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EXISTENCE, MULTIPLICITY AND CONCENTRATION OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS

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Abstract. Using variational methods we establish existence and multiplicity of positive solutions for the following class of quasilinear problems

$$-\Delta_p u + \lambda V(x)|u|^{p-2}u = \mu|u|^{p-2}u + |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N$$

where $\Delta_p u$ is the p-Laplacian operator, $2 \le p < N, p^* = pN/(N-p)$, $\lambda, \mu \in (0,\infty)$ and $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying some hypothesis.

1. Introduction

We are concerned with the existence of positive solutions for the following class of quasilinear elliptic problems

$$(\mathrm{P}_{\lambda,\mu}) \qquad \begin{cases} -\Delta_p u + \lambda V(x)|u|^{p-2}u = \mu|u|^{p-2}u + |u|^{p^*-2}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $2 \le p < N$, $p^* = pN/(N-p)$, $\lambda, \mu \in (0,\infty)$ and $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function. Nonlinear equations involving the

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p-Laplacian Δ_p have been studied extensively in the last years, see for example [1]–[4], [14]–[16] and the references cited in these works. In this paper we study the problems $(\mathrm{P}_{\lambda,\mu})$ with V verifying the following hypotheses:

(H_1) $V \geq 0, \Omega = \operatorname{int}, V^{-1}(0)$ is a nonempty bounded set with smooth boundary.

(H_2) There exists $M_0 > 0$ such that $\mathcal{L}\{x \in \mathbb{R}^N : V(x) \leq M_0\} < \infty$ where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Such hypotheses were firstly posed to the potentials of a class of Schrödinger equations by Bartsch and Wang in the paper [5]. See also [6], [11] and [12]. Motivated by [6] and [11], we are here interested in the following problems related to $(\mathrm{P}_{\lambda,\mu})$:

- Existence of least energy solutions for large λ.
- The concentration behaviour of the solutions as $\lambda \to \infty$.
- Multiplicity of solutions involving the Lusternick–Schineralmann category of Ω.

Here by *a least energy solution* we understand a positive solution with minima energy over all nontrivial solutions of $(\mathrm{P}_{\lambda,\mu})$. By concentration behaviors we describe tendencies of solutions u_λ of $(\mathrm{P}_{\lambda,\mu})$ as $\lambda \to \infty$. Precisely, letting (D_μ) denote the limit problem

$$(\mathrm{D}_\mu) \qquad \begin{cases} -\Delta_p u = \mu|u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

we say that the solutions (u_n) of $(\mathrm{P}_{\lambda_n,\mu})$ will be *concentrate at a solution u of* (D_μ) if a subsequence converges strongly to u in $W^{1,p}(\mathbb{R}^N)$ as $\lambda_n \to \infty$.

We say that a sequence (u_n) of solutions of $(\mathrm{P}_{\lambda_n,\mu})$ *concentrates at a solution* u of (D_μ) if along a subsequence it converges to u strongly in $W^{1,p}(\mathbb{R}^N)$ as $\lambda_n \to \infty$.

The paper is organized as follows. In Section 2 we shall fix some notations and give several technical results. Section 3 is devoted to prove the existence of positive solution for $(\mathrm{P}_{\lambda,\mu})$, the main result reads as follows:

Theorem A. *Assume* (H_1) *and* (H_2) *hold and* $N \geq p^2$. *Then, for every* $0 < \mu < \mu_1$, *there exists* $\lambda_\mu > 0$ *such that* $(\mathrm{P}_{\lambda,\mu})$ *has at least energy solution* u_λ *for each* $\lambda \geq \lambda(\mu)$.

Here by μ_1 we denote the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. In Section 4 we shall study the concentrate behavior of the solutions found in the Theorem A, and the main result is:

THEOREM B. *Every sequence of solutions* (u_n) *of* $(\mathrm{P}_{\lambda_n,\mu})$ *such that* $\mu \in (0,\mu_1), \lambda_n \to \infty$ *and* $I_{\lambda_n,\mu}(u_n) \to c < 1/NS^{N/p}$ *as* $n \to \infty$ *concentrates at a solution of* (D_μ).

In the above theorem, S is the best Sobolev constant of the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, given by

$$S = \inf_{u \in W^{1,p}\setminus\{0\}} \frac{|\nabla u|_p^p}{|u|_{p^*}^p},$$

and $I_{\lambda,\mu}$ is functional related to $(\mathrm{P}_{\lambda,\mu})$ given by

$$I_{\lambda,\mu}(u) = \int_{\mathbb{R}^N} \bigg(\frac{1}{p}|\nabla u|^p + \frac{1}{p}(\lambda V(x) - \mu)|u|^p - \frac{1}{p^*}|u|^{p^*}\bigg)dx.$$

In Section 5, we conclude the paper by showing a result of multiplicity which is related to the Lusternick–Schineralmann category of Ω denoted by $\mathrm{cat}(\Omega)$. The result is the following:

THEOREM C. *Assume* (H_1) *and* (H_2) *hold and that* $N \geq p^2$. *Then there exist* $0 < \mu^* < \mu_1$ *and for each* $0 < \mu < \mu^*$ *two numbers* $\Lambda(\mu) > 0$ *and* $0 < c(\mu) < 1/NS^{(N/p)}$ *such that, if* $\lambda \geq \Lambda(\mu)$, *then* $(\mathrm{P}_{\lambda,\mu})$ *has at least* $\mathrm{cat}(\Omega)$ *solutions with energy* $I_{\lambda,\mu} \leq c(\mu)$.

