functions, and $a > 0$ and a_{ij} satisfy

$$a_{ij}\xi_i\xi_j \equiv \frac{\partial \mathcal{F}(u_x, u_{xx})}{\partial u_{ij}} \xi_i \xi_j > 0, \quad |\xi| = 1. \tag{2.3}$$

A standard reasoning, based on the maximum principle for parabolic equations, yields

$$u_t(z) \geq \min_{\partial' Q} u_t \equiv \nu_2 \quad \text{if } g_t \leq 0 \text{ in } Q, \tag{2.4}$$

$$u_t(z) \leq \max_{\partial' Q} u_t \equiv \mu_1 \quad \text{if } g_t \geq 0 \text{ in } Q. \tag{2.5}$$

Let us also find a majorant for u_t without the hypothesis that $g_t \geq 0$. For this purpose we introduce the function $v = we^{-\lambda t}$, $\lambda > 0$. By (2.2),

$$-\frac{1}{\sqrt{1+u_x^2}}(v_t + \lambda v) + a_{ij} v_{ij} + b_i v_i = g_t e^{-\lambda t}. \tag{2.6}$$

If v attains its maximum on $\overline{Q}$ at $(x^0, t^0) \in Q$ then at this point we have $v_t \geq 0$ and $a_{ij} v_{ij} + b_i v_i \leq 0$, which, together with (2.6), leads to

$$\lambda v(x^0, t^0) \leq -e^{-\lambda t^0} g_t \sqrt{1+u_x^2}\big|_{(x^0,t^0)},$$

and then

$$u_t(x,t) \leq e^{\lambda t} \sup_{(y,\tau)\in Q_t} \left\{ -\frac{1}{\lambda} e^{-\lambda \tau} g_\tau(y,\tau) \sqrt{1+u_y^2(y,\tau)} \right\}, \quad Q_t = \Omega \times (0,t).$$

If the point (x^0, t^0) of the maximum of v on $\overline{Q}$ happens to lie on $\partial' Q$, then $v(x,t) \le v(x^0, t^0)$ for any (x,t) and

$$u_t(x,t) \le e^{\lambda t} \sup_{(y,\tau)\in\partial' Q_t} \{u_\tau(y,\tau) e^{-\lambda\tau}\}.$$

So for both cases, using the arbitrariness of $\lambda > 0$, we get

$$u_t(x,t) \le \inf_{\lambda>0} \Big\{ e^{\lambda t} \max \Big[\sup_{(y,\tau)\in Q_t} \Big(-\frac{1}{\lambda} e^{-\lambda\tau} g_\tau(y,\tau) \sqrt{1+u_y^2(y,\tau)} \Big), \sup_{(y,\tau)\in\partial' Q_t} (u_\tau(y,\tau) e^{-\lambda\tau}) \Big] \Big\} \equiv \mu_2(t), \tag{2.7}$$

for all $x \in \overline{\Omega}$ and $t \in [0,T]$.

We sum up these conclusions in the following theorem.

THEOREM 2. *Let u be a smooth solution of* (2.1) *with smooth $\mathcal{F}$ and g, and $\mathcal{F}$ satisfying* (2.3). *Then:*

(a)
$$\nu_2 \equiv \min_{\partial' Q} u_t \le u_t(x,t) \le \max_{\partial' Q} u_t = \mu_1, \quad (x,t) \in \overline{Q},$$
if $g_t \equiv 0$;

(b)
$$\nu_2 \equiv \min_{\partial' Q} u_t \le u_t(x,t) \le \mu_2(t), \quad (x,t) \in \overline{Q},$$
with $\mu_2(t)$ from (2.7), *if $g_t \le 0$.*

For a complete investigation of the problem (1), (2), we need to have some bounds for the values of $f(k(u))(z)$, $z \in \overline{Q}$. In virtue of (1), the equality $f(k(u)) = u_t/\sqrt{1+u_x^2} + g$ and Theorem 2 with $\mathcal{F}(u_x, u_{xx}) = f(k(u))$ we come to the following conclusions.

If $\inf_Q g \ge 0$, $\nu_1 \ge 0$ and $g_t \le 0$ in Q, then

$$\begin{aligned} f(k(u)) &= \frac{u_t}{\sqrt{1+u_x^2}} + g \ge \frac{1}{\sqrt{1+u_x^2}} (\nu_2 + \inf_Q g) \\ &\ge \frac{\nu_1}{\sqrt{1+M_1^2}} \equiv \nu_3, \end{aligned} \tag{2.8}$$

with M_1 from (4) of Sec. 1. Under the same hypothesis about g the majorant $\mu_2(t)$ in (2.7) is nonnegative and therefore

$$f(k(u)) = \frac{u_t}{\sqrt{1+u_x^2}} + g \le \mu_2(t) + \sup_Q g \le \sup_{t\in[0,T]} \mu_2(t) + \sup_Q g \equiv \mu_3. \tag{2.9}$$

So, if $\inf_Q g \ge 0$, $\nu_1 > 0$ and $g_t \le 0$ we have

$$0 < \nu_3 \le f(k(u))(z) \le \mu_3. \tag{2.10}$$

Note that if $\nu_2 = \min_{\partial' Q} u_t \ge 0$ then instead of (2.8) there is another minorant for f that does not require the condition $\inf_Q g \ge 0$, namely,

$$f(k(u)) \ge \frac{\nu_2}{\sqrt{1+M_1^2}} + \inf_Q g \equiv \widetilde{\nu}_3 \quad \text{if } \nu_2 = \min_{\partial' Q} u_t \ge 0. \tag{2.11}$$

Under the conditions of Theorem 1, we have thus found some positive bounds ν_3 and μ_3 for $f(k(u))(z)$. The inequality $f(k(u))(z) \geq \nu_3 > 0$ guarantees that $k(u)(z)$ in the course of evolution never leaves the cone Γ_m of ellipticity for $f = f_m$.

