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TOPOLOGICAL METHODS FOR BOUNDARY VALUE PROBLEMS INVOLVING DISCRETE VECTOR ϕ-LAPLACIANS

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Abstract. In this paper, using Brouwer degree arguments, we prove some existence results for nonlinear problems of the type

$$-\nabla[\phi(\Delta x_m)] = g_m(x_m, \Delta x_m) \quad (1 \le m \le n-1),$$

submitted to Dirichlet, Neumann or periodic boundary conditions, where $\phi(x) = |x|^{p-2}x$ ($p > 1$) or $\phi(x) = x/\sqrt{1-|x|^2}$ and $g_m\colon \mathbb{R}^N \to \mathbb{R}^N$ ($1 \le m \le n-1$) are continuous nonlinearities satisfying some additional assumptions.

1. Introduction and notation

In this paper, using Brouwer degree arguments, we prove some existence results for nonlinear problems of the type

$$-\nabla[\phi(\Delta x_m)] = g_m(x_m, \Delta x_m) \quad (1 \le m \le n-1),$$

submitted to Dirichlet, Neumann or periodic boundary conditions, where functions $g_m\colon \mathbb{R}^N \to \mathbb{R}^N$ ($1 \le m \le n-1$) are continuous and the discrete vector ϕ-Laplacian operator is defined as follows.

Let n, N be positive integers, $0 < a \le \infty$ and $\phi\colon B(a) \subset \mathbb{R}^N \to \mathbb{R}^N$ be a homeomorphism such that $\phi(0) = 0$. (In what follows $B(\rho)$ denotes an open

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ball with center in zero and radius ρ). For any $x = (x_0, \dots, x_n) \in [\mathbb{R}^N]^{n+1}$ we define

$$\Delta x_m = x_{m+1} - x_m, \quad (0 \le m \le n-1)$$

and if $|\Delta x_m| < a$ $(0 \le m \le n-1)$ we define

$$\nabla[\phi(\Delta x_m)] = \phi(\Delta x_m) - \phi(\Delta x_{m-1}), \quad (1 \le m \le n-1).$$

Discrete vector p-Laplacian generated by the homeomorphism

$$h_p : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto |x|^{p-2}x \quad (p > 1),$$

and the relativistic discrete operator generated by

$$\phi : B(1) \subset \mathbb{R}^N \to \mathbb{R}^N, \quad x \to \frac{x}{\sqrt{1 - |x|^2}}.$$

are important special cases. Here and in what follows $|\cdot|$ denotes the Euclidean norm generated by the Euclidean scalar product $(\cdot \mid \cdot)$.

In the p-Laplacian case ($\phi = h_p$), our main results (Theorems 2.4 and 2.8) are discrete versions of some interesting results from [5] (see also [4]). It is worth to point out that the main tool used in [5] is the Leray–Schauder a priori estimation method applied to some fixed point operators acting in the Sobolev space $W^{1,p}$. In the discrete case, we use a different strategy based on the main properties of the Brouwer degree: the homotopy invariance, existence property, Borsuk's theorem. Note also that Corollary 2.5 is a discrete version of [6, Theorem 7.1]. For interesting applications of Brouwer degree to nonlinear difference equations the reader can consult [8].

In the singular case ($a < \infty$) our main result (Theorem 3.4) is a discrete version of [1, Theorem 5] and the particular case $N = 1$ and $A = 0 = B$ is considered in [2].

If $\Omega \subset X$ is an open subset of a finite dimensional normed space X, $x_0 \in X$ and $S : \overline{\Omega} \to X$ is a continuous function such that $x_0 \notin S(\partial\Omega)$, then $d_B[S, \Omega, x_0]$ denotes the Brouwer degree of S with respect to Ω and x_0. For the definition and properties of the Brouwer degree see [3], [7].

