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# GRAPH APPROXIMATIONS OF SET-VALUED MAPS UNDER CONSTRAINTS

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Abstract. In the paper we study the existence of constrained graph approximations of set-valued maps with non-convex values. We prove, in particular, that any open neighbourhood of the graph of a map satisfying the so-called topological tangency assumptions contains a graph of constrained continuous single-valued map provided that the domain is finite-dimensional.

## 1. Introduction

Let $X$, $Y$ be metric spaces, $\varphi\colon X \multimap Y$ be an *upper semicontinuous* set-valued map with *compact* values. Let $C \subset X \times Y$ be a *constraint set*, i.e. we assume that for any $x \in X$,

$$\varphi(x) \cap C(x) \neq \emptyset, \tag{1.1}$$

where $C(x) := \{y \in Y \mid (x,y) \in C\}$. The constraint set determines the set valued map $C(\,\cdot\,)\colon X \ni x \multimap C(x) \in Y$.

We shall address the *constrained approximation problem.* Namely, given an open neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}\,(\varphi)$ of $\varphi$, find a continuous map $f\colon X \to Y$ such that $\mathrm{Gr}\,(f) \subset \mathcal{U} \cap C$. Thus we are looking for $\mathcal{U}$-*approximation* $f$ of $\varphi$

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(i.e. $\mathrm{Gr}(f) \subset \mathcal{U}$) being a *selection* of the map $C(\,\cdot\,)$ (i.e. $f(x) \in C(x)$ for any $x \in X$) simultaneously. If $A \subset X$, then a continuous map $f\colon A \to Y$ such that $\mathrm{Gr}(f) \subset \mathcal{U} \cap C$ will be called $\mathcal{U} \cap C$-*approximation* (on $A$).

The natural motivation to consider the problem may be the following. If $X$ is a smooth manifold embedded in a Banach space $Y$, and for any $x \in X$, $C(x) := T_x X$ is the tangent space of $X$ at $x$, then $C = TX$ is the tangent bundle. Moreover, if $\varphi$ is *weakly tangent* to $X$, i.e. condition (1.1) holds, and $\mathcal{U}$ is an open neighbourhood of $\mathrm{Gr}(\varphi)$, then the question of the existence of $\mathcal{U} \cap C$-approximations concerns the availability of a vector field on $X$ such that $\mathrm{Gr}(f) \subset \mathcal{U}$. Therefore it seems that the existence of constrained graph approximations is important for the study of differential inclusions on manifolds. More generally, we can consider fiberwise spaces $X$ and $Y$ over a base space $B$ and $C$ is their fiberwise product. Then $\mathcal{U} \cap C$-approximations coincides with fiberwise maps over $B$ being $\mathcal{U}$-approximations of $\varphi$. Other examples of constraint sets will be given below. Some applications will be given elsewhere (e.g. [20]).

Recall that in case of the lack of constraints, i.e. if $C = X \times Y$, the problem has been intensively studied by many authors (see [12] and references therein). For example, a classical result due to Cellina ([9]), says that if $Y$ is a normed space and $\varphi$ has convex values, then for any $\varepsilon > 0$, there is a $B(\mathrm{Gr}(\varphi), \varepsilon)$-approximation on $X$ (1). Among many results concerning the approximability of maps with not necessarily convex values (see [11], [7], [8], [14] and references therein), we recall the following Kryszewski theorem that will be necessary for our purposes ([13], comp. [23]): If $Y \in \mathrm{LC}^n$ and $\varphi(x)$ has $\mathrm{UV}^n$-property in $Y$ for any $x \in X$ (see Section 3.1), then for any open neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$, there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{V}$-approximation on $A$ such that $\dim(X \setminus A) \le n+1$, extends to a $\mathcal{U}$-approximation on $X$.

If constraint set $C$ is a proper subset of $X \times Y$, Ben-El-Mechaiekh and Kryszewski obtained in [3] the Cellina–Michael-type result, i.e. if $Y$ is a Banach space, $\varphi$ has convex values and the map $C(\,\cdot\,)\colon X \multimap Y$ is lower semicontinuous with closed and convex values, then for any $\varepsilon > 0$ there is a $B(\mathrm{Gr}(\varphi), \varepsilon) \cap C$-approximation on $X$. In [17] similar results for set-valued maps with so-called $\alpha$-convex values, defined on the finite dimensional space $X$, are obtained.

If $C(\,\cdot\,)$ is still lower semicontinuous with closed convex values, but $\varphi(x)$ is contractible or has $\mathrm{UV}^n$-property in $Y$ for any $x \in X$, then constrained approximation problem may have no solution (see Example 3.14).

(1) If values of the map $\varphi$ are convex and compact, then there is a $\mathcal{U}$-approximation for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$.

In the paper we provide conditions sufficient for the existence of constrained graph approximation in general situation without assumptions concerning convexity of values of $\varphi$ and $C(\,\cdot\,)$. For that reason, we introduce a condition (C) describing the behavior of the set-valued map $\varphi$ with regard to the constraint set $C$ and obtain Theorem 3.1 and Corollary 3.2. In Section 3.1 we provide the fiberwise UV-property, which allows to generalize the above Kryszewski's result (Corollary 3.13). Section 3.2 is devoted to the proof of the approximation result. Moreover, we study different variants of condition (C). In case if $C$ is the fiberwise product of $X$ and $Y$ we obtain the existence of fiberwise $\mathcal{U}$-approximations (Corollary 3.18). In Section 3.4 we investigate the problem if sufficiently close $\mathcal{U}\cap C$-approximations are homotopic.

## 2. Preliminaries

From now on by a *space* we mean a *metric space* and by a map a *continuous transformation* of spaces. Given a space $X$ and $A\subset X$, $\operatorname{cl} A$ stands for the *closure* of $A$.

Let $B$ a space. A *fiberwise space over* $B$ (or just a space over $B$) is a pair $(X,p)$ consisting of a space $X$ and a map $p\colon X\to B$, called the *projection* ([2]). $B$ is called the *base* space of $X$. Note that if $(X,p)$ is a space over $B$ and $X_0\subset X$, then the pair $(X_0,p_0)$, where $p_0:=p|_{X_0}\colon X_0\to B$ is the restriction of $p$ to $X_0$, is a *subspace* over $B$ of $(X,p)$. Similarly, given $A\subset B$, $(X_A,p_A):=(p^{-1}(A),p|_{p^{-1}(A)})$ is the space over $A$. By a *trivial space over* $B$ we mean the product $X\times B$ with the usual projection $\pi_B\colon X\times B\to B$ onto $B$, i.e. $\pi_B(x,b)=b$ for $x\in X$ and $b\in B$ ([3]).

Let $(X,p)$ and $(Y,q)$ be spaces over $B$. We say that a map $f\colon X\to Y$ is a *map over* $B$ if for each $b\in B$, $f$ transforms the *fiber* $X_b:=p^{-1}(b)$ into the fiber $Y_b:=q^{-1}(b)$, i.e. $q\circ f=p$ ([4]). It is clear that $f\colon X\to Y$ is a map over $B$ if and only if the *graph* $\operatorname{Gr}(f):=\{(x,f(x))\mid x\in X\}$ of $f$ is a subset of the *fiberwise product* $X\times_B Y:=\{(x,y)\in X\times Y\mid p(x)=q(y)\}$.

By a *homeomorphism* (resp. an *embedding*, a *closed embedding*, a *retraction*) over the base space $B$ we understand a map over $B$ being a homeomorphism (resp. an embedding, a closed embedding, a retraction).

Given spaces $X$, $Y$, by a *set-valued map* $\varphi$ from $X$ into $Y$ (written $\varphi\colon X\multimap Y$) we mean a map assigning to any $x\in X$ a *nonempty* (not necessarily closed) subset $\varphi(x)$ of $Y$ and, by the *graph* of $\varphi$, the set $\operatorname{Gr}(\varphi):=\{(x,y)\in X\times Y\mid y\in\varphi(x)\}$. Note that $\operatorname{Gr}(\varphi)$ may be regarded as a space over $X$ with the usual

([2]) If the projection $p$ is recognized from the context, then we say simply that $X$ is a space over $B$.

([3]) Sometimes it is more convenient to consider the product $B\times X$ as a trivial space over $B$.

([4]) Since we do not require $p$ to be surjective, fibers may be empty.

projection $\pi_X \colon \mathrm{Gr}(\varphi) \to X$ onto $X$. We say that a set-valued map $\varphi$ is *upper semicontinuous* (resp. *lower semicontinuous*) if for any closed (resp. open) set $U \subset Y$, the *preimage* $\varphi^{-1}(U) := \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is closed (resp. open); $\varphi$ is *continuous* if it is upper and lower semicontinuous. Recall that the graph $\mathrm{Gr}(\varphi)$ of an upper semicontinuous map $\varphi$ with closed values is closed; $\varphi$ is upper semicontinuous and has compact values if and only if for each $x \in X$ and a sequence $(x_n, y_n) \in \mathrm{Gr}(\varphi)$ such that $x_n \to x$, there exists a subsequence $(y_{n_k})$ such that $y_{n_k} \to y \in \varphi(x)$ (this means that a projection $\pi_X$ is a *proper* map, i.e. for each compact $K \subset X$, $\pi_X^{-1}(K)$ is compact). A set-valued map $\varphi$ is lower semicontinuous if for given $x \in X$, $y \in \varphi(x)$ and a sequence $(x_n)$ convergent to $x$, there is a sequence $y_n \in \varphi(x_n)$ such that $y_n \to y$. It is clear that $\varphi$ is lower semicontinuous if and only if the projection $\pi_X$ is open. Moreover $p \colon X \to B$ is an open map if and only if $p(X)$ is open in $B$ and the set-valued map $B \supset p(X) \ni b \multimap p^{-1}(b) \subset X$ is lower semicontinuous. Hence $p \colon X \to B$ is an open map if and only if for any $(x, b) \in X \times B$ such that $p(x) = b$ and for any sequence $\{b_n\} \subset B$ such that $b_n \to b$, there is a sequence $\{x_n\} \subset X$ such that $x_n \to x$ and for almost all $n$, $p(x_n) = b_n$.

By a *selection* of $\varphi$ we mean a map $f \colon X \to Y$ such that $f(x) \in \varphi(x)$ for any $x \in X$. Note that if $(X, p)$, $(Y, q)$ are spaces over $B$, then $f \colon X \to Y$ is a map over $B$ if and only if $f$ is a selection of the set valued map $\varphi(x) := q^{-1}(p(x))$, $x \in X$.

Given a space $X$, by $\dim(X)$ we denote the *covering dimension* of $X$. Denote by $\mathcal{M}$ the class of all *closed pairs* of (metric) spaces, i.e. $(Z, Z_0) \in \mathcal{M}$ if $Z$ is a space and $Z_0 \subset Z$ is closed. The following subclass of $\mathcal{M}$

$$\mathcal{M}_n^c := \{(Z, Z_0) \in \mathcal{M} \mid \dim(Z \setminus Z_0) \leq n + 1\}, \quad n \geq 0,$$

will be of a special importance for us.

Now, for a reader's convenience we shall recall some relevant concepts and results obtained in [19].

Lemma 2.1. *If $(Z, Z_0) \in \mathcal{M}_n^c$, then for any open cover $\mathcal{W}$ of the space $Z$, there exists a sequence of closed subspaces $Z_0 \subset Z_1 \subset \ldots \subset Z_{n+2} = Z$, such that for $i = 0, \ldots, n+1$,*

$$Z_{i+1} = Z_i \cup \bigcup_{\alpha \in I_i} B_i^\alpha,$$

*where $I_i$ is a set of indices, and for any $\alpha \in I_i$, $B_i^\alpha$ is a closed set contained in an element of the cover $\mathcal{W}$. The sets $\{B_i^\alpha \cap (Z \setminus Z_0)\}_{\alpha \in I_i}$ are pairwise separated by open neighbourhoods* ([5]).

([5]) Subsets $\{B_i\}_{i \in I}$ of a space $Z$ are pairwise separated by open neighbourhoods, if for any $i, i' \in I$, $i \neq i'$, there are open neighbourhoods $U \supset B_i$ and $V \supset B_{i'}$ such that $U \cap V = \emptyset$.

Let $\mathcal{C}$ be a subclass of $\mathcal{M}$. We say that a space $Y$ is an *absolute neighbourhood extensor* (resp. *absolute extensor*) *for the class* $\mathcal{C}$, written $Y \in \mathrm{ANE}(\mathcal{C})$ (resp. $Y \in \mathrm{AE}(\mathcal{C})$), if for any pair $(Z, Z_0) \in \mathcal{C}$, any map $f_0: Z_0 \to Y$ admits a (continuous) extension $f: U \to Y$ onto an open neighbourhood $U$ of $Z_0$ (resp. $f: Z \to Y$). It is well-known that the class $\mathrm{ANE}(\mathcal{M})$ (resp. $\mathrm{AE}(\mathcal{M})$) coincides with the class of absolute neighbourhood retracts (resp. absolute retracts) ([5], [16]). It is clear that $\mathrm{ANE}(\mathcal{M}) \subset \mathrm{ANE}(\mathcal{M}_n^c)$ and $\mathrm{AE}(\mathcal{M}) \subset \mathrm{AE}(\mathcal{M}_n^c)$ since $\mathcal{M}_n^c \subset \mathcal{M}$.