In the next section we will make use of

$$\nu_4 \leq f(k(u)) \leq \mu_4, \quad z \in \overline{Q}, \tag{2.12}$$

for the solution $u(z)$.

3. Bounds for second derivatives

Let u be a smooth admissible solution of (1) with f satisfying (5), and let the majorants M_k in (4) and constants ν_4, μ_4 in (2.12) be known. We require additionally that

$$S_1(k) = \sum_{i=1}^{n} k_i \geq \Phi(f(k)), \quad k \in \mathcal{D}, \tag{3.1}$$

with a nondecreasing continuous function $\Phi : \mathbb{R}^1 \to \mathbb{R}^1$. This and (2.12) imply

$$S_1(k(u))(z) \geq \Phi(\nu_4) \equiv c_1, \quad z \in \overline{Q}. \tag{3.2}$$

If we find a majorant c_2 in

$$\sup_{z \in Q} k_i(u)(z) \leq c_2, \quad i = 1, \dots, n, \tag{3.3}$$

then we can conclude from (3.2) and (3.3) that

$$-k_i(u)(z) \leq \sum_{j \neq i} k_j(u)(z) - c_1 \leq (n-1)|c_2| + |c_1|, \quad i = 1, \dots, n. \tag{3.4}$$

Now a majorant c in

$$\sup_{Q} |u_{xx}| \leq c \tag{3.5}$$

is calculated elementarily.

To find c_2 in (3.3), we use the invariance of the left-hand side of (1) with respect to the choice of cartesian coordinates in the space $\mathbb{E}^{n+1}$ of variables (x, x_0). As above, we consider t in $u(x,t)$ as a parameter and associate with $u(\,\cdot\,, t)$ the surface $\mathcal{T}_t$ in $\mathbb{E}^{n+1}$ determined by the equation

$$x_0 = u(x,t), \quad x \in \overline{\Omega}.$$

Fix a point $P^0 \in \mathcal{T}_{t^0}$ in $\mathbb{E}^{n+1}$ with coordinates $(x^0, u^0 = u(x^0, t^0))$, $x^0 \in \Omega$, $t^0 \in (0, T]$, and denote by ν^0 the unit normal to $\mathcal{T}_{t^0}$ at P^0 directed according to increasing x_0, i.e.,

$$\nu^0 = \nu(\mathcal{T}_{t^0})(P^0) = \sum_{i=1}^{n} \left(\frac{-u_{x_i}}{\sqrt{1+u_x^2}} \right)(x^0, t^0) e_i + \left(\frac{1}{\sqrt{1+u_x^2}} \right)(x^0, t^0) e_0.$$

Here $e_1, \dots, e_n, e_0$ is the orthogonal basis in $\mathbb{E}^{n+1}$ corresponding to the coordinates $x_1, \dots, x_n, x_0$. Choose another orthogonal basis $\xi_1, \dots, \xi_n, \xi_0$ with $\xi_0 = \nu^0$. The matrix $B = (b_{\alpha\beta})$ with $b_{\alpha\beta} = (\varepsilon_\alpha, e_\beta)$, $\alpha, \beta = 1, \dots, n, 0$, is orthogonal and its elements $b_{0\beta} = (\varepsilon_0, e_\beta) = (\nu^0, e_\beta)$ are fixed. Later we will define the other rows of B in a proper way, always keeping the orthogonality of B. For now they are taken arbitrary.

Denote by $(y_1, \dots, y_n, y_0)$ the cartesian coordinates in $\mathbb{E}^{n+1}$ corresponding to $\xi_1, \dots, \xi_n$, ξ_0, so that we have in $\mathbb{E}^{n+1}$ the relation

$$\sum_{i=1}^{n} (x_i - x_i^0) e_i + (x_0 - u^0) e_0 = \sum_{i=1}^{n} y_i \xi_i + y_0 \xi_0. \tag{3.6}$$

The surface $\mathcal{T}_t$ with t near t^0 can be represented near P^0 by the equation

$$y_0 = v(y,t), \quad y = (y_1, \dots, y_n), \tag{3.7$_1$}$$

where the function v is determined by the identity

$$\sum_{i=1}^{n} (x_i - x_i^0) e_i + (u(x,t) - u^0) e_0 = \sum_{i=1}^{n} y_i \xi_i + v(y,t) \xi_0. \tag{3.7$_2$}$$

