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NODAL SOLUTIONS FOR A NONHOMOGENEOUS ELLIPTIC EQUATION WITH SYMMETRY

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Abstract. We consider the semilinear problem $-\Delta u + \lambda u = |u|^{p-2}u + f(u)$ in Ω, $u = 0$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2 < p < 2^* = 2N/(N-2)$ and $f(t)$ behaves like $t^{p-1-\varepsilon}$ at infinity. We show that if Ω is invariant by a nontrivial orthogonal involution then, for $\lambda > 0$ sufficiently large, the equivariant topology of Ω is related with the number of solutions which change sign exactly once. The results are proved by using equivariant Lusternik–Schnirelmann theory.

1. Introduction

Consider the problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda \geq 0$, $2 < p < 2^* := 2N/(N-2)$ and the function $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies

(f_1) $\lim_{t\to\infty} f(t)/t^{p-1} = 0$;

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(f_2) there exists $\gamma > 0$ such that

$$\frac{d}{dt}\left(\frac{f(t)}{t^{1+\gamma}}\right) \geq 0 \quad \text{for any } t > 0;$$

(f_3) $f(t) \geq 0$ for any $t > 0$.

We are interested in investigating the effect of the topology of Ω on the number of solutions of (1.1). The starting point of our study is the paper of Benci and Cerami [4], where the authors considered $f \equiv 0$ and proved that (1.1) possesses at least $\operatorname{cat}(\Omega)$ positive solutions provided λ is large enough or p is close to 2^*. Here, $\operatorname{cat}(\Omega)$ stands the usual Lusternik–Schnirelmann category of $\overline{\Omega}$ in itself. The result for λ large was extended for nonhomogeneous nonlinearities by the same authors in [5]. Since the work [4], multiplicity results for problems like (1.1) involving the category have been intensively studied (see [6], [7], [11] for subcritical, and [16], [14], [2], [1] for critical nonlinearities).

In the aforementioned works, the authors obtained positive solutions. Castro, Cossio and Neuberger considered in [10] a slightly different class of nonlinearities and proved that the problem possesses a solution which changes sign exactly once. This means that the solution u is such that $\Omega \setminus u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other. In [3], Bartsch obtained infinite nodal solutions for (1.1). Motivated by these works and by a recent paper of Castro and Clapp [9], we are interested in relating the topology of Ω with the number of solutions which change sign exactly once.

We deal with the problem

$$(\mathrm{P}_\lambda^\tau) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\tau x) = -u(x) & \text{for all } x \in \Omega, \end{cases}$$

where $\tau\colon \mathbb{R}^N \to \mathbb{R}^N$ is a linear orthogonal transformation such that $\tau \neq \mathrm{id}$, $\tau^2 = \mathrm{id}$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain such that $\tau\Omega = \Omega$. Since we are looking for nodal solutions we suppose that f is odd, that is,

$$(\mathrm{f}_4) \qquad f(-t) = -f(t) \quad \text{for any } t \in \mathbb{R}.$$

Before to state our main results, we would like to quote the paper [8], where Cao, Li and Zhong proved that, under (f_1)–(f_4), the problem without symmetry (1.1) has at least $\operatorname{cat}(\Omega)$ positive solutions. Quite recently, Furtado [13] considered the problem (P_λ^τ) for $f \equiv 0$ and proved that, if $\lambda \geq 0$ is fixed and p is sufficiently close to 2^*, then there exists an effect of the equivariant topology of Ω on the number of solutions which change sign exactly once. In view of this and the results of [4], [5], [8], it is natural to ask if the same kind of result holds for the nonhomogeneous symmetric problem (P_λ^τ) when p is fixed and λ

is large. In this paper we give an affirmative answer to this question by proving the following result.