Recall that for $n \geq 0$ a space $K$ is *locally $n$-connected* (written $K \in \mathrm{LC}^n$) if for any $y \in K$, any open neighbourhood $U$ of $y$ in $K$ contains an open neighbourhood $V$ of $y$ in $K$ such that for any $0 \leq k \leq n$, every map $f_0: S^k \to V$ extends to $f: D^{k+1} \to U$, or equivalently, every map $f_0: S^k \to V$ is homotopically trivial in $U$ (here and below $S^k$ and $D^{k+1}$ stand for the unit sphere and the unit closed ball in $\mathbb{R}^{k+1}$ for $k \geq 0$ and $S^{-1} = \emptyset$, $D^0 = \{0\}$). A space $K$ is *$n$-connected* (written $K \in C^n$) if for all $0 \leq k \leq n$, every map $f_0: S^k \to K$ extends to $f: D^{k+1} \to K$. In other words $C^n = \mathrm{AE}(\mathcal{S}_n)$ where

$$\mathcal{S}_n := \{(D^{k+1}, S^k) \mid -1 \leq k \leq n\}.$$

The classical Kuratowski–Dugundji extension theorem (see [16]) asserts that for any integer $n \geq 0$,

$$\mathrm{ANE}(\mathcal{M}_n^c) = \mathrm{LC}^n, \qquad \mathrm{AE}(\mathcal{M}_n^c) = \mathrm{LC}^n \cap C^n.$$

In what follows, we recall the similar characterization of fiberwise absolute (neighbourhood) extensors given in [19].

Let again $\mathcal{C}$ be a subclass of the class $\mathcal{M}$ of all closed pairs of spaces.

Definition 2.2. We say that a space $(Y, q)$ over $B$ is a (*fiberwise*) *absolute neighbourhood extensor over $B$ for the class* $\mathcal{C}$, written $(Y, q) \in \mathrm{ANE}_B(\mathcal{C})$, if for every space $(Z, s)$ over $B$, $(Z, Z_0) \in \mathcal{C}$ and a map $f_0: Z_0 \to Y$ over $B$, there is an open neighbourhood $U$ of $Z_0$ in $Z$ and a map $f: U \to Y$ over $B$ such that the following diagram

$$\begin{tikzcd} Z_0 \arrow[r, "f_0"] \arrow[dr, "\subset"'] & Y \arrow[r, "q"] & B \\ & U \arrow[u, dotted, "f"] \arrow[ur, "s|_U"'] & \end{tikzcd}$$

is commutative; in other words $f_0$ admits an *extension* to a map $f: U \to Y$ over $B$. A space $(Y, q)$ over $B$ is an *absolute extensor over $B$ for the class* $\mathcal{C}$ (written $(Y, q) \in \mathrm{AE}_B(\mathcal{C})$) if for every space $(Z, s)$ over $B$, $(Z, Z_0) \in \mathcal{C}$, any map $f_0: Z_0 \to Y$ admits an extension $f: Z \to Y$ over $B$.

It is clear that $\mathrm{AE}_B(\mathcal{M}) \subset \mathrm{AE}_B(\mathcal{M}_n^c)$ and $\mathrm{ANE}_B(\mathcal{M}) \subset \mathrm{ANE}_B(\mathcal{M}_n^c)$. Suppose that $\varphi$ is a lower semicontinuous map. Following again the paper [19], if $Y$ is a Banach space, and the set-valued map $\varphi$ has closed and convex values, then

$(\mathrm{Gr}\,(\varphi), \pi_X) \in \mathrm{AE}_X(\mathcal{M})$). Moreover, if the family $\{\varphi(x) \mid x \in X\}$ is *equi-locally $n$-connected* (comp. [22]), i.e. if for any $(x, y) \in \mathrm{Gr}\,(\varphi)$, every open neighbourhood $U$ of $y$ in $Y$ contains an open neighbourhood $V$ of $y$ in $Y$ such that for all $x' \in X$ and $k \le n$, any map $f_0: S^k \to \varphi(x') \cap V$ admits an extension $f: D^{k+1} \to \varphi(x') \cap U$, then $(\mathrm{Gr}\,(\varphi), \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$. Besides $(\mathrm{Gr}\,(\varphi), \pi_X) \in \mathrm{AE}_X(\mathcal{M}_n^c)$ if and only if $(\mathrm{Gr}\,(\varphi), \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and $\varphi(x) \in C^n$ for all $x \in X$. It is easy to obtain the following fact.

Proposition 2.3. *If $(Y, q) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, then $q$ is an open map.*

Given a space over $B$, $(Y, q)$ and $Y_0 \subset Y$. We say that a pair $(Y, Y_0)$ has a *lifting property over $B$ with respect to the class $\mathcal{M}_n^c$* (written $(Y, Y_0) \in \mathrm{LP}_B(\mathcal{M}_n^c)$), if for any pair $(Z, Z_0) \in \mathcal{M}_n^c$, where $(Z, s)$ is a space over $B$, and for any map $g: Z \to Y$ over $B$ such that $g(Z_0) \subset Y_0$, there is a *lifting over $B$*, $f: Z \to Y_0$ of the map $g$, i.e. $f$ is a map over $B$ such that $f|_{Z_0} = g|_{Z_0}$.

In the following result we characterize the class $\mathrm{ANE}_B(\mathcal{M}_n^c)$ in terms of the lifting property over $B$. The equivalence of conditions (a) and (c) is given in [19], and condition (b) can be obtained by the similar methods used in the proof of Theorem 2.9 in [19]. For the completeness we provide the the proof of the equivalence of (b) with (a) and (c) in Appendix.

Theorem 2.4. *The following assertions are equivalent:*

(a) $(Y, q) \in \mathrm{ANE}_B(\mathcal{M}_n^c)$,

(b) *for any embedding over $B$ of $Y$ in an arbitrary space $T$ over $B$ (6), for any continuous function $\varepsilon: Y \to (0, \infty)$, there is an open neighbourhood $U$ of $Y$ such that for any pair $(Z, Z_0) \in \mathcal{M}_n^c$, where $(Z, s)$ is a space over $B$, and for any map $g: Z \to U$ over $B$ such that $g(Z_0) \subset Y$, there is a lifting $f: Z \to Y$ over $B$ of the map $g$, such that $d(f(z), g(z)) < \varepsilon(f(z))$ for any $z \in Z$, where $d$ is the given metric on $T$.*

(c) *for any embedding over $B$ of $Y$ into a space $T$ over $B$, there is an open neighbourhood $U$ of $Y$ in $T$ such that $(U, Y) \in \mathrm{LP}_B(\mathcal{M}_n^c)$.*

## 3. Graph approximations of set-valued maps under constraints

The main result of this paper is the following.

Theorem 3.1. *Let $n \ge 0$, $X$, $Y$ be metric spaces, $\varphi: X \multimap Y$ be an upper semicontinuous set-valued map with compact values and $C \subset X \times Y$ such that $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$. Then for any open neighbourhood $\mathcal{U}$ of the graph*

(6) We mean a map $i: Y \to T$ over $B$ such that $i$ transforms homeomorphically $Y$ onto $i(Y)$. Then we treat $(Y, q)$ as a subspace over $B$ of $T$.

*$\mathrm{Gr}(\varphi)$ there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{V} \cap C$-approximation on $A$ such that $\dim(X \setminus A) \leq n+1$, extends to $\mathcal{U} \cap C$-approximation on $X$, provided that the tangency condition is satisfied:*

(C) *for any $x_0 \in X$ and for any open neighbourhood $U$ of $\varphi(x_0)$, there are an open neighbourhood $V \subset U$ of $\varphi(x_0)$ and an open neighbourhood $W$ of $x_0$ such that for any $x \in W$, for any $-1 \leq k \leq n$, every map $f_0: S^k \to V \cap C(x)$ admits an extension $f: D^{k+1} \to U \cap C(x)$.*

The proof of will be performed in Section 3.2.

Now observe that we easily obtain an answer to the problem given in Introduction. Indeed, taking $A = \emptyset$ in Theorem 3.1 we obtain the following corollary.

Corollary 3.2. *Let $n \geq 0$, $X$, $Y$ be metric spaces, $\dim(X) \leq n+1$, $\varphi: X \multimap Y$ be an upper semicontinuous set-valued map with compact values and $C \subset X \times Y$ such that $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$. If tangency condition* (C) *holds, then for any open neighbourhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$, *there is a $\mathcal{U} \cap C$-approximation on $X$.*

Below we provide situations when $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and tangency condition (C) is satisfied.

Remark 3.3. (a) Let $C(\,\cdot\,): X \multimap Y$ be a lower semicontinuous set-valued map with closed and convex values in Banach space $Y$. In view of the Michael selection theorem (see [21], [19]), $(C, \pi_X) \in \mathrm{AE}_X(\mathcal{M}) \subset \mathrm{ANE}_X(\mathcal{M}_n^c)$ for any $n \geq 0$. Moreover, if $\varphi$ has convex values and $\varphi(x) \cap C(x) \neq \emptyset$ for any $x \in X$, then tangency condition (C) is satisfied. Indeed, any open neighbourhood $U$ of $\varphi(x_0)$ contains an open and convex neighbourhood $V$ of $\varphi(x_0)$ and, by the lower semicontinuity of $C(\,\cdot\,)$, $V \cap C(x)$ is nonempty and convex for all $x$ from some open neighbourhood of $x_0$. Thus Theorem 3.1 provides $\mathcal{U} \cap C$-approximations in the convex situation.

(b) In case of lack of constraints Theorem 3.1 coincides with the Kryszewski result given in Introduction. Indeed, if $C = X \times Y$, then $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ if and only if $Y \in \mathrm{ANE}(\mathcal{M}_n^c) = \mathrm{LC}^n$ (see [19]). Moreover, in this situation, tangency condition (C) means that values of $\varphi$ has $\mathrm{UV}^n$-property in $Y$ (see Section 3.1).

Observe that if condition (C) holds, then the assumption (1.1) is satisfied, i.e. $C$ is a constraint set, since if $k = -1$ then we obtain the non-emptiness of $U \cap C(x)$ for any $x \in X$ and open $U \supset \varphi(x)$. In applications, usually ones consider situations when $X$ is a subset of a Banach space $Y$ and $C(x)$ denotes the tangent space (or cone) at $x \in X$ and assumption (1.1) is called (weak) tangency of $\varphi$ to $X$ (e.g. [10]). Therefore we say that (C) is tangent condition. Note that (C) contains a topological structure of values of $\varphi$ and describes the local behavior of the map $\varphi$ with respect to constraint set $C$. In Section 3.1 we introduce some auxiliary concepts that will explain the role and the meaning of assumption (C).

**3.1. A fiberwise** UV**-property.** Let $(Y,q)$ be a space over the base $B$ and $Y_0 \subset Y$. Following the paper [19], we say that the pair $(Y,Y_0)$ has the (*fiberwise*) *extension property over $B$ for the class $\mathcal{C}$* (written $(Y,Y_0) \in \mathrm{EP}_B(\mathcal{C})$) if for any space $(Z,s)$ over $B$, $(Z,Z_0) \in \mathcal{C}$ such that $s(Z) \subset q(Y_0)$, any fiberwise map $f_0\colon Z_0 \to Y_0$ over $B$ admits an extension $f\colon Z \to Y$ over $B$, i.e. the following diagram

$$\begin{tikzcd}
 & Y_0 \arrow[dr, "\subset"] \arrow[drr, bend left, "q|_{Y_0}"] & & \\
Z_0 \arrow[ur, "f_0"] \arrow[dr, "\subset"'] & & Y \arrow[r, "q"] & B \\
 & Z \arrow[ur, dotted, "f"] \arrow[urr, bend right, "s"'] & &
\end{tikzcd}$$

is commutative.

Definition 3.4. We say that a nonempty subset $K \subset Y$ has *a* (*fiberwise*) UV*-property in $Y$ over $B$ for the class $\mathcal{C}$* (written $K \in \mathrm{UV}_B(Y;\mathcal{C})$) if for any open neighbourhood $U$ of the set $K$ in $Y$, there is an open neighbourhood $V \subset U$ of $K$ such that $(U,V) \in \mathrm{EP}_B(\mathcal{C})$.

It is clear that if $q$ is a surjection, then $Y \in \mathrm{UV}_B(Y;\mathcal{C})$ if and only if $(Y,Y) \in \mathrm{EP}_B(\mathcal{C})$ or, equivalently $(Y,q) \in \mathrm{AE}(\mathcal{C})$.