In fact, it follows from (3.7$_2$) that

$$x_i - x_i^0 = \sum_{j=1}^{n} b_{ji} y_j + b_{0i} v(y,t), \quad i = 1, \dots, n, \tag{3.8$_1$}$$

$$u(x,t) - u^0 = \sum_{i=1}^{n} b_{i0} y_i + b_{00} v(y,t), \tag{3.8$_2$}$$

and

$$y_i = \sum_{j=1}^{n} b_{ij} (x_j - x_j^0) + b_{i0} (u(x,t) - u^0), \quad i = 1, \dots, n, \tag{3.9$_1$}$$

$$v(y,t) = \sum_{j=1}^{n} b_{0j} (x_j - x_j^0) + b_{00} (u(x,t) - u^0). \tag{3.9$_2$}$$

Relations (3.9$_k$) determine the functions

$$Y_i(x,t) = \sum_{j=1}^{n} b_{ij} (x_j - x_j^0) + b_{i0} (u(x,t) - u^0), \quad i = 1, \dots, n, \tag{3.10}$$

if we consider $u : \overline{Q} \to \mathbb{R}^1$ as a given function, and they also determine the functions X_j such that

$$y_i = b_{ij}(X_j(y,t) - x_j^0) + b_{i0}[u(X(y,t),t) - u^0], \quad i = 1, \dots, n, \tag{3.11$_1$}$$

for all y and t near $y = 0$, $t = t^0$. The function v from (3.7_1) is determined by the equality

$$v(y,t) = b_{0j}[X_j(y,t) - x_j^0] + b_{00}[u(X(y,t),t) - u^0]. \tag{3.11$_2$}$$

By the orthogonality of B, from (3.11_1), (3.11_2), we get the equalities

$$X_i(y,t) - x_i^0 = \sum_{j=1}^{n} b_{ji} y_j + b_{0i} v(y,t), \quad i = 1, \dots, n, \tag{3.11$_1$}$$

$$u(X(y,t),t) - u^0 = \sum_{i=1}^{n} b_{i0} y_i + b_{00} v(y,t), \tag{3.12$_2$}$$

which are identities near $y = 0$ and $t = t^0$. We can consider them also as the identities

$$x_i - x_i^0 = \sum_{j=1}^{n} b_{ji} Y_j(x,t) + b_{0i} v(Y(x,t),t), \tag{3.13$_1$}$$

$$u(x,t) - u^0 = \sum_{i=1}^{n} b_{i0} Y_i(x,t) + b_{00} v(Y(x,t),t) \tag{3.13$_2$}$$

with respect to (x,t) near (x^0,t^0), since the vector-valued function $Y \equiv (Y_1, \dots, Y_n)(x,t)$ is inverse to $X = (X_1, \dots, X_n)(y,t)$.

So, for t close to t^0, the surface $\mathcal{T}_t$, near $P^0 = (x^0, u^0)$, has the equation

$$y_0 = v(y,t), \tag{3.14}$$

where v is determined by (3.11_2). Note that the new variables y and v depend on z^0, but we will choose z^0 at a step and will not change it after that. Therefore, we do not indicate explicitly the dependence of the new variables on z^0.

The quotient $u_t/\sqrt{1+u_x^2}$ is invariant with respect to the change of variables $(x,u,t) \to (y,v,t)$, i.e.,

$$\frac{u_t}{\sqrt{1+u_x^2}}(x,t) = \frac{v_t}{\sqrt{1+v_y^2}}(y,t) \tag{3.15$_1$}$$

for the corresponding $(x,t) \leftrightarrow (y,t)$, since both sides of (3.15_1) give the velocity of the shift of $\mathcal{T}_t$ in the direction of the normal $\nu(\mathcal{T}_t)$ to $\mathcal{T}_t$. This will also follow from our subsequent calculations.

The analytical expressions for the principal curvatures $k_i(\mathcal{T}_t)$ of $\mathcal{T}_t$ in variables (x, x_0) and (y, y_0) also coincide, so we have

$$f(k(u))(x,t) = f(k(v))(y,t). \tag{3.15$_2$}$$

But $g(x,t)$ in the new variables will depend on v, namely,

$$\begin{aligned} &g(x_1,\dots,x_n,t) \\ &\quad = g(x_1^0 + b_{j1}y_j + b_{01}v(y,t),\dots,x_n^0 + b_{jn}y_j + b_{0n}v(y,t),t) \equiv \widehat{g}(y,t,v). \end{aligned} \tag{3.16}$$

So, equation (1) in the new variables has the form

$$-\frac{v_t(y,t)}{\sqrt{1+v_y^2}} + f(k(v))(y,t) = \widehat{g}(y,t,v). \tag{3.17}$$

Later, we will have to differentiate (3.17) with respect to y_m and twice with respect to y_1. For this purpose, let us calculate these derivatives for $\widehat{g}$:

$$\widehat{g}_{y_m} = g_{x_i}\frac{\partial x_i}{\partial y_m} = g_{x_i}(b_{mi} + v_{y_m}b_{0i}) \tag{3.18$_1$}$$

and

$$\widehat{g}_{y_1y_1} = g_{x_ix_j}(b_{1j} + v_{y_1}b_{0j})(b_{1i} + v_{y_1}b_{0i}) + g_{x_i}v_{y_1y_1}b_{0i}. \tag{3.18$_2$}$$