Theorem 1.1. *Suppose* $p \in (2, 2^*)$ *and* f *satisfies* (f_1)–(f_4). *Then there exists* $\overline{\lambda}(p)$ *such that, for all* $\lambda \geq \overline{\lambda}(p)$, *the problem* $(\mathrm{P}_\lambda^\tau)$ *has at least* $\tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau)$ *pairs of solutions which change sign exactly once.*

Here, $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ and τ-cat is the G_τ-equivariant Lusternik–Schnirelmann category for the group $G_\tau = \{\mathrm{id}, \tau\}$ (see Section 3). There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ with $\tau = -\mathrm{id}$. In this case $\mathrm{cat}(\mathbb{S}^{N-1}) = 2$ whereas $\tau\text{-cat}(\mathbb{S}^{N-1}) = N$. Thus, as a consequence of Theorem 1.1 we have

Corollary 1.2. *Suppose* $p \in (2, 2^*)$ *and* f *satisfies* (f_1)–(f_4). *Assume further that* Ω *is symmetric with respect to the origin,* $0 \notin \Omega$ *and there is an odd map* $\varphi: \mathbb{S}^{N-1} \to \Omega$. *Then there exists* $\overline{\lambda}(p)$ *such that, for all* $\lambda \geq \overline{\lambda}(p)$, *the problem* (1.1) *has at least* N *pairs of odd solutions which change sign exactly once.*

The above results complement those of [9] where the authors considered the critical semilinear problem

$$-\Delta u = \lambda u + |u|^{2^*-2} u, \quad u \in H_0^1(\Omega), \qquad u(\tau x) = -u(x) \quad \text{in } \Omega,$$

and obtained the same results for $\lambda > 0$ small enough. They also complement the results of [8] since we obtain nodal solutions under the same hypothesis on f, as well the aforementioned works which deal only with positive solutions.

The paper is organized as follows. In Section 2 we present the abstract framework of the problem and some technical results. Section 3 is devoted to recalling some facts about equivariant Lusternik–Schnirelmann theory. The main results are proved in Section 4.

2. Functional setting and some technical results

Throughout this paper, we denote by H the Sobolev space $H_0^1(\Omega)$ endowed with the norm

$$\|u\| = \left\{ \int_\Omega |\nabla u|^2 dx \right\}^{1/2}$$

and by $|u|_s$ the $L^s(\Omega)$-norm of a function $u \in L^s(\Omega)$. For simplicity of notation, we write only $\int_\Omega u$ instead of $\int_\Omega u(x)\,dx$.

We start by noting that the involution τ of Ω induces an action on H, which we also denote by τ, in the following way: for each $u \in H$ we define $\tau u \in H$ by

$$(\tau u)(x) = -u(\tau x). \tag{2.1}$$

If we set $H^\tau = \{u \in H : \tau u = u\}$ as being the subspace of τ-invariant functions, it follows from the above expression that any function $u \in H^\tau$ satisfies the symmetry condition which appears in $(\mathrm{P}^\tau_\lambda)$.

It is well known that the nontrivial weak solutions of the problem (1.1) are precisely the nontrivial critical points of the C^2-functional $E_\lambda\colon H \to \mathbb{R}$ given by

$$E_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2) - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega F(u),$$

where $F(t) = \int_0^t f(s)\,ds$ is the primitive of f. All of them belong to the Nehari manifold of E_λ given by

$$\begin{aligned}\mathcal{N}_\lambda &= \{u \in H \setminus \{0\} : \langle E'_\lambda(u), u\rangle = 0\}\\ &= \left\{u \in H \setminus \{0\} : \|u\|^2 + \lambda |u|_2^2 = |u|_p^p + \int_\Omega f(u)u\right\}.\end{aligned}$$

In order to obtain τ-invariant solutions, we will look for critical points of E_λ restricted to the τ-invariant Nehari manifold

$$\mathcal{N}^\tau_\lambda = \{u \in \mathcal{N}_\lambda : \tau u = u\} = \mathcal{N}_\lambda \cap H^\tau.$$

By using conditions $(f_2) - (f_4)$ we can check that

$$0 \le (2+\gamma)F(t) \le f(t)t, \tag{2.2}$$

for any $t \in \mathbb{R}$. Thus, if $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned}E_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega |u|^p + \frac{1}{2}\int_\Omega f(u)u - \int_\Omega F(u)\\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega |u|^p + \left(\frac{1}{2} - \frac{1}{2+\gamma}\right)\int_\Omega f(u)u > 0,\end{aligned}$$

and therefore the following minimization problems are well defined

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} E_\lambda(u) \quad \text{and} \quad m^\tau_\lambda = \inf_{u \in \mathcal{N}^\tau_\lambda} E_\lambda(u).$$

By using the symmetry of the problem $(\mathrm{P}^\tau_\lambda)$ we can obtain the following relation between the two minimizers defined above.