Our methods to the problems are variational. The solutions are obtained from critical points of $I_{\lambda,\mu}$ on its Nehari manifold. Since the problem is posed on $\mathbb{R}^N$ and the imbedding of $W^{1,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is not compact, we analyze the Palais–Smale sequences with the aid of the parameter λ. We adapt an argument similar to that of Brézis and Nirenberg [10] to deal with the critical nonlinearity. By letting μ small and λ large we connect the multiplicity of solutions with the topology of Ω; the idea here may go back to the work of Benci and Cerami [7] (see also, e.g. [6], [11] and [20]). In addition, since the p-Laplacian operator Δ_p is nonlinear, it is clear that the arguments for general $p \geq 2$ are more subtle than that for $p = 2$.

2. Notations and technical results

From now on we always assume that (H_1)–(H_2) hold and that $N \geq p^2$. We denote by $|\cdot|_q$ and $\|\cdot\|_{1,p}$ the usual norms in the Banach spaces $L^q(\mathbb{R}^N)$ for $q \in [1,\infty]$ and $W^{1,p}(\mathbb{R}^N)$ respectively, and by μ_1 the first eigenvalue of the following problem

$$\begin{cases} -\Delta_p u = \eta |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Let

$$E = \bigg\{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\bigg\}$$

be the Banach space endowed with the norm

$$\|u\| = \left(\|u\|_{1,p}^p + \int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^{1/p}$$

which is equivalent to each of the norms

$$\|u\|_\lambda = \left(\|u\|_{1,p}^p + \lambda\int_{\mathbb{R}^N} V(x)|u|^p dx\right)^{1/p} \quad \text{for } \lambda > 0.$$

Lemma 2.1. *Let $\lambda_n \geq 1$ and $u_n \in E$ be such that $\lambda_n \to \infty$ and $\|u_n\|_{\lambda_n}^p < K$ for some positive constant K. Then there is $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^p(\mathbb{R}^N)$.*

Proof. Since $\|u_n\|^p \leq \|u_n\|_{\lambda_n}^p < K$ we may assume that $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L_{\rm loc}^p(\mathbb{R}^N)$. Set $C_m = \{x : |x| \leq m,\ V(x) \geq 1/m\}, m \in \mathbb{N}$. Then

$$\int_{C_m} |u_n|^p \leq m\int_{C_m} V(x)|u_n|^p \leq \frac{mK}{\lambda_n} \to 0 \quad \text{as } n \to \infty$$

for every m. This implies that $u(x) = 0$ for a.e. $x \in \mathbb{R}^N \setminus \Omega$. Hence, since $\partial\Omega$ is smooth, $u \in W_0^{1,p}(\Omega)$.

We now show that $u_n \to u$ in $L^p(\mathbb{R}^N)$. Let $F = \{x \in \mathbb{R}^N : V(x) \leq M_0\}$ with M_0 as in (H_2). Then

$$\int_{F^c} |u_n|^p \leq \frac{1}{\lambda_n M_0}\int_{F^c} \lambda_n V(x)|u_n|^p \leq \frac{K}{\lambda_n M_0} \to 0 \quad \text{as } n \to \infty.$$

Setting $B_R = \{x \in \mathbb{R}^N : |x| \leq R\}$, and choosing $r \in (1, N/(N-p))$, $r' = r/(r-1)$, we have

$$\int_{B_R^c \cap F} |u_n - u|^p \leq |u_n - u|_{pr}^p \mathcal{L}(B_R^c \cap F)^{1/r'} \leq c\|u_n - u\|^p \mathcal{L}(B_R^c \cap F)^{1/r'} \to 0$$

as $R \to \infty$ due to (H_2). Finally, since $u_n \to u$ in $L_{\rm loc}^p(\mathbb{R}^N)$,

$$\int_{B_R} |u_n - u|^p dx \quad \text{as } n \to \infty$$

from where follows $u_n \to u$ in $L^p(\mathbb{R}^N)$. □

Hereafter we denote by $L_\lambda : W^{1,p}(\mathbb{R}^N) \to (W^{1,p}(\mathbb{R}^N))'$ the operator given by

$$\langle L_\lambda u, v\rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + \lambda V(x)|u|^{p-2}uv)\,dx$$

and the number

$$\gamma_\lambda = \inf\left\{\int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x)|u|^p)\,dx;\ u \in E,\ |u|_p = 1\right\}.$$

It is easy to check that γ_λ is a nondecreasing function in λ.