Using (3.10), we also calculate the relations between $u_i \equiv u_{x_i}$ and $v_j \equiv v_{y_j}$. From (3.13$_2$), it follows that

$$u_j = b_{i0}\frac{\partial Y_i}{\partial x_j} + b_{00}v_i\frac{\partial Y_i}{\partial x_j} = (b_{i0} + b_{00}v_i)(b_{ij} + u_jb_{i0}),$$

and from this and orthogonality of B we get

$$u_j(1 - b_{i0}b_{i0} - b_{00}b_{i0}v_i) = u_j(b_{00} - b_{i0}v_i)b_{00} = b_{i0}b_{ij} + b_{00}b_{ij}v_i = (-b_{0j} + b_{ii}v_i)b_{00}$$

and therefore

$$u_j = \frac{-b_{0j} + b_{ij}v_i}{b_{00} - b_{i0}v_i}, \quad j = 1,\dots,n. \tag{3.19}$$

Introducing the vector fields

$$\nu(v) = -\frac{v_i}{\sqrt{1+v_y^2}}\varepsilon_i + \frac{1}{\sqrt{1+v_y^2}}\varepsilon_0, \tag{3.20}$$

we rewrite (3.19) in the form

$$u_j = -\frac{(\nu(v), e_j)}{(\nu(v), e_0)}, \quad j = 1,\dots,n. \tag{3.21}$$

From (3.21), it follows that

$$1 + u_x^2 = (\nu(v), e_0)^{-2} \tag{3.22$_1$}$$

and

$$\frac{1}{\sqrt{1+u_x^2}} = (\nu(v), e_0) = \frac{1}{\sqrt{1+v_y^2}}(b_{00} - b_{j0}v_j). \tag{3.22$_2$}$$

For v_t we have from (3.11$_2$),

$$\begin{aligned} v_t &= b_{0j}\frac{\partial X_j}{\partial t} + b_{00}\bigg(u_j\frac{\partial X_j}{\partial t} + u_t\bigg) = (b_{0j} + b_{00}u_j)b_{0j}v_t + b_{00}u_t \\ &= (1 - b_{00}^2 + b_{00}b_{0j}u_j)v_t + b_{00}u_t. \end{aligned}$$

After reduction of similar terms this gives

$$u_t = (b_{00} - b_{0j}u_j)v_t$$

and from this and (3.21) we get

$$u_t = \frac{(\nu(v), \varepsilon_0)}{(\nu(v), e_0)}v_t.$$

Using this relation, (3.22$_2$) and (3.20), we find

$$u_t(\nu(v), e_0) = \frac{u_t}{\sqrt{1+u_x^2}} = v_t(\nu(v), \varepsilon_0) = \frac{v_t}{\sqrt{1+v_y^2}},$$

i.e. (3.15$_1$).

Now we start to calculate the majorant c_2 in (3.3). Let $z^0 = (x^0, t^0)$ be a point of $\overline{Q}$ where the maximum M of all functions

$$h(\eta)k_i(u)(z), \quad i = 1, \dots, n,\ z \in \overline{Q}, \tag{3.23$_1$}$$

is realized. Here $h(\,\cdot\,)$ is a smooth function of

$$\eta = \frac{1}{\sqrt{1+u_x^2}}. \tag{3.23$_2$}$$

It will be chosen later and defined on the interval

$$[b_1, 1], \quad b_1 = \frac{1}{\sqrt{1+M_1^2}}. \tag{3.23$_3$}$$

It is sufficient to consider the case when $z^0 \in Q$, since for $z^0 \in \partial' Q$ a majorant for $h(\eta)k_i(u)$, and therefore for $k_i(u)$, is given by (4).

So, let $z^0 = (x^0, t^0)$ lie in Q. We now use the new variables y, t, v described above, choosing for the origin of coordinates (y, y_0) the point $P^0 = (x^0, u^0 \equiv u(x^0, t^0))$ of $\mathbb{E}^{n+1}$ and as ε_0 the unit normal

$$\nu^0 = -\sum_{i=1}^{n}\frac{\overset{\circ}{u}_i}{\sqrt{1+\overset{\circ}{u}{}_x^2}}e_i + \frac{1}{\sqrt{1+\overset{\circ}{u}{}_x^2}}e_0, \quad \overset{\circ}{u}_i \equiv u_{x_i}(x^0, t^0),$$

to $\mathcal{T}_{t^0} \subset \mathbb{E}^{n+1}$ at the point P^0. We direct the other basis vectors $\varepsilon_1, \dots, \varepsilon_n$ in $\mathbb{E}^{n+1}$ along the lines of the principal curvatures of $\mathcal{T}_{t^0}$ at P^0, and enumerate them in such a way that $k_1(u)(z^0) \ge k_i(u)(z^0)$, $i = 2, \dots, n$.

It is sufficient to consider the case when

$$k_1(u)(z^0) \equiv v_{y_1 y_1}(0, t^0) > 0. \tag{3.24}$$

In the new variables the functions from (3.23_1) have the form

$$h\biggl(\frac{1}{\sqrt{1+v_y^2}}(b_{00} - b_{j0} v_j)\biggr) k_i(v)(y,t), \quad i = 1, \dots, n. \tag{3.25}$$

They are defined in the vicinity

$$\widehat{Q}_\varepsilon = \{(y,t) : |y| \le \varepsilon,\ t \in [t^0 - \varepsilon, t^0]\}$$

of $z^0 \leftrightarrow (y = 0, t = t^0) \equiv \widehat{z}^0$ and have their local maximum at $\widehat{z}^0$.

