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EXISTENCE RESULTS FOR THE p-LAPLACIAN EQUATION WITH RESONANCE AT THE FIRST TWO EIGENVALUES

Ming-Zheng Sun

Abstract. In this paper, by a space decomposition we will study the existence and multiplicity for the p-Laplacian equation with resonance at the first two eigenvalues.

1. Introduction

In this paper, we consider the boundary value problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < \infty$, and assume that

(f_0) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfying the growth condition:

$$|f(x,t)| \leq c(1 + |t|^{q-1}), \quad \text{for all } x \in \Omega,\ t \in \mathbb{R},$$

for some $c > 0$ and $q \in [1, p^*)$, where $p^* = Np/(N-p)$ if $p < N$ and $p^* = \infty$ if $N \leq p$.

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Let $W_0^{1,p}(\Omega)$ be the Sobolev space endowed with the norm

$$\|u\| = \bigg(\int_\Omega |\nabla u|^p\,dx\bigg)^{1/p}.$$

Under the condition (f_0), it is well known that the weak solutions of (1.1) correspond to the critical points of the functional $I\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx,$$

where $F(x,u) = \int_0^u f(x,t)\,dt$.

In recent years, there are many papers that have studied the equation (1.1) with the non-resonant or resonant conditions. For example, in order to obtain the existence of the solutions, the authors in [10], [13], [20] study the case

$$\lim_{|u|\to\infty} \frac{pF(x,u)}{|u|^p} < \lambda_1, \quad \text{uniformly for } x \in \Omega,$$

and the paper [1], [3], [23] has used the following condition

$$\lambda_1 \preceq l(x) = \liminf_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} \le \limsup_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = k(x) < \lambda_2,$$

uniformly for $x \in \Omega$, where λ_1 and λ_2 are the first and second eigenvalues of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, respectively (see [19]), and $\lambda_1 \preceq l(x)$ means that $\lambda_1 \le l(x)$ and the strict inequality holds on a set of positive measure.

Equation (1.1) is called a resonant problem at the first eigenvalue if

$$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = \lambda_1, \quad \text{uniformly for } x \in \Omega. \tag{1.2}$$

In [18], the authors have obtained the existence of multiple solutions of equation (1.1) with (1.2) and the following non-quadratic condition

$$\lim_{|u|\to\infty} (uf(x,u) - pF(x,u)) = -\infty, \quad \text{uniformly for } x \in \Omega.$$

Moreover, under the condition (1.2) and

$$\lim_{|u|\to\infty} (uf(x,u) - pF(x,u)) = \infty, \quad \text{uniformly for } x \in \Omega,$$

the paper [20] has proved that I is coercive, and the same result can also be found in [1], [18] which assume that

$$\lim_{|u|\to\infty} \Big(F(x,u) - \frac{1}{p}\lambda_1|u|^p\Big) = -\infty, \quad \text{uniformly for } x \in \Omega.$$

With other versions of the non-quadratic conditions, a lot of papers have studied the case

$$\lambda_1 \le a(x) = \liminf_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} \le \limsup_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = b(x) < \lambda_2, \tag{1.3}$$

uniformly for $x \in \Omega$, see for example [22], [24], [25]. In addition, for the solvability of resonant (1.1) with the Landesman–Lazer type conditions, we refer to [9], [14] and references therein.

Since the function f depends on x, the aim of our paper is to study the equation (1.1) with the condition:

(f_1) there exists a constant $M > 0$ such that

$$a(x) \le \frac{f(x,u)}{|u|^{p-2}u} \le b(x), \quad \text{for } |u| \ge M,\ x \in \Omega,$$

where a and b are continuous functions.