Remark 3.5. Observe that if $\varphi\colon X \multimap Y$ is a set-valued map such that $\mathrm{Gr}\,(\varphi) \in \mathrm{UV}_X(X \times Y;\mathcal{C})$, then for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ there is an open neighbourhood $\mathcal{V}$ of $\mathrm{Gr}\,(\varphi)$ such that any $\mathcal{V}$-approximation on $A$ such that $(X,A) \in \mathcal{C}$, extends to a $\mathcal{U}$-approximation on $X$. In particular, if $(X,\emptyset) \in \mathcal{C}$, then for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ there is a $\mathcal{U}$-approximation on $X$ (comp. Corollary 3.13). We regard $X \times Y$ as a space over $X$ with the usual projection $\pi\colon X \times Y \to X$. Indeed, if $f\colon A \to Y$ is a $\mathcal{V}$-approximation, then $\widehat{f}\colon A \to \mathcal{V}$ given by the formula: $\widehat{f}(x) = (x,f(x))$ for any $x \in A$, is a well-defined map over $X$, where $(A,\mathrm{id}_X|_A)$ is a space over $X$. Thus we obtain the conclusion.

In what follows, we investigate UV-property in the non-fiberwise situation, i.e. when $B$ is a singleton. In this case we write $\mathrm{EP}(\mathcal{C}) = \mathrm{EP}_B(\mathcal{C})$, and $\mathrm{UV}(Y;\mathcal{C}) = \mathrm{UV}_B(Y;\mathcal{C})$.

It is clear that in view of the inclusions $\mathcal{S}_n \subset \mathcal{M}_n^c \subset \mathcal{M}$, we have that

$$\mathrm{UV}(Y;\mathcal{M}) \subset \mathrm{UV}(Y;\mathcal{M}_n^c) \subset \mathrm{UV}(Y;\mathcal{S}_n).$$

UV-properties for the classes $\mathcal{M}$ and $\mathcal{S}_n$ are well-known in the literature. Namely, in view of the Hyman theorem (see [15]), a compact set $K$ is *cell-like* (i.e. there is an embedding $i: K \to Y$ into an ANR $Y$ such that $i(K)$ is contractible in each of its neighbourhoods) if and only if there is an embedding $i: K \to Y$ into an ANR $Y$ such that $i(K) \in \mathrm{UV}(Y; \mathcal{M})$. Hence, if $Y \in ANR$, then a compact set $K \subset Y$ is a cell-like set if and only if $K \in \mathrm{UV}(Y; \mathcal{M})$. Moreover, if $K$ is a convex subset of a Banach space $Y$, then $K \in \mathrm{UV}(Y; \mathcal{M})$.

Furthermore, the class of compact subsets of a given space $Y$ having $\mathrm{UV}^n$-*property in* $Y$, by the very definition, coincides with the class $\mathrm{UV}(Y; \mathcal{S}_n)$. We write $K \in \mathrm{UV}^n$, if $K \in \mathrm{UV}(Y; \mathcal{S}_n)$ for some embedding of $K$ into $Y \in \mathrm{LC}^n$. It can be shown that if $K \in \mathrm{UV}^n$, then $K \in \mathrm{UV}(Y; \mathcal{S}_n)$ for any embedding of $K$ into $Y \in \mathrm{LC}^n$ (it follows from [6, Lemma 2])

Recall that a compact set $K$ has $\mathrm{UV}^\omega$-*property in* $Y$ (written $K \in \mathrm{UV}^\omega(Y)$) provided that $K \in \mathrm{UV}(Y; \mathcal{S}_n)$ for any $n \geq 0$. Moreover, if a compact set $K \in \mathrm{UV}^\omega(Y)$, then $K$ is acyclic with respect to Čech cohomology with coefficients in any abelian group (see [18]). It is clear that $Y \in \mathrm{LC}^n$ if and only if $\{y\} \in \mathrm{UV}(Y; \mathcal{S}_n)$ for any $y \in Y$.

The class $\mathrm{UV}(Y; \mathcal{S}_n)$ is substantially larger than the class $\mathrm{UV}(Y; \mathcal{M}_n^c)$. Indeed, let $Z$ denotes a set $\{0\} \cup \{1/n \mid n = 1, 2, \dots\} \subset \mathbb{R}$ and

$$Y := \{(tz, 1-t) \mid z \in Z,\ t \in [0,1]\} \subset \mathbb{R}^2.$$

Then $\{0\} \times [0,1] \in \mathrm{UV}^\omega(Y) \subset \mathrm{UV}(Y; \mathcal{S}_n)$ and $\{0\} \times [0,1] \notin \mathrm{UV}(Y; \mathcal{M}_0^c) \subset \mathrm{UV}(Y; \mathcal{M}_n^c)$ (comp. the example in Section 4 in [13]). However, in case of $Y \in \mathrm{LC}^n$, the above-mentioned classes coincide, i.e. $\mathrm{UV}(Y; \mathcal{S}_n) = \mathrm{UV}(Y; \mathcal{M}_n^c)$ (see Corollary 3.7).

From [16], $Y \in \mathrm{LC}^n$ if and only if for any $y \in Y$, $\{y\} \in \mathrm{UV}(Y; \mathcal{M}_n^c)$, and which was indicated in the Preliminaries, it is equivalent to the condition $Y \in \mathrm{ANE}(\mathcal{M}_n^c)$. If $(Y, q)$ is a space over $B$, then the similar result holds (see [19]): $(Y, q) \in \mathrm{ANE}_B(\mathcal{M}_n^c)$ if and only if $\{y\} \in \mathrm{UV}_B(Y; \mathcal{M}_n^c)$ for any $y \in Y$.

The constraint set $C \subset X \times Y$ may be regarded as a space over $X$ with the usual projection $\pi_X: C \to X$.

Theorem 3.6. *Let $K$ be a subset of $X \times Y$, $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ be a space over $X$ such that $K \cap C \neq \emptyset$. If for any open neighbourhood $U$ of $K$ there is an open neighbourhood $V \subset U$ of $K$ such that for any $x \in \pi_X(V \cap C)$*

$$((U \cap C)(x), (V \cap C)(x)) \in \mathrm{EP}(\mathcal{S}_n),$$

*then for any open neighbourhood $U$ of $K$ there is an open neighbourhood $V \subset U$ of $K$ such that $(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c)$.*

Proof. Suppose that $Y$ is a normed space. We show that for any open neighbourhood $U$ of $K$ and for any (continuous) function $\varepsilon: U \cap C \to (0, \infty)$, there are an open neighbourhood $V \subset U$ of $K$ and a function $\delta\colon V \cap C \to (0,\infty)$ such that $\delta(x,y) \le \varepsilon(x,y)$ for all $(x,y) \in V \cap C$, and if

$$\mathcal{U} := \bigcup_{(x,y)\in U\cap C} B((x,y),\varepsilon(x,y)), \qquad \mathcal{V} := \bigcup_{(x,y)\in V\cap C} B((x,y),\delta(x,y)),$$

then the following condition is satisfied:

$$(\mathcal{U}(x),\mathcal{V}(x)) \in \mathrm{EP}(\mathcal{S}_n) \tag{3.1}$$

for any $x \in W := \pi_X(V \cap C)$. Let $U$ and $\varepsilon$ be given as above. Then there is an open neighbourhood $V \subset U$ of $K$ such that

$$((U\cap C)(x),(V\cap C)(x)) \in \mathrm{EP}(\mathcal{S}_n) \tag{3.2}$$

for any $x \in W$. Observe that $(V \cap C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, since $V \cap C$ is an open subset of $C$ (see [19]). In view of Theorem 2.4, we can be find an open neighbourhood $U_0$ of $V \cap C$ such that condition $(b)$ is satisfied. Moreover, using a standard paracompactness argument and a partition of unity, we find a (continuous) function $\delta\colon V \cap C \to (0,\infty)$ such that

$$\mathcal{V} = \bigcup_{(x,y)\in V\cap C} B((x,y),\delta(x,y)) \subset U_0.$$

Let $0 \le k \le n$, $x \in W$ and $g\colon S^k \to \mathcal{V}(x)$. Now we show that $g$ is homotopically trivial in $\mathcal{U}(x)$. Notice that $\widehat{g}\colon \{x\}\times S^k \to \mathcal{V} \subset U_0$, where $\widehat{g}(x,z) = g(z)$ for any $z \in S^k$, is a map over $X$, and according to condition $(b)$ of Theorem 2.4, there is a lifting $\widehat{f}\colon \{x\}\times S^k \to V \cap C$ over $X$ such that, for each $z \in S^k$,

$$d(\widehat{f}(x,z),\widehat{g}(x,z)) < \varepsilon(\widehat{f}(x,z)).$$

By condition (3.2), a map $f\colon S^k \to (V\cap C)(x)$, given by the formula

$$f(z) := \pi_Y(\widehat{f}(x,z)) \quad \text{for } z \in S^k,$$

is homotopically trivial in $(U \cap C)(x) \subset \mathcal{U}(x)$.

Define a homotopy $H\colon S^k \times [0,1] \to Y$ such that

$$H(z,t) := tf(z) + (1-t)g(z) \quad \text{for any } z \in S^k.$$

Observe that

$$\|H(z,t) - f(z)\| \le \|f(z) - g(z)\| < \varepsilon(\widehat{f}(x,z)) = \varepsilon(x,f(z))$$

for any $z \in S^k$ (7). Hence $(x,H(z,t)) \in B((x,f(z)),\varepsilon(x,f(z))) \subset \mathcal{U}$ and $H(z,t) \subset \mathcal{U}(x)$ for any $(z,t) \in S^k \times [0,1]$. Therefore the map $g$ is homotopically

(7) We consider a metric in $X \times Y$ given by the formula

$$d((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,y_1), d_Y(x_2,y_2)\},$$

where $d_X$ and $d_Y$ stand for metrics in $X$ and $Y$ respectively.

trivial in $\mathcal{U}(x)$, hence there is an extension $g'\colon D^{k+1} \to \mathcal{U}(x)$ of the map $g$, which completes proof of condition (3.1).

Let $U$ be an open neighbourhood of $K$. Then $(U \cap C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, and taking into account condition (b) of Theorem 2.4, we find a function $\varepsilon_{n+1}\colon U \cap C \to (0,\infty)$ such that if

$$\mathcal{U}_{n+1} := \bigcup_{(x,y)\in U\cap C} B((x,y),\varepsilon_{n+1}(x,y)),$$

then the following condition holds

$$(\mathcal{U}_{n+1}, U \cap C) \in \mathrm{LP}_X(\mathcal{M}_n^c). \tag{3.3}$$

Hence we obtain a sequence of open neighbourhoods $V := U_0 \subset U_1 \subset \ldots \subset U_{n+1} := X \times Y$ of $K$ and a sequence of functions $\{\varepsilon_i\colon U_i \cap C \to (0,\infty)\}_{0\le i\le n+1}$ such that $\varepsilon_0(x,y) \le \varepsilon_1(x,y) \le \ldots \le \varepsilon_{n+1}(x,y)$ for any $(x,y) \in U_0 \cap C$ and assigning

$$\mathcal{U}_i := \bigcup_{(x,y)\in U_i\cap C} B((x,y),\varepsilon_i(x,y)),$$

$0 \le i \le n$, the following condition is satisfied

$$(\mathcal{U}_{i+1}(x), \mathcal{U}_i(x)) \in \mathrm{EP}(\mathcal{S}_n) \tag{3.4}$$

for any $x \in W := \pi_X(V \cap C)$ and $0 \le i \le n$.

Let $(Z,s)$ be a space over $B$ and $Z_0$ its closed subset such that $(Z,Z_0) \in \mathcal{M}_n^c$ and $s(Z) \subset \pi_X(V \cap C) = W$. Let $f_0\colon Z_0 \to V \cap C$ be a map over $X$. Define a map $G_i\colon Z \multimap Y$ by the formula

$$G_i(z) := \mathcal{U}_i(s(z)) = \{y \in Y \mid (s(z),y) \in \mathcal{U}_i\}$$

for any $z \in Z$ and $0 \le i \le n+1$. The above set-valued maps satisfies the assumptions of Bielawski's theorem [4, Theorem 1.1]. In fact, the graph $\mathrm{Gr}\,(G_i) := G^{-1}(\mathcal{U}_i)$ is open, where $G\colon Z \times Y \to X \times Y$ is a map such that $G(z,y) := (s(z),y)$ for $(z,y) \in Z \times Y$. Moreover, in view of (3.4), any map $g_0\colon S^i \to G_i(z)$ extends to a map $g\colon D^{i+1} \to G_{i+1}(z)$ for any $0 \le i \le n$, $z \in Z$. Observe that a map $\pi_Y \circ f_0$ is a selection of the map $G_0$, thus, in view of the Bielawski's theorem, there is an extension $\widehat{f}\colon Z \to Y$ of $\pi_Y \circ f_0$ such that $\widehat{f}$ is a selection of $G_{n+1}$. Let $g\colon Z \to \mathcal{U}_{n+1}$ be a map over $X$ such that $g(z) := (s(z), \widehat{f}(z))$ for any $z \in Z$. Observe that $g|_{Z_0} = f_0$, and by the lifting property (3.3), we get a map $f\colon Z \to U \cap C$ over $X$ being an extension of $f_0$, which completes the proof in case $Y$ is a normed space.

