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SECOND NOETHER-TYPE THEOREM FOR THE GENERALIZED VARIATIONAL PRINCIPLE OF HERGLOTZ

Bogdana Georgieva — Ronald Guenther

Dedicated to the memory of Professor O. A. Ladyzhenskaya

Abstract. The generalized variational principle of Herglotz defines the functional, whose extrema are sought, by a differential equation rather than by an integral. For such functionals the classical Noether theorems are not applicable. First and second Noether-type theorems which do apply to the generalized variational principle of Herglotz were formulated and proved. These theorems contain the classical first and second Noether theorems as special cases. We published the first Noether-type theorem previously in this journal. Here we prove the second Noether-type theorem and show that it reduces to the classical second Noether theorem when the Herglotz variational principle reduces to the classical variational principle.

1. Introduction

In 1918, Emmy Noether [12], [13] proved two remarkable theorems relating symmetry groups of a variational integral to properties of its associated Euler–Lagrange equations. For modern derivations and discussions of these theorems see Logan [11], Olver [14], Bluman and Kumei [1]. In the first of these theorems, Noether shows how each one-parameter variational symmetry group gives rise to a conservation law of the Euler–Lagrange equations. Conservation of energy, for example, comes from invariance of the system under time translations, while

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conservation of linear and angular momenta reflect invariance of the system under spatial translations and rotations. This first theorem gives a one-to-one correspondence between symmetry groups and conservation laws. Noether's second theorem relates an infinite-dimensional symmetry group of the functional to an identity involving the corresponding Euler–Lagrange equations.

Noether's theorems are applicable only to the classical variational principle, in which the functional is defined by an integral. The generalized variational principle of Herglotz defines the functional, whose extrema are sought, by a differential equation rather than an integral. For such functionals the classical Noether's theorems are not applicable. We formulated and proved two Noether-type theorems which do apply to the Generalized Variational Principle, and contain the first and second Noether theorems as special cases. See Georgieva and Guenther [4]. This theorem gives explicit conserved quantities for non-conservative (and conservative) systems described by the Generalized Variational Principle corresponding to symmetries of the functional under an n-parameter symmetry group. Here we prove the second Noether-type theorem. It gives an identity, which reduces to the identity provided by the classical second Noether theorem in the case when the functional is defined by an integral.

The Generalized Variational Principle was proposed by Gustav Herglotz in 1930. See Herglotz [8] and [9]. It generalizes the classical variational principle by defining the functional, whose extrema are sought, by a differential equation. Herglotz's original idea was published in 1979 in his collected works; this publication was supervised by Schwerdtfeger, see Gesammelte Schriften, Vandenhoeck and Ruprecht, Gӧttingen. Immediately thereafter, Scherdtfeger and R. B. Guenther published Herglotz's Vorlesungen über die Mechanik der Kontinua which appeared in the series Teubner-Archive zur Mathematik, B. G. Teubner Verlagsgesellschaft, Leipzig, 1985. Herglotz reached the idea of the Generalized Variational Principle through his work on contact transformations and their connections with Hamiltonian systems and Poisson brackets. His work was motivated by ideas from S. Lie, C. Carathéodory and other researchers. An important reference on the Generalized Variational Principle is The Herglotz Lectures on Contact Transformations and Hamiltonian Systems published in 1996 by R. B. Guenther, J. A. Gottsch and C. M. Guenther [7].

The variational principle of Herglotz is important for a number of reasons:

(1) The solutions of the equations, which give the extrema of the functional defined by the generalized variational principle, when written in terms of x_i and $p_i = \partial L/\partial \dot{x}_i$, determine a family of *contact transformations*. This family is a one-parameter group in a certain case. See Guenther et al [7] as well as Caratheodory [2] and Eisenhart [3].

(2) The generalized variational principle gives a *variational* description of nonconservative processes.

(3) For a process, conservative or nonconservative, which can be described with the generalized variational principle, one can systematically derive conserved quantities by application of the first Noether-type theorem. See Georgieva and Guenther [4].

(4) For any process described by the generalized variational principle the second Noether-type theorem, proved in this paper, can be applied to derive a non-trivial identity involving the so called generalized Euler–Lagrange equations.