Let $\lambda_1(a)$ be the first eigenvalue of the equation

$$-\Delta_p u - a(x)|u|^{p-2}u = \lambda|u|^{p-2}u$$

with the Dirichlet boundary value, it is well known that $\lambda_1(a)$ is simple and isolated (see for example [19]), then the second eigenvalue

$$\lambda_2(a) = \inf\{\lambda > \lambda_1(a) \mid \lambda \text{ is the eigenvalue of } -\Delta_p - a(x) \text{ on } W_0^{1,p}(\Omega)\}$$

is well defined. By the monotonicity of $\lambda_1(a)$ (see [11]) and $\lambda_2(b)$ (see [2]), the condition (1.3) implies that

$$\lambda_1(a) \le 0 < \lambda_2(b).$$

For the first eigenfunction $\varphi_1(a) > 0$, if we assume $V = \operatorname{span}\{\varphi_1(a)\}$, and denote by

$$V^\perp = \left\{u \in W_0^{1,p}(\Omega) \;\middle|\; \int_\Omega (\varphi_1(a))^{p-1} u \, dx = 0\right\},$$

then we have

$$W_0^{1,p}(\Omega) = V \oplus V^\perp. \tag{1.4}$$

Moreover, from [14], we know that there exists $\overline{\lambda}(a) \in (\lambda_1(a), \lambda_2(a)]$ such that

$$\int_\Omega (|\nabla u|^p - a(x)|u|^p)\, dx \ge \overline{\lambda}(a) \int_\Omega |u|^p \, dx, \quad \text{for any } u \in V^\perp.$$

Similarly, we can define $\lambda_1(b)$, $\varphi_1(b)$ and $\overline{\lambda}(b)$.

Now, we state the assumptions

(f_2) $\displaystyle\lim_{|u|\to\infty} \int_\Omega \Big(F(x,u) - \frac{1}{p} b(x)|u|^p\Big)\, dx = -\infty,$

(f_3) $\displaystyle\lim_{|u|\to\infty} (uf(x,u) - pF(x,u)) = -\infty,$

and the main result in this paper is the followings:

Theorem 1.1. *Assume that* (f_0) *and* (f_1) *hold. If one of the following conditions is satisfied,*

(a) $\lambda_1(b) > 0$,
(b) $\lambda_1(b) \geq 0$ *and* (f_2) *holds,*
(c) $\lambda_1(a) < 0 < \overline{\lambda}(b)$,
(d) $\lambda_1(a) \leq 0 \leq \overline{\lambda}(b)$ *and* (f_3) *holds,*

then equation (1.1) *has at least one solution.*

Remark 1.2. (1) For the case $p = 2$, we can take $\overline{\lambda}(b) = \lambda_2(b)$, and the results of (c) and (d) can be found in [16], [17]. Since the spectrum of $-\Delta_p$ in the general case $p \neq 2$ is still being established, it remains an open question whether the $\overline{\lambda}(b)$ in our theorem can be replaced by $\lambda_2(b)$.

(2) The proof of our theorem is based on the linking theorem. There are two difficulties when one wants to treat the condition (f_1). One is the Palais–Smale condition for I and the other is to construct linking sets. For the case $a = \lambda_1$ and $b = \lambda_2$, we can decompose the space $W_0^{1,p}(\Omega)$ as $W_0^{1,p}(\Omega) = E_1 \oplus E_1^{\perp}$ where $E_1 = \mathrm{Ker}(-\Delta_p - \lambda_1)$. But in our case, we have to give a decomposition of the space $W_0^{1,p}(\Omega)$ according to the eigenfunctions of different functions a and b (see Lemma 3.2). For $p = 2$, this method of space decomposition has been used by [16], [JS] and the paper [12] which studies the periodic boundary value problem.

The paper is organized as follows: In Section 2, we will prove that the functional I satisfies the Palais–Smale condition. In Section 3, we give a decomposition lemma for $W_0^{1,p}(\Omega)$, which is the basis of the proof of Theorem 1.1. In Section 4, we are interested in finding the nontrivial solutions of equation (1.1). In the sequel, the letter C will be used to denote various positive constants whose exact value is irrelevant.

2. The Palais–Smale condition

In this section, we will prove the following Palais–Smale condition for I.

Definition 2.1. The functional I is said to satisfy the Palais–Smale condition at the level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ for short) if every sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ with

$$I(u_n) \to c, \quad (\|u_n\| + 1)I'(u_n) \to 0, \quad \text{as } n \to \infty, \tag{2.1}$$

possesses a convergent subsequence. I satisfies the (PS) if I satisfies $(\mathrm{PS})_c$ at any $c \in \mathbb{R}$.