Lemma 2.2. *For each $\mu \in (0,\mu_1)$ there is $\lambda(\mu) > 0$ such that*

$$\gamma_\lambda \geq \frac{(\mu+\mu_1)}{2} \quad \text{for all } \lambda \geq \lambda(\mu).$$

Consequently, there exists $\alpha_\mu > 0$ such that

$$\alpha_\mu \|u\|_\lambda^p \leq \int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda V(x) - \mu)|u|^p)\,dx \quad \text{for all } u \in E \text{ and } \lambda \geq \lambda(\mu).$$

Proof. Assume by contradiction that there exists a sequence $\lambda_n \to \infty$ such that

$$\gamma_{\lambda_n} < (\mu+\mu_1)/2 \quad \text{for all } n \in \mathbb{N}$$

and

$$\gamma_{\lambda_n} \to \tau \leq (\mu+\mu_1)/2 \quad \text{as } n \to \infty.$$

Let $u_n \in E$ be such that $|u_n|_p = 1$ and $\langle L_{\lambda_n} u_n, u_n\rangle = \tau + o_n(1)$. Since

$$\|u_n\|_{\lambda_n}^p = \int_{\mathbb{R}^N} (|\nabla u_n|^p + (1+\lambda_n V(x))|u_n|^p)\,dx$$

we have

$$\|u_n\|_{\lambda_n}^p \leq 2(1+\mu_1)$$

for all n large. By Lemma 2.1 there is $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } E \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\mathbb{R}^N).$$

Therefore

$$|u|_p = 1 \quad \text{and} \quad \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \geq \int_{\mathbb{R}^N} |\nabla u|^p\,dx$$

so

$$\int_\Omega (|\nabla u|^p - \tau|u|^p)\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^p - \tau|u_n|^p)\,dx$$

which implies

$$\int_\Omega (|\nabla u|^p - \tau|u|^p)\,dx \leq \liminf_{n\to\infty} (\langle L_{\lambda_n} u_n, u_n\rangle - \tau) = 0$$

and thus

$$\int_\Omega |\nabla u|^p\,dx \leq \tau \int_\Omega |u|^p dx = \tau < \mu_1$$

obtaining this way a contradiction. □

Consider the functional

$$I_{\lambda,\mu}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x)|u|^p - \mu|u|^p)\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}\,dx$$

that is,

$$I_{\lambda,\mu}(u) = \frac{1}{p}(\langle L_\lambda u, u\rangle - \mu|u|_p^p) - \frac{1}{p^*}|u|_{p^*}^{p^*}.$$

Then $I_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and critical points of $I_{\lambda,\mu}$ are solutions of

$$-\Delta_p u + \lambda V(x)|u|^{p-2}u = \mu|u|^{p-2}u + |u|^{p^*-2}u, \quad u \in W^{1,p}(\mathbb{R}^N).$$

Recall that a sequence $(u_n) \subset E$ is called a $(\mathrm{PS})_c$ sequence for $I_{\lambda,\mu}$, if $I_{\lambda,\mu}(u_n) \to c$ and $I'_{\lambda,\mu}(u_n) \to 0$ as $n \to \infty$. $I_{\lambda,\mu}$ is said to satisfy the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence contains a convergent subsequence.

Lemma 2.3. *If $\mu \in (0,\mu_1)$ and $\lambda \geq \lambda(\mu)$, the functional $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition for all $c < 1/NS^{(N/p)}$.*

Proof. By definition,

$$I_{\lambda,\mu}(u_n) - \frac{1}{p^*}I'_{\lambda,\mu}(u_n)u_n = \frac{1}{N}(\langle L_\lambda u_n, u_n\rangle - \mu|u_n|_p^p) \tag{2.1}$$

and

$$I_{\lambda,\mu}(u_n) - \frac{1}{p}I'_{\lambda,\mu}(u_n)u_n = \frac{1}{N}|u_n|_{p^*}^{p^*}. \tag{2.2}$$

Using Lemma 2.2 and (2.1), we get that u_n is a bounded sequence in E.