Lemma 2.1. *For any $\lambda \ge 0$, we have that $2m_\lambda \le m^\tau_\lambda$.*

Proof. Let $u \in \mathcal{N}^\tau_\lambda$ and set $u^\pm = \max\{\pm u, 0\}$. Since $u \in H^\tau$, we can use (2.1) to conclude that u is negative in $\tau(A)$ whenever u is positive in some subset $A \subset \Omega$. We claim that

$$\int_\Omega f(u^\pm)u^\pm = \frac{1}{2}\int_\Omega f(u)u, \quad \|u^\pm\|^2 = \frac{1}{2}\|u\|^2 \quad \text{and} \quad |u^\pm|_s^s = \frac{1}{2}|u|_s^s, \tag{2.3}$$

for any $2 \leq s < 2^*$. Indeed, if we set $\Omega^+ = \{x \in \Omega : u(x) > 0\}$, we can use (2.1) to verify that $\Omega^- = \{x \in \Omega : u(x) < 0\} = \tau(\Omega^+)$. Recalling that $u = u^+ - u^-$ and f is an odd function, we obtain

$$\int_\Omega f(u)u = \int_{\Omega^+} f(u^+)u^+ - \int_{\Omega^-} f(-u^-)u^- = \int_\Omega f(u^+)u^+ + \int_\Omega f(u^-)u^-. \tag{2.4}$$

Moreover, since $\tau = \tau^{-1}$, we can use a change of variables to conclude that

$$\begin{aligned}\int_\Omega f(u^+)u^+ &= \int_{\Omega^+} f(u(x))u(x)\,dx = \int_{\tau^{-1}(\Omega^+)} f(u(\tau y))u(\tau y)\,dy \\ &= \int_{\Omega^-} f(-u(y))(-u(y))\,dy = \int_\Omega f(u^-)u^-.\end{aligned}$$

This and (2.4) imply the first equality in (2.3). The other ones can be proved in a similar way.

Since F is even and $F(0) = 0$, we can argue as above to conclude that

$$\int_\Omega F(u) = \int_\Omega F(u^+ - u^-) = \int_\Omega F(u^+) + \int_\Omega F(u^-).$$

Moreover, since $u \in \mathcal{N}_\lambda^\tau$, it follows from (2.3) that $u^\pm \in \mathcal{N}_\lambda$. Thus, we can use the above equation and (2.3) to get

$$E_\lambda(u) = E_\lambda(u^+) + E_\lambda(u^-) \geq 2m_\lambda,$$

which concludes the proof of the lemma. $\square$

In what follows we denote by $\|E_\lambda'(u)\|_*$ the norm of the derivative of the restriction of E_λ to $\mathcal{N}_\lambda^\tau$ at u, which is defined by (see [18, Section 5.3])

$$\|E_\lambda'(u)\|_* = \min_{\theta\in\mathbb{R}} \|E_\lambda'(u) - \theta J_\lambda'(u)\|_{(H^\tau)^*},$$

where $(H^\tau)^*$ is the dual space of H^τ and $J_\lambda\colon H^\tau \to \mathbb{R}$ is given by

$$J_\lambda(u) = \|u\|^2 + \lambda|u|_2^2 - |u|_p^p - \int_\Omega f(u)u.$$

LEMMA 2.2. *If u is a critical point of E_λ restricted to $\mathcal{N}_\lambda^\tau$, then $E_\lambda'(u) = 0$ in the dual space of H.*

PROOF. By definition, there exits $\theta \in \mathbb{R}$ such that $\langle E_\lambda'(u) - \theta J_\lambda'(u), \phi\rangle = 0$, for all $\phi \in H^\tau$. Since $u \in \mathcal{N}_\lambda^\tau$, we can take $\phi = u$ to get $\theta\langle J_\lambda'(u), u\rangle = 0$.