Now let $Y$ be an arbitrary (metric) space. In view of the Arens–Eells theorem (see [2]) there is a closed embedding $i_E\colon Y \to E$ into normed space $E$. Let $i\colon X \times Y \to X \times E$ be a closed embedding given as follows: $i(x,y) := (x, i_E(y))$ for any $(x,y) \in X \times Y$. Let $K' := i(K)$ and $C' := i(C)$. We show that for any open neighbourhood $U'$ in $X \times E$ of $K'$ there is an open neighbourhood $V' \subset U'$ in $X \times E$ of $K'$ such that

$$((U' \cap C')(x), (V' \cap C')(x)) \in \mathrm{EP}(\mathcal{S}_n). \tag{3.5}$$

for any $x \in \pi_X(V' \cap C')$, where $\pi_X$ denotes the standard projection of $X \times E$ on $X$, too. Let $U'$ be an open neighbourhood in $X \times E$ of $K'$. Then, there is an open neighbourhood $V$ in $X \times Y$ of $K$ such that

$$((U' \cap C')(x), (i(V) \cap C')(x)) \in \mathrm{EP}(\mathcal{S}_n) \tag{3.6}$$

for any $x \in W := \pi_X(V \cap C) = \pi_X(i(V) \cap C')$. Since $(U' \cap C', \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and by condition $(c)$ of Theorem 2.4, there is an open neighbourhood $U''$ of $U' \cap C'$ such that

$$(U'', U' \cap C') \in \mathrm{LP}_X(\mathcal{M}_n^c). \tag{3.7}$$

Using a partition of unity, we find a function $\varepsilon: U' \cap C' \to (0, \infty)$ such that

$$B((x,y), \varepsilon(x,y)) \subset U'' \tag{3.8}$$

for any $(x,y) \in U' \cap C'$. Observe that $(i(V) \cap C', \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, hence for the function $\varepsilon|_{i(V) \cap C'}$, we find an open neighbourhood $V''$ in $X \times E$ of $i(V) \cap C'$ satisfying condition (b) of Theorem 2.4. Let $V' := V'' \cap (W \times E)$, $0 \leq k \leq n$, $x \in W = \pi_X(V' \cap C')$ and $g: S^k \to (V' \cap C')(x)$ be a map. Then $\widehat{g}: \{x\} \times S^k \to V''$, where $\widehat{g}(x,z) = g(z)$ for any $z \in S^k$, is a map over $X$, hence $\widehat{f}: \{x\} \times S^k \to i(V) \cap C'$ is a map over $X$ such that

$$d(\widehat{f}(x,z), \widehat{g}(x,z)) < \varepsilon(\widehat{f}(x,z)).$$

By condition (3.6), a map $f: S^k \to (i(V) \cap C')(x)$ given by the formula: $f(z) := \pi_E(\widehat{f}(x,z))$ for any $z \in S^k$, is homotopically trivial in $(U' \cap C')(x) \subset U''(x)$.

Similarly as above, we define a homotopy $H: S^k \times [0,1] \to E$ such that $H(z,t) := tf(z) + (1-t)g(z)$ for any $z \in S^k$. Then, by condition (3.8), we get the inclusions

$$(x, H(z,t)) \in B((x, f(z)), \varepsilon(x, f(z))) \subset U'',$$

and hence $H(z,t) \subset \mathcal{U}(x)$ for any $(z,t) \in S^k \times [0,1]$. Then $g$ is homotopically trivial in $U''(x)$, and hence, there is an extension $g': D^{k+1} \to U''(x)$ of the map $g$. By condition (3.7), we easily obtain a map $g'': D^{k+1} \to (U' \cap C')(x)$ being an extension of $g$. Therefore we has shown that condition (3.5) is satisfied.

Observe that in view of the first part of the proof, we obtain that for any open neighbourhood $U'$ in $X \times E$ of $K'$, there is an open neighbourhood $V' \subset U'$ in $X \times E$ of $K'$ such that $(U' \cap C', V' \cap C') \in \mathrm{EP}_X(\mathcal{M}_n^c)$, where we put $K' := i(K)$ and $C' := i(C)$. Let $U$ be an open neighbourhood of $K$. Since

$(i(U) \cap C', \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and by condition (c) of Theorem 2.4, there is an open neighbourhood $U'$ of $i(U) \cap C'$ such that

$$(U', i(U) \cap C') \in \mathrm{LP}_X(\mathcal{M}_n^c).$$

Let $V' \subset U'$ be an open neighbourhood of $K'$ such that $(U' \cap C', V' \cap C') \in \mathrm{EP}_X(\mathcal{M}_n^c)$. If $V := i^{-1}(V')$, then it is easy to check that $(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c)$, which completes the proof of the theorem. $\square$

Corollary 3.7. *Let $Y \in \mathrm{LC}^n$ and $K$ be a subset of $Y$. Then $K \in \mathrm{UV}(Y; \mathcal{S}_n)$ if and only if $K \in \mathrm{UV}(Y; \mathcal{M}_n^c)$.*

Proof. Let $K \in \mathrm{UV}(Y; \mathcal{S}_n)$. In Theorem 3.6 we take $X := \{x_0\}$ and $C := X \times Y$. Then $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, since $Y \in \mathrm{LC}^n = \mathrm{ANE}(\mathcal{M}_n^c)$. Hence, if $K \in \mathrm{UV}(Y; \mathcal{S}_n)$, then $\{x_0\} \times K \in \mathrm{UV}(\{x_0\} \times Y; \mathcal{S}_n)$, and in view of Theorem 3.6, we obtain that $\{x_0\} \times K \in \mathrm{UV}(\{x_0\} \times Y; \mathcal{M}_n^c)$. Thus $K \in \mathrm{UV}(Y; \mathcal{M}_n^c)$. $\square$

Corollary 3.8. *$(C, \pi_X) \in \mathrm{AE}_X(\mathcal{M}_n^c)$ if and only if $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and $C(x) \in C^n$ for every $x \in X$.*

Proof. It is clear that if $(C, \pi_X) \in \mathrm{AE}_X(\mathcal{M}_n^c)$, then $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and $C(x) \in \mathrm{AE}(\mathcal{M}_n^c) \subset C^n$ for any $x \in X$. In order to prove the converse implication, it sufficient to apply Theorem 3.6 to a set $K := X \times Y$. $\square$

**3.2. Main results – proofs.** As before, we assume that $\varphi\colon X \multimap Y$ is an upper semicontinuous set-valued map with compact values, $C$ is a constraint set (see (1.1)) and $(C, \pi_X)$ is a space over $X$.

Theorem 3.9. *If $\pi_X\colon C \to X$ is an open map, then the following conditions are equivalent:*

- (A) *for any $x \in X$ and for any open neighbourhood $U$ of $\{x\} \times \varphi(x)$, there is an open neighbourhood $V \subset U$ of $\{x\} \times \varphi(x)$ such that $(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c)$.*
- (B) *for any open neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$ there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$ such that $(\mathcal{U} \cap C, \mathcal{V} \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c)$.*

Before we prove Theorem 3.9 we need the following lemmas.

Lemma 3.10. *For any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, for any open covers $\mathcal{W}$ and $\{L_x\}_{x \in X}$ of $X$ such that $x \in L_x$ for any $x \in X$, for any family $\{V_x\}_{x \in X}$ of open subsets of $Y$ such that $\varphi(x) \subset V_x$ for any $x \in X$, there are an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and open cover $\mathcal{W}'$ of $X$ and a refinement of $\mathcal{W}$ such that:*

(a) *$W' \times \mathcal{V}(W') \subset \mathcal{U}$ for any $W' \in \mathcal{W}'$* [8]

[8] If $\mathcal{V} \subset X \times Y$ and $A \subset X$, then we write $\mathcal{V}(A) := \{y \in Y \mid (x, y) \in \mathcal{V},\ x \in A\}$.

(b) *for any $W' \in \mathcal{W}'$ there are $W \in \mathcal{W}$ and $x \in W$ such that $W' \subset W \cap L_x$ and $\mathcal{V}(W') \subset V_x$.*

Proof. Let $\mathcal{U}$ be an open neighbourhood of $\mathrm{Gr}\,(\varphi)$ an let $\mathcal{W}$, $\{L_x\}_{x\in X}$ and $\{V_x\}_{x\in X}$ be given as above. Since $\varphi$ is an upper semicontinuous set-valued map, for every $x \in X$, there are open neighbourhoods $L_x^1$ of $x$ and $U_x^1$ of $\varphi(x)$ such that

$$\varphi(L_x^1) \subset U_x^1 \quad \text{and} \quad L_x^1 \times U_x^1 \subset \mathcal{U}. \tag{3.9}$$

Moreover, we can assume that $U_x^1 \subset V_x$, the cover $\{L_x^1\}_{x\in X}$ is a refinement of $\mathcal{W}$ and $L_x^1 \subset L_x$ for any $x \in X$. Since $X$ is a paracompact space, there is a locally finite cover $\{L_x^2\}_{x\in X}$ of $X$ being a refinement of $\{L_x^1\}_{x\in X}$, where $L_x^2 \subset L_x^1$ for any $x \in X$. Let

$$U^2(z) := \bigcap \{U_x^1 \mid x \in X \text{ and } z \in L_x^2\}$$

for every $z \in X$. Taking into account the locally finiteness of the cover $\{L_x^2\}_{x\in X}$ we obtain that the set $U^2(z)$ is open in $Y$. Moreover, if $z \in L_x^2$, then

$$\varphi(z) \subset \varphi(L_x^2) \subset \varphi(L_x^1) \subset U_x^1.$$

Hence $\varphi(z) \subset U^2(z)$. Again, taking into account the locally finiteness of the cover $\{L_x^2\}_{x\in X}$, a set

$$\mathcal{U}^2 := \bigcup_{z\in X} \{z\} \times U^2(z)$$

is an open neighbourhood of $\mathrm{Gr}\,(\varphi)$. Moreover,

$$U^2(z) \subset U_x^1 \quad \text{for any } x \in X,\ z \in L_x^2. \tag{3.10}$$

Now we recall the following lemma (see [13, Lemma 3.2] or [1, Lemmas A.8 and A.10]): *If $\varphi$ is an upper semicontinuous set-valued map with compact values, then for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$, there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ and an open cover $\mathcal{W}$ of the space $X$ such that $\mathcal{V}(U) \subset \mathcal{U}(x)$ for any $U \in \mathcal{W}$ and $x \in U$.*

Applying the above lemma for the open neighbourhood $\mathcal{U}^2$ of $\mathrm{Gr}\,(\varphi)$ we find an open neighbourhood $\mathcal{V} \subset \mathcal{U}^2$ of $\mathrm{Gr}\,(\varphi)$ and an open cover $\mathcal{W}^2$ of $X$ such that

$$\mathcal{V}(W^2) \subset \mathcal{U}^2(z) = U^2(z) \quad \text{for any } W^2 \in \mathcal{W}^2 \text{ and } z \in W^2. \tag{3.11}$$

Finally, we put $\mathcal{W}' := \{W^2 \cap L_x^2 \mid W^2 \in \mathcal{W}^2,\ x \in X,\ W^2 \cap L_x^2 \neq \emptyset\}$. Then $\mathcal{W}'$ is an open cover of $X$ and refines $\{L_x^2\}_{x\in X}$. Hence $\mathcal{W}'$ is a refinement of the cover $\mathcal{W}$. Moreover, by conditions (3.0)–(3.10) we obtain that for any $z \in W^2 \cap L_x^2$ the following inclusions hold:

$$(W^2 \cap L_x^2) \times \mathcal{V}(W^2 \cap L_x^2) \subset (W^2 \cap L_x^2) \times \mathcal{U}^2(z) \subset (W^2 \cap L_x^2) \times U_x^1 \subset L_x^1 \times U_x^1 \subset \mathcal{U}.$$

Then condition (a) is satisfied. Moreover, $W^2 \cap L^2_x \subset L^1_x \subset L_x$, $x \in L^1_x$, the cover $\{L^1_x\}_{x \in X}$ is a refinement of $\mathcal{W}$ and

$$\mathcal{V}(W^2 \cap L^2_x) \subset U^1_x \subset V_x$$

for any $x \in X$. Therefore condition (b) holds. $\square$

Lemma 3.11. *Let $\pi_X: C \to X$ be an open map and suppose that the following condition is satisfied:*

(∗) *for any $x \in X$ and for any open neighbourhood $U$ of $\{x\} \times \varphi(x)$, there is an open neighbourhood $V \subset U$ of $\{x\} \times \varphi(x)$ such that $(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}^c_n)$.*

*Then for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, for any open cover $\mathcal{W}$ of $X$, there are an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and an open cover $\mathcal{W}'$ of $X$ which refines $\mathcal{W}$ such that*

$$(\mathcal{U} \cap C \cap \pi_X^{-1}(W'), \mathcal{V} \cap C \cap \pi_X^{-1}(W')) \in \mathrm{EP}_X(\mathcal{M}^c_n)$$

*for any $W' \in \mathcal{W}'$.*

Proof. Let $\mathcal{U}$ be an open neighbourhood of $\mathrm{Gr}(\varphi)$ and $\mathcal{W}$ be an open cover $X$. Applying Lemma 3.10, we obtain an open neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and an open cover $\mathcal{W}_1$, a refinement of $\mathcal{W}$, such that

$$W \times \mathcal{U}'(W) \subset \mathcal{U} \quad \text{for every } W \in \mathcal{W}_1. \tag{3.12}$$

