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ON SELECTION THEOREMS WITH DECOMPOSABLE VALUES

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Abstract. The main result of the paper asserts that for every separable measurable space $(T, \mathfrak{F}, \mu)$, where $\mathfrak{F}$ is the σ-algebra of measurable subsets of T and μ is a nonatomic probability measure on $\mathfrak{F}$, every Banach space E and every paracompact space X, each dispersible closed-valued mapping $F : x \rightsquigarrow L_1(T, E)$ of X into the Banach space $L_1(T, E)$ of all Bochner integrable functions $u : T \to E$, admits a continuous selection. Our work generalizes some results of Gončarov and Tol'stonogov.

1. Introduction

Let $(T, \mathfrak{F}, \mu)$ be a separable measurable space, where $\mathfrak{F}$ is the σ-algebra of measurable subsets of T and μ is a nonatomic probability measure on $\mathfrak{F}$. For an arbitrary Banach space E we shall denote by $L_1(T, E)$ the Banach space of all Bochner integrable functions $u : T \to E$.

By the classical isomorphism theorem [6], each separable measurable space $(T, \mathfrak{F}, \mu)$ with a nonatomic probability measure μ is isomorphic to the space $(I, \mathfrak{L}, m)$ of Lebesgue measurable sets on the interval $I = [0, 1]$. Therefore all separable measurable spaces $(T, \mathfrak{F}, \mu)$ and the corresponding Banach spaces $L_1(T, E)$ will be hereafter identified with the space $(I, \mathfrak{L}, m)$ of Lebesgue measurable sets on the interval I and the corresponding Banach space $L_1(I, E)$.

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A subset $K \subset L_1(T,E)$ is said to be *decomposable*, if for each measurable set $A \in \mathfrak{F}$ the following holds:

$$u\chi_A + v\chi_{T\setminus A} \in K \quad \text{for all } u, v \in K. \tag{1}$$

A typical example of a decomposable set is

$$L_1(T, E_0) = \{g \in L_1(T,E) \mid g(t) \subset E_0\},$$

where E_0 is any subset of E. The open ball $B(f,a)$ is an example of an indecomposable set: if $T_0 \subset T$ and $e \in E$ are such that $\mu(T_0) = 1/2$ and $|e| = 3a/2$, then the functions $u = f + e\chi_{T_0}$ and $v = f + e\chi_{T\setminus T_0}$ belong to $B(f,a)$, however the distance between the functions $u\chi_{T_0} + v\chi_{T\setminus T_0} = f + e\chi_T$ and f equals $3a/2 > a$.

Decomposable sets appear in the theory of differential inclusions [10], where the problem of existence of solutions in many cases reduces to the problem of existence of selections of multivalued mappings with decomposable values. In the pioneering work [3] an original method for studying such selection problems was proposed, via an application of the Michael convex-valued selection theorem, because a certain relationship was observed between the notions of convexity and decomposability.

Nevertheless, it is impossible to adapt the proof of the convex-valued selection theorem directly to our situation. The reason is a big difference between the mapping which associates to each set its convex hull (it is continuous in the Hausdorff metric on sets), and the one which associates to each set its decomposable hull (it fails to be continuous).

For example, the decomposable hull $\operatorname{Dec}(B(f,a))$ of any ball $B(f,a)$ coincides with the entire space $L_1(T,E)$. Therefore the Michael selection theorem was used in [3] indirectly, via the Lyapunov theorem [7] on convexity of the set of values of vector measures. It turned out that this method was inconvenient, mainly because of very complicated proofs. Subsequently, stronger results on selections of mappings with decomposable images were obtained [1], [2], [5], [8]. However, the proofs remained very complicated.

In an attempt to return to the original idea of the convex-valued Michael selection theorem we choose here a more economical construction of the decomposable hull $\operatorname{Dec}(A)$ of sets A, namely the *dispersibly decomposable* hull $\operatorname{Disp}(A) \subset \operatorname{Dec}(A)$ using the notion of the *dispersibly decomposable* sets.

All decomposable sets belong to the class of dispersibly decomposable sets and also (which is more important) all open and closed balls are dispersibly decomposable, since $\operatorname{Disp}(B(f,a)) = B(f,a)$ and $\operatorname{Disp}(\operatorname{Cl} B(f,a)) = \operatorname{Cl} B(f,a)$. Precisely the latter fact enables us to apply the well–developed techniques developed for the Michael convex-valued selection theorem and to prove the following selection theorem for the multivalued mappings with uniformly dispersed values (the so-called *dispersible* multivalued mappings):

Theorem 1.1. *Let $(T, \mathfrak{F}, \mu)$ be a separable measurable space, E a Banach space, X a paracompact space and $L_1(T, E)$ the space of all Bochner integrable functions $u : T \to E$. Then each dispersible closed-valued mapping $F : X \rightsquigarrow L_1(T, E)$ admits a continuous selection.*

Remarks. (1) The proof of this theorem does not use the difficult Lyapunov theorem and follows only the classical scheme of the proof of the Michael theorem. This theorem holds also beyond the class of separable (measurable spaces) T if X is separable and paracompact. However, in this case the proof becomes very complicated. Moreover, in the applications we usually have the separability of T. Therefore the nonseparable case will be considered in a separate paper.

(2) In earlier works a notion of *weakly decomposable* sets was defined, which lies between the notions of the decomposable sets and the dispersible sets, introduced in the present paper.

Since every lower semicontinuous multivalued mapping with decomposable values is dispersible, the proof of the theorem [3] for the case of separable measurable space $(T, \mathfrak{F}, \mu)$ can be simplified. Moreover, all known multivalued mappings applicable in the theory of differential inclusions [1], [3], [5], are dispersible. In the present paper we construct new classes of dispersible multivalued mappings.