By using (f_2)–(f_4) we can check that

$$f(t)t - f'(t)t^2 \leq -\gamma f(t)t \leq 0 \quad \text{for any } t \in \mathbb{R}.$$

This and the definition of J_λ imply that

$$\begin{aligned}\langle J'_\lambda(u), u\rangle &= 2\|u\|^2 + 2\lambda|u|_2^2 - p|u|_p^p - \int_\Omega \{f'(u)u^2 + f(u)u\} \\ &= (2-p)|u|_p^p + \int_\Omega \{f(u)u - f'(u)u^2\} < 0.\end{aligned}$$

Thus $\theta = 0$ and therefore $\langle E'_\lambda(u), \phi\rangle = 0$ for all $\phi \in H^\tau$. The result follows from the principle of symmetric criticality [15] (see also [18, Theorem 1.28]). □

Let V be a Banach space, M be a C^1-manifold of V and $I: V \to \mathbb{R}$ a C^1-functional. We recall that I restricted to M satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset M$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. We end this section by stating the compactness property satisfied by E_λ.

Lemma 2.3. *The functional E_λ restricted to $\mathcal{N}_\lambda^\tau$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.*

Proof. Since we are dealing with a subcritical nonlinearity, the proof follows from the boundedness of Ω, the Ambrosetti–Rabinowitz condition in (2.2) and standard arguments (see [5]). We omit the details. □

3. Equivariant Lusternik–Schnirelmann theory

We recall in this section some facts about equivariant Lusternik–Schnirelmann theory. An involution on a topological space X is a continuous function $\tau_X: X \to X$ such that τ_X^2 is the identity map of X. A subset A of X is called τ_X-invariant if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y, respectively, then an equivariant map is a continuous function $f: X \to Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1: X \to Y$ are equivariantly homotopic if there is an homotopy $\Theta: X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\tau_X(x), t) = \tau_Y(\Theta(x,t))$, for all $x \in X, t \in [0,1]$.

Definition 3.1. The equivariant category of an equivariant map $f: X \to Y$, denoted by (τ_X, τ_Y)-cat(f), is the smallest number k of open invariant subsets $X_1, \ldots, X_k$ of X which cover X and which have the property that, for each $i = 1, \ldots, k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i: X_i \times [0,1] \to Y$ such that $\Theta_i(x,0) = f(x)$, $\Theta_i(x,1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta_i(\tau_X(x), t) = \tau_Y(\Theta_i(x,t))$ for every $x \in X_i$, $t \in [0,1]$. If no such covering exists we define (τ_X, τ_Y)-cat$(f) = \infty$.

If A is a τ_X-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion map we write

$$\tau_X\text{-cat}_X(A) = (\tau_X, \tau_X)\text{-cat}(\iota) \quad \text{and} \quad \tau_X\text{-cat}(X) = \tau_X\text{-cat}_X(X).$$

The following properties can be verified.

LEMMA 3.2.

(a) *If* $f: X \to Y$ *and* $h: Y \to Z$ *are equivariant maps then*

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \le \tau_Y\text{-cat}(Y).$$

(b) *If* $f_0, f_1: X \to Y$ *are equivariantly homotopic then*

$$(\tau_X, \tau_Y)\text{-cat}(f_0) = (\tau_X, \tau_Y)\text{-cat}(f_1).$$

Let $\tau_a: V \to V$ be the antipodal involution $\tau_a(u) = -u$ on the vector space V. Equivariant Lusternik–Schnirelmann category provides a lower bound for the number of pairs $\{u, -u\}$ of critical points of an even functional, as stated in the following abstract result (see [12, Theorem 1.1], [17, Theorem 5.7]).

THEOREM 3.3. *Let* $I: M \to \mathbb{R}$ *be an even* C^1*-functional on a complete symmetric* $C^{1,1}$*-submanifold* M *of some Banach space* V*. Assume that* I *is bounded below and satisfies (PS)*$_c$ *for all* $c \le d$*. Then, if* $I^d = \{u \in M : I(u) \le d\}$*, the functional* I *has at least* $\tau_a\text{-cat}_{I^d}(I^d)$ *antipodal pairs* $\{u, -u\}$ *of critical points with* $I(\pm u) \le d$*.*

4. Proofs of the main results

By standard regularity theory we know that if u is a solution of $(\mathrm{P}_\lambda^\tau)$, then it is of class C^1. We say it changes sign k times if the set $\{x \in \Omega : u(x) \neq 0\}$ has $k+1$ connected components. By (2.1), if u is a nontrivial solution of problem $(\mathrm{P}_\lambda^\tau)$ then it changes sign an odd number of times. More specifically, we have the following relation between the number of nodal regions of a solution and its energy.