In view of condition (∗), for any $x \in X$ there are open neighbourhoods $L_x$ of $x$ and $V_x$ of $\varphi(x)$ such that

$$((X \times U'(x)) \cap C, (L_x \times V_x) \cap C) \in \mathrm{EP}_X(\mathcal{M}^c_n). \tag{3.13}$$

Observe that by the openness of the map $\pi_X: C \to X$, we can assume that

$$L_x = \pi_X((L_x \times V_x) \cap C)$$

for any $x \in X$. Again, applying Lemma 3.10,b we find an open cover $\mathcal{W}'$ and an open neighbourhood $\mathcal{V} \subset \mathcal{U}'$ of $\mathrm{Gr}(\varphi)$ such that for any $W' \in \mathcal{W}'$ there are $W \in \mathcal{W}_1$ and $x \in W$ such that $W' \subset W \cap L_x$ and

$$\mathcal{V}(W') \subset V_x. \tag{3.14}$$

Let $W' \in \mathcal{W}'$. We show that

$$(\mathcal{U} \cap C \cap \pi_X^{-1}(W'), \mathcal{V} \cap C \cap \pi_X^{-1}(W')) \in \mathrm{EP}_X(\mathcal{M}^c_n).$$

Let $(Z, s)$ be a space over $X$ such that $s(Z) \subset W' = \pi_X(\mathcal{V} \cap C \cap \pi_X^{-1}(W'))$, let $(Z, Z_0) \in \mathcal{M}^c_n$ and $f_0: Z_0 \to \mathcal{V} \cap C \cap \pi_X^{-1}(W')$ be a map over $X$. Observe that condition (3.14) implies that for some $W \in \mathcal{W}_1$ and $x \in X$ we have the following inclusions

$$s(Z) \subset W' \subset W \cap L_x \subset L_x = \pi_X(L_x \times V_x \cap C),$$

and

$$\pi_Y \circ f_0(Z_0) \subset \mathcal{V}(s(Z_0)) \subset V_x.$$

Hence $f_0(Z_0) \subset (L_x \times V_x) \cap C$ and, in view of (3.13), there is a map $f\colon Z \to X \times \mathcal{U}'(x) \cap C$ over $X$ being an extension of $f_0$. Moreover from (3.12) we get inclusions

$$f(Z) \subset s(Z) \times \mathcal{U}'(x) \subset W \times \mathcal{U}'(W) \subset \mathcal{U}.$$

Then we obtain that $f(Z) \subset \mathcal{U} \cap C \cap \pi_X^{-1}(W')$, which completes the proof. $\square$

PROOF OF THEOREM 3.9. Let $\pi_X\colon C \to X$ be an open map and suppose that condition (A) is satisfied. Let $\mathcal{U}$ be an open neighbourhood of $\mathrm{Gr}\,(\varphi)$. Applying Lemma 3.11, we obtain a sequence of neighbourhoods $\mathcal{V}_0 := \mathcal{U}, \mathcal{V}_1, \dots, \mathcal{V}_{n+2}$ of $\mathrm{Gr}\,(\varphi)$ and a sequence of open covers $\mathcal{W}_0 := \{X\}, \mathcal{W}_1, \dots, \mathcal{W}_{n+2}$ of $X$ such that $\mathcal{V}_{k+1} \subset \mathcal{V}_k$, $\mathcal{W}_{k+1}$ is a refinement of $\mathcal{W}_k$ and

$$(\mathcal{V}_k \cap C \cap \pi_X^{-1}(W), \mathcal{V}_{k+1} \cap C \cap \pi_X^{-1}(W)) \in \mathrm{EP}_X(\mathcal{M}_n^c) \tag{3.15}$$

for any $W \in \mathcal{W}_{k+1}$ and $k = 0, \dots, n+1$. Put $\mathcal{V} := \mathcal{V}_{n+2}$. Let $(Z, s)$ be a space over $X$, $Z_0$ its closed subset such that $(Z, Z_0) \in \mathcal{M}_n^c$, $f_0\colon Z_0 \to \mathcal{V} \cap C$ be a map over $X$. Applying Lemma 2.1 to the cover $\{s^{-1}(W)\}_{W \in \mathcal{W}_{n+2}}$, we find a sequence of closed subspaces $Z_0 \subset Z_1 \subset \ldots \subset Z_{n+2} = Z$ having the properties enlisted in this lemma.

For each $i = 0, \dots, n+2$ we construct a map $f_i\colon Z_i \to \mathcal{V}_{n+2-i} \cap C$ over $X$ such that $f_{i+1}(z) = f_i(z)$ for any $i = 0, \dots, n+1$ and $z \in Z_i$.

The map $f_0$ is given. Suppose that for some $i \in \{0, \dots, n+1\}$, a map $f_i$ satisfying the above conditions is constructed. Let $\alpha \in I_i$. Observe that

$$(B_i^\alpha, Z_i \cap B_i^\alpha) \in \mathcal{M}_n^c,$$

and $s(B_i^\alpha) \subset W = \pi_X(\mathcal{V}_{n+2-i} \cap C \cap \pi_X^{-1}(W))$ for some $W \in \mathcal{W}_{n+2-i}$. Hence, according to condition (3.15), there is a map $f_i^\alpha\colon B_i^\alpha \to \mathcal{V}_{n+1-i} \cap C$ over $X$ being an extension of $f_{i|Z_i \cap B_i^\alpha}$. Now define a map $f_{i+1} : Z_{i+1} \to \mathcal{V}_{n+1-i} \cap C$ by the formula:

$$f_{i+1}(z) = \begin{cases} f_i(z) & \text{if } z \in Z_i, \\ f_i^\alpha(z) & \text{if } z \in B_i^\alpha,\ \alpha \in I_i. \end{cases}$$

Then $f_{i+1}$ is a well-defined continuous map over $X$, since the family $\{B_i^\alpha \cap (Z \setminus Z_0)\}_{\alpha \in I_i}$ is pairwise separated by open neighbourhoods. Moreover, $f_{i+1}(z) = f_i(z)$ for any $z \in Z_i$. By induction, $f := f_{n+2}\colon Z \to \mathcal{U} \cap C$ is a required extension of $f_0$.

Now suppose that condition (B) is satisfied. Let $x \in X$ and $U$ be an open neighbourhood of $\{x\} \times \varphi(x)$. Since $\varphi$ is an upper semicontinuous map with compact values, there is an open neighbourhood $W$ of $x$ such that

$$W \times \varphi(W) \subset U.$$

Let $W_0$ be an open neighbourhood of $x$ such that $\operatorname{cl} W_0 \subset W$. Then

$$\mathcal{U} := U \cup ((X \setminus \operatorname{cl} W_0) \times Y)$$

is an open neighbourhood of $\operatorname{Gr}(\varphi)$. By condition $(B)$ we find an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\operatorname{Gr}(\varphi)$ such that

$$(\mathcal{U} \cap C, \mathcal{V} \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c). \tag{3.16}$$

Let $V$ be an open neighbourhood of $\{x\} \times \varphi(x)$ such that $V \subset U \cap \mathcal{V}$ and $\pi_X(V) \subset W_0$. We show that $(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c)$. Let $(Z, s)$ be a space over $X$, $Z_0$ its closed subset such that $(Z, Z_0) \in \mathcal{M}_n^c$ and $s(Z) \subset \pi_X(V \cap C)$. Let $f_0\colon Z_0 \to V \cap C$ be a map over $X$. Then $f_0(Z_0) \subset \mathcal{V} \cap C$ and in view of (3.16) there is a map $f\colon Z \to \mathcal{U} \cap C$ over $X$ being an extension of $f_0$. Observe that

$$f(Z) \subset s(Z) \times \mathcal{U}(s(Z)) \subset W_0 \times \mathcal{U}(W_0) \subset U.$$

Then $f(Z) \subset U \cap C$, and the proof is completed. □

Now we are going to prove Theorem 3.1.

Lemma 3.12. *Let $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$. Then tangency condition* (C) *is equivalent to condition* (A).

Proof. Observe that tangency condition (C) is equivalent to the following condition:

- for any $x_0 \in X$ and for any open neighbourhood $U$ of $\{x_0\} \times \varphi(x_0)$, there is an open neighbourhood $V \subset U$ of $\{x_0\} \times \varphi(x_0)$ such that

$$((U \cap C)(x), (V \cap C)(x)) \in \mathrm{EP}(\mathcal{S}_n) \quad \text{for any } x \in \pi_X(V \cap C).$$

Hence, in view of Theorem 3.6, if $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, then tangency condition (C) implies condition (A) of Theorem 3.9.

Now, let condition (A) holds. Fix $x_0 \in X$ and let $U$ be an open neighbourhood of $\varphi(x_0)$. Let $U' := X \times U$. Then, by condition (A), there is an open neighbourhood $V'$ of $\{x_0\} \times \varphi(x_0)$ such that

$$(U' \cap C, V' \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c). \tag{3.17}$$

There are open neighbourhoods $V \subset U$ of $\varphi(x_0)$ and $W$ of $x_0$ such that

$$\pi_X((W \times V) \cap C) = W \quad \text{and} \quad W \times V \subset V'.$$

Let $x \in W$, $(Z, Z_0) \in \mathcal{S}_n$ and let $f_0: Z_0 \to V \cap C(x)$ be a map. Then $\widehat{f_0}: \{x\} \times Z_0 \to (W \times V) \cap C$ defined as follows: $\widehat{f_0}(x, z) := (x, f_0(z))$ for any $z \in Z_0$, is a map over $X$, where the projection $s$ of $\{x\} \times Z$ is given by formula: $s(x, z) := x$. Observe that $(\{x\} \times Z, \{x\} \times Z_0) \in \mathcal{M}_n^c$ and $s(\{x\} \times Z) \subset \pi_X(U)$. Then, taking into account condition (3.17), there is an extension $\widehat{f}: \{x\} \times Z \to U' \cap C$ over $X$ of $\widehat{f_0}$. Moreover, a map $f: Z \to U \cap C(x)$ defined by formula: $f(z) := \pi_Y(\widehat{f}(x, z))$ for any $z \in Z$, is an extension of $f_0$. Therefore condition (C) is satisfied. □

Proof of Theorem 3.1. By the Lemma 3.12, if $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, then tangency condition (C) implies condition (A) of Theorem 3.9. In order to complete the proof, observe that condition (B) implies the conclusion of Theorem 3.1. Indeed, let $\mathcal{U}$ and $\mathcal{V}$ be such that

$$(\mathcal{U} \cap C, \mathcal{V} \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c). \tag{3.18}$$

Let $f_0: A \to Y$ be a $\mathcal{V} \cap C$-approximation on $A$ such that $(X, A) \in \mathcal{M}_n^c$. Then a map $\widehat{f_0}: X \to \mathcal{V} \cap C$ given by $\widehat{f}(x) := (x, f(x))$ for any $x \in X$, is a well-defined map over $X$. By condition (3.18), there is an extension $\widehat{f}: X \to \mathcal{U} \cap C$ over $X$ of $f_0$. Hence $f := \pi_Y \circ \widehat{f}$ is a $\mathcal{U} \cap C$-approximation on $X$ being an extension of $f_0$. □

**3.3. Consequences of main results.** The following result follows from Theorem 3.9.

Corollary 3.13.

(a) $\mathrm{Gr}(\varphi) \in \mathrm{UV}_X(X \times Y; \mathcal{M}_n^c)$ *if and only if* $\varphi(x) \in \mathrm{UV}(Y; \mathcal{M}_n^c)$ *for every* $x \in X$.

(b) *If* $\varphi(x) \in \mathrm{UV}(Y; \mathcal{M}_n^c)$ *for any* $x \in X$, *then for any open neighbourhood* $\mathcal{U}$ *of the graph* $\mathrm{Gr}(\varphi)$, *there is an open neighbourhood* $\mathcal{V} \subset \mathcal{U}$ *of the graph* $\mathrm{Gr}(\varphi)$ *such that for any subset* $A \subset X$, $\dim(X \setminus A) \leq n + 1$, *every* $\mathcal{V}$*-approximation on* $A$ *extends to a* $\mathcal{U}$*-approximation on* $X$.

Proof. It is not difficult to check that if $C = X \times Y$, then condition (A) is equivalent to the following one: $\varphi(x) \in \mathrm{UV}(Y; \mathcal{M}_n^c)$ for any $x \in X$. Moreover, the projection $\pi_X: X \times Y \to X$ is open. On the other hand, condition (B) is equivalent to the following condition: $\mathrm{Gr}(\varphi) \in \mathrm{UV}_X(X \times Y; \mathcal{M}_n^c)$. Therefore (a) follows from Theorem 3.9. The assertion (b) is a direct consequence of (a) (see Remark 3.5). □

Corollary 3.13 is a generalization of the Kryszewski result given in Introduction. In fact, having the additional assumption $Y \in \mathrm{LC}^n$, in view of Corollary 3.7, we obtain $\varphi(x) \in \mathrm{UV}(Y; \mathcal{S}_n)$ if an only if $\varphi(x) \in \mathrm{UV}(Y; \mathcal{M}_n^c)$ for any $x \in X$. Hence if $Y \in \mathrm{LC}^n$, $\varphi(x) \in \mathrm{UV}(Y; \mathcal{S}_n)$, then for any open neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$, there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$

such that any $\mathcal{V}$-approximation on $A$ such that $\dim(X \setminus A) \leq n+1$, extends to a $\mathcal{U}$-approximation on $X$.