2. The p-Laplacian case

Dirichlet boundary value problems. Let $f_m : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function $(1 \le m \le n-1)$ and consider the following nonlinear Dirichlet boundary-value problem involving the discrete vector p-Laplacian

$$-\nabla[h_p(\Delta x_m)] = f_m(x_m) \quad (1 \le m \le n-1), \quad x_0 = 0 = x_n. \tag{2.1}$$

First of all notice that the solutions of (2.1) can be seen as the zeros of the continuous mapping $F : V_N^{n-1} \to [\mathbb{R}^N]^{n-1}$ defined by

$$F_m(x) = \nabla[h_p(\Delta x_m)] + f_m(x_m), \quad (1 \le m \le n-1),$$

where the $(n-1)N$-dimensional vector space V_N^{n-1} is defined by

$$V_N^{n-1} = \{x \in [\mathbb{R}^N]^{n+1} : x_0 = 0 = x_n\}.$$

We endow the vector space V_N^{n-1} with the norm $||\cdot||_p$ defined by

$$||x||_p = \left[\sum_{m=1}^{n-1} |x_m|^p\right]^{1/p}, \quad (x \in V_N^{n-1}).$$

On the other hand, the mapping $||\cdot||_{p,\Delta}$ defined by

$$||(x_0,\dots,x_n)||_{p,\Delta} = \left[\sum_{m=0}^{n-1} |\Delta x_m|^p\right]^{1/p}, \quad ((x_0,\dots,x_n) \in V_N^{n-1})$$

is also a norm on V_N^{n-1}. We introduce the eigenvalue-like constant

$$\lambda_1 = \inf\left\{\frac{\displaystyle\sum_{m=0}^{n-1} |\Delta x_m|^p}{\displaystyle\sum_{m=1}^{n-1} |x_m|^p} : (x_0,\dots,x_n) \in [\mathbb{R}^N]^{n+1} \setminus \{0\},\ x_0 = 0 = x_n\right\}. \tag{2.2}$$

In the next Lemma we prove that the constant λ_1 is strictly positive and we obtain a Poincaré type inequality.

Lemma 2.1. *The constant λ_1 defined in* (2.2) *is strictly positive and*

$$\lambda_1 ||x||_p^p \le ||x||_{p,\Delta}^p \quad \textit{for all } x \in V_N^{n-1}. \tag{2.3}$$

Proof. From the definition of λ_1 it follows that

$$\lambda_1 = \inf\{||x||_{p,\Delta}^p : x \in V_N^{n-1},\ ||x||_p = 1\},$$

and using that $||\cdot||_p$ and $||\cdot||_{p,\Delta}$ are norms on V_N^{n-1} it follows that there exist $x \in V_N^{n-1}$ such that $||x||_p = 1$ and $\lambda_1 = ||x||_{p,\Delta}^p$. Hence, $\lambda_1 > 0$ and (2.3) follows immediately from the definition of λ_1. $\square$

In the next Lemma we prove a summation by parts type formula for vectors belonging to V_N^{n-1}.

Lemma 2.2. *We have that*

$$-\sum_{m=1}^{n-1} (x_m \mid \nabla[h_p(\Delta x_m)]) = \sum_{m=0}^{n-1} |\Delta x_m|^p \quad \textit{for all } (x_0,\dots,x_n) \in V_N^{n-1}. \tag{2.4}$$

Proof. Let $(x_0,\dots,x_n) \in V_N^{n-1}$ be fixed. For all $1 \le m \le n-1$ we have that

$$\begin{aligned}(x_m \mid \nabla[h_p(\Delta x_m)]) = {}& |\Delta x_m|^{p-2}(x_m \mid x_{m+1}) + |\Delta x_{m-1}|^{p-2}(x_{m-1}|x_m) \\ & - |\Delta x_m|^{p-2}|x_m|^2 - |\Delta x_{m-1}|^{p-2}|x_m|^2.\end{aligned}$$

On the other hand, for all $0 \le m \le n-1$ we have that

$$|\Delta x_m|^p = |\Delta x_m|^{p-2}[|x_m|^2 - 2(x_m \mid x_{m+1}) + |x_{m+1}|^2].$$