(5) In a previous paper (Georgieva, Guenther and Bodurov [5]) we have shown that the variational principle of Herglotz can be extended to several independent variables. The new principle reduces to the classical variational principle with several independent variables. A first Noether-type theorem which applies to the new variational principle was formulated and proved in the same paper.

2. The generalized variational principle of Herglotz

Let $x = (x^1, \dots, x^n)$ be a set of functions $x^k = x^k(t)$ on the interval $0 \le t \le s$ which are at least twice differentiable. Consider the differential equation

$$\frac{dz}{dt} = L(t, x, \dot{x}, z), \quad 0 \le t \le s \tag{2.1}$$

where t is the independent variable, z is the dependent variable, $\dot{x} = dx/dt$, and $L(t, x, \dot{x}, z)$ is a given function, which we will call the Lagrangian. For any arbitrary but fixed set of functions $x(t)$ and a fixed initial value for z at $t = 0$ the solution of the differential equation (2.1) depends both on t and on $x(t)$. If we take $t = s =$ const. then the solution $z = z[x; s]$ is a functional of the set of functions $x^k(t)$.

The generalized variational principle of Herglotz is as follows:

Let the functional $z = z[x; s]$ of $x(t)$ be given by a differential equation of the form (2.1) *and let the functions $\eta = (\eta^1(t), \dots, \eta^n(t))$ have continuous first derivatives and satisfy the boundary conditions*

$$\eta(0) = \eta(s) = 0 \tag{2.2}$$

but otherwise be arbitrary. Then the value of the functional $z[x; s]$ is an extremum for functions $x(t)$ which satisfy the condition

$$\left. \frac{d}{d\varepsilon} z[x + \varepsilon\eta; s] \right|_{\varepsilon=0} = 0. \tag{2.3}$$

It should be observed that when a variation $\varepsilon\eta$ is applied to x equation (2.1) must be solved with the same fixed initial condition $z(0)$ at $t = 0$ and the solution evaluated at the same fixed final $t = s$ for all varied argument functions $x + \varepsilon\eta$.

Herglotz proved that condition (2.3) is fulfilled when the functions $x^k(t)$ are solutions of the equations

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k} = 0, \quad k = 1, \dots, n \tag{2.4}$$

which he called the *generalized Euler–Lagrange equations*.

If the Lagrangian L does not depend on z and $x^k(t)$ are given functions equation (2.1) can be integrated immediately

$$z(s) - z(0) = \int_0^s L(t, x, \dot{x})\, dt$$

showing that now $z[x; s] = z(s)$ is a conventional functional of $x^k(t)$. This justifies naming L the Lagrangian. At the same time equations (2.4) reduce to the classical Euler–Lagrange equations. Hence, the Herglotz variational principle contains the classical variational principle as a special case.

3. Second Noether-type theorem for the generalized variational principle of Herglotz

The following theorem extends the second Noether theorem so that it applies to the generalized variational principle of Herglotz. This new theorem, which we call the *second Noether-type theorem*, provides an identity for each infinite dimensional symmetry group of the functional $z[x; s]$ as defined by (2.1).

Theorem 3.1. *Let the infinite-dimensional Lie group*

$$\begin{aligned} \overline{t} &= \phi(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t)), \\ \overline{x}^k &= \psi^k(t, x, p(t), p^{(1)}(t), \ldots, p^{(r)}(t)), \quad k = 1, \ldots, n, \end{aligned} \tag{3.1}$$

which depends on the function $p(t) \in \mathrm{C}^{r+2}$ and its derivatives $p^{(i)} = d^i p/dt^i$, subject to the conditions $\overline{t} = t$ and $\overline{x}^k = x^k$ when $p(t) = p^{(1)}(t) = \ldots = p^{(r)}(t) = 0$, be a symmetry group of the functional $z[x; s]$ defined by the differential equation (2.1). *Then the identity*

$$\widetilde{X}^k(EQ_k) - \widetilde{U}(EQ_k \dot{x}^k) = 0 \tag{3.2}$$

holds. Here $\widetilde{U}$ and $\widetilde{X}^k$ are the adjoints of the linear differential operators