This Palais–Smale type condition was introduced by G. Cerami in [6], and it was shown that this condition suffices to get the linking theorem (see [4]).

LEMMA 2.2. *Under the assumptions of Theorem* 1.1*, the functional I satisfies the* (PS) *condition.*

PROOF. *Case 1.* We will show that the functional I is coercive on $W_0^{1,p}(\Omega)$. Since $\lambda_1(b) > 0$ and $b \in C(\overline{\Omega})$, we have

$$\begin{aligned}\int_\Omega |\nabla u|^p\,dx &= \int_\Omega [|\nabla u|^p - b(x)|u|^p]\,dx + \int_\Omega b(x)|u|^p\,dx \\ &\le \int_\Omega [|\nabla u|^p - b(x)|u|^p]\,dx + C\int_\Omega |u|^p\,dx \\ &\le \int_\Omega [|\nabla u|^p - b(x)|u|^p]\,dx + \frac{C}{\lambda_1(b)}\int_\Omega [|\nabla u|^p - b(x)|u|^p]\,dx \\ &\le C\int_\Omega [|\nabla u|^p - b(x)|u|^p]\,dx,\end{aligned}$$

then there is a constant $\delta > 0$ such that

$$\int_\Omega [|\nabla u|^p - b(x)|u|^p]\,dx \ge \delta \int_\Omega |\nabla u|^p\,dx, \quad \text{for any } u \in W_0^{1,p}(\Omega). \tag{2.2}$$

From (f_0) and (f_1), we get that

$$F(x,u) \le \frac{1}{p} b(x)|u|^p + C, \tag{2.3}$$

this together with (2.2) implies that

$$\begin{aligned}I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx \\ &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{1}{p}\int_\Omega b(x)|u|^p\,dx - \int_\Omega \Big(F(x,u) - \frac{1}{p}b(x)|u|^p\Big)\,dx \\ &\ge \frac{\delta}{p}\int_\Omega |\nabla u|^p\,dx - C.\end{aligned}$$

Then we get that $I(u) \to \infty$ as $\|u\| \to \infty$, this proves the case.

Case 2. We will also show that the functional I is coercive on $W_0^{1,p}(\Omega)$.

By contradiction, we assume that there are a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ and a constant C_0 such that

$$I(u_n) \le C_0, \quad \text{as } \|u_n\| \to \infty. \tag{2.4}$$

Set $v_n = u_n/\|u_n\|$, then there exists a $v \in W_0^{1,p}(\Omega)$ such that, passing if necessary to a subsequence,

$$\begin{cases} v_n \rightharpoonup v & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \to v & \text{strongly in } L^p(\Omega), \\ v_n \to v & \text{for a.e. } x \in \Omega. \end{cases}$$

Using (2.3) and (2.4), we have

$$\frac{C_0}{\|u_n\|^p} \ge \frac{1}{p}\int_\Omega (|\nabla v_n|^p - b(x)|v_n|^p)\,dx - \frac{C}{\|u_n\|^p},$$

which implies that

$$\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^p\,dx \le \int_\Omega b(x)|v|^p\,dx. \tag{2.5}$$

Moreover, since $\lambda_1(b)\ge 0$, from the lower semi-continuity of the norm we get

$$\int_\Omega b(x)|v|^p\,dx \le \int_\Omega |\nabla v|^p\,dx \le \liminf_{n\to\infty}\int_\Omega |\nabla v_n|^p\,dx,$$

this together with (2.5) gives $\|v_n\|\to\|v\|$, as $n\to\infty$. Since $W_0^{1,p}(\Omega)$ is uniformly convex, we have $v_n\to v$ in $W_0^{1,p}(\Omega)$, as $n\to\infty$ with $\|v\|=1$ and

$$\int_\Omega b(x)|v|^p\,dx = \int_\Omega |\nabla v|^p\,dx.$$

With no loss generally, we assume that $\lambda_1(b)=0$, then we can take $v=\pm\varphi_1(b)$, which implies that $|u_n(x)|\to\infty$ almost everywhere in Ω.