Now observe that if tangency condition (C) is satisfied, then:

(D) for any $x \in X$ and for any open neighbourhood $U$ of $\varphi(x)$, there is an open neighbourhood $V \subset U$ of $\varphi(x)$ such that $(U \cap C(x), V \cap C(x)) \in \mathrm{EP}(\mathcal{S}_n)$.

Contrary to (C) concerns a local behavior of $\varphi$ with regard to constraint set $C$, the above condition is a type of the pointwise tangency. The following example shows that condition (D) is not sufficient to get the existence of $\mathcal{U} \cap C$-approximations.

Example 3.14. Let $X := [0,1]$, $Y := \mathbb{R}^2$,

$$L = \mathrm{conv}\,(\{(0,0),(1,-1)\}) \cup \mathrm{conv}\,(\{(1,-1),(1,0)\}),$$

and for any $x \in X$, $\varphi(x) := L$,

$$C(x) = \begin{cases} \mathrm{conv}\,(\{(0,0),(1,1-x)\}) & \text{if } x \in [0,1), \\ (1,0) & \text{if } x = 1. \end{cases}$$

Then the map $C(\,\cdot\,)\colon X \multimap Y$ is lower semicontinuous with closed and convex values (hence $(C,\pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$) and $\varphi$ is upper semicontinuous with contractible values. For any $x \in X$, $\varphi(x) \cap C(x)$ is a singleton and any open neighbourhood $U$ of $\varphi(x)$ contains an open neighbourhood $V$ such that $V \cap C(x)$ is contractible. Therefore condition (D) is satisfied. Let $\mathcal{U}$ be an open neighbourhood of $\mathrm{Gr}\,(\varphi)$. We find an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ such that there are no $\mathcal{V} \cap C$-approximations. Indeed, given $\mathcal{U} \supset \mathrm{Gr}\,(\varphi)$. There is $\varepsilon > 0$ such that $B(L,\varepsilon) \cap ([0,1] \times [0,1])$ has two path-connected components and $\mathcal{V} := [0,1] \times B(L,\varepsilon) \subset \mathcal{U}$. Let $f\colon X \to Y$ be a $\mathcal{V} \cap C$-approximation on $X$. Since

$$\mathrm{Gr}\,(f) \subset \mathcal{V} \cap C \subset \mathcal{V} \cap ([0,1] \times [0,1] \times [0,1])$$

then $f(X) \subset B(L,\varepsilon) \cap ([0,1] \times [0,1])$ and $f(0)$, $f(1)$ belong to different path-connected components, which leads to contradiction.

However we shall show that if the *strong tangency* holds, i.e. if $\varphi(x) \subset C(x)$ for any $x \in X$ (or equivalently, $\mathrm{Gr}\,(\varphi) \subset C$), or if constraint set $C$ is closed in $X \times Y$, then the *pointwise tangency condition* (D) is sufficient for the existence of $\mathcal{U} \cap C$-approximations.

Lemma 3.15. *Let* $\mathrm{Gr}\,(\varphi) \subset C$ *or* $C$ *be a closed subset of* $X \times Y$. *If* $(C,\pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, *then the following conditions are equivalent:*

(a) *the pointwise tangency condition* (D),
(b) $\varphi(x) \cap C(x) \in \mathrm{UV}^n$ *for any* $x \in X$,
(c) *tangency condition* (C).

Proof. Let $(C,\pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and suppose that either $\mathrm{Gr}(\varphi) \subset C$ or $C$ is closed in $X \times Y$.

(a) $\Rightarrow$ (b). Suppose that $\mathrm{Gr}(\varphi) \subset C$, then condition (D) is equivalent to the following one: $\varphi(x) = \varphi(x) \cap C(x) \in \mathrm{UV}(C(x);\mathcal{S}_n)$ for any $x \in X$. Moreover, if $C$ is closed in $X \times Y$, then for any $x \in X$ the set $C(x)$ is closed in $Y$. Hence for any open neighbourhood $U$ of $\varphi(x) \cap C(x)$ in $C(x)$ there is an open neighbourhood $U'$ of $\varphi(x)$ in $Y$ such that $U' \cap C(x) = U$. Now it is easy to see that if $C$ is closed in $X \times Y$, then condition (D) is equivalent to the following condition: $\varphi(x) \cap C(x) \in \mathrm{UV}(C(x);\mathcal{S}_n)$ for any $x \in X$. Since $C(x) \in \mathrm{ANE}(\mathcal{M}_n^c) = \mathrm{LC}^n$ and $\varphi(x) \cap C(x)$ is compact, for any $x \in X$, $\varphi(x) \cap C(x) \in \mathrm{UV}(C(x);\mathcal{S}_n)$ if and only if $\varphi(x) \cap C(x) \in \mathrm{UV}^n$.

(b) $\Rightarrow$ (c). Fix $x_0 \in X$ and $K := \varphi(x_0)$. Let $K \cap C(x_0) \in \mathrm{UV}^n$. We show that for any open neighbourhood $U$ of $\{x_0\} \times K$, there is an open neighbourhood $V \subset U$ of $\{x_0\} \times K$ such that

$$(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c). \tag{3.19}$$

Assume that $Y \in \mathrm{LC}^n$. Then we obtain that $K \cap C(x_0) \in \mathrm{UV}(Y;\mathcal{S}_n)$ and, by Corollary 3.7,

$$K \cap C(x_0) \in \mathrm{UV}(Y;\mathcal{M}_n^c). \tag{3.20}$$

Let $U$ be an open neighbourhood of $\{x_0\} \times K$. Since $(U \cap C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, in view of condition $(c)$ from Theorem 2.4, there is an open neighbourhood $U_0$ of $U \cap C$ such that

$$(U_0, U \cap C) \in \mathrm{LP}_X(\mathcal{M}_n^c). \tag{3.21}$$

Since $K \cap C(x_0)$ is compact, there are open neighbourhoods $U_1 \supset K \cap C(x_0)$ and $W \ni x_0$ such that

$$W \times U_1 \subset U_0. \tag{3.22}$$

In view of (3.20) we get an open neighbourhood $V_1 \supset K \cap C(x_0)$ such that $(U_1, V_1) \in \mathrm{EP}(\mathcal{M}_n^c)$. Since $K \subset C(x_0)$ or $C$ is a closed subset of $X \times Y$, there is an open neighbourhood $V$ of $\{x_0\} \times K$ such that

$$V \cap C \subset W \times V_1.$$

Let $(Z,s)$ be a space over $X$ and $Z_0$ its closed subset such that $(Z,Z_0) \in \mathcal{M}_n^c$ and $s(Z) \subset \pi_X(V \cap C)$. Let $f_0\colon Z_0 \to V \cap C$ be a map over $X$. Then $\pi_Y \circ f_0\colon Z_0 \to V_1$ is a well-defined map. Since $(U_1,V_1) \in \mathrm{EP}(\mathcal{M}_n^c)$, there is an extension $f_1\colon Z \to U_1$ over $X$ of $\pi_Y \circ f_0$. Taking into account condition (3.22), a map $g\colon Z \to U_0$ given by the formula: $g(z) = (s(z), f_1(z))$ for $z \in Z$, is a well-defined map over $X$. Moreover, by condition (3.21), there is a lifting $f\colon Z \to U \cap C$ over $X$ such that $f(z) = g(z) = f_0(z)$ for any $z \in Z$. Hence a map $f$ is a required extension of $f_0$.

Now, let $Y$ be a (metric) space. The Arens-Eells theorem implies that there is a closed embedding $i_E: Y \to E$ into a normed space $E$. Let $i: X \times Y \to X \times E$ be a closed embedding defined as follows: $i(x,y) := (x, i_E(y))$ for any $(x,y) \in X \times Y$. Let $K' := i_E(K)$ and $C' := i(C)$. Hence, if $\{x_0\} \times K \subset C$, then $\{x_0\} \times K' \subset C'$, and if $C$ is closed in $X \times Y$, then $C'$ is closed in $X \times E$. Besides

$$K' \cap C'(x_0) = i_E(K \cap C(x_0)) \in \mathrm{UV}^n.$$

Since $E \in \mathrm{LC}^n$ and in view of the first part of the proof of the implication (b) $\Rightarrow$ (c), we get that for any open neighbourhood $U'$ in $X \times E$ of $K'$ there is an open neighbourhood $V' \subset U'$ in $X \times E$ of $K'$ such that $(U' \cap C', V' \cap C') \in \mathrm{EP}_X(\mathcal{M}_n^c)$. Let $U$ be an open neighbourhood of $K$. Since $(i(U) \cap C', \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, in view of condition $(c)$ in Theorem 2.4, there is an open neighbourhood $U'$ of $i(U) \cap C'$ such that

$$(U', i(U) \cap C') \in \mathrm{LP}_X(\mathcal{M}_n^c).$$

Let $V' \subset U'$ be an open neighbourhood of $\{x_0\} \times K'$ such that $(U' \cap C', V' \cap C') \in \mathrm{EP}_X(\mathcal{M}_n^c)$. Put $V := i^{-1}(V')$. Then we easily check that $(U \cap C, V \cap C) \in \mathrm{EP}_X(\mathcal{M}_n^c)$, which completes the proof of condition (3.19). Since $x_0$ was arbitrary chosen, we obtain condition (A) of Theorem 3.9. Hence, in view of Lemma 3.12, tangency condition (C) is satisfied. □

In view of Lemma 3.15 we obtain the following approximation result.

Theorem 3.16. *Let $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and one of the following conditions is satisfied:*

(a) *$\varphi(x) \subset C(x)$ and $\varphi(x) \in \mathrm{UV}^n$ for any $x \in X$,*
(b) *$C$ is closed subset of $X \times Y$ and $\varphi(x) \cap C(x) \in \mathrm{UV}^n$ for any $x \in X$.*

*Then, for any open neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$, there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{V} \cap C$-approximation on $A$ such that $\dim(X \setminus A) \le n+1$, extends to a $\mathcal{U} \cap C$-approximation on $X$.*

Theorem 3.16 allows to obtain a following variant of Brodskii's result ([6, Theorem 3]).

Corollary 3.17. *Let $\dim(X) \le n+1$, $A$ be a closed subset of $X$, $Y$ is a complete space, $\varphi: X \multimap Y$ is an upper semicontinuous map with compact values such that $\varphi(x) \in \mathrm{UV}^n$ for any $x \in X$. Let $C(\,\cdot\,): X \multimap Y$ be a set-valued map with closed values such that $\pi_X: C \to X$ is locally $n$-soft, $\varphi(x) \subset C(x)$ for any $x \in X$, and $\varphi(x) = C(x)$ is a singleton for any $x \in A$. Then for any open neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$ there a $\mathcal{U} \cap C$-approximation $f: X \to Y$ such that $\{f(x)\} = \varphi(x) = C(x)$ for any $x \in A$.*

Proof. Since $\pi_X: C \to X$ is locally $n$-soft (see [6]), by the very definition, $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n)$, where $\mathcal{M}_n$ stands for a class of pairs $(Z, Z_0) \in \mathcal{M}$ such that $\dim(X) \le n+1$. Then $\pi_X : C \to X$ is an open map, and hence the set-valued map $C(\,\cdot\,): X \multimap Y$ is a lower semicontinuous with closed values. The similar arguments as in the proof of Theorem 2.4 in [19] ([9]) shows that for every space $(Z,s)$ over $X$, a family $\{\{z\} \times \{\varphi(s(z))\}\ |z \in Z\}$ is equi-locally $n$-connected. In view of Theorem 2.4 in [19], we obtain that $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$. Moreover, $\varphi(x) \subset C(x)$ for any $x \in X$, and $(X, A) \in \mathcal{M}_n^c$. Thus the conclusion of the Brodskiĭ's result follows from Theorem 3.16. □

Now suppose that $(X,p)$ and $(Y,q)$ are fiberwise spaces over $B$. We are going to investigate a problem of the existence of graph approximations, provided that constraint set $C$ is a fiberwise product of $X$ and $Y$, i.e. $C = X \times_B Y$.

Let $\mathcal{U}$ be an open neighbourhood of $\mathrm{Gr}\,(\varphi)$, $A$ be a subset of $X$. We say that $f: A \to Y$ is a *fiberwise $\mathcal{U}$-approximation over $B$ on $A$*, if $f$ is a $\mathcal{V}$-approximation being a fiberwise map over $B$, i.e. $\mathrm{Gr}\,(f) \subset \mathcal{V} \cap (X \times_B Y)$.