$$\begin{aligned} U &= \frac{\partial \phi}{\partial p} + \frac{\partial \phi}{\partial p^{(1)}} \frac{d}{dt} + \ldots + \frac{\partial \phi}{\partial p^{(r)}} \frac{d^r}{dt^r}, \\ X^k &= \frac{\partial \psi^k}{\partial p} + \frac{\partial \psi^k}{\partial p^{(1)}} \frac{d}{dt} + \ldots + \frac{\partial \psi^k}{\partial p^{(r)}} \frac{d^r}{dt^r}, \quad k = 1, \ldots, n, \end{aligned} \tag{3.3}$$

evaluated at $p(t) = p^{(1)}(t) = \ldots = p^{(r)}(t) = 0$; Q_k denote the generalized Euler–Lagrange expressions in (2.4)

$$Q_k = \frac{\partial L}{\partial x^k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial \dot{x}^k}, \quad k = 1, \ldots, n, \tag{3.4}$$

$$E = \exp\biggl(- \int_0^t \frac{\partial L}{\partial z}\, d\theta \biggr). \tag{3.5}$$

PROOF. By Lie theory we know that close to the identity transformation the action of the group (3.1) is the same as the action of the infinitesimal transformation corresponding to (3.1). This infinitesimal transformation is

$$\bar{t} = t + \varepsilon U p, \quad \overline{x}^k = x^k + \varepsilon X^k p, \quad k = 1, \dots, n \tag{3.6}$$

where U and X^k are defined by (3.3). For notational simplicity we set

$$\tau = Up, \quad \xi^k = X^k p \tag{3.7}$$

with which the infinitesimal transformation (3.6) is

$$\bar{t} = t + \varepsilon\tau, \quad \overline{x}^k = x^k + \varepsilon\xi^k, \quad k = 1, \dots, n. \tag{3.8}$$

The transform of the differential equation (2.1) under (3.6), or equivalently (3.8) is

$$\frac{d\overline{z}}{d\bar{t}} = L\left(\bar{t}, \overline{x}(\bar{t}), \frac{d\overline{x}(\bar{t})}{d\bar{t}}, \overline{z}\right)$$

which can be written in more detail as

$$\frac{d\overline{z}}{dt} = L\left(\bar{t}, \overline{x}(\bar{t}), \frac{d\overline{x}(\bar{t})}{d\bar{t}}, \overline{z}\right)\frac{d\bar{t}}{dt} = L\left(t + \varepsilon\tau, x + \varepsilon\xi, \frac{d\overline{x}}{d\bar{t}}, \overline{z}\right)\frac{d}{dt}(t + \varepsilon\tau). \tag{3.9}$$

Differentiate (3.9) with respect to ε, set $\varepsilon = 0$ and denote

$$\zeta(t) = \frac{d}{d\varepsilon}\overline{z}[\overline{x}; \bar{t}]\bigg|_{\varepsilon=0} \tag{3.10}$$

to obtain the differential equation

$$\frac{d\zeta}{dt} = L\frac{d\tau}{dt} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^k}\xi^k + \frac{\partial L}{\partial \dot{x}^k}\frac{d}{d\varepsilon}\frac{d\overline{x}^k}{d\bar{t}}\bigg|_{\varepsilon=0} + \frac{\partial L}{\partial z}\zeta \tag{3.11}$$

for the variation ζ of the functional z produced by the transformation (3.8). We need to express the term

$$\frac{d}{d\varepsilon}\frac{d\overline{x}^k}{d\bar{t}}\bigg|_{\varepsilon=0}$$

with the non-transformed variables, only. To do this we observe that

$$\frac{d\overline{x}^k}{dt} = \frac{d\overline{x}^k}{d\bar{t}}\frac{d\bar{t}}{dt}$$

which on account of (3.8) becomes

$$\frac{dx^k}{dt} + \varepsilon\frac{d\xi^k}{dt} = \frac{d\overline{x}^k}{d\bar{t}}\left(1 + \varepsilon\frac{d\tau}{dt}\right). \tag{3.12}$$