The following theorem, which substantially generalizes [5], is an easy consequence of Theorem 1.1:

Theorem 1.2. *Let $(T, \mathfrak{F}, \mu)$ be a separable measurable space and X a paracompact space. Let $F : X \rightsquigarrow L_1(T, E)$ be a dispersible closed-valued mapping and $\{G_i : X \rightsquigarrow L_1(T, E)\}_{i \in \mathbb{N}}$ a sequence of dispersible multivalued mappings with open graphs such that $\mathrm{N}(G_i(x); a_i) \subset G_{i+1}(x)$, where the sequence $\{a_i\}$ does not depend on x. If for every point $x \in X$, $\Phi(x) = F(x) \cap G(x) \neq \emptyset$, where $G(x) = \bigcup_{i=1}^{\infty} G_i(x)$, then the multivalued mapping $\Phi : X \rightsquigarrow L_1(T, E)$, $x \mapsto \Phi(x)$, admits a continuous selection.*

The class of dispersibly convex sets is substantially bigger than the classes of decomposable and weakly decomposable sets. This fact enables further applications of Theorems 1.1 and 1.2. We add the following to already listed dispersibly convex sets:

(i) the closed and open balls corresponding to the so-called *dispersible* seminorms in $L_1(T, E)$. This class of seminorms contains decomposable seminorms as well as the most useful scalar compact seminorms [4];

(ii) the sets $\{u \in L_1([0,1], E) \mid \int \pi(|u|)\, dt \leq 1\}$, where $\pi(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ is a continuous mapping such that $\pi(x) \leq x$, if $x \leq 1$.

2. Dispersible partitions of the interval

By a *partition* $\mathcal{P}$ of the interval $I = [0, 1]$ we mean its representation as a union $\bigcup_{i=0}^n P_i$, where P_i are measurable subsets of I such that $m(P_i \cap P_j) = 0$ for all $i \neq j$. The *mesh* of the partition $\mathcal{P}$ is the number $\mathrm{mesh}(\mathcal{P}) = \sup\{m(P_i) \mid 1 \le i \le n\}$. Each sequence of points $0 = t_0 < t_1 < \ldots < t_m = 1$ divides the interval $[0, 1]$ into m smaller subintervals I_k and generates the so–called *linear* partition $\mathcal{L} = \{I_k\}_{k=0}^{m-1}$.

Definition 2.1. A partition $\mathcal{P} = \{P_i\}_{i=0}^n$ of the interval I is said to be s-*dispersible* where $s = (\kappa_0, \ldots, \kappa_n)$, $\sum_{i=0}^n \kappa_i = 1$, $\kappa_i \ge 0$, is a point of the n-simplex Δ^n, if

(1) $m(P_i \cap J) = \kappa_i m(J)$ for every subinterval $J \subset I$.

Proposition 2.2. *$\mathcal{P}$ is an s-dispersible partition of the interval I if and only if $s \in (\Delta^n)^{(0)}$.*

In this way the notion of an s-dispersible partition of the interval is trivialized. However, changing equality (1) by an approximate equality leads to useful constructions which substantially simplify proofs of Theorems 1.1 and 1.2.

Definition 2.3. Let $\sigma > 0$ and $s = (\kappa_0, \ldots, \kappa_n) \in \Delta^n$ be fixed. A partition $\mathcal{P} = \{P_i\}_{i=0}^n$ of the interval I is said to be σ-*approximatively* s-*dispersible* if

(2) $|m(P_i \cap J) - \kappa_i m(J)| < \sigma$ for each $0 \le i \le n$ and each subinterval $J \subset I$.

We shall denote the equality (2) by $m(P_i \cap J) \overset{\sigma}{\approx} \kappa_i m(J)$. Note that in order to prove (2) it suffices to verify (2) only for a finite number of intervals J. The following theorem supplies a sufficient amount of σ-approximatively s-dispersible partitions.

Theorem 2.4. *For every $\sigma > 0$ and $s \in \Delta^n$ there exists a σ-approximatively s-dispersible partition of the interval I.*

The basic geometric idea of the proof of this theorem is to divide each interval $[k, k + 1]$, lying in a sufficiently large interval $[0, N]$, into $n + 1$ subintervals of length equal to the corresponding barycentric coordinates of the point $s \in \Delta^n$. Then the ith subintervals of all intervals $[k, k + 1]$ are joint into a union and subsequently shrunk to the origin by a factor $1/N$, to obtain the ith element of the desired partition.

Proof of Theorem 2.4. Let $\mathcal{L}$ be the linear partition of interval I, determined by the points $0 = t_0 < t_1 < \ldots < t_m = 1$ with $\mathrm{mesh}(\mathcal{L}) < \sigma$. Divide each interval $[t_k, t_{k+1}]$ by the points $t_k = t_{k0} \le t_{k1} \le \ldots \le t_{kn} \le t_{k(n+1)} = t_{k+1}$ so that the lengths are proportional to $(\kappa_0, \ldots, \kappa_n)$:

$$(t_{k1} - t_{k0})/\kappa_0 = (t_{k2} - t_{k1})/\kappa_1 = \ldots = (t_{k+1} - t_{kn})/\kappa_n.$$

In particular, this means that if $\kappa_i = 0$, then $t_{ki} = t_{k(i+1)}$. Then the elements P_i of the desired partition $\mathcal{P}$ are defined as the union of m pairwise disjoint subintervals

$$P_i = \coprod_{k=0}^{m-1} [t_{ki}, t_{k(i+1)}], \quad i = 0, \dots, n.$$

Since the partition $\mathcal{P}$ depends on $\mathcal{L}, s \in \Delta^n$ and σ, we shall denote $\mathcal{P} = \mathcal{L}(s;\sigma)$ and shall call $\mathcal{L}(s;\sigma)$ the *special* partition, generated by the linear partition $\mathcal{L}$ and $s \in \Delta^n, \sigma > 0$.