By (f_2) it follows

$$\lim_{n\to\infty}\int_\Omega \Big(F(x,u_n)-\frac{1}{p}b(x)|u_n|^p\Big)\,dx=-\infty,$$

then we have

$$\begin{aligned} I(u_n) &= \frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx - \int_\Omega F(x,u_n)\,dx \\ &= \frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx - \frac{1}{p}\int_\Omega b(x)|u_n|^p\,dx - \int_\Omega \Big(F(x,u_n)-\frac{1}{p}b(x)|u_n|^p\Big)\,dx \\ &\ge -\int_\Omega \Big(F(x,u_n)-\frac{1}{p}b(x)|u_n|^p\Big)\,dx \to \infty, \quad \text{as } n\to\infty. \end{aligned}$$

This is a contradiction with (2.4).

Case 3. We assume that $\{u_n\}\subset W_0^{1,p}(\Omega)$ and satisfies (2.1), by (f_0) it suffices to show that $\{u_n\}$ is bounded (see [10]).

By contradiction, we assume $\|u_n\|\to\infty$ as $n\to\infty$. Let $z_n=u_n/\|u_n\|$, then there exists $z\in W_0^{1,p}(\Omega)$ such that, passing if necessary to a subsequence,

$$\begin{cases} z_n \rightharpoonup z & \text{weakly in } W_0^{1,p}(\Omega), \\ z_n \to z & \text{strongly in } L^p(\Omega), \\ z_n \to z & \text{for a.e. } x\in\Omega. \end{cases}$$

Let $g_n(x)=f(x,u_n)/\|u_n\|^{p-1}$, then g_n is bounded in $L^{p'}(\Omega)$ with $1/p+1/p'=1$, and for a subsequence, we assume that

$$g_n \rightharpoonup g \quad \text{weakly in } L^{p'}(\Omega). \tag{2.6}$$

The proofs of the following two claims are similar to Lemmas 2.6 and 2.7 in the paper [3], respectively.

CLAIM 1. $g=0$ almost everywhere in $\Omega\setminus A$, where $A=\{x\in\Omega \mid z(x)\neq 0\}$.

CLAIM 2. Set

$$m(x) = \begin{cases} \dfrac{g(x)}{|z(x)|^{p-2}z(x)} & \text{on } A, \\ a(x) & \text{on } \Omega \setminus A, \end{cases}$$

then we have

$$a(x) \leq m(x) \leq b(x), \quad \text{a.e. in } \Omega. \tag{2.7}$$

CLAIM 3. $z_n \to z$ in $W_0^{1,p}(\Omega)$ and z is a nontrivial solution of the equation

$$\begin{cases} -\Delta_p u = m(x)|u|^{p-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{2.8}$$

Indeed, from (2.1), for any $\phi \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla z_n|^{p-2}\nabla z_n \nabla\phi\, dx - \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}\phi\, dx = o(1)\|\phi\|. \tag{2.9}$$

Let $\phi = z_n - z$, it is easy to see that

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}(z_n - z)\, dx = 0,$$

this together with (2.9) gives

$$\lim_{n\to\infty} \int_\Omega |\nabla z_n|^{p-2}\nabla z_n \nabla(z_n - z)\, dx = 0.$$

From the fact that $-\Delta_p$ is of type S^+ (see [10]), we conclude that $z_n \to z$ in $W_0^{1,p}(\Omega)$ with $\|z\| = 1$.

Using (2.6) we deduce that

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}\phi\, dx = \int_\Omega g\phi\, dx,$$

then from (2.9) and our claims we have

$$\int_\Omega |\nabla z|^{p-2}\nabla z \nabla\phi\, dx = \int_\Omega m(x)|z|^{p-2}z\phi\, dx,$$

which implies the equation (2.8).

By (2.7), the monotonicity of $\lambda_1(a)$ (see [11]) and $\lambda_2(b)$ (see [2]) gives

$$\lambda_1(m) \leq \lambda_1(a) < 0, \quad \lambda_2(m) \geq \lambda_2(b) \geq \overline{\lambda}(b) > 0,$$

then 0 is not an eigenvalue of $-\Delta_p - m(x)$, which contradicts the equation (2.8).