Setting $\varepsilon = 0$ in the last equation produces

$$\frac{d\overline{x}^k}{d\bar{t}} = \frac{dx^k}{dt} = \dot{x}^k. \tag{3.13}$$

Next, we differentiate (3.12) with respect to ε, set $\varepsilon = 0$ and use (3.13) to find

$$\frac{d}{d\varepsilon}\frac{d\overline{x}^k}{d\overline{t}}\bigg|_{\varepsilon=0} = \frac{d\xi^k}{dt} - \dot{x}^k\frac{d\tau}{dt}.$$

Inserting this expression into equation (3.11) we get

$$\frac{d\zeta}{dt} = L\frac{d\tau}{dt} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^k}\xi^k + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\xi^k}{dt} - \dot{x}^k\frac{d\tau}{dt}\right) + \frac{\partial L}{\partial z}\zeta. \tag{3.14}$$

This is an equation for ζ of the form

$$\frac{d\zeta}{dt} = A + \frac{\partial L}{\partial z}\zeta, \quad 0 \le t \le s \tag{3.15}$$

where $\partial L/\partial z$ and

$$A = L\frac{d\tau}{dt} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^k}\xi^k + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\xi^k}{dt} - \dot{x}^k\frac{d\tau}{dt}\right)$$

are known functions of t since $x^k(t)$ are assumed to be known. The solution $\zeta(s)$ of equation (3.15), evaluated at $t = s$, is

$$\zeta(s)\,\exp\left(-\int_0^s \frac{\partial L}{\partial z}d\theta\right) - \zeta(0) = \int_0^s A\,\exp\left(-\int_0^t \frac{\partial L}{\partial z}d\theta\right)dt.$$

According to the definition of the functional z the initial value $z(0)$ is fixed and independent of x and, hence, independent of ε. Consequently $\zeta(0) = 0$. Furthermore, $\zeta(s) = 0$ because the functional $z[x; s]$ is invariant under the transformation (3.8) by the hypothesis of Theorem 3.1. Thus we have

$$\begin{aligned}\int_0^s A\,\exp\left(-\int_0^t \frac{\partial L}{\partial z}d\theta\right)dt \\ = \int_0^s E\left(L\frac{d\tau}{dt} + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^k}\xi^k + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\xi^k}{dt} - \dot{x}^k\frac{d\tau}{dt}\right)\right)dt = 0\end{aligned} \tag{3.16}$$

where for brevity we have set

$$E = \exp\left(-\int_0^t \frac{\partial L}{\partial z}d\theta\right). \tag{3.5}$$

Next we integrate by parts the terms involving $d\tau/dt$, $d\xi^k/dt$, take into account that

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x^k}\dot{x}^k + \frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k + \frac{\partial L}{\partial z}\dot{z}$$

and perform some algebraic manipulations to obtain

$$\begin{aligned}E\left(\left(L - \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\tau + \frac{\partial L}{\partial \dot{x}^k}\xi^k\right)\Bigg|_{t=0}^{t=s} \\ + \int_0^s E\left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k}\right)(\xi^k - \dot{x}^k\tau)\,dt = 0.\end{aligned}$$

Since $p(t)$ is arbitrary, we may choose $p(t)$ such that

$$(3.17)\quad p(0) = p^{(1)}(0) = \ldots = p^{(r)}(0) = 0, \quad p(s) = p^{(1)}(T) = \ldots = p^{(r)}(T) = 0$$

which makes $\tau = Up = 0$ and $\xi^k = X^k p = 0$. Then the first term in the above expression vanishes and the remaining term can be written as

$$\int_0^s E\,Q_k(X^k - \dot{x}^k U\,)\,p\,dt = 0$$

with the use of the notational abbreviations (3.3)–(3.5) and (3.7). Using the adjoint operators $\widetilde{X}^k$ and $\widetilde{U}$ the last equation becomes

$$\int_0^s (\widetilde{X}^k(E\,Q_k) - \widetilde{U}(E\,Q_k\,\dot{x}^k))\,p(t)\,dt + [\ldots]_{t=0}^{t=s} = 0$$