COROLLARY 3.18. *Let $(X,p)$, $(Y,q)$ be spaces over $B$, $(Y,q) \in \mathrm{ANE}_B(\mathcal{M}_n^c)$ and suppose that $\varphi: X \multimap Y$ is upper semicontinuous with compact valued such that $\varphi(x) \cap q^{-1}(p(x)) \in \mathrm{UV}^n$ for any $x \in X$. Then, for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ such that any fiberwise $\mathcal{V}$-approximation over $B$ on $A$ such that $\dim(X \setminus A) \le n+1$ extends to a fiberwise $\mathcal{U}$-approximation over $B$ on $X$. In particular, if $\dim(X) \le n+1$, then for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}\,(\varphi)$ there is a fiberwise $\mathcal{U}$-approximation over $B$ on $X$.*

PROOF. Let $(Y,q) \in \mathrm{ANE}_B(\mathcal{M}_n^c)$. We show that $(X \times_B Y, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$, where by the pair $(X \times_B Y, \pi_X)$ we mean a subspace over $X$ of the trivial space $(X \times Y, \pi_X)$. Let $(Z, Z_0) \in \mathcal{M}_n^c$ and let $(Z,s)$ be a space over $X$, $f_0: Z_0 \to X \times_B Y$ be a map over $X$. Thus a map $\pi_Y \circ f_0: Z_0 \to Y$ is over $B$, where $Z$ is a space over $B$ together with a projection $p \circ s: Z \to B$. Hence there is an extension $f_1: U \to Y$ over $B$ to an open neighbourhood $U$ of $Z_0$. Define a map $f: U \to X \times_B Y$ by the formula: $f(z) := (s(z), f_1(z))$ for any $z \in U$. It is clear that $f$ is a well-defined map over $X$. Moreover, $f$ is an extension over $X$ of $f_0$. Since $C = X \times_B Y$ is a closed subset of $X \times Y$, in view of Theorem 3.16, we easily obtain the assertions. □

**3.4. Homotopy properties of graph approximations of set-valued maps under constraints.** When studying the properties and homotopy invariants of set-valued maps, it is important to know homotopy properties of approximations ([11], [13]). Therefore, in case of the presence of constraints, we show that sufficiently close $\mathcal{U} \cap C$-approximations are homotopic.

Again we assume that $\varphi: X \multimap Y$ is upper semicontinuous with compact values and $C \subset X \times Y$ is a constraint set.

([9]) It is sufficient to replace the class $\mathcal{M}_n^c$ by the class $\mathcal{M}_n$ in the proof of [19, Theorem 2.4].

Theorem 3.19. *Suppose that the projection $\pi_X\colon C\to X$ is an open map and condition (A) is satisfied. Then, for any open neighbourhood $\mathcal{U}$ of* $\mathrm{Gr}\,(\varphi)$*, there is an open neighbourhood $\mathcal{V}\subset\mathcal{U}$ of* $\mathrm{Gr}\,(\varphi)$ *such that the following condition is satisfied:*

(E) *If $\dim(X\setminus A)\le n$, $f,g\colon X\to Y$ are $\mathcal{V}\cap C$-approximations on $X$ such that $h\colon A\times[0,1]\to Y$ is a homotopy joining $f|_A$ with $g|_A$ and $h(\,\cdot\,,t)$ is a $\mathcal{V}\cap C$-approximation on $A$ for any $t\in[0,1]$, then there is a homotopy $H\colon X\times[0,1]\to Y$ joining $f$ with $g$ such that $H|_{A\times[0,1]}=h$ and $H(\,\cdot\,,t)$ is a $\mathcal{U}\cap C$-approximation on $X$ for any $t\in[0,1]$.*

Proof. For simplicity we use the notation $X':=[0,1]\times X$ instead of $X\times[0,1]$. Let $\pi\colon[0,1]\times X\to X$ be a projection of $[0,1]\times X$ on $X$, $\varphi'\colon=\varphi\circ\pi\colon X'\to Y$, and $C':=[0,1]\times C$. We show that condition according to (A) is satisfied for the map $\varphi'$ and constraint set $C'$. Namely:

(F) for any $(t,x)\in[0,1]\times X$ and for any open neighbourhood $U'$ of $\{(t,x)\}\times\varphi(x)$ there is an open neighbourhood $V'\subset U'$ of $\{(t,x)\}\times\varphi(x)$ such that
$$(U'\cap C',V'\cap C')\in\mathrm{EP}_{X'}(\mathcal{M}_n^c),$$
where $(C',\pi_{X'})$ is a space over $X'$ with the projection given by
$$\pi_{X'}(t,x,y)=(t,x)\quad\text{for any } (t,x,y)\in C'.$$

Let $(t,x)$ and $U'$ be as above. Then there exists open neighbourhoods $U_t$ of $t$ and $U$ of $\{x\}\times\varphi(x)$ such that $U_t\times U\subset U'$. By condition (A) we find an open neighbourhood $V$ of $\{x\}\times\varphi(x)$ such that
$$(U\cap C,V\cap C)\in\mathrm{EP}_X(\mathcal{M}_n^c).\tag{3.23}$$
Let $V':=U_t\times V\supset\{(t,x)\}\times\varphi(x)$. We show that $(U'\cap C',V'\cap C')\in\mathrm{EP}_{X'}(\mathcal{M}_n^c)$. Let $(Z,Z_0)\in\mathcal{M}_n^c$, $(Z,s)$ be a space over $X'$ such that $s(Z)\subset\pi_{X'}(V'\cap C')$. Then we obtain that $s^{[0,1]}(Z)\subset U_t$ and $s^X(Z)\subset V\cap C$, where $s^{[0,1]}\colon Z\to[0,1]$ and $s^X\colon Z\to X$ are maps such that $s(z)=(s^{[0,1]}(z),s^X(z))$ for every $z\in Z$. Let $f_0\colon Z_0\to V'\cap C'$ be a map over $X'$. Similarly as above $f_0^{[0,1]}\colon Z_0\to[0,1]$ and $f_0^C\colon Z_0\to C$ are maps such that $f_0(z)=(f_0^{[0,1]}(z),f_0^C(z))$ for any $z\in Z_0$. Then $f_0^C$ is a map over $X$ such that $f_0^C(Z_0)\subset V\cap C$. In view of (3.23), there is an extension $f^C\colon Z\to U\cap C$ over $X$ of $f_0^C$. Observe that the map $f^{[0,1]}\colon Z\to U_t$ given by the formula: $f^{[0,1]}(z)=s^{[0,1]}(z)$ for any $z\in Z$, is an extension of $f_0^{[0,1]}$. Finally, the map $f\colon Z\to U_t\times(U\cap C)\subset U'\cap C'$ defined as follows: $f(z):=(f^{[0,1]}(z),f^C(z))$ for any $z\in Z$, is an extension over $X'$ of $f_0$, which completes the proof of condition (F).

Let $\mathcal{U}$ be an open neighbourhood of $\mathrm{Gr}\,(\varphi)$. Then $\mathcal{U}':=[0,1]\times\mathcal{U}$ is an open neighbourhood of $\mathrm{Gr}\,(\varphi')=[0,1]\times\mathrm{Gr}\,(\varphi)$. Hence, in view of Theorem 3.9, there is an open neighbourhood $\mathcal{V}' \subset \mathcal{U}'$ of $\mathrm{Gr}(\varphi')$ such that

$$(\mathcal{U}' \cap C', \mathcal{V}' \cap C') \in \mathrm{EP}_{X'}(\mathcal{M}_n^c). \tag{3.24}$$

The compactness of $[0,1]$, implies the existence of an open neighbourhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that $[0,1] \times \mathcal{V} \subset \mathcal{V}'$.

Let $(X, A) \in \mathcal{M}_{n-1}^c$, $f, g: X \to Y$ be $\mathcal{V} \cap C$-approximations over $X$ such that $h: [0,1] \times A \to Y$ is a homotopy joining $f|_A$ with $g|_A$, and $h(t, \cdot)$ is a $\mathcal{V} \cap C$-approximation on $A$, for any $t \in [0,1]$. Put $A' := \{0,1\} \times X \cup [0,1] \times A$. Then $(X', A') \in \mathcal{M}_n^c$ and let $f_0: A' \to \mathcal{V}' \cap C'$ be a map over $X'$ defined as follows: $f_0(t,x) = (t, x, h(t,x))$ if $(t,x) \in [0,1] \times A$, and $f_0(0,x) = (0, x, f(x))$, $f_0(1,x) = (1, x, g(x))$ for any $x \in X$. According to (3.24) there is an extension $f: X' \to \mathcal{U}' \cap C'$ over $X'$ of the map $f_0$. It is easy to check that $H := \pi_Y \circ f: X' = [0,1] \times X \to Y$ is a required homotopy. $\square$

Corollary 3.20. *Suppose that* $\dim(X) \le n$, $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ *and tangency condition* (C) *is satisfied. Then for any open neighbourhood* $\mathcal{U}$ *of* $\mathrm{Gr}(\varphi)$ *there is an open neighbourhood* $\mathcal{V} \subset \mathcal{U}$ *of* $\mathrm{Gr}(\varphi)$ *such that for any* $\mathcal{V} \cap C$*-approximations* $f$, $g: X \to Y$ *on* $X$*, there is a homotopy* $H: X \times [0,1] \to Y$ *joining* $f$ *with* $g$ *such that* $H(\,\cdot\,, t)$ *is a* $\mathcal{U} \cap C$*-approximation on* $X$ *for any* $t \in [0,1]$.

Proof. Suppose that $\dim(X) \le n$, $(C, \pi_X) \in \mathrm{ANE}_X(\mathcal{M}_n^c)$ and tangency condition (C) holds. Then the map $\pi_X: C \to X$ is open and, in view of Lemma 3.12, we obtain that condition $(A)$ is satisfied. Hence, according to Theorem 3.19, condition (E) holds. Thus for the pair $(X, \emptyset) \in \mathcal{M}_{n-1}^c$ we obtain the conclusion. $\square$

Observe that under the assumptions of Corollary 3.20, in view of Theorem 3.1, for any open neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ there is a $\mathcal{U} \cap C$-approximation.

## 4. Appendix

In Appendix we are going to provide the proof of Theorem 2.4.

Lemma 4.1.

(a) *Let* $Y$ *be a subset of a space* $T$*, and for any* $y \in Y$*,* $V_y$ *is an open neighbourhood of* $y$ *in* $T$*. Then there is a continuous function* $\varepsilon: Y \to (0, \infty)$*, and for any* $y \in Y$ *there is* $m(y) \in Y$ *such that*

$$B(y, \varepsilon(y)) \subset V_{m(y)}.$$

(b) *For any continuous function* $\varepsilon: Y \to (0,\infty)$ *there is a continuous function* $\delta: Y \to (0, \infty)$ *such that for any* $y \in Y$

$$\mathrm{st}(B(y, \delta(y)), \{B(y', \delta(y')) \mid y' \in Y\}) \subset B(y, \varepsilon(y)) \ (^{10}).$$

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(10) $\mathrm{st}(A, \mathcal{W})$ denotes the star of a set $A$ in a cover $\mathcal{W}$.

Proof. (a) For any $y \in Y$ there is $\varepsilon_y > 0$ such that $B(y, \varepsilon_y) \subset V_y$. In view of the paracompactness of $W_0 := \bigcup_{y\in Y} B(y, \varepsilon_y/2)$ there exists a locally finite open refinement $\mathcal{W} = \{W_\alpha\}_{\alpha\in\mathcal{A}}$ of the cover $\{B(y, \varepsilon_y/2)\}_{y\in Y}$. Then for any $\alpha \in \mathcal{A}$ we fix $y_\alpha$ such that $W_\alpha \subset B(y_\alpha, \varepsilon_{y_\alpha}/2)$.

Again, in view of the paracompactness of $W_0$, there is a partition of unity $\{\lambda_\alpha\colon W_0 \to [0, 1] \mid \alpha \in \mathcal{A}\}$ such that $\operatorname{supp}(\lambda_\alpha) \subset W_\alpha$ for any $\alpha \in \mathcal{A}$. Then we define a continuous function $\varepsilon\colon Y \to (0, \infty)$ by the formula:

$$\varepsilon(y) := \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(y)\varepsilon_{y_\alpha}/2,$$

for any $y \in Y$. Note that for any $y \in Y$ there is $\alpha$ such that $y \in W_\alpha$ and $\varepsilon(y) \le \varepsilon_{y_\alpha}/2$. Then

$$B(y, \varepsilon(y)) \subset B(y, \varepsilon_{y_\alpha}/2) \subset B(y_\alpha, \varepsilon_{y_\alpha}) \subset V_{y_\alpha}.$$

Moreover, we obtain the following transformation $Y \ni y \mapsto m(y) := y_\alpha \in Y$.