The verification of (2) consists of two steps:

(a) $J = [t_l, t_p] \subset I$:

$$\begin{aligned} m(P_i \cap J) &= m\Big(\coprod_{k=l}^{p-1} [t_{ki}, t_{k(i+1)}]\Big) \\ &= \sum_{k=l}^{p-1} m([t_{ki}, t_{k(i+1)}]) = \kappa_i \sum_{k=l}^{p-1} m([t_k, t_{k+1}]) = \kappa_i m(J). \end{aligned}$$

(b) $J = [a, b]$, $t_{l-1} < a < t_l$, $t_p < b < t_{p+1}$:

$$m(P_i \cap J) \overset{2\sigma}{\approx} m(P_i \cap [t_l, t_p]) \overset{2\sigma}{\approx} \kappa_i m(J). \qquad \square$$

We now introduce new notions to generalize the preceeding theorem. Let Λ be a linearly ordered set. A family $\mathcal{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ of measurable subsets $P_\lambda \subset I$ is said to be a Λ-*partition* if the set $\{\lambda \mid P_\lambda \neq \emptyset\}$ is finite and $m(P_\lambda \cap P_{\lambda'}) = 0$ for all $\lambda \neq \lambda'$. Equip the set $\mathfrak{P}$ of all Λ-partitions with the metric $d(\mathcal{P}, \mathcal{P}') = \sup_\lambda m(P_\lambda \vartriangle P'_\lambda)$.

Let Δ be a full polyhedron equipped with a metric topology. Let the vertices of Δ be indexed by the linearly ordered set Λ. If $\Delta^{(0)} = \{v_\lambda\}_{\lambda \in \Lambda}$, then

$$\Delta = \Big\{ s = \sum_{\lambda \in \Lambda} \kappa_\lambda v_\lambda \,\Big|\, \text{all } \kappa_\lambda \geq 0, \sum_{\lambda \in \Lambda} \kappa_\lambda = 1$$
$$\text{and there exist finitely many indices } \lambda_0 < \ldots < \lambda_p$$
$$\text{such that } \kappa_{\lambda_i} > 0 \text{ and } \sum_{i=0}^{p} \kappa_{\lambda_i} = 1 \Big\}.$$

For a given point $s = \{\kappa_\lambda\} \in \Delta$ and $\sigma > 0$, a Λ-partition $\mathcal{P} = \{P_\lambda\}$ of the interval I is said to be σ-*approximatively* s-*dispersible* if $m(P_\lambda \cap J) \overset{\sigma}{\approx} \kappa_\lambda m(J)$ for arbitrary $\lambda \in \Lambda$ and any subinterval $J \subset I$.

Theorem 2.5. *For each $\sigma > 0$ and $s = \sum_{i=0}^{p} \kappa_{\lambda_i} v_{\lambda_i} \in \Delta$, where $\lambda_0 < \ldots < \lambda_p$, $\sum_{i=0}^{p} \kappa_{\lambda_i} = 1$, there exists a σ-approximatively s-dispersible Λ-partition of the interval I.*

Proof. The verification of the assertion is analogous to the proof of Theorem 2.4 and we shall thus give only an outline. Fix the linear partition $\mathcal{L}$ of interval I, defined by points $0 = t_0 < t_1 < \ldots < t_m = 1$ with $\operatorname{mesh}(\mathcal{L}) < \sigma$. Divide each interval $[t_k, t_{k+1}]$ by points $t_k = t_{k\lambda_0} \leq t_{k\lambda_1} \leq \ldots \leq t_{k\lambda_p} \leq t_{k\lambda_{p+1}} = t_{k+1}$, proportionally to the numbers $(\kappa_{\lambda_0}, \ldots, \kappa_{\lambda_p})$:

$$(t_{k\lambda_1} - t_{k\lambda_0})/\kappa_{\lambda_0} = (t_{k\lambda_2} - t_{k\lambda_1})/\kappa_{\lambda_1} = \ldots = (t_{k+1} - t_{k\lambda_p})/\kappa_{\lambda_p}.$$

Then the elements P_λ of the desired Λ-partition $\mathcal{P} = \mathcal{L}(s;\sigma)$ are defined by the following formula;

$$(*) \qquad P_\lambda = \begin{cases} \emptyset & \text{if } \lambda \neq \lambda_i \text{ for all } 0 \leq i \leq p, \\ \coprod_{k=0}^{m-1} [t_{k\lambda_i}, t_{k\lambda_{i+1}}] & \text{if } \lambda = \lambda_i. \end{cases} \qquad \square$$

Theorem 2.6. *Let X be a paracompact space and $\sigma : X \to (0,\infty)$ a continuous function. Let the index set Λ of a locally finite covering $\omega = \{V_\lambda\}_{\lambda\in\Lambda} \in \operatorname{cov} X$ be linearly ordered. Denote by $s : X \to \mathcal{N}\langle\omega\rangle$ the canonical mapping of the nerve of the covering ω, which lies in the polyhedron Δ. Then there exists a continuous mapping $\Pi : X \to \mathfrak{P}$ into the space $\mathfrak{P}$ of all Λ-partitions, such that:*

(c) $\Pi(x) = \{\Pi(x)_\lambda\}_{\lambda\in\Lambda}$ *is a $\sigma(x)$-approximatively $s(x)$-dispersible Λ-partition of the interval I, for all $x \in X$;*

(d) *If $x \notin U_\lambda$, then $m(\Pi(x)_\lambda) = 0$.*

Proof. For each point $x \in X$ denote by m_x the integer part of the number $1/\sigma(x) > 0$ and introduce the linear partition $\mathcal{L}_x$ of the interval I, defined by the points $0 = t_{x0} < t_{x1} < \ldots < t_{xm_x} = 1$ such that

(3) $|t_{x1} - t_{x0}| = \ldots = |t_{x(m_x-1)} - t_{x(m_x-2)}| = \sigma(x)$,

(4) $|t_{xm_x} - t_{x(m_x-1)}| \leq \sigma(x)$.