The same local maximum $M > 0$ and also at the same $\widehat{z}^0$ is realized by the function

$$\Psi(y,t) = \biggl[h(\eta) \frac{v_{11}}{\gamma(v)(1+v_1^2)}\biggr](y,t), \quad \gamma(v) = \sqrt{1+v_y^2}, \tag{3.26_1}$$

where

$$\eta = \frac{1}{1+v_y^2}(b_{00} - b_{j0} v_j) = \frac{1}{1+u_x^2} \tag{3.26_2}$$

and, as above, $v_i = v_{y_i}$, $v_{ij} = v_{y_i y_j}$. In contrast to $k_i(v)$, the smoothness of Ψ depends only on the smoothness of v, and therefore at the maximum point $\widehat{z}^0$ of Ψ we have

$$(\ln \Psi)_{y_i} = 0, \quad (\ln \Psi)_t \ge 0 \quad \text{and} \quad (\ln \Psi)_{y_i y_i} \le 0. \tag{3.27}$$

Let us calculate (3.27) at $\widehat{z}^0$, keeping in mind that at $\widehat{z}^0$ we have

$$v_i = 0, \quad k_i(v) = v_{ii}, \quad v_{ij} = v_{ii} \delta_j^i. \tag{3.28}$$

First, we calculate

$$\begin{gathered} \partial_{y_i} \gamma(v) = \frac{v_k v_{ki}}{\gamma(v)}, \quad \partial_{y_i} \frac{1}{\gamma(v)} = -\frac{v_k v_{ki}}{\gamma(v)^3}, \\ \eta_i \equiv \eta_{y_i} = -\frac{v_k v_{ki}}{\gamma(v)^3}(b_{00} - b_{j0} v_j) - \frac{1}{\gamma(v)} b_{j0} v_{ji}. \end{gathered} \tag{3.29_1}$$

From these equalities, it follows that at $\widehat{z}^0$,

$$\partial_{y_i} \gamma(v) = 0, \quad \partial_{y_i} \frac{1}{\gamma(v)} = 0, \quad \partial^2_{y_i y_i} \gamma(v) = v_{ii}^2, \quad \eta_i = -b_{i0} v_{ii} \tag{3.29_2}$$

and also

$$\eta_{ii} \equiv \eta_{y_i y_i} = -v_{ii}^2 b_{00} - b_{j0} v_{jii}, \quad \text{where } v_{jii} \equiv v_{y_j y_i y_i}. \tag{3.29_3}$$

Now, computing (3.27) and bearing in mind (3.29_k), we obtain at $\widehat{z}^0$ the relations

$$(3.30_1)\qquad (\ln\Psi)_{y_i} = \frac{h'}{h}\eta_i + \frac{v_{11i}}{v_{11}} - \frac{v_k v_{ki}}{1+v_y^2} - \frac{2v_1 v_{1i}}{1+v_1^2} = 0,$$

$$(3.30_2)\qquad 0 \ge (\ln\Psi)_{y_i y_i} = \frac{h'}{h}\eta_{ii} + \left(\frac{h'}{h}\right)'\eta_i^2 + \frac{v_{11ii}}{v_{11}} - \frac{v_{11i}^2}{v_{11}^2} - \frac{\sum_{k=1}^n v_{ki}^2}{1+v_y^2} - \frac{2v_{1i}^2}{1+v_1^2},$$

$$(3.30_3)\qquad 0 \le (\ln\Psi)_t = \frac{h'}{h}\eta_t - \frac{v_{11t}}{v_{11}}.$$

These relations and (3.28) give us

$$(3.31_1)\qquad \frac{v_{11i}}{v_{11}} = -\frac{h'}{h}\eta_i = \frac{h'}{h}b_{i0}v_{ii},$$

$$(3.31_2)\qquad \frac{v_{11ii}}{v_{11}} \le \frac{h'}{h}(v_{ii}^2 b_{00} + b_{j0}v_{jii}) - \left[\left(\frac{h'}{h}\right)' - \left(\frac{h'}{h}\right)^2\right] b_{i0}^2 v_{ii}^2 + v_{ii}^2 + 2v_{1i}^2$$
$$= v_{ii}^2\left\{b_{00}\frac{h'}{h} - b_{i0}\left[\left(\frac{h'}{h}\right)' - \left(\frac{h'}{h}\right)^2\right] + 1\right\} + \frac{h'}{h}b_{j0}v_{jii} + 2v_{1i}^2$$

and

$$(3.31_3)\qquad -\frac{v_{11t}}{v_{11}} \le -\frac{h'}{h}b_{j0}v_{jt}.$$

Let us now make use of the equation (3.17) for v and the representation

$$(3.32)\qquad f(k(v)) = F(A(v)),$$

where

$$A(v) = \frac{1}{\gamma(v)}v_{(yy)}, \quad \gamma(v) = \sqrt{1+v_y^2},$$

and

$$(3.33)\qquad v_{(yy)} = g(v_y)^{-1/2} v_{yy} g(v_y)^{-1/2}$$

(see the end of Sec. 1). The numbers $k_i(v)(y,t)$ are the eigenvalues of the matrix $A(v)(y,t)$. The elements of the matrix $g(v_y)^{-1/2}$ have the form

$$(g(v_y)^{-1/2})_{ij} = \delta^i_j - \frac{v_i v_j}{\gamma(v)(1+\gamma(v))},$$

and the elements $v_{(ij)}$ of the matrix $v_{(yy)}$ are

$$(3.34)\qquad v_{(ij)} = v_{ij} - \frac{v_i v_k v_{kj}}{\gamma(v)(1+\gamma(v))} - \frac{v_j v_k v_{ki}}{\gamma(v)(1+\gamma(v))} + \frac{v_i v_j v_k v_l v_{kl}}{[\gamma(v)(1+\gamma(v))]^2}.$$