To prove that (u_n) has a strongly convergent subsequence in E, we assume that $\lambda(\mu)$ verifies the following inequality $\lambda(\mu) \geq \mu/M_0$, thus

$$\lambda M_0 - \mu \geq 0 \quad \text{for all } \lambda \in [\lambda(\mu),\infty). \tag{2.3}$$

Since (u_n) is a bounded in E, we may assume without loss of generality that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } E,\\ u_n &\to u && \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N),\\ u_n(x) &\to u(x) && \text{a.e. in } x \in \mathbb{R}^N. \end{aligned}$$

Moreover, using the same arguments developed in Garcia Azorero and Peral Alonso [14], Gueda and Veron [16] and Alves [1], we have

$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \text{in } L^p(\mathbb{R}^N),\ i = 1,\ldots,N.$$

The above informations imply that u is a weak solution of

$$-\Delta_p u + \lambda V(x)|u|^{p-2}u = \mu|u|^{p-2}u + |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N.$$

Let $w_n = u_n - u$. By the Brézis and Lieb Lemma [9], we have

$$|V^{1/p}u_n|_p^p = |V^{1/p}u|_p^p + |V^{1/p}w_n|_p^p + o_n(1), \tag{2.4}$$

$$|u_n|_{p^*}^{p^*} = |u|_{p^*}^{p^*} + |w_n|_{p^*}^{p^*} + o_n(1). \tag{2.5}$$

Moreover, using a lemma proved by Alves in [2], we also have

$$\int_{\mathbb{R}^N} ||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u - |\nabla w_n|^{p-2}\nabla w_n|^{p/(p-1)}\,dx = o_n(1). \tag{2.6}$$

From (2.4)–(2.6) together $I'_{\lambda,\mu}(u_n) \to 0$ follow

$$(\langle L_\lambda w_n, w_n\rangle - \mu|w_n|_p^p) - |w_n|_{p^*}^{p^*} = o_n(1). \tag{2.7}$$

By the last equality, up to a subsequence, we can assume that

$$\lim_{n\to\infty}(\langle L_\lambda w_n, w_n\rangle - \mu|w_n|_p^p) = l \quad \text{and} \lim_{n\to\infty}|w_n|_{p^*}^{p^*} = l \le Nc < S^{N/p}.$$

As in the proof of Lemma 2.1 one shows that

$$\int_F |w_n|^p\,dx \to 0 \quad \text{as } n\to\infty$$

where $F = \{x\in\mathbb{R}^N : V(x)\le M_0\}$. Using the inequality (2.3)

$$S|w_n|_{p^*}^{p^*} \le |\nabla w_n|_p^p \le |\nabla w_n|_p^p + \int_{F^c}(\lambda V(x)-\mu)|w_n|^p\,dx$$

hence

$$S|w_n|_{p^*}^{p^*} \le (\langle L_\lambda w_n, w_n\rangle - \mu|w_n|_p^p) + \mu\int_F |w_n|^p\,dx,$$

or equivalently

$$S|w_n|_{p^*}^{p^*} \le (\langle L_\lambda w_n, w_n\rangle - \mu|w_n|_p^p) + o_n(1).$$

Passing to the limit in the last inequality, we obtain $Sl^{(p/p^*)} \le l$. Since $l < S^{(N/p)}$ it follows $l = 0$, hence $w_n \to 0$ in E. □

3. Existence of positive solutions

The main objective of this section is to prove the Theorem A. We begin recalling the definition of the Nehari manifold $\mathcal{M}_{\lambda,\mu}$ related to the functional $I_{\lambda,\mu}$ given by

$$\mathcal{M}_{\lambda,\mu} = \{u\in E\setminus\{0\} : I'_{\lambda,\mu}(u)u = 0\}.$$

Note that by well know arguments, we have that following equality

$$c_{\lambda,\mu} = \inf_{u\in\mathcal{M}_{\lambda,\mu}} I_{\lambda,\mu}(u) = \frac{1}{N}\inf_{v\in\mathcal{V}}(\langle L_\lambda u, u\rangle - \mu|u|_p^p)^{N/p}$$

where $\mathcal{V} = \{v\in E : |v|_{p^*} = 1\}$.

Using arguments explored by Benci and Cerami [7], we have the following result:

Proposition 3.1. *Let $u\in\mathcal{M}_{\lambda,\mu}$ be a critical point of $I_{\lambda,\mu}$ with $I_{\lambda,\mu}(u) < 2c_{\lambda,\mu}$. Then u does not change sign,hence, we can assume that it is a positive function of* $(\mathrm{P}_{\lambda,\mu})$.

Below, for every domain $\mathcal{D}\subset\mathbb{R}^N$, we consider the functional

$$I_{\mu,\mathcal{D}}(u) = \frac{1}{p}\int_{\mathcal{D}}(|\nabla u|^p - \mu|u|^p)\,dx - \frac{1}{p^*}\int_{\mathcal{D}}|u|^{p^*}\,dx = \frac{1}{p}(\langle L_0 u, u\rangle - \mu|u|_p^p) - \frac{1}{p^*}|u|_{p^*}^{p^*}$$

on $W_0^{1,p}(\mathcal{D})$. Its Nehari manifold is

$$\mathcal{M}_{\mu,\mathcal{D}} = \{u \in W_0^{1,p}(\mathcal{D}) \setminus \{0\} : \langle L_0 u, u\rangle - \mu|u|_p^p = |u|_{p^*}^{p^*}\}$$

and

$$c(\mu,\mathcal{D}) = \inf_{u\in\mathcal{M}_{\mu,\mathcal{D}}} I_{\mu,\mathcal{D}}(u) = \frac{1}{N}\inf_{v\in\mathcal{V}_\mathcal{D}} (\langle L_0 u, u\rangle - \mu|u|_p^p)^{N/p}$$

where $\mathcal{V}_\mathcal{D} = \{v \in W_0^{1,p}(\mathcal{D}) : |v|_{p^*} = 1\}$.