Clearly, $\operatorname{mesh} \mathcal{L}_x \leq \sigma$. It is easy to prove that

(5) if $1/\sigma(x_0)$ is not an integer then $m_x = m_{x_0}$, and the functions $t_{xk}, k \leq m_{x_0}$ are continuous in a sufficiently small neighbourhood $\mathcal{O}(x_0)$ of the point x_0; and

(6) if $1/\sigma(x_0)$ is an integer m_{x_0}, then $m_x \leq m_{x_0} + 1$, and the function t_{xk} is continuous in a sufficiently small neighbourhood $\mathcal{O}(x_0)$ of the point x_0, for all $k \leq m_{x_0}$.

Let $\Pi(x) = \mathcal{L}_x(s(x);\sigma(x))$, the latter given by the formula $(*)$. Since the preimage of the open star of vertex $\langle V_\lambda\rangle$ under the mapping s lies in V_λ, it follows by Theorem 2.5 that the partition $\Pi(x)$ satisfies conditions (c) and (d).

To verify that Π is continuous at the point $x_0 \in X$ it suffices, because of the local finiteness of the covering ω, to construct a neighbourhood $\mathcal{O}(x_0)$ and finitely many indices $\lambda_0 < \lambda_1 < \ldots < \lambda_p$ such that $\mathcal{O}(x_0) \cap V_{\lambda_i} \neq \emptyset$ for all $0 \leq i \leq p$, and $\mathcal{O}(x_0) \cap V_{\lambda_i} = \emptyset$ for all $\lambda \notin \{\lambda_0, \ldots, \lambda_p\}$. Then $s(x) = \sum_{i=0}^{p} \kappa_{\lambda_i}(x)\langle V_{\lambda_i}\rangle$.

Divide the interval $[t_{xk}, t_{x(k+1)}]$ by points $t_{xk} = t_{xk\lambda_0} \leq t_{xk\lambda_1} \leq \ldots \leq t_{xk\lambda_p} \leq t_{xk\lambda_{p+1}} = t_{xk+1}$ into subintervals with lengths proportional to the numbers $(\kappa_{\lambda_0}(x), \ldots, \kappa_{\lambda_p}(x))$. It is easy to verify that

(7) the functions $t_{xk\lambda_i}$ are continuous in a sufficiently small neighbourhood $\mathcal{O}(x_0)$ of the point x_0 for all $k \leq m_{x_0}$ and $i \leq p$.

By definition $(*)$ we have

$$\Pi(x)_{\lambda_i} = \coprod_{k=0}^{m_x-1} [t_{xk\lambda_i}, t_{xk\lambda_{i+1}}] \quad \text{and} \quad \Pi(x_0)_{\lambda_i} = \coprod_{k=0}^{m_x-1} [t_{x_0k\lambda_i}, t_{x_0k\lambda_{i+1}}].$$

Denote

$$\coprod_{k=0}^{m_{x_0}-1} \left([t_{xk\lambda_i}, t_{x_0k\lambda_i}] \cup [t_{xk\lambda_{i+1}}, t_{x_0k\lambda_{i+1}}]\right)$$

by $D(x_0, x)$. It follows from (7) that

(8) the values of $D(x_0, x)$ can be made arbitrarily small in a sufficiently small neighbourhood $\mathcal{O}(x_0)$ of the point x_0.

Observe that if $1/\sigma(x_0)$ is not an integer, then $\Pi(x)_{\lambda_i} \bigtriangleup \Pi(x_0)_{\lambda_i} \subset D(x_0, x)$ for a sufficiently small neighbourhood $\mathcal{O}(x_0)$ of the point x_0. If on the other hand, the number $1/\sigma(x_0)$ is an integer m_{x_0}, then

$$\Pi(x)_{\lambda_i} \bigtriangleup \Pi(x_0)_{\lambda_i} \subset \begin{cases} D(x_0, x) & \text{if } m_x = m_{x_0}, \\ D(x_0, x) \cup [t_{x(m_x-1)}, 1] & \text{if } m_x = m_{x_0} + 1, \end{cases}$$

for a sufficiently small neighbourhood $\mathcal{O}(x_0)$ of the point x_0. By (8) the value of $m(\Pi(x)_{\lambda_i} \bigtriangleup \Pi(x_0)_{\lambda_i})$ can be made to be arbitrarily close to zero by choosing an even smaller neighbourhood $\mathcal{O}(x_0)$. Therefore, the value of $d(\Pi(x), \Pi(x_0)) = \sup_{\lambda_i} m(\Pi(x)_{\lambda_i} \bigtriangleup \Pi(x_0)_{\lambda_i})$ can also be made arbitrarily close to zero. The continuity of Π at the point x_0 is thus proved. □

Consider the map $\Pi : X \to \mathfrak{P}$, constructed in Theorem 2.6. It follows by the definition of metric on $\mathfrak{P}$ that the characteristic function $\chi_{\Pi(x)_\lambda}$ is a continuous mapping of the space X into $\mathrm{L}_1([0,1], E)$ for all λ. By the local finiteness of ω, it is easy to prove that the map $\Psi : X \to \mathrm{L}_1([0,1], E)$, defined by $\Psi(x) = \sum_{\lambda \in \Lambda} \psi(\lambda)\chi_{\Pi(x)_\lambda}$, where $\psi : \Lambda \to \mathrm{L}_1([0,1], E)$ is an arbitrary map, is continuous. Two remarks concerning the definition of map Ψ:

(e) since the measure of the intersection $\Pi(x)_\lambda \cap \Pi(x)_{\lambda'}$ is zero, the map Ψ can be defined there to be arbitrary, and

(f) since $\Pi(x)_\lambda \neq \emptyset$ only for finitely many $\lambda \in \Lambda$, the sum in the definition of Ψ is finite.