It is also known that (a) convexity of $f(k)$ with respect to k implies convexity of $F(A)$ with respect to A, and (b) at a point $A \in M^{n\times n}_{\mathrm{sym}}$ with A diagonal, the matrix $\partial F(A)/\partial A$ is diagonal with elements $\partial F(A)/\partial A_{ij}$. (These facts have been noticed and used, for example, in [1], [2].)

At $\widehat{z}^0$ we have

$$\gamma(v) = 1, \quad A(v) = v_{(yy)} = v_{yy} = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{nn} \end{pmatrix}, \quad k_i = v_{ii},$$
$$\frac{\partial F(A)}{\partial A_{ij}} = \frac{\partial F(A)}{\partial A_{ii}}\delta_i^j = \frac{\partial f(k)}{\partial k_i}\delta_j^i \equiv f^i(k)\delta_j^i. \tag{3.35}$$

Let us introduce the notation

$$F^i = \frac{\partial F(A)}{\partial A_{ii}}(\widehat{z}^0) = f^i(k)(\widehat{z}^0) \equiv f^i. \tag{3.36}$$

Below, we also use

$$\partial_{y_k} v_{(ij)} = v_{ijk},$$
$$\partial^2_{y_1 y_1} v_{(ii)} = v_{ii11} - 2v_{i1}v_{11}v_{1i} = v_{ii11} - 2v_{11}^3\delta_i^1 \quad \text{at } \widehat{z}^0, \tag{3.37}$$

where $v_{ijkl} = \partial^4_{y_i y_j y_k y_l} v$.

We rewrite equation (3.17) in the form

$$-v_t + \gamma(v)F\left(\frac{1}{\gamma(v)}v_{(yy)}\right) = \widehat{g}(y,t,v)\gamma(v) \tag{3.38}$$

and differentiate it with respect to y_m:

$$-v_{tm} + \gamma\frac{\partial F(A)}{\partial A_{ij}}\partial_{y_m}\left(\frac{1}{\gamma}v_{(ij)}\right) + \partial_{y_m}\gamma F = \partial_{y_m}(\widehat{g}\gamma), \tag{3.39}$$

where $A = \frac{1}{\gamma(v)}v_{(yy)}$. At $\widehat{z}^0$, (3.39) has the form

$$-v_{tm} + F^i v_{(ii)m} = \partial_{y_m}\widehat{g}. \tag{3.40}$$

Now we differentiate (3.39) for $m = 1$ with respect to y_1 and write the result at $\widehat{z}^0$, keeping in mind (3.28) and (3.35):

$$\begin{aligned} -v_{t11} + F^i v_{(ii)11} + \frac{\partial^2 F(v_{(yy)})}{\partial v_{(ij)}\partial v_{(kl)}}v_{(ij)1}v_{(kl)1} + \gamma_{11}(F - F^i v_{ii}) \\ = \partial^2_{y_1 y_1}\widehat{g} + \widehat{g}\gamma_{11}. \end{aligned} \tag{3.41}$$

Here we have used the notations

$$v_{(ij)1} = \partial_{y_1} v_{(ij)}, \quad v_{(il)11} = \partial^2_{y_1} v_{(il)}, \quad \gamma_{11} = \partial^2_{y_1}\gamma,$$

and later we will take into account that $\gamma_{11} = v_{11}^2 > 0$ at $\widehat{z}^0$. By concavity of $F(A)$, (3.41) implies the inequality

$$j_1 \equiv -v_{t11} + F^i v_{(ii)11} + v_{11}^2(F - F^i v_{ii}) \geq \partial^2_{y_1 y_1}\widehat{g} + \widehat{g}v_{11}^2 \equiv j_2. \tag{3.42}$$

By (3.31_k) and (3.18_k),

$$\partial_{y_m}\widehat{g}(\widehat{z}^0) = b_{mi}g_i(z^0), \quad \text{where } g_i = \partial_{x_i}g, \tag{3.43$_1$}$$

and

$$j_2 = g_{ij}(z^0)b_{1i}b_{1j} + g_i(z^0)b_{0i}v_{11}(\widehat{z}^0) + g(z^0)v_{11}^2(\widehat{z}^0), \qquad \text{where } g_{ij} = \partial^2_{x_ix_j}g. \tag{3.43$_2$}$$

So, (3.40) gives the equality

$$-v_{tm} + F^i v_{(ii)m} = b_{mi}g_i(z^0) \quad \text{at } \widehat{z}^0. \tag{3.44}$$

Using (3.31_3) and (3.37), we deduce from (3.42) that

$$\begin{aligned} j_2 \le j_1 &\le -\frac{h'}{h}b_{j0}v_{jt}v_{11} + F^i(v_{ii11} - 2v_{11}^3\delta_i^1) + v_{11}^2(F - F^iv_{ii}) \\ &= -\frac{h'}{h}b_{j0}v_{jt}v_{11} + F^iv_{ii11} - 2F^1v_{11}^3 + v_{11}^2(F - F^iv_{ii}). \end{aligned}$$