(b) Let $\varepsilon\colon Y \to (0, \infty)$ be a continuous function and $\mathcal{W}$ be an open cover of

$$\bigcup_{y\in Y} (B(y, \varepsilon(y)/4) \cap \varepsilon^{-1}(\varepsilon(y)/2, \infty))$$

being a star refinement of

$$\{B(y, \varepsilon(y)/4) \cap \varepsilon^{-1}(\varepsilon(y)/2, \infty) \mid y \in Y\}.$$

For any $y \in Y$ there is $\delta_y > 0$ such that $B(y, \delta_y)$ is a subset of some $W_y \in \mathcal{W}$. Let $W_0 := \bigcup_{y\in Y} B(y, \delta_y)$ and $\{\lambda_\alpha\colon W_0 \to [0, 1] \mid \alpha \in \mathcal{A}\}$ be a locally finite partition of unity refined into a cover $\{B(y, \delta_y) \mid y \in Y\}$ of the paracompact space $W_0$. For any $\alpha \in \mathcal{A}$ we fix $y_\alpha \in Y$ such that

$$\operatorname{supp}(\lambda_\alpha) \subset B(y_\alpha, \delta_{y_\alpha}).$$

Let $\delta\colon Y \to (0, \infty)$ be defined as follows:

$$\delta(y) := \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(y)\delta_{y_\alpha}/2,$$

for any $y \in Y$. The function $\delta$ is well-defined and continuous. Moreover, for any $y \in Y$ there is $y_\alpha$ such that $\delta(y) \le \delta_{y_\alpha}/2$, and thus

$$\begin{aligned}\operatorname{st}(B(y, \delta(y)), \{B(y', \delta(y')) \mid y' \in Y\}) &\subset \operatorname{st}(B(y_\alpha, \delta_{y_\alpha}), \{B(y', \delta(y')) \mid y' \in Y\}) \\ &\subset \operatorname{st}(W_{y_\alpha}, \mathcal{W}) \subset B(y', \varepsilon(y')/4) \cap \varepsilon^{-1}(\varepsilon(y')/2, \infty).\end{aligned}$$

In order to finish the proof note that $y \in B(y', \varepsilon(y')/4) \cap \varepsilon^{-1}(\varepsilon(y')/2, \infty)$, and if $z \in B(y', \varepsilon(y')/4)$, then

$$d(z, y) \le d(z, y') + d(y', y) \le \varepsilon(y')/4 + \varepsilon(y')/4 = \varepsilon(y')/2 < \varepsilon(y). \qquad \square$$

Theorem 4.2. *The following conditions are equivalent:*

(a) $(Y,q) \in \mathrm{ANE}_B(\mathcal{M}_n^c)$;

(b) *for any* $y \in Y$*, any open neighbourhood* $U$ *of* $y$ *contains an open neighbourhood* $V$ *of* $y$ *such that* $(U,V) \in \mathrm{EP}_B(\mathcal{M}_n^c)$;

(c) *for any continuous function* $\varepsilon: Y \to (0,\infty)$ *there is a continuous function* $\delta: Y \to (0,\infty)$ *such that* $\delta(y) \le \varepsilon(y)$ *and* $(B(y,\varepsilon(y)), B(y,\delta(y))) \in \mathrm{EP}_B(\mathcal{M}_n^c)$ *for all* $y \in Y$.

Proof. The equivalence of conditions (a) and (b) is proven in [19, Theorem 2.6]. The implication (c) $\Rightarrow$ (b) is straightforward. We show (b) $\Rightarrow$ (c). Let $\varepsilon: Y \to (0,\infty)$ be a continuous function. For any $y \in Y$ let

$$W_y := \{y' \in Y \mid B(y,\varepsilon(y)/2) \subset B(y',\varepsilon(y'))\},$$

and let $\delta_y > 0$ be such that $(B(y,\varepsilon(y)/2), B(y,\delta_y)) \in \mathrm{EP}_B(\mathcal{M}_n^c)$.

Let $\{\lambda_\alpha: Y \to [0,1] \mid \alpha \in \mathcal{A}\}$ be a locally finite partition of unity refined into $\{W_y \cap B(y,\delta_y/2) \mid y \in Y\}$. For any $\alpha \in \mathcal{A}$ we fix $y_\alpha \in Y$ such that

$$\mathrm{supp}(\lambda_\alpha) \subset W_{y_\alpha} \cap B(y_\alpha, \delta_{y_\alpha}/2).$$

Let $\delta: Y \to (0,\infty)$ be defined as follows:

$$\delta(y) := \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(y)\delta_{y_\alpha}/2,$$

for any $y \in Y$. Note that the function $\delta$ is continuous and well-defined. Moreover, for any $y \in Y$ there is $y_\alpha$ such that $\delta(y) \le \delta_{y_\alpha}/2$ and $\lambda_\alpha(y) \ne 0$, and thus $B(y,\delta(y)) \subset B(y_\alpha, \delta_{y_\alpha})$. Furthermore $B(y_\alpha, \varepsilon(y_\alpha)/2) \subset B(y,\varepsilon(y))$, which completes the proof. $\square$

Proof of Theorem 2.4. Since the implication (b) $\Rightarrow$ (c) is clear it is sufficient to show the implication (a) $\Rightarrow$ (b). Recall that the proof of the equivalence of conditions (a) and (c) is given in [19].

Let $(Y,q) \in \mathrm{ANE}_B(\mathcal{M}_n^c)$. We may assume that $Y \subset T$ and $q: T \to B$ is the projection of $T$ such that $q|_Y$ is an open map (see Proposition 2.3). Let $\varepsilon: Y \to (0,\infty)$ be a continuous function. For any $y \in Y$ let $V'_y$ be an open neighbourhood of $y$ in $T$ such that

$$V'_y \cap Y = \{y' \in Y \mid B(y,\varepsilon(y)/2) \subset B(y',\varepsilon(y'))\}. \tag{4.1}$$

In view of Lemma 4.1(a) there is a function $\varepsilon_0: Y \to (0,\infty)$ such that for any $y \in Y$ there is $m(y) \in Y$ and the following inclusion holds

$$B(y,\varepsilon_0(y)) \subset B(m(y), \varepsilon(m(y))/2) \cap V'_{m(y)}. \tag{4.2}$$

Applying conditions (c) of Theorem 4.2 and (b) of Lemma 4.1, for any $i = 0,\dots,n+1$ we find continuous functions $\varepsilon_i: Y \to (0,\infty)$ and $\delta_i: Y \to (0,\infty)$ such that for any $y \in Y$ if $U^i_y := B(y, \varepsilon_i(y))$, $V^i_y := B(y, \delta_i(y))$, $\mathcal{U}^i := \{U^i_y \mid y \in Y\}$ then the following conditions are satisfied:

$$(U^i_y \cap Y, V^i_y \cap Y) \in \mathrm{EP}_B(\mathcal{M}^c_n) \tag{4.3}$$

$$\mathrm{st}(U^i_y, \mathcal{U}^i) \subset V^{i-1}_y, \quad \text{if } i \geq 1. \tag{4.4}$$

Since $q|_Y$ is an open map, then $V^{n+1}_y \cap q^{-1}(q(V^{n+1}_y \cap Y))$ is an open neighbourhood of $y$ in $T$. Thus

$$U := \bigcup_{y \in Y} (V^{n+1}_y \cap q^{-1}(q(V^{n+1}_y \cap Y)))$$

is an open neighbourhood of $Y$. Let $(Z, Z_0) \in \mathcal{M}^c_n$, where $(Z, s)$ is a space over $B$. Let $g\colon Z \to U$ be a map over $B$. For any $y \in Y$ we assign

$$W_y := g^{-1}(V^{n+1}_y \cap q^{-1}(q(V^{n+1}_y \cap Y))).$$

Note that $\{W_y\}_{y \in Y}$ is an open cover of $Z$. In view of Lemma 2.1 we obtain a sequence of closed subspaces $Z_0 \subset Z_1 \subset \ldots \subset Z_{n+2} = Z$ satisfying the enlisted conditions in the lemma with respect to the cover $\{W_y\}_{y \in Y}$.

For any $i = 0, \ldots, n+1$ and for any $\alpha \in I_i$, we choose a point $y(i, \alpha) \in Y$ such that $B^\alpha_i \subset W_{y(i,\alpha)}$.

For any $i = 0, \ldots, n+2$, we construct a map $f_i\colon Z_i \to Y$ over $B$ satisfying the following conditions for any $i \geq 1$:

$$f_i(z) = f_{i-1}(z) \qquad \text{for any } z \in Z_{i-1}, \tag{4.5}$$

$$f_i(B^\alpha_{i-1}) \cup V^{n+1}_{y(i-1,\alpha)} \subset U^{n+2-i}_{y(i-1,\alpha)} \quad \text{for any } \alpha \in I_{i-1}. \tag{4.6}$$

Let $f_0 := g|_{Z_0}$. Suppose that for some $i \in \{0, \ldots, n+1\}$, we have constructed maps $\{f_k \mid k = 0, \ldots, i\}$ over $B$ satisfying the above conditions. Let $\alpha \in I_i$. We claim that the inclusion $f_i(Z_i \cap B^\alpha_i) \subset V^{n+1-i}_{y(i,\alpha)}$ holds. Indeed, if $i = 0$, then the inclusion is clear. Let $i \geq 1$ and $I(i, \alpha) := \{(k, \alpha') \mid k = 0, \ldots, i-1,\ \alpha' \in I_k,\ B^{\alpha'}_k \cap B^\alpha_i \neq \emptyset\}$. Note that

$$\begin{aligned} f_i(Z_i \cap B^\alpha_i) &\subset f_i(Z_0 \cap B^\alpha_i) \cup \bigcup_{k=0}^{i-1} \bigcup_{\alpha' \in I_k} f_i(B^{\alpha'}_k \cap B^\alpha_i) \\ &\subset V^{n+1}_{y(i,\alpha)} \cup \bigcup_{(k,\alpha') \in I(i,\alpha)} (f_{k+1}(B^{\alpha'}_k) \cup V^{n+1}_{y(k,\alpha')}). \end{aligned} \tag{4.7}$$

Let $(k, \alpha') \in I(i, \alpha)$. In view of (4.6) we obtain that

$$f_{k+1}(B^{\alpha'}_k) \cup V^{n+1}_{y(k,\alpha')} \subset U^{n+1-k}_{y(k,\alpha')} \subset U^{n+2-i}_{y(k,\alpha')}.$$

Since $B^{\alpha'}_k \cap B^\alpha_i \neq \emptyset$, we get $V^{n+1}_{y(k,\alpha')} \cap V^{n+1}_{y(i,\alpha)} \neq \emptyset$ and $V^{n+1}_{y(i,\alpha)} \cap U^{n+2-i}_{y(k,\alpha')} \neq \emptyset$. Therefore taking into account conditions (4.7) and (4.4) we obtain that

$$f_i(Z_i \cap B^\alpha_i) \subset \mathrm{st}(V^{n+1}_{y(i,\alpha)}, \mathcal{U}^{n+2-i}) \subset \mathrm{st}(V^{n+2-i}_{y(i,\alpha)}, \mathcal{U}^{n+2-i}) \subset V^{n+1-i}_{y(i,\alpha)}.$$

Since $(B_i^\alpha, Z_i\cap B_i^\alpha)\in\mathcal{M}_n^c$ and

$$s(B_i^\alpha)\subset s(W_{y(i,\alpha)})=q(g(W_{y(i,\alpha)}))\subset q(V_{y(i,\alpha)}^{n+1}\cap Y)\subset q(V_{y(i,\alpha)}^{n+1-i}\cap Y),$$

then in view of (4.3) there is a map $f_i^\alpha\colon B_i^\alpha\to U_{y(i,\alpha)}^{n+1-i}\cap Y$ over $B$ being an extension over $B$ of the map $f_{i|Z_i\cap B_i^\alpha}$. Now we can define a map $f_{i+1}\colon Z_{i+1}\to Y$ as follows:

$$\begin{aligned} f_{i+1}(z) &= f_i^\alpha(z), \quad \text{if } z\in B_i^\alpha \text{ and } \alpha\in I_i,\\ f_{i+1}(z) &= f_i(z), \quad \text{if } z\in Z_i. \end{aligned}$$

Observe that $f_{i+1}$ is a continuous map over $B$, since sets $\{B_i^\alpha\cap(U\setminus Z_0)\}_{\alpha\in I_i}$ are pairwise separated by open neighbourhoods. Moreover, $f_{i+1}$ satisfies conditions (4.5), (4.6). Thus $f:=f_{n+2}\colon Z=Z_{n+2}\to Y$ is a required extension over $B$ of $f_0$. Note that if $z\in Z\setminus Z_0$, then there are $i$ and $\alpha\in I_i$ such that $z\in B_i^\alpha$. Hence in view of conditions (4.6) and (4.2) we obtain that

$$f(z)=f_{i+1}(z)\in U_{y(i,\alpha)}^{n+1-i}\subset B(y_0,\varepsilon(y_0)/2)\cap V'_{y_0},$$

where $y_0:=m(y(i,\alpha))$. Since $f(z)\in Y$, then by (4.1) we get that

$$B(y_0,\varepsilon(y_0)/2)\subset B(f(z),\varepsilon(f(z))).$$

Note that $g(z)\in V_{y(i,\alpha)}^{n+1}\subset B(y_0,\varepsilon(y_0)/2)\cap V'_{y_0}$. Then $g(z)\in B(f(z),\varepsilon(f(z)))$, which completes the proof. $\square$

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