Case 4. By contradiction, we assume that $\{u_n\} \subset W_0^{1,p}(\Omega)$ and satisfies (2.1), but $\|u_n\| \to \infty$ as $n \to \infty$. Let $z_n = u_n/\|u_n\|$, then there exists $z \in W_0^{1,p}(\Omega)$ such that, passing if necessary to a subsequence,

$$\begin{cases} z_n \rightharpoonup z & \text{weakly in } W_0^{1,p}(\Omega), \\ z_n \to z & \text{strongly in } L^p(\Omega), \\ z_n \to z & \text{for a.e. } x \in \Omega. \end{cases}$$

From (f_0) and (f_3), it is easy to show that

$$F(x,u) \le C|u|^p + C. \tag{2.10}$$

Combining (2.1) and (2.10), we obtain that

$$\frac{1}{p}\|u_n\|^p - C\|u_n\|_p^p - C \le C,$$

which implies that

$$\frac{1}{p} - C\|z\|_p^p \le 0,$$

so $z \neq 0$. If we define $\Omega' = \{x \in \Omega \mid z(x) \neq 0\}$, then we have

$$\operatorname{mes}(\Omega') > 0, \quad |u_n(x)| \to \infty, \quad \text{as } n \to \infty,\ x \in \Omega',$$

which implies that

$$\lim_{n\to\infty} (pF(x,u_n) - u_n f(x,u_n)) = \infty, \quad x \in \Omega'.$$

From the Fatou's lemma we conclude that

$$\lim_{n\to\infty} \int_\Omega (pF(x,u_n) - u_n f(x,u_n))\,dx = \infty.$$

However, using (2.1), it follows that

$$\lim_{n\to\infty} \int_\Omega (pF(x,u_n) - u_n f(x,u_n))\,dx = -pc.$$

This contradiction completes the proof. □

3. Proof of the Theorem 1.1

In this section, we will first give a decomposition of $W_0^{1,p}(\Omega)$ which is the basis of the linking theorem. We recall the following lemma:

LEMMA 3.1 ([21]). *Let E be a vector space such that for subspaces X and Y, $E = X \oplus Y$. If Y is finite dimensional and Z is a subspace of E such that $X \cap Z = \{0\}$ and $\dim(Y) = \dim(Z)$ then $E = X \oplus Z$.*

Let $\varphi_1(a)$ and $\varphi_1(b)$ be the first eigenfunctions of $\lambda_1(a)$ and $\lambda_1(b)$, respectively. If we set $E_1 = \operatorname{span}\{\varphi_1(a)\}$ and $E_2 = \operatorname{span}\{\varphi_1(b)\}$, then similar to (1.4) we have

$$W_0^{1,p}(\Omega) = E_1 \oplus E_1^\perp, \quad W_0^{1,p}(\Omega) = E_2 \oplus E_2^\perp.$$

LEMMA 3.2. *If the continuous functions $a(x) \le b(x)$ for $x \in \Omega$ satisfying*

$$\lambda_1(a) \le 0 \le \overline{\lambda}(b),$$

then we have that $W_0^{1,p}(\Omega) = E_1 \oplus E_2^{\perp}$.

PROOF. From the Lemma 3.1, we only need to prove that $E_1 \cap E_2^{\perp} = \{0\}$. With no loss generally, we assume that $\{x \in \Omega \mid a(x) \ne b(x)\}$ is not empty, so it is easy to see that if $u \in \mathrm{Ker}(-\Delta_p - a) \cap \mathrm{Ker}(-\Delta_p - b)$, then we get $u = 0$.

For any $u_0 \in E_1 \cap E_2^{\perp}$, by the assumptions, we get

$$\begin{aligned} 0 \ge \lambda_1(a) \int_\Omega |u_0|^p \, dx &= \int_\Omega (|\nabla u_0|^p - a(x)|u_0|^p) \, dx \\ &\ge \int_\Omega (|\nabla u_0|^p - b(x)|u_0|^p) \, dx \ge \overline{\lambda}(b) \int_\Omega |u_0|^p \, dx \ge 0, \end{aligned}$$

which implies that $u_0 \in \mathrm{Ker}\,(-\Delta_p - a) \cap \mathrm{Ker}\,(-\Delta_p - b)$, then $u_0 = 0$. $\square$

Now, we are ready to give the proof of our theorem.