From this, (3.31_2) and $F^i > 0$ we get the inequality

$$\begin{aligned} j_2 \le\ & -\frac{h'}{h}b_{j0}v_{jt}v_{11} + F^iv_{ii}^2v_{11}\left(b_{00}\frac{h'}{h} + 1\right) \\ & - \sum_{i=1}^n F^iv_{ii}^2v_{11}b_{i0}^2\left[\left(\frac{h'}{h}\right)' - \left(\frac{h'}{h}\right)^2\right] \\ & + \frac{h'}{h}b_{j0}v_{11}F^iv_{jii} + v_{11}^2(F - F^iv_{ii}) \\ =\ & \frac{h'}{h}b_{j0}v_{11}(-v_{jt} + F^iv_{jii}) + F^iv_{ii}^2v_{11}\left(b_{00}\frac{h'}{h} + 1\right) \\ & - \sum_{i=1}^n F^iv_{ii}^2v_{11}b_{i0}^2\left[\left(\frac{h'}{h}\right)' - \left(\frac{h'}{h}\right)^2\right] + v_{11}^2(F - F^iv_{ii}). \end{aligned} \tag{3.45}$$

Using (3.44) and (3.37), we exclude from (3.45) the terms with v_{jt} and v_{jii} and obtain an inequality containing only the space derivatives of v of the first and second orders. Namely,

$$\begin{aligned} j_2 &= g_{ij}b_{1i}b_{ij} + g_ib_{0i}v_{11} + gv_{11}^2 \\ &\le \frac{h'}{h}b_{j0}v_{11}g_ib_{ji} - \sum_{i=1}^n F^iv_{ii}^2v_{11}b_{i0}^2\left[\left(\frac{h'}{h}\right)' - \left(\frac{h'}{h}\right)^2\right] \\ &\quad - H_0F^iv_{ii}^2v_{11} + v_{11}^2(F - F^iv_{ii}), \end{aligned} \tag{3.46}$$

where

$$H \equiv H(\eta) = -b_{00}\frac{h'(\eta)}{h(\eta)} - 1 \tag{3.47$_1$}$$

and

$$(3.47_2)\quad H_0 = H(\eta^0), \quad \eta^0 = \eta(\widehat{z}^0) = \frac{1}{\sqrt{1+u_x^2(z^0)}} \in [b_1, 1], \quad b_1 = \frac{1}{\sqrt{1+M_1^2}}.$$

We choose for $h(\eta)$ the solution

$$(3.48)\qquad h(\eta) = \frac{1}{\eta - b}, \quad b \in (0, b_1),$$

of the equation $(h'/h)' - (h'/h)^2 = 0$ (precisely this function h was used in [2] for the estimation of second derivatives of solutions to the stationary problem (1), (2)). Such an h is positive on $[b_1, 1]$,

$$H_0 = H(\eta^0) = \frac{b_{00}}{\eta^0 - b} - 1 = \frac{\eta^0}{\eta^0 - b} - 1 = \frac{b}{\eta^0 - b} \geq \frac{b}{1-b} \geq 0$$

as $b_{00} = (\varepsilon_0, e_0) = 1/\sqrt{1+\overset{\circ}{u}{}_x^2} = \eta^0$, and, in addition,

$$\left|\frac{h'(\eta^0)}{h(\eta^0)}\right| = \frac{1}{\eta^0 - b} \leq \frac{1}{b_1 - b}.$$

By all this, we obtain from (3.46) the relations

$$\begin{aligned}(3.49)\qquad v_{11}^2(g - F + F^i v_{ii} &+ H_0 F^i v_{ii}^2 v_{11}^{-1}) \\ &\leq -g_{ji} b_{1i} b_{1j} - g_i b_{0i} v_{11} + \frac{1}{b_1 - b} v_{11} |b_{j0} g_i b_{ji}| \\ &\leq c_3(1 + v_{11}),\end{aligned}$$

with a $c_3 = c_3(b)$ under control. Let us introduce the functions

$$(3.50)\qquad j_3(k, b) = -f(k) + f^i(k) k_i + \frac{b}{1-b} f^i(k) k_i^2 k_1^{-1}$$

and

$$(3.51)\qquad j_4(z^0, b) = g(z^0) + j_3(k(u)(z^0), b).$$

We consider them for $b \in (0, b_1)$ and for

$$\begin{aligned}(3.52)\quad k \in \widehat{\Gamma} &= \widehat{\Gamma}(\nu_4, \mu_4) \\ &= \{k : k \in \Gamma,\ \nu_4 \leq f(k) \leq \mu_4,\ k_1 \geq 1, k_1 \leq k_i,\ i = 1, \ldots, n\},\end{aligned}$$

where Γ is a domain of ellipticity of f, i.e. where (5) of Sec. 1 is satisfied.