LEMMA 4.1. *If* u *is a solution of problem* $(\mathrm{P}_\lambda^\tau)$ *which changes sign* $2k-1$ *times, then* $E_\lambda(u) \ge k m_\lambda^\tau$*. In particular, if* u *is a nontrivial solution of* $(\mathrm{P}_\lambda^\tau)$ *such that* $E_\lambda(u) < 2m_\lambda^\tau$*, then* u *changes sign exactly once.*

PROOF. The set $\{x \in \Omega : u(x) > 0\}$ has k connected components $A_1, \ldots, A_k$. Let $u_i(x) = u(x)$ if $x \in A_i \cup \tau A_i$ and $u_i(x) = 0$, otherwise. We have that

$$\begin{aligned} 0 =&\langle E_\lambda'(u), u_i\rangle = \int_\Omega (\nabla u \cdot \nabla u_i + \lambda u u_i - |u|^{p-2} u u_i - f(u) u_i) \\ =&\|u_i\|^2 + \lambda |u_i|_2^2 - |u_i|_p^p - \int_\Omega f(u_i) u_i. \end{aligned}$$

Thus, $u_i \in \mathcal{N}_\lambda^\tau$ for all $i = 1, \ldots, k$, and $E_\lambda(u) = E_\lambda(u_1) + \ldots + E_\lambda(u_k) \ge k m_\lambda^\tau$, as desired. □

Given $r > 0$, we define the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < r\} \quad \text{and} \quad \Omega_r^- = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega \cup \Omega^\tau) \ge r\}.$$

From now on we fix $r > 0$ sufficiently small in such way that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences. We also define the barycenter map $\beta : H \setminus \{0\} \to \mathbb{R}^N$ by setting

$$\beta(u) = \frac{\int_\Omega x \cdot |\nabla u(x)|^2 \, dx}{\int_\Omega |\nabla u(x)|^2 \, dx}.$$

Let $E_{\lambda,r} \colon H_0^1(B_r(0)) \to \mathbb{R}$ be defined as

$$E_{\lambda,r}(u) = \frac{1}{2} \int_{B_r(0)} (|\nabla u|^2 + \lambda u^2) - \frac{1}{p} \int_{B_r(0)} |u|^p - \int_{B_r(0)} F(u),$$

and set

$$m_{\lambda,r} = \inf_{u \in \mathcal{N}_{\lambda,r}} E_{\lambda,r}(u),$$

where $\mathcal{N}_{\lambda,r}$ stands the Nehari manifold of $E_{\lambda,r}$. The following lemma is an important tool for the proof of Theorem 1.1.

Lemma 4.2. *For any fixed $p \in (2, 2^*)$ there exists $\overline{\lambda}(p)$ such that, for any $\lambda \geq \overline{\lambda}(p)$, there hold*

(a) $m_{\lambda,r} < 2m_\lambda$,
(b) *if $u \in \mathcal{N}_\lambda$ and $E_\lambda(u) \leq m_{\lambda,r}$, then $\beta(u) \in \Omega_r^+$.*

Proof. See [8, Corollary 3.20 and Lemma 3.24]. □

For any given $d > 0$ we set $E_\lambda^d = \{u \in \mathcal{N}_\lambda^\tau : E_\lambda(u) \leq d\}$. By using the second statement of the above lemma we are able to prove the following result.