PROOF OF THE THEOREM 1.1. (a) and (b). Since in each case the functional I is coercive on $W_0^{1,p}(\Omega)$, the existence of a solution is trivial.

(c) Now, we want to prove that:

(1) $I(u) \to -\infty$ as $\|u\| \to \infty,\ u \in E_1$.

From (f_0) and (f_1), if we set $G(x,u) = F(x,u) - (1/p)a(x)|u|^p$, then

$$G(x,u) \ge -C. \tag{3.1}$$

Since $\lambda_1(a) < 0$ and $\dim(E_1) < \infty$, (3.1) gives that

$$\begin{aligned} I(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u) \, dx \\ &= \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{1}{p} \int_\Omega a(x)|u|^p \, dx - \int_\Omega G(x,u) \, dx \\ &\le \frac{\lambda_1(a)}{p} \int_\Omega |u|^p \, dx + C \le -C\|u\|^p + C, \end{aligned}$$

then $I(u) \to -\infty$ as $u \in E_1$ and $\|u\| \to \infty$.

(2) $I(u)$ is bounded from below on $E_2^{\perp}$.

Similarly, if we set $G_1(x,u) = F(x,u) - (1/p)b(x)|u|^p$, then $G_1(x,u) \le C$, which implies that, for any $u \in E_2^{\perp}$,

$$\begin{aligned} I(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u) \, dx \\ &= \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{1}{p} \int_\Omega b(x)|u|^p \, dx - \int_\Omega G_1(x,u) \, dx \\ &\ge \frac{\overline{\lambda}(b)}{p} \int_\Omega |u|^p \, dx - C \ge -C, \end{aligned}$$

so $I(u)$ is bounded from below on $E_2^{\perp}$.

(3) Now, we fix an R such that $\sup_{u\in\partial B(R)\cap E_1} I(u) \le \beta - 1$, where $\beta = \inf_{u\in E_2^\perp} I(u)$, and $B(R) = \{u \in W_0^{1,p}(\Omega) \mid \|u\| \le R\}$. Set

$$\begin{aligned} \Gamma &= \{\gamma : B(R)\cap E_1 \to W_0^{1,p}(\Omega) \mid \gamma(u) = u \text{if } u \in E_1,\ \|u\| = R\},\\ c &= \inf_{\gamma\in\Gamma} \max_{u\in B(R)} I(u). \end{aligned}$$

Since $\partial B(R)\cap E_1$ and $E_2^\perp$ are linking and the (PS) condition holds for I, $c \ge \beta$ is a critical value of I (see [7]). So there is a critical point $u_0 \in W_0^{1,p}(\Omega)$ such that $I(u_0) = c$. The proof of this case is finished.

(d) Similar to (c) we only need to prove that $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in E_1$.

Indeed, we still write $G(x,u) = F(x,u) - (1/p)a(x)|u|^{p-2}u$, and $g(x,u) = f(x,u) - a(x)|u|^{p-2}u$, then using ($f_3$) we have

$$\lim_{|u|\to\infty} (g(x,u)u - pG(x,u)) = -\infty,$$

which implies that (see [18])

$$\lim_{|u|\to\infty} G(x,u) = \infty, \quad \text{for } x \in \Omega. \tag{3.2}$$

Then for any $u \in E_1$, (3.2) and the fact $\dim(E_1) < \infty$ give that

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx\\ &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{1}{p}\int_\Omega a(x)|u|^p\,dx - \int_\Omega G(x,u)\,dx\\ &= \frac{\lambda_1(a)}{p}\int_\Omega |u|^p\,dx - \int_\Omega G(x,u)\,dx \to -\infty, \quad \text{as } \|u\| \to \infty. \qquad \square \end{aligned}$$