3. Dispersible sets

Definition 3.1. A subset $K \subset \mathrm{L}_1([0,1],E)$ is said to be *dispersibly convex* if for each $\varepsilon > 0$ and for arbitrary functions $u_i \in K$, $i = 0, \dots, n$, there exists $\sigma > 0$ such that for any point $s \in \Delta^n$ and for each σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$ the function $u = \sum_{i=0}^n u_i \chi_{P_i}$ lies in the ε-neighbourhood of K.

Remark. Let $\mathcal{T} : (T, \mathfrak{F}, \mu) \to ([0,1], \mathfrak{L}, m)$ be the isomorphism of a separable measurable space endowed with a nonatomic probability measure μ into the space of Lebesgue measurable subsets of the interval $I = [0,1]$ constructed by the procedure explained in [6, Chapter VIII]. Then there exists an isomorphism

$$\mathfrak{T} : L_1(T,E) \to L_1(I,E)$$

of Banach spaces which preserves the decomposable hull:

$$\mathfrak{T}(u\chi_A) = \mathfrak{T}(u)\chi_{\mathcal{T}(A)}.$$

A subset $K \subset \mathrm{L}_1(T,E)$ is said to be *dispersibly convex* if the set $\mathfrak{T}(K) \subset L_1(I,E)$ is dispersibly convex in $L_1(I,E)$. Obviously the notion of dispersible convexity for separable measurable spaces depends on the choice of the isomorphism $\mathcal{T}$.

It is easy to prove that

(a) Any intersection of dispersible sets is dispersible; and
(b) Any linear combination $\alpha K + \beta H$ of dispersible sets is dispersible.

Definition 3.2. The smallest dispersibly convex set containing the set $K \subset \mathrm{L}_1([0,1],E)$ is said to be the *dispersible hull* of the set K.

It follows from (a) that the dispersible hull of the set K coincides with the intersection of all dispersibly convex sets containing the set K. It would be useful to find a constructive description of the dispersible hull of a set. The following simple fact will be useful in the sequel:

Lemma 3.3. *If $\mathcal{P} = \{P_i\}_{i=0}^n$ is a partition of the interval I, and $\rho(u_i, v_i) < a, 0 \le i \le n$, then*

$$\rho\bigg(\sum_{i=0}^n u_i\chi_{P_i}, \sum_{i=0}^n v_i\chi_{P_i}\bigg) < (n+1)a.$$

Lemma 3.3 and the definition of dispersibly convex sets imply the following:

Lemma 3.4. *If K is a dispersibly convex set then its closure* $\operatorname{Cl} K$ *is also dispersibly convex.*

The most important result concernes open and closed balls which turn out to be dispersibly convex. By Lemma 3.4 it suffices to verify this fact only for open balls:

THEOREM 3.5. *For each function $f \in L_1(T, E)$ and each $a > 0$ the open ball*

$$B(f,a) = \left\{ g \,\middle|\, |f-g| = \int \|f(t) - g(t)\| \,\partial t < a \right\}$$

is a dispersible set.

Theorem 3.5 can be easily deduced from the following theorem.

THEOREM 3.6. *For arbitrary functions $u_i, v_i \in L_1([0,1], E)$, $i = 0, \dots, n$, such that $|u_i - v_i| < A_i$ for all i, there exists a $\sigma > 0$ such that for any point $s \in \Delta^n$ and any σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$ the distance between functions $u = \sum_{i=0}^n u_i \chi_{P_i}$ and $v = \sum_{i=0}^n v_i \chi_{P_i}$ is less than $\sum \kappa_i A_i$.*

PROOF. Consider first the case when the functions u_i and v_i are piecewise constant. Let $\mathcal{L} = \{I_k\}_{k=0}^m$ be a linear partition of $I = [0,1]$ such that $u_i = c_{ik} \in E$ on I_k and $v_i = d_{ik} \in E$ on I_k. Hence,

$$u_i = \sum_{k=0}^m c_{ik} \chi_{I_k} \quad \text{and} \quad v_i = \sum_{k=0}^m d_{ik} \chi_{I_k}.$$

Since $|u_i - v_i| < A_i$, it follows that

$$|u_i - v_i| = \sum_{k=0}^m \|c_{ik} - d_{ik}\| m(I_k) = B_i < A_i.$$

Let $B = \min_i (A_i - B_i)$ and $\sigma < B/((\sum\sum \|c_{ik} - d_{ik}\|) + 1)$. Consider an arbitrary σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$, where $s = \{\kappa_i\} \in \Delta^n$. Since $m(P_i \cap I_k) \overset{\sigma}{\approx} \kappa_i m(I_k)$, it follows that

$$\begin{aligned}
|u - v| &= \sum_{i=0}^n \sum_{k=0}^m (\|c_{ik} - d_{ik}\| \kappa_i m(P_i \cap I_k)) \\
&\overset{\Theta}{\approx} \sum_{i=0}^n \sum_{k=0}^m (\|c_{ik} - d_{ik}\| \kappa_i m(I_k)) \\
&= \sum_{i=0}^n \kappa_i \left(\sum_{k=0}^m \|c_{ik} - d_{ik}\| m(I_k) \right) = \sum_{i=0}^n \kappa_i B_i,
\end{aligned}$$

where $\Theta = \sum\sum(\|c_{ik} - d_{ik}\|\kappa_i \sigma)$. From $\Theta < \sigma \sum\sum \|c_{ik} - d_{ik}\| < B < \sum_{i=0}^n \kappa_i (A_i - B_i)$ it follows that $|u - v| < \sum_{i=0}^n \kappa_i A_i$.