Lemma 4.3. *For any fixed $p \in (2, 2^*)$, let $\overline{\lambda}(p)$ be given by Lemma 4.2. Then*

$$\tau_a\text{-cat}(E_\lambda^{2m_{\lambda,r}}) \geq \tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau), \quad \text{for any } \lambda \geq \overline{\lambda}(p).$$

Proof. We claim that, for any $\lambda \geq \overline{\lambda}(p)$ fixed, there exist two maps

$$\Omega_r^- \xrightarrow{\alpha_\lambda} E_\lambda^{2m_{\lambda,r}} \xrightarrow{\gamma_\lambda} \Omega_r^+$$

such that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$, $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$, and $\gamma_\lambda \circ \alpha_\lambda$ is equivariantly homotopic to the inclusion map $\Omega_r^- \hookrightarrow \Omega_r^+$. Assuming the claim and recalling that the maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences, we can use Lemma 3.2 to get

$$\tau_a\text{-cat}(E_\lambda^{2m_{\lambda,r}}) \geq \tau\text{-cat}_{\Omega_r^+}(\Omega_r^-) = \tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau).$$

In order to prove the claim we follow [9]. Let $v_\lambda \in \mathcal{N}_{\lambda,r}$ be a positive radial function such that $E_{\lambda,r}(v_\lambda) = m_{\lambda,r}$. We define $\alpha_\lambda \colon \Omega_r^- \to E_\lambda^{2m_{\lambda,r}}$ by

$$\alpha_\lambda(x) = v_\lambda(\,\cdot - x) - v_\lambda(\,\cdot - \tau x). \tag{4.1}$$

It is clear that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$. Furthermore, since v_λ is radial and τ is an isometry, we have that $\alpha_\lambda(x) \in H^\tau$. The definition of Ω_r^- implies that

$|x - \tau x| \geq 2r$ for any $x \in \Omega_r^-$. Thus, we can check that $E_\lambda(\alpha_\lambda(x)) = 2m_{\lambda,r}$ and $\alpha_\lambda(x) \in \mathcal{N}_\lambda^\tau$. All this considerations show that α_λ is well defined.

Given $u \in E_\lambda^{2m_{\lambda,r}}$ we can proceed as in the proof of Lemma 2.1 to conclude that $u^+ \in \mathcal{N}_\lambda$ and $2E_\lambda(u^+) = E_\lambda(u) \leq 2m_{\lambda,r}$. It follows from Lemma 4.2(b) that $\gamma_\lambda \colon E_\lambda^{2m_{\lambda,r}} \to \Omega_r^+$ given by $\gamma_\lambda(u) = \beta(u^+)$ is well defined. A simple calculation shows that $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$. Moreover, using (4.1) and the fact that v_λ is radial we get

$$\gamma_\lambda(\alpha_\lambda(x)) = \frac{\int_{B_r(x)} y \cdot |\nabla v_\lambda(y-x)|^2\, dy}{\int_{B_r(x)} |\nabla v_\lambda(y-x)|^2\, dy} = \frac{\int_{B_r(0)} (y+x) \cdot |\nabla v_\lambda(y)|^2\, dy}{\int_{B_r(0)} |\nabla v_\lambda(y)|^2\, dy} = x,$$

for any $x \in \Omega_r^-$. This concludes the proof. □

We are now ready to present the proof of our main results.

Proof of Theorem 1.1. Let $p \in (2, 2^*)$ and $\overline{\lambda}(p)$ be given by the Lemma 4.2. For any $\lambda \geq \overline{\lambda}(p)$, we can apply Theorem 3.3 and Lemma 4.3 to obtain $\tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau)$ pairs $\pm u_i$ of critical points of the even functional E_λ restricted to $\mathcal{N}_\lambda^\tau$ verifying

$$E_\lambda(\pm u_i) \leq 2m_{\lambda,r} < 4m_\lambda \leq 2m_\lambda^\tau,$$

where we have used Lemma 4.2(a) and Lemma 2.1. It follows from Lemmas 2.2 and 4.1 that $\pm u_i$ are solutions of $(\mathrm{P}_\lambda^\tau)$ which change sign exactly once. □

Proof of Corollary 1.2. Let $\tau \colon \mathbb{R}^N \to \mathbb{R}^N$ be given by $\tau(x) = -x$. It is proved in [9, Corollary 3] that our assumptions imply $\tau\text{-cat}(\Omega) \geq N$. Since $0 \notin \Omega$, $\Omega^\tau = \emptyset$. It suffices now to apply Theorem 1.1. □

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