4. Multiplicity results of equation (1.1)

Now, we are interested in finding multiple nontrivial solutions of equation (1.1). First, let us recall some results of Morse theory that will be used below, for details, we refer to [7]. Let X be a real Banach space and $\Phi \in C^1(X,\mathbb{R})$ and satisfies the Palais–Smale condition. Let $K = \{u \in X \mid \Phi'(u) = 0\}$ be the critical set of Φ. Let $u \in K$ be an isolated critical point with $\Phi(u) = c \in \mathbb{R}$, and U be an isolated neighbourhood of u, i.e. $K \cap U = \{u\}$. The group

$$C_*(\Phi,u) = H_*(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad * = 0,1,\dots,$$

is called the $*$-th critical group of Φ at u, where $\Phi^c = \{u \in X \mid \Phi(u) \le c\}$, $H_*(\,\cdot\,,\,\cdot\,)$ are the singular relative homology groups with a coefficient group G. By the excision property of the homology groups, the critical groups are independent of the choices of U, then they are well defined. In particular, if u, v are the critical points of Φ and $C_q(\Phi,u) \ne C_q(\Phi,v)$ for some q then $u \ne v$.

Our result in this section reads as follows.

THEOREM 4.1. *Under the assumptions* (c) *or* (d) *of Theorem* 1.1*, if the following condition holds,*

(f$_4$) $f(x,0)=0$ *and there is a continuous function* $l(x)$ *such that*

$$\lim_{|u|\to 0}\frac{pF(x,u)}{|u|^p}\le l(x)\quad \textit{with } \lambda_1(l)>0,\ x\in\Omega,$$

then equation (1.1) *has one nontrivial solution.*

REMARK 4.2. Obviously, (f$_4$) is weaker than the condition

$$\lim_{|u|\to 0}\frac{pF(x,u)}{|u|^p}=l(x)\preceq\lambda_1,\quad x\in\Omega,$$

which implies that 0 is a local minimum of I (see [8], [18]).

LEMMA 4.3. *Under our conditions,* 0 *is a local minimum of the functional* I.

PROOF. Since $\lambda_1(l)>0$, there exists a constant $\varepsilon>0$ such that $\lambda_1(l+\varepsilon)>0$ (see for example [15]). From (f$_4$), there is a $\delta=\delta(\varepsilon)$ such that

$$F(x,t)\le\frac{1}{p}(l(x)+\varepsilon)|t|^p,\quad \text{for } |t|\le\delta,\ x\in\Omega.$$

Moreover, for $p<s\le p^*$ we can find $C>0$ such that

$$F(x,t)\le C|t|^s,\quad \text{for } |t|>\delta,\ x\in\Omega.$$

Then we get

$$F(x,t)\le\frac{1}{p}(l(x)+\varepsilon)|t|^p+C|t|^s,\quad \text{for } t\in\mathbb{R},\ x\in\Omega. \tag{4.1}$$

Similar to (2.2), combining (4.1) and the embedding theorem, we have

$$\begin{aligned} I(u)&=\frac{1}{p}\int_\Omega|\nabla u|^p\,dx-\int_\Omega F(x,u)\,dx\\ &\ge\frac{1}{p}\int_\Omega|\nabla u|^p\,dx-\frac{1}{p}\int_\Omega(l(x)+\varepsilon)|u|^p\,dx-\int_\Omega C|u|^s\,dx\\ &\ge C\|u\|^p-C\|u\|_s^s\ge C\|u\|^p-C\|u\|^s>0,\end{aligned}$$

as $0<\|u\|\ll 1$, which implies that 0 is a local minimum of I. □

PROOF OF THE THEOREM 4.1. From Lemma 4.2, we obtain that

$$C_*(I,0)=\delta_{*,0}G.$$

Using the results in [5], the solution u_0 obtained by Theorem 1.1 satisfies

$$C_1(I,u_0)\ne 0.$$

Hence u_0 is the nontrivial critical point of I. □

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MING-ZHENG SUN
College of Science
North China University of Technology
Beijing 100144, P.R. CHINA

E-mail address: smz@pku.edu.cn