Lemma 3.2. *If $\mu \in (0,\mu_1)$, and $\lambda \geq \lambda(\mu)$ then*

$$\frac{1}{N}(\alpha_\mu S)^{N/p} \leq c_{\lambda,\mu} < c(\mu,\Omega) < \frac{1}{N}S^{N/p}.$$

Proof. By Lemma 2.2,

$$\alpha_\mu\|v\|_{W^{1,p}}^p \leq \alpha_\mu\|v\|_\lambda^p \leq \langle L_\lambda v, v\rangle - \mu|v|_p^p.$$

Using the definitions of the numbers S, $c_{\lambda,\mu}$ and $c(\mu,\Omega)$, we have the following inequalities

$$\frac{1}{N}(\alpha_\mu S)^{N/p} \leq c_{\lambda,\mu} \leq c(\mu,\Omega).$$

From the results showed by Guedda and Veron in [16], we know that

$$c(\mu,\Omega) < \frac{1}{N}S^{N/p} \quad \text{for all } \mu \in (0,\mu_1)$$

and $c(\mu,\Omega)$ is achieved at some $u_0 > 0$ with $u_0 \in W_0^{1,p}(\Omega)\cap C(\overline{\Omega})$. Therefore $c_{\lambda,\mu} < c(\mu,\Omega)$, because otherwise would be also achieved at u_0 which vanish outside Ω. From Harnack's inequality (see Trudinger [19]) follows that $u_0 \equiv 0$ in $\mathbb{R}^N$, contradicting the fact that u_0 is positive on Ω. □

Proof of Theorem A. Let (u_n^λ) be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathcal{M}_{\lambda,\mu}$. Then by Ekeland's variational principle (see Ekeland [13]), we may assume that it is a (PS) sequence. It follows from Proposition 3.1 and Lemmas 2.3 and 2.4 that a subsequence converges to a least energy solution u_λ of $(\mathrm{P}_{\lambda,\mu})$. □

4. Concentration of the solutions

Now we prove Theorem B. We need two technical results. The first one is the following (cf. Alves, Carrião and Medeiros [3])

Lemma 4.1. *Let $F \in C^2(\mathbb{R},\mathbb{R}_+)$ a convex and even function such $F(0) = 0$ and $f(s) = F'(s) \geq 0$ for all $s \in [0,\infty)$. Then, for all $\phi,\varphi \geq 0$ we have*

$$|F(\phi-\varphi) - F(\phi) - F(\varphi)| \leq 2(f(\phi)\varphi + f(\varphi)\phi).$$

Proof. Indeed, we have two cases to be considered. If $\varphi \leq \phi$, by convexity we have

$$\frac{F(\varphi)-F(0)}{\varphi - 0} \leq f(\phi),$$

that is, $F(\varphi) \le f(\phi)\varphi$. On the other hand, since $f' = F'' \ge 0$ we have that f is nondecreasing and consequently

$$|F(\phi-\varphi) - F(\phi)| \le \varphi \int_0^1 f(\phi - t\varphi)\, dt \le \varphi f(\phi).$$

Therefore,

$$|F(\phi-\varphi) - F(\phi) - F(\varphi)| \le 2\varphi f(\phi). \tag{4.1}$$

If $\phi \le \varphi$, we repeat the above argument to find

$$|F(\phi-\varphi) - F(\phi) - F(\varphi)| \le 2\phi f(\varphi). \tag{4.2}$$

From (4.1)–(4.2) the lemma follows. □

The second one reads as

PROPOSITION 4.2. *Let u_n be a sequence of solutions related to* $(\mathrm{P}_{\lambda_n,\mu})$ *with* $\lambda_n \to \infty$. *Then, if $w_n = u_n - u$ where u is the weak limit of u_n in E, we have*

$$\langle L_{\lambda_n} u_n, u_n\rangle = \langle L_0 u, u\rangle + \langle L_{\lambda_n} w_n, w_n\rangle + o_n(1).$$