It follows that

$$\begin{aligned}
&\sum_{m=1}^{n-1} (x_m \mid \nabla[h_p(\Delta x_m)]) \\
&\quad = \sum_{m=1}^{n-1} [|\Delta x_m|^{p-2}(x_m \mid x_{m+1}) + |\Delta x_{m-1}|^{p-2}(x_{m-1} \mid x_m)] \\
&\qquad - \sum_{m=1}^{n-1} [|\Delta x_m|^{p-2}|x_m|^2 + |\Delta x_{m-1}|^{p-2}|x_m|^2] \\
&\quad = \sum_{m=1}^{n-1} |\Delta x_m|^{p-2}(x_m \mid x_{m+1}) + \sum_{m=0}^{n-2} |\Delta x_m|^{p-2}(x_m \mid x_{m+1}) \\
&\qquad - \sum_{m=1}^{n-1} |\Delta x_m|^{p-2}|x_m|^2 - \sum_{m=0}^{n-2} |\Delta x_m|^{p-2}|x_{m+1}|^2 \\
&\quad = - \sum_{m=0}^{n-1} \{|\Delta x_m|^{p-2}[|x_m|^2 - 2(x_m|x_{m+1}) + |x_{m+1}|^2]\} = - \sum_{m=0}^{n-1} |\Delta x_m|^p,
\end{aligned}$$

and the proof is completed. □

Now, we consider the homotopy $\mathcal{F}\colon [0,1] \times V_N^{n-1} \to [\mathbb{R}^N]^{n-1}$ defined by

$$\mathcal{F}_m(x) = \nabla[h_p(\Delta x_m)] + \lambda f_m(x_m), \quad (1 \le m \le n-1).$$

Notice that $\mathcal{F}(1, \cdot\,) = F$ and $\mathcal{F}(0, \cdot\,)$ is the discrete vector p-Laplacian operator, which is odd. On the other hand for $\lambda \in [0,1]$ one has that $x \in V_N^{n-1}$ is a zero of $\mathcal{F}(\lambda, \cdot\,)$ if and only if x is a solution of the Dirichlet boundary-value problem

$$-\nabla[h_p(\Delta x_m)] = \lambda f_m(x_m) \quad (1 \le m \le n-1), \quad x_0 = 0 = x_n. \tag{2.5}$$

In the next Lemma we obtain a priori estimations for the possible zeros of $\mathcal{F}$.

Lemma 2.3. *If*

$$\limsup_{|x|\to\infty} \frac{(x|f_m(x))}{|x|^p} < \lambda_1 \quad \text{for all } 1 \le m \le n-1, \tag{2.6}$$

holds, then there exists $\rho > 0$ such that any possible zero (λ, x) of $\mathcal{F}$ satisfies $||x||_p < \rho$.

Proof. Let $(\lambda, x) \in [0,1] \times V_N^{n-1}$ be such that $\mathcal{F}(\lambda, x) = 0$. It follows that $x = (x_0, \dots, x_n)$ is a solution of (2.5). Multiplying (2.5) by x_m, summing from

1 to $n-1$ and using Lemma 2.2 it follows that

$$\sum_{m=0}^{n-1} |\Delta x_m|^p = \lambda \sum_{m=1}^{n-1} (x_m \mid f_m(x_m)). \tag{2.7}$$

Using (2.6) and the continuity of f_m it follows that there exists $\sigma \in (0, \lambda_1)$ and $k_1 > 0$ such that

$$(y \mid f_m(y)) \leq \sigma |y|^p + k_1 \quad \text{for all } y \in \mathbb{R}^N,\ 1 \leq m \leq n-1. \tag{2.8}$$

From (2.3), (2.7) and (2.8) we deduce that

$$\lambda_1 ||x||_p^p \leq \sigma ||x||_p^p + (n-1)k_1.$$

Hence, using that $\sigma \in (0, \lambda_1)$ and $p > 1$ it follows that there exists $\rho > 0$ such that $||x||_p < \rho$. □

Theorem 2.4. *If* (2.6) *holds, then* (2.1) *has at least one solution.*

Proof. Using Lemma 2.3 and the invariance of the Brouwer degree under homotopy it follows that

$$d_B[\mathcal{F}(1, \cdot\,), B(\rho), 0] = d_B[\mathcal{F}(0, \cdot\,), B(\rho), 0]. \tag{2.9}$$