where the boundary terms $[\ldots]_{t=0}^{t=s} = 0$ vanish due to (3.17). Recognizing that $p(t)$ is arbitrary we arrive at the final result

$$(3.2)\qquad \widetilde{X}^k(E\,Q_k) - \widetilde{U}(E\,Q_k\,\dot{x}^k) = 0$$

which concludes the proof of the theorem. □

Observe that $E = 1$ when L does not depend on z. Then the identity (3.2) reduces to the identity provided by the second Noether theorem, namely,

$$\widetilde{X}^k\left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k}\right) - \widetilde{U}\left(\left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k}\right)\dot{x}^k\right) = 0.$$

Thus, we see that when the generalized variational principle of Herglotz reduces to the classical variational principle Theorem 3.1 reduces to the classical second Noether theorem.

4. Concluding discussion

Motivated by the important applications of the classical second Noether theorem, we are currently preparing a paper on the second Noether-type theorem for the Herglotz variational principle in the case of several independent variables. Extending the classical variational principle with one independent variable to several independent variables is trivial, but this is not so with Herglotz principle: the functional in the Herglotz variational principle with one independent variable is defined by an ordinary differential equation; in the case of several independent variables it is defined by an integro-differential equation. In a previous paper we extended the variational principle of Herglotz to the case of several independent variables and proved a first Noether-type theorem. See Georgieva et all, [5]. The classical second Noether theorem in the case of several independent variables has applications in classical field theories, gauge theory, general relativity and other areas. In the near future we will publish a paper proving the second Noether-type theorem for the Herglotz principle in the case of several independent variables together with some applications of it.

References

[1] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer–Verlag, New York, 1989.

[2] C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, ed. 2, Chelsea Publishing, New York, 1989.

[3] L. P. Eisenhart, *Continuous Groups of Transformations*, Princeton Univ. Press, 1933.

[4] B. Georgieva and R. Guenther, *First Noether-type theorem for the generalized variational principle of Herglotz*, Topol. Methods Nonlinear Anal. **20** (2002), 261–273.

[5] B. Georgieva, R. Guenther and T. Bodurov, *Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem*, J. Math. Phys. **44** (2003), 3911–3927.

[6] H. Goldstein, *Classical Mechanics*, ed. 2, Addison-Wesley Publishing, 1981.

[7] R. B. Guenther, J. A. Gottsch and C. M. Guenther, *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems*, Juliusz Center for Nonlinear Studies, Toruń (Poland), 1996.

[8] G. Herglotz, *Gesammelte Schriften* (H. Schwerdtfeger, ed.), Vandenhoeck & Ruprecht, Göttingen, 1979.

[9] ______, *Berührungstransformationen*, Lectures at the University of Göttingen, Göttingen, 1930.

[10] S. Lie, *Die Theorie der Integralinvarianten ist ein Korollar der Theorie der Differentialinvarianten*, Leipz. Berich. **3** (1897), 342–357; Gesammelte Abhandlungen, vol. 6, Tuebner, Leipzig, 1927, pp. 649–663.

[11] J. D. Logan, *Invariant Variational Principles*, Academic Press, New York, 1977.

[12] E. Noether, *Invariante Variationsprobleme*, Nachr. König. Gesell. Wissen. Göttingen, Matt.-Phys. Kl. (1918), 235–257; English transl. in Transport Theory Statist. Phys. **1** (1971), 186–207.

[13] ______, *Invarianten beliebiger Differentialausdrücke*, Nachr. König. Gesell. Wissen. Göttingen, Matt.-Phys. Kl. (1918), 37–44.

[14] P. J. Olver, *Applications of Lie Groups to Differential Equations*, ed. 2, Springer–Verlag, New York, 1993.

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Bogdana Georgieva
Department of Mathematical Sciences
Pacific University
Forest Grove, Oregon 97116, USA

E-mail address: bogdana@pacificu.edu

Ronald Guenther
Department of Mathematics
Oregon State University
Corvallis, Oregon 97331, USA

E-mail address: guenth@math.orst.edu