PROOF. Using Lemma 4.1 with $F(u) = |u|^p$ $(p \ge 2)$, $\phi = u_n$ and $\varphi = u$, we get

$$|u_n|^p + |u|^p - 2p\Theta_n \le |w_n|^p \le |u_n|^p + |u|^p + 2p\Theta_n \tag{4.3}$$

where $\Theta_n = |u_n|^{p-2}u_n u + |u|^{p-2} u u_n$. Repeating the same arguments explored in the proof of Lemma 2.1, we observe that $u \in W_0^{1,p}(\Omega)$, thus

$$\int_{\mathbb{R}^N} V(x)\Theta_n\, dx = 0$$

and, by (4.3),

$$\int_{\mathbb{R}^N} V(x)|u_n|^p\, dx = \int_{\mathbb{R}^N} V(x)|w_n|^p\, dx.$$

The last equality and Brézis and Lieb's Lemma imply

$$\langle L_{\lambda_n} u_n, u_n\rangle = \langle L_0 u, u\rangle + \langle L_{\lambda_n} w_n, w_n\rangle + o_n(1). \qquad \square$$

PROOF OF THEOREM B. Let (u_n) be a sequence of solutions of $(\mathrm{P}_{\lambda_n,\mu})$, $\mu \in (0,\mu_1)$, $\lambda_n \to \infty$ such that

$$NI_{\lambda_n,\mu}(u_n) = \langle L_{\lambda_n} u_n, u_n\rangle - \mu |u_n|_p^p \to Nc < S^{N/p}.$$

Then, it follows from Lemmas 2.1 and 2.2 that there exists a $u \in W_0^{1,p}(\Omega)$ such that along a subsequence $u_n \rightharpoonup u$ weakly in E and

$$u_n \to u \quad \text{in } L^p(\mathbb{R}^N). \tag{4.4}$$

Since u_n is a solution of $(\mathrm{P}_{\lambda_n,\mu})$, we have, for all $v \in E$, the following equality:

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla v + \lambda_n V(x)|u_n|^{p-2}u_n v - \mu|u_n|^{p-2}u_n v = \int_{\mathbb{R}^N} |u_n|^{p^*-2}u_n v.$$

Using the Concentration–Compactness Principle by Lions [17], and similar arguments found in [14] and [1], we have that

$$u_n \to u \quad \text{in } L^{p^*}_{\mathrm{loc}}(\Omega)$$

which implies

$$u_n \to u \quad \text{in } W^{1,p}_{\mathrm{loc}}(\Omega).$$

If $v \in W^{1,p}_0(\Omega)$ then $\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n v\,dx = 0$ for all $n \in \mathbb{N}$. So, letting $n \to \infty$ in the above equality yields

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v - \mu|u|^{p-2}uv = \int_{\mathbb{R}^N} |u|^{p^*-2}uv \quad \text{for all } v \in W^{1,p}_0(\Omega).$$

This implies that u is a solution of (D_μ). Setting $w_n = u_n - u$, by Proposition 4.2 and Brézis and Lieb's Lemma

$$(\langle L_{\lambda_n} w_n, w_n\rangle - \mu|w_n|_p^p) - |w_n|_{p^*}^{p^*} = o_n(1).$$

We claim that $|w_n|_{p^*} \to 0$. Assume by contradiction that $|w_n|_{p^*}^{p^*} \to l > 0$. Then, since

$$S|w_n|_{p^*}^p \le |\nabla w_n|_p^p \le (\langle L_{\lambda_n} w_n, w_n\rangle - \mu|w_n|_p^p) + o_n(1)$$

we have

$$S|w_n|_{p^*}^p \le |w_n|_{p^*}^{p^*} + o_n(1).$$

Using the fact that $|u_n|_{p^*}^{p^*} \ge |w_n|_{p^*}^{p^*} + o_n(1)$, we get

$$S^{N/p} \le \lim_{n\to\infty} |u_n|_{p^*}^{p^*} = Nc < S^{N/p},$$

which is a contradiction. Therefore, $|w_n|_{p^*} \to 0$ and $\langle L_{\lambda_n} w_n, w_n\rangle - \mu|w_n|_p^p \to 0$ which, jointly with (4.4), implies $\langle L_{\lambda_n} w_n, w_n\rangle \to 0$ consequently,

$$\int_{\mathbb{R}^N} (|\nabla w_n|^p + \lambda_n V|w_n|^p) \to 0. \tag{4.5}$$

Now the combination of (4.4) and (4.5) shows that $u_n \to u$ in E finishing the proof. □

Corollary 4.3. *For each $\mu \in (0,\mu_1)$, $\lim_{\lambda\to\infty} c_{\lambda,\mu} = c(\mu,\Omega)$.*

Proof. By Lemma 3.2, $c_{\lambda,\mu} \to c \le c(\mu,\Omega) < (1/N)S^{N/p}$ and, by Theorem A, $c_{\lambda,\mu}$ is achieved for $\lambda \ge \lambda(\mu)$. So Theorem B implies that c is achieved by $I_{\mu,\Omega}$ on $\mathcal{M}_{\mu,\Omega}$. Hence, $c \ge c(\mu,\Omega)$. □

5. Multiplicity of solutions involving $\operatorname{cat}(\Omega)$

In this section we prove Theorem C which establishes the existence of multiply solutions related with category of set Ω.