On the other hand $\mathcal{F}(0, \cdot\,)$ is odd, so the Borsuk theorem implies that

$$d_B[\mathcal{F}(0, \cdot\,), B(\rho), 0] \neq 0,$$

which together with (2.9) imply that

$$d_B[\mathcal{F}(1, \cdot\,), B(\rho), 0] \neq 0.$$

Hence, using the existence property of the Brouwer degree, we deduce that $\mathcal{F}(1, \cdot\,)$ has at least one zero which is also a solution of (2.1). □

An immediate consequence is the following

Corollary 2.5. *Let A_m $(1 \leq m \leq n-1)$ be a $N \times N$-matrix. If there exists $\sigma \in (0, \lambda_1)$ and $R > 0$ such that*

$$(x \mid A_m x) \leq \sigma |x|^2 \quad \textit{for all } |x| \geq R,\ 1 \leq m \leq n-1,$$

then the Dirichlet boundary-value problem

$$-\nabla[h_p(\Delta x_m)] = A_m[h_p(x_m)] + l_m \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n$$

has at least one solution for any $(l_1, \dots, l_{n-1}) \in [R^N]^{n-1}$.

Periodic boundary value problems. Let $f_m: \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function satisfying

$$\limsup_{|x|\to\infty} \frac{(x|f_m(x))}{|x|^p} < 0 \quad (1 \le m \le n-1), \tag{2.10}$$

and consider the following nonlinear periodic boundary-value problem involving the discrete vector p-Laplacian

$$-\nabla[h_p(\Delta x_m)] = f_m(x_m) \quad (1 \le m \le n-1), \quad x_0 = x_n,\ \Delta x_0 = \Delta x_{n-1}. \tag{2.11}$$

Note that, from (2.10) it follows that there exists $\sigma_1 \in (0,1)$ and $\sigma_2 > 0$ such that

$$(x \mid f_m(x)) \le -\sigma_1|x|^p + \sigma_2 \quad \text{for all } x \in \mathbb{R}^N\ (1 \le m \le n-1), \tag{2.12}$$

and that the solutions of (2.11) can be seen as the zeros of the continuous mapping $F: U_N^{n-1} \to [\mathbb{R}^N]^{n-1}$ defined by

$$F_m(x) = \nabla[h_p(\Delta x_m)] + f_m(x_m), \quad (1 \le m \le n-1),$$

where the $(n-1)N$-dimensional vector space U_N^{n-1} is defined by

$$\begin{aligned} U_N^{n-1} =& \{(x_0, \dots, x_n) \in [\mathbb{R}^N]^{n+1} : x_0 = x_n,\ \Delta x_0 = \Delta x_{n-1}\} \\ =& \{(x_0, \dots, x_n) \in [\mathbb{R}^N]^{n+1} : x_0 = (x_1 + x_{n-1})/2 = x_n\}. \end{aligned}$$

We endow the vector space U_N^{n-1} with the norm $\|\cdot\|_p$.

Lemma 2.6. *If $(x_0, \dots, x_n) \in U_N^{n-1}$, then*

$$\sum_{m=1}^{n-1} (\nabla[h_p(\Delta x_m)]|x_m) \le 0.$$

Proof. One has that

$$\begin{aligned} &\sum_{m=1}^{n-1} (\nabla[h_p(\Delta x_m)]|x_m) \\ &\quad = (h_p(\Delta x_1) - h_p(\Delta x_0)|x_1) + \dots + (h_p(\Delta x_{n-1}) - h_p(\Delta x_{n-2})|x_{n-1}) \\ &\quad = (h_p(\Delta x_1)|x_1) - (h_p(\Delta x_0)|x_1) + (h_p(\Delta x_2)|x_2) - (h_p(\Delta x_1)|x_2) + \dots \\ &\qquad + (h_p(\Delta x_{n-1})|x_{n-1}) - (h_p(\Delta x_{n-2})|x_{n-1}) \\ &\quad = -(h_p(\Delta x_0)|x_1) - (h_p(\Delta x_1)|\Delta x_1) - \dots \\ &\qquad - (h_p(\Delta x_{n-2})|\Delta x_{n-2}) + (h_p(\Delta x_{n-1})|x_{n-1}) \\ &\quad = -\sum_{m=1}^{n-2} |\Delta x_m|^p - \frac{|\Delta x_{n-1}|^{p-2}|x_{n-1} - x_1|^2}{2} \le 0. \end{aligned}$$

$\square$

Lemma 2.7. *If* $L: U_N^{n-1} \to [\mathbb{R}^N]^{n-1}$ *is the odd continuous function defined by*

$$L_m(x) = \nabla[h_p(\Delta x_m)] - h_p(x_m), \quad (1 \le m \le n-1),$$

then $d_B[L, B(\rho), 0] \neq 0$ *for all* $\rho > 0$.