In the general case we shall approximate the functions u_i, v_i by piecewise-constant functions $\widehat{u}_i, \widehat{v}_i \in L_1(I, E)$ so that $|u_i - \widehat{u}_i| + |v_i - \widehat{v}_i| < D/(n+1)^2$, where $D = \min_i |A_i - C_i|$, and $C_i = |u_i - v_i| < A_i$. By Lemma 3.3, we have

$$|u - \widehat{u}| + |v - \widehat{v}| < \frac{(n+1)D}{(n+1)^2} = \frac{D}{(n+1)}.$$

Since for piecewise-constant functions $\widehat{u}_i, \widehat{v}_i$ the inequality

$$|\widehat{u}_i - \widehat{v}_i| < \frac{D}{(n+1)^2} + C_i$$

holds, it follows by the previous considerations that there exists $\sigma > 0$ such that for each point $s \in \Delta^n$ and each σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$, the distance between functions $\widehat{u} = \sum_{i=0}^n \widehat{u}_i \chi_{P_i}$ and $\widehat{v} = \sum_{i=0}^n \widehat{v}_i \chi_{P_i}$ is at most $\sum \kappa_i(D/(n+1)^2 + B_i) = \sum \kappa_i B_i + D/(n+1)^2$. As a result, we get:

$$|u - v| \leq \frac{D}{(n+1)} + \frac{D}{(n+1)}^2 + \sum \kappa_i B_i < \sum \kappa_i B_i + D < \sum \kappa_i A_i. \qquad \square$$

Corollary 3.7. *Each a-neighbourhood $N(K;a) = \{g \in L_1([0,1],E) \mid \rho(g,K) < a\}$ of a dispersibly convex set K is dispersibly convex.*

Remark. The proof, analogous to the proof of Theorem 3.6, shows that the sets $\{u \in L_1([0,1],E) \mid \int \sqrt{|u|}\,dt \leq 1\}$ as well as the balls with respect to the seminorm $q(u) = |\int_T u(t)\,dt|$ are dispersibly convex.

4. Dispersible multivalued mappings

Hereafter X will be a paracompact space and Y the space $L_1(I,E)$.

Definition 4.1. A multivalued mapping $F : X \rightsquigarrow Y$ is said to be *dispersible* if for each $x_0 \in X$ and $a > 0$, each point $s \in \Delta^n$ and each functions $u_0, u_1, \dots, u_n \in F(x_0)$ there exist a neighbourhood $\mathcal{O}(x_0)$ of the point x_0 and a number $\sigma > 0$ such that for any σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$, the function $\sum_{i=0}^n u_i \chi_{P_i}$ is contained in $N(F(x),a)$, for every point $x \in \mathcal{O}(x_0)$.

It is easy to prove the following result by means of Theorem 3.6:

Proposition 4.2. *A multivalued mapping $F : X \rightsquigarrow Y$ is dispersible if and only if for each $x_0 \in X$ and $a > 0$, an arbitary point $s \in \Delta^n$ and arbitrary functions $u_0, \dots, u_n \in N(F(x_0);a)$ there exist a neighbourhood $\mathcal{O}(x_0)$ of the point x_0 and a number $\sigma > 0$ such that for each σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$ the function $\sum_{i=0}^n u_i \chi_{P_i}$ is contained in $N(F(x),a)$, for every point $x \in \mathcal{O}(x_0)$.*

It follows from Definition 4.1 that the mapping F is lower semicontinuous and the values $F(x)$ are dispersible convex at each point $x \in X$. In general, the converse does not hold. It is easy to verify by means of Theorem 3.6 that a lower semicontinuous mapping $F : X \rightsquigarrow Y$ is dispersible if for each point $x \in X$ the value $F(x)$ is a decomposable set. Moreover, the dispersibility is preserved under many operations:

Proposition 4.3. *If multivalued mappings $F, H : X \rightsquigarrow Y$ are dispersible then*

(a) *the multivalued mapping $\Phi : X \rightsquigarrow Y$, $\Phi(x) = F(x) + H(x)$ is dispersible, and*
(b) *the multivalued mapping $\Phi : X \rightsquigarrow Y$, $\Phi(x) = \operatorname{Cl} F(x)$ is dispersible.*

Definition 4.4. A multivalued mapping $H : X \rightsquigarrow Y$ is said to be *openly generated* if it satisfies the following condition:

(a) For every points $x_0 \in X$ and $y_0 \in H(x_0)$ there exist neighbourhoods $O(x_0)$, $O(y_0)$ such that $O(y_0) \subset H(x)$, for all $x \in O(x_0)$.

Clearly, H is an openly generated mapping if and only if its graph $\{(x, H(x)) \mid x \in X\}$ is an open subset of $X \times Y$. This is the reason that the openly generated mappings are also called also mappings with *open graphs*. Openly generated mappings are always lower semicontinuous.