Following the arguments of Benci and Cerami [7], Since Ω is a bounded smooth domain of $\mathbb{R}^N$, we may fix $r > 0$ small enough such that

$$\Omega_{2r}^+ = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < 2r\} \quad \text{and} \quad \Omega_r^- = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$$

are homotopically equivalent to Ω. Moreover, we may assume that $B_r = \{x \in \mathbb{R}^N : |x| < r\} \subset \Omega$. We write $c(\mu, r) = c(\mu, B_r)$. Then, arguing as in the proof of Lemma 3.2, we have that

$$c(\mu, \Omega) < c(\mu, r) < \frac{1}{N} S^{N/p} \quad \text{for } 0 < \mu < \mu_1.$$

By Talenti [18], we know that the numbers $c(0, G)$ with $G \subset \mathbb{R}^N$ are independing of G, in the sense that $c(0, G) = (1/N)S^{N/p}$. Moreover, in Alves and Ding [4, Lemma 2.4] it was proved that

$$\lim_{\mu \to 0} c(\mu, G) = \frac{1}{N} S^{N/p}. \tag{5.1}$$

For $0 \neq u \in L^{p^*}(\Omega)$ we consider its center of mass

$$\beta(u) = \frac{\int_\Omega |u|^{p^*} x \, dx}{\int_\Omega |u|^{p^*} \, dx}.$$

Using the same arguments explored by Alves and Ding in [4, Lemma 3.3], we have the following result

Lemma 5.1. *There exists a $\mu^* = \mu^*(r) \in (0, \mu_1)$ such that, for $0 < \mu < \mu^*$,*

(a) $c(\mu, r) < 2c(\mu, \Omega)$,
(b) $\beta(u) \in \Omega_r^+$ *for every* $u \in \mathcal{M}_{\mu,\Omega}$ *with* $I_{\mu,\Omega}(u) \leq c(\mu, r)$.

As in Bartsch and Wang [6], we choose $R > 0$ with $\overline{\Omega} \subset B_R$ and set

$$\xi(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq R, \\ R/t & \text{for } R \leq t. \end{cases}$$

Define

$$\beta_0(u) = \frac{\int_\Omega |u|^{p^*} \xi(|x|) x \, dx}{\int_\Omega |u|^{p^*} \, dx} \quad \text{for } u \in L^{p^*}(\mathbb{R}^N) \setminus \{0\}.$$

Lemma 5.2. *There exist $\mu^* = \mu^*(r) \in (0, \mu_1)$ and for each $0 < \mu < \mu^*$ a number $\Lambda(\mu) \geq \lambda(\mu)$ with the following properties:*

(a) $c(\mu, r) < 2c_{\lambda,\mu}$ *for all* $\lambda \geq \Lambda(\mu)$, *and*
(b) $\beta_0(u) \in \Omega_{2r}^+$ *for all* $\lambda \geq \Lambda(\mu)$ *and all* $u \in \mathcal{M}_{\lambda,\mu}$ *with* $I_{\lambda,\mu} \leq c(\mu, r)$.

Proof. Assertion (a) follows immediately from Lemma 5.1 and Corollary 4.3. We now prove (b). Assume, by contradiction, that for μ arbitrarily small there is a sequence (u_n) such that $u_n \in \mathcal{M}_{\lambda_n,\mu}$, $\lambda_n \to \infty$, $I_{\lambda_n,\mu}(u_n) \to c \le c(\mu,r)$ and $\beta_0(u_n) \notin \Omega^+_{2r}$. Then, by Lemma 2.1, there is $u \in W^{1,p}_0(\Omega)$ such that $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^p(\mathbb{R}^N)$. We distinguish two cases:

Case 1. $|u|^{p^*}_{p^*} \le \langle L_0 u, u\rangle - \mu|u|^p_p$.

Let $w_n = u_n - u$. Since $V(x) = 0$ for $x \in \Omega$, as before, we have

$$\langle L_{\lambda_n} u_n, u_n\rangle - \mu|u_n|^p_p = \langle L_0 u, u\rangle - \mu|u|^p_p + \langle L_{\lambda_n} w_n, w_n\rangle - \mu|w_n|^p_p + o_n(1).$$

Using the fact that $u_n \in \mathcal{M}_{\lambda_n,\mu}$,

$$\langle L_{\lambda_n} w_n, w_n\rangle - \mu|w_n|^p_p \le |w_n|^{p^*}_{p^*} + o_n(1).$$

We claim that $|w_n|_{p^*} \to 0$. Assume by contradiction that $|w_n|^{p^*}_{p^*} \to l > 0$. Then, since

$$S|w_n|^p_{p^*} \le |\nabla w_n|^p_p \le \langle L_{\lambda_n} w_n, w_n\rangle - \mu|w_n|^p_p + o_n(1)$$

that is,

$$S|w_n|^p_{p^*} \le |w_n|^{p^*}_{p^*} + o_n(1).$$

Recalling that $|u_n|^{p^*}_{p^*} \ge |w_n|^{p^*}_{p^*}$ follows that

$$S^{N/p} \le \lim_{n\to\infty} |u_n|^{p^*}_{p^*} = Nc < S^{N/p},$$

which a contradiction. Consequently, $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$ and, therefore, $\beta_0(u_n) \to \beta(u)$. But, since $I_{\mu,\Omega}(u) \le c(\mu,r)$, it follows from Lemma 5.1 that $\beta(u) \in \Omega^+_r$. This contradicts our assumptions that $\beta_0(u_n) \notin \Omega^+_{2r}$.