If we can guarantee a positive minorant ν_5 in

$$(3.53)\qquad j_4(z^0, b) \geq \nu_5 > 0$$

for some $b \in (0, b_1)$ and all $z^0 \in Q$, then we obtain from (3.49) the estimate

$$(3.54)\qquad k_1(z^0) \leq c_4, \quad c_4 = \frac{c_3 + \sqrt{c_3^2 + 4c_3\nu_5}}{4\nu_5}.$$

Let us define the following characteristic of f:

$$\nu_6(b) = \inf_{k\in\widehat{\Gamma}} j_3(k,b), \quad b \in (0, b_1).$$

If

$$\inf_Q g + \nu_6(b) \equiv \nu_7 > 0, \tag{3.55}$$

then (3.53) holds with $\nu_5 = \nu_7$. If $f(k)$ is a 1-homogeneous function of k, then

$$j_3(k,b) = \frac{b}{1-b} f^i(k) k_i^2 k_1^{-1} > 0, \quad k \in \widehat{\Gamma}, \tag{3.56$_1$}$$

and

$$\nu_6(b) = \frac{b}{1-b} \inf_{k\in\widehat{\Gamma}} f^i(k) k_i^2 k_1^{-1}.$$

Thus, the inequality (3.55) will be satisfied if

$$\inf_Q g > 0. \tag{3.56$_2$}$$

For $f(k) = f_m(k) = S_m(k)^{1/m}$, $m > 1$, we have the estimate

$$f_m^i(k) \equiv \frac{\partial f_m(k)}{\partial k_i} \geq \frac{1}{m} \cdot \frac{f_m(k)}{f_1(k)} \quad \text{for all } k \in \Gamma_m, \tag{3.57}$$

which is easily derived from the consequence $S_m(k)/S_1(k) \leq \partial_{k_i} S_m(k)$ of the fact that the ratio $S_m(k)/S_1(k)$ is an increasing function of any k_i ([9]). Using it and

$$\frac{\sum_j k_j^2}{\sum_j k_j} k_1^{-1} \geq \frac{1}{\sqrt{n}}$$

we obtain the estimates

$$j_3(k,b) = \frac{b}{1-b} f_m^i(k) k_i^2 k_1^{-1} \geq \frac{b}{1-b} \cdot \frac{1}{m\sqrt{n}} f_m(k) \geq \frac{b}{1-b} \cdot \frac{\nu_4}{m\sqrt{n}} \tag{3.58$_1$}$$

for k in

$$\Gamma_m(\nu_4, \mu_4) = \{k : k \in \Gamma_m,\ \nu_4 \leq f_m(k) \leq \mu_4\}. \tag{3.58$_2$}$$

Under the hypothesis of Theorem 1, we have proved the positivity of ν_4, and therefore condition (3.53) for $f = f_m$ will be satisfied if

$$\inf_Q g + \frac{b}{1-b} \cdot \frac{\nu_4}{m\sqrt{n}} \equiv \nu_8 > 0. \tag{3.59}$$

Let us mention that in the stationary case

$$f(k)(x) = g(x), \quad u|_{\partial\Omega} = \varphi, \quad x \in \Omega \subset \mathbb{R}^n, \tag{3.60}$$

we have

$$j_4 = g - f(k) + f^i(k) k_i + \frac{b}{1-b} f^i(k) k_i^2 k_1^{-1} \geq f^i(k) k_i, \quad k \in \Gamma,\ k_1 > 0,$$

and the hypothesis

$$f^i(k)k_i \geq c_0 > 0 \tag{3.61}$$

for $k \in \Gamma(\nu_4, \mu_4) \equiv \{k : k \in \Gamma,\ \nu_4 \leq f(k) \leq \mu_4\}$ just corresponds to hypothesis (8) from Introduction of [2]. It guarantees (3.53) with $\nu_5 = c_0$.

Finally, let us show how to calculate a majorant c_2 in (3.3), having (3.54) at hand. If

$$\sup_{z\in Q,\, i=1,\dots,n} \frac{k_i(u)}{\eta - b} \equiv M$$

is achieved at a point $z^0 \in Q$, then we have found the estimate (3.54) and hence

$$M = \Psi(z^0) = (h(\eta)k_1)(z^0) \leq \frac{c_4}{b_1 - b}.$$

In this case, for all $z \in \overline{Q}$ and any $i = 1, \dots, n$,

$$\frac{c_4}{b_1 - b} \geq M \geq \left(\frac{k_i(u)}{\eta - b}\right)(z) \geq \frac{k_i(u)(z)}{1 - b}.$$

In the other case, when the supremum M is achieved at $\partial' Q$, it does not exceed a constant c_5, determined by majorants M_1 and M_2 of $\sup_Q |u_x|$ and $\sup_{\partial' Q} |u_{xx}|$, which we suppose in this work to be known. Hence

$$k_i(u)(z) \leq c_5(1 - b).$$

Thus, in any case we have

$$\sup_{z\in Q,\, i=1,\dots,n} k_i(u)(z) \leq c_2 = (1 - b) \max\left\{\frac{c_4}{b_1 - b}, c_5\right\}. \tag{3.62}$$

So we have proved the following theorem:

THEOREM 3. *Let u be an admissible solution of* (1)*, for which constants ν_4 and μ_4 in* (2.12) *and majorants M_1 and M_2 for $\sup_Q |u_x|$ and $\sup_{\partial' Q} |u_{xx}|$ respectively are known. If we also know a positive minorant ν_5 in* (3.53)*, then we can calculate a majorant c for $\sup_Q |u_{xx}|$.*

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