A large class of openly generated dispersible multivalued mappings can be constructed by the following method:

Proposition 4.5. *Let B be a dispersibly convex set such that $tB \subset B$, for all $0 < t < 1$ and $\{0\} \in \operatorname{Int} B$. If $\varphi : X \to (0, \infty)$ is a lower semicontinuous singlevalued function then the multivalued mapping $H : X \rightsquigarrow Y$, $H(x) = \varphi(x)B$ is dispersible. If, in addition B is open, then H is openly generated.*

Proof. First consider the case when B is open. To verify that H is dispersible consider the functions $u_0, \dots, u_n \in H(x_0)$ constructed in the following manner $u_i = \varphi(x_0)b_i$, where $b_i \in B$. Since B is open we may assume that $u_i = tb_i$, where $t < \varphi(x_0)$. Since the function φ is lower semicontinuous, there exists $\alpha > 0$ such that $O(x_0) = \{x \mid \varphi(x) > t + \alpha\}$ is a neighbourhood of the point x_0. We claim that the open set $(t + \alpha)B$ which is a neighbourhood of $u_0 \in H(x_0)$, is contained in $H(x)$ for $x \in O(x_0)$. Indeed,

$$\frac{t+\alpha}{\varphi(x)} < 1 \;\Rightarrow\; \frac{t+\alpha}{\varphi(x)} B \subset B \;\Rightarrow\; (t+\alpha)B \subset \varphi(x)B = H(x).$$

Therefore the mapping H is openly generated.

Since the interior of B contains the point 0, it follows that $tN(B;\beta) \subset (t + \alpha)B$, for a certain number $\beta > 0$. Therefore, we get:

(1) $tN(B;\beta) \subset H(x)$, for every $x \in O(x_0)$.

Since B is dispersible, there exists a number $\sigma > 0$, such that for each point $s \in \Delta^n$ and each σ-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^n$, the function $b = \sum_{i=0}^n b_i \chi_{P_i}$ is contained in $N(B;\beta)$. Then $\sum_{i=0}^n u_i \chi_{P_i} = t \sum_{i=0}^n b_i \chi_{P_i} \in tN(B;\beta)$. However, by (1) the last set is contained in $H(x)$, for all $x \in O(x_0)$. This proves that the mapping H is dispersible and openly generated.

In the case when B is not open we consider the mapping $H'(x) = \varphi(x) \operatorname{Int} B$, which has already been proved to be dispersible. By Proposition 4.3(b) the mapping $H(x) = \operatorname{Cl} H'(x)$ is also dispersible. □

The following proposition is proved analogously:

Proposition 4.6. *Let B be a dispersibly convex open set such that $tB \subset B$, for all $0 < t < 1$, $\{0\} \in \operatorname{Int} B$. If $\varphi : X \to (0, \infty)$ is a lower semicontinuous singlevalued function, the multivalued mapping $F : X \rightsquigarrow Y$ is dispersible, and the intersection $\Phi(x) = F(x) \cap \varphi(x)B$ is nonempty, then the multivalued mapping $\Phi : X \rightsquigarrow Y$ is dispersible.*

5. Proofs of Theorems 1.1 and 1.2

Without loss of generality we can assume that $(T, \mathfrak{F}, \mu) = ([0,1], \mathfrak{L}, m)$ and $L_1(T, E) = L_1([0,1], E) = Y$.

Proposition 5.1. *Let X be a paracompact space and $F : X \rightsquigarrow Y$ a dispersible multivalued mapping. Then for each $\varepsilon > 0$ there exists an ε-selection $\Psi : X \to Y$ of the mapping F.*

Proof. Let $\{N(y, \varepsilon) \mid y \in Y\}$ be a covering of the space Y by open ε-balls. Denote the preimage $F^{-1}(N(y, \varepsilon))$ by W_y. Then for each $x \in W_y$ the set $F(x) \cap N(y; \varepsilon)$ is nonempty or equivalently, $y \in N(F(x); \varepsilon)$.

Let an open locally finite covering $\omega = \{V_\lambda\}_{\lambda \in \Lambda}$ of the space X be a star refinement of the open covering $\{W_y \mid y \in Y\}$. Let $\operatorname{St}(V_\lambda; \omega) \subset W_{\psi(\lambda)}$, where $\psi : \Lambda \to Y$ is a mapping. Then $\operatorname{Cl} V_\lambda \subset W_{\psi(\lambda)}$. Choose a linear ordering on the index set Λ. For each point $x \in X$ fix all those elements $V_0, \dots, V_{n_x}$ of the covering ω whose closure contains the point x, ordered by the ordering of their indices. If $\operatorname{St}(V_i; \omega) \subset W_{\psi_i}$, then obviously

(1) $x \in \operatorname{Cl} V_i \subset W_{\psi_i}$.

The neighbourhood $\mathcal{O}_1(x)$ of the point x intersects only the elements $V_0, \dots V_{n_x}$ of the covering and is contained in $\cap W_{\psi_i}$. Because of (1) all points $\{\psi_i\}_{i=0}^{n_x}$ lie in $N(F(x'); \varepsilon)$, for all $x' \in \mathcal{O}_1(x)$.

Now apply the hypothesis that the mapping F is dispersible. By Proposition 4.3 there exist a neighbourhood $\mathcal{O}_2(x) \subset \mathcal{O}_1(x)$ and a number $\sigma_x > 0$ such that

(2) For arbitrary functions $\{\psi_i\}_{i=0}^{n_x} \subset N(F(x), \varepsilon)$, for each point $s \in \Delta^{n_x}$ and for an arbitary σ_x-approximatively s-dispersible partition $\mathcal{P} = \{P_i\}_{i=0}^{n_x}$, the function $u_x = \sum_{i=0}^{n_x} \psi_i \chi_{P_i}$ lies in $N(F(x'); \varepsilon)$, for each point $x' \in \mathcal{O}_2(x)$.