Case 2. $|u|^{p^*}_{p^*} > \langle L_0 u, u\rangle - \mu|u|^p_p$.

In this case $tu \in \mathcal{M}_{\mu,\Omega}$ for some $t \in (0,1)$ and, therefore,

$$c(\mu,\Omega) \le I_{\mu,\Omega}(tu) = \frac{t^p}{N}(\langle L_0 u, u\rangle - \mu|u|^p_p) \le \lim_{n\to\infty} I_{\lambda_n,\mu}(u_n) \le c(\mu,r).$$

Since, by (5.1),

$$\lim_{\mu\to 0} c(\mu,\Omega) = \lim_{\mu\to 0} c(\mu,r) = \frac{1}{N} S^{N/p},$$

we have that for each $\epsilon > 0$,

$$\Big|\lim_{n\to\infty} I_{\lambda_n,\mu}(u_n) - I_{\mu,\Omega}(tu)\Big| < \frac{\epsilon}{2N} \quad \text{for all } \mu \in (0,\mu^*).$$

Consequently, there is a $n(\mu)$ large enough such that

$$\big||u_{n(\mu)}|^{p^*}_{p^*} - |tu|^{p^*}_{p^*}\big| < \varepsilon$$

which implies

$$\big|\beta_0(u_{n(\mu)}) - \beta(tu)\big| < r.$$

From Lemma 5.1, $\beta(tu) \in \Omega_r^+$, consequently by the last inequality $\beta_0(u_{n(\mu)}) \in \Omega_{2r}^+$, which is a contradiction. □

We will apply the following result of [11] to prove Theorem C.

Proposition 5.3. *Let $I: M \to \mathbb{R}$ be an even C^1−functional on a complete symmetric $C^{1,1}$-submanifold $M \subset X \setminus 0$ of some Banach space X. Assume that I is bounded below and satisfies the Palais–Smale condition* $(\mathrm{PS})_c$ *for all $c \le b$. Further, assume that there are maps*

$$i: Z \to I^{\le b} \quad \text{and} \quad \beta_0: I^{\le b} \to W$$

where $I^{\le b} = \{u \in M : I(u) \le b\}$, whose compositions $\beta_0 i$ is a homotopy equivalence, and that $\beta_0(u) = \beta_0(-u)$ for all $u \in M \cap I^{\le b}$. Then I has at least $\mathrm{cat}(Z)$ *pairs $\{u, -u\}$ of critical points with $I(u) = I(-u) \le b$.*

Proof of Theorem C. We are going to apply Proposition 5.3. Take $X = E$, $Z = \Omega_r^-$ and $W = \Omega_{2r}^+$. For $0 < \mu \le \mu^*$ and $\lambda \ge \Lambda(\mu)$ we consider $I = I_{\lambda,\mu}$, $M = \mathcal{M}_{\lambda,\mu}$ and $b = c(\mu, r)$. As mentioned before, $b < (1/N)S^{N/p}$, hence by Lemma 2.3 $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition for all $c \le b$. Clearly $I_{\lambda,\mu}(u) = I_{\lambda,\mu}(-u)$. Take $\alpha = \beta_0$ defined above. Lemma 5.2 shows that it is well defined from $I_{\lambda,\mu}^{\le c(\mu,r)}$ into $\mathcal{M}_{\lambda,\mu}$. By definition $\beta_0(u) = \beta_0(-u)$. Let $u_r \in W_0^{1,p}(B_r) \subset E$ be a minimizer of I_{μ,B_r} on $\mathcal{M}_{\mu,B_r}$ with $u_r > 0$, radially symmetric. We define the map i by setting $i(x) = u_r(\cdot - x)$. Since $i(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ for every $x \in \Omega_r^-$, we have $i(x) \in \mathcal{M}_{\lambda,\mu}$ and $I_{\lambda,\mu}(i(x)) = I_{\mu,B_r}(u_r) = c(\mu, r)$. The radially symmetry implies that $\beta_0(i(x)) = x$ for every $x \in \Omega_r^-$. Now it follows from Proposition 5.3 that $(\mathrm{P}_{\lambda,\mu})$ has at least $\mathrm{cat}(\Omega)$ solutions, finishing the proof. □

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