Let $\tau = \{U_\beta\}$ be an open locally finite covering of the space X which is a star refinement of the covering $\{\mathcal{O}_2(x) \mid x \in X\}$. Let $\operatorname{St}(U_\beta; \tau) \subset \mathcal{O}_2(x_\beta)$. Obviously, the function

$$x \in X \mapsto \min\{\sigma_{x_\beta} \mid x \in U_\beta \subset \mathcal{O}_2(x_\beta)\} = d(x) > 0$$

is lower semicontinuous. By the Dowker theorem [9] there exists a continuous function $\sigma : X \to \mathbb{R}^+$ such that $0 < \sigma(x) < d(x)$, for all $x \in X$.

Consider the canonical mapping $s : X \to \mathcal{N}\langle\omega\rangle$ into the nerve of the covering ω, lying in the full polyhedron Δ. By Theorem 2.6 there exists a continuous map $\Pi : X \to \mathfrak{P}$, for given s and σ, such that

(c) $\Pi(x) = \{\Pi(x)_\lambda\}_{\lambda\in\Lambda}$ is a $\sigma(x)$-approximatively $s(x)$-dispersible Λ-partition of the interval I for all $x \in X$, and

(d) if $x \notin V_\lambda$ then $m(\Pi(x)_\lambda) = 0$.

We claim that the desired ε-selection is the mapping $\Psi : X \to Y$,

$$\Psi(x) = \sum_{\lambda\in\Lambda} \psi(\lambda)\chi_{\Pi(x)_\lambda}.$$

We have proved at the end of the Section 3 that Ψ is continuous. It remains to prove that $\Psi(x) \in N(F(x), \varepsilon)$, for all $x \in X$. Let $x \in U_\beta \subset \mathcal{O}_2(x_\beta)$. Since $\sigma(x) < d(x) \le \sigma_{x_\beta}$, the $\sigma(x)$-approximatively $s(x)$-dispersible partition $\Pi(x)$ is σ_{x_β}-approximatively $s(x)$-dispersible. From (2) it follows that

$$\Psi(x) = \sum_{\lambda\in\Lambda} \psi(\lambda)\chi_{\Pi(x)_\lambda} = \sum_{i=0}^{n_x} \psi_i\chi_{\Pi(x)_i} \in N(F(x'); \varepsilon),$$

for all $x' \in \mathcal{O}_2(x_\beta)$, therefore also for $x \in \mathcal{O}_2(x_\beta)$. $\square$

The rest of this section is devoted to the proof of Theorem 1.1 concerning selections of dispersible mappings.

Theorem 5.2. *Let $F, F_1 : X \rightsquigarrow Y$ be lower semicontinuous dispersible multivalued mappings (not necessarily with closed values). If for each point $x \in X$ the inclusion $\mathrm{Cl}\, F_1(x) \subset F(x)$ holds, then the mapping F admits a continuous selection.*

Corollary 5.3. *If the values of a dispersible multivalued mapping $F : X \rightsquigarrow Y$ are closed then the mapping F admits a continuous selection.*

Proof of Theorem 5.2. Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $\varepsilon_{k+1} < \varepsilon_k/2$ and $\varepsilon_1 = 1$. By Proposition 5.1 the mapping F_1 admits an ε_1-selection $s_1 : X \to Y$. By Theorem 3.5 and Proposition 4.5, the open ε_k-ball $B_k = N(0, \varepsilon_k)$ is a dispersibly convex set and the mapping $s_1(x) + B_1 = \mathrm{N}(s_1(x), \varepsilon_1)$ is dispersible and openly generated.

It follows by Proposition 4.6 that the mapping $F_2(x) = F_1(x) \cap N(s_1(x), \varepsilon_1) \neq \emptyset$ is dispersible. By Proposition 5.1, it admits an ε_2-selection $s_2 : X \to Y$.

Analogously, multivalued mappings $F_{k+1}(x) = F_k(x) \cap N(s_k(x), \varepsilon_k) \neq \emptyset$ as well as their ε_{k+1}-selections $s_{k+1} : X \to Y$, $k = 2, 3, \dots$ can be constructed.

Since $F_k(x) \subset N(s_{k-1}(x), \varepsilon_{k-1})$, it follows that $\operatorname{diam} F_k(x) \leq 2\varepsilon_{k-1}$. Because of the inequalities

$$\begin{aligned}\rho(s_k(x), s_{k+1}(x)) \leq \rho(s_k(x), F_k(x)) + \rho(s_{k+1}(x), F_k(x)) + \operatorname{diam}(F_k(x))\\ \leq 2\varepsilon_k + 2\varepsilon_{k-1},\end{aligned}$$

the sequence $\{s_k\}$ is a Cauchy sequence in $Y = L_1(T, E)$. Completeness of Y implies that this sequence converges to a mapping $\psi : X \to Y$. The mapping $\psi(x) \in \operatorname{Cl} F_1(x) \subset F(x)$ is precisely the desired selection of mapping F. □

We remark that the theorem on the extension of a partial selection of a dispersible mapping $F : X \rightsquigarrow L_1(T, E)$ defined on closed subspace of paracompact spaces X, to a global selection is also valid and its proof is a standard reduction to the proof of Theorem 1.1.

Proof of Theorem 1.2. We shall only give a sketch of the proof. Let $X_k = \{x \in X \mid F(x) \cap G_k(x) \neq \emptyset\}$. It is obvious that $X_i \subset X_{i+1}$ and $\cup X_k = X$. We can thus assume without lossing generality that X_k are closed.

The mappings $F \cap G_k$ are dispersible and defined on X_k and $\operatorname{Cl}(F \cap G_k) \subset F \cap G_{k+1}$. By means of Theorem 5.2, it is possible to construct selections $s_k : X_k \to Y$ of the mappings $F \cap G_{k+1}$ such that $s_k|_{X_{k-1}} = s_{k-1}$. The desired selection $s : X \to Y$ of the mapping $F \cap G$ is then given by $s|_{X_k} = s_k$. □

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