Proof. Assume that $(x_0, \dots, x_n)$ solves the problem

$$L(x_0, \dots, x_n) = 0, \quad (x_0, \dots x_n) \in U_N^{n-1}.$$

It follows that

$$\sum_{m=1}^{n-1} \{(\nabla[h_p(\Delta x_m)] | x_m) - (h_p(x_m)|x_m))\} = 0.$$

Using Lemma 2.6 we deduce that

$$\sum_{m=1}^{n-1} |x_m|^p = \sum_{m=1}^{n-1} (\nabla[h_p(\Delta x_m)] | x_m) \le 0,$$

and $x_0 = \dots = x_n = 0$. Now the result follows from Borsuk's theorem. □

Theorem 2.8. *If* (2.10) *holds, then* (2.11) *has at least one solution.*

Proof. Let $\mathcal{H}: [0,1] \times U_N^{n-1} \to [\mathbb{R}^N]^{n-1}$ be the homotopy

$$\mathcal{H}_m(x) = \nabla[h_p(\Delta x_m)] + \lambda f_m(x_m) - (1-\lambda) h_p(x_m) \quad (1 \le m \le n-1).$$

It is clear that

$$\mathcal{H}(0, \cdot\,) = L, \quad \mathcal{H}(1, \cdot\,) = F.$$

Let also $(\lambda, x) \in [0,1] \times U_N^{n-1}$ be such that $\mathcal{H}(\lambda, x) = 0$. Using (2.12) and Lemma 2.6 we deduce that

$$\begin{aligned} 0 &\le \lambda \sum_{m=1}^{n-1} (x_m | f_m(x_m)) - (1-\lambda) ||x||_p^p \\ &\le -\lambda \sigma_1 ||x||_p^p + (n-1)\sigma_2 - (1-\lambda) ||x||_p^p. \end{aligned}$$

Hence, $||x||_p < \rho$ for any $\rho > ((n-1)\sigma_2/\sigma_1)^{1/p}$. Using Lemma 2.7 and the invariance under homotopy of the Brouwer degree, it follows that

$$d_B[F, B(\rho), 0] = d_B[L, B(\rho), 0], \quad d_B[F, B(\rho), 0] \neq 0,$$

for $\rho > ((n-1)\sigma_2/\sigma_1)^{1/p}$. Then, using the existence property of the Brouwer degree it follows that F has a zero which is also a solution of (2.11). □

Neumann boundary value problems. Let $f_m: \mathbb{R}^N \to \mathbb{R}^N$ $(1 \le m \le n-1)$ be a continuous function satisfying (2.10) and consider the following nonlinear Neumann boundary-value problem involving the discrete vector p-Laplacian

$$-\nabla[h_p(\Delta x_m)] = f_m(x_m) \quad (1 \le m \le n-1), \quad \Delta x_0 = 0 = \Delta x_{n-1}. \tag{2.13}$$

Using the same strategy like in he periodic case one can prove that (2.13) has at least one solution.

3. The ϕ-Laplacian case with singular ϕ

Let n, N be positive integers, $a > 0$, $\phi: B(a) \subset \mathbb{R}^N \to \mathbb{R}^N$ be a homeomorphism such that $\phi(0) = 0$; we call it *singular*. The space $[\mathbb{R}^N]^{n+1}$ will be endowed we the norm

$$||x||_\infty = \sum_{m=0}^{n} |x_m| \quad (x \in [\mathbb{R}^N]^{n+1}).$$

Nonhomogeneous Neumann boundary value problems. Let A and B in $\mathbb{R}^N$ be fixed.

Lemma 3.1. *Let $l_1, \dots, l_{n-1} \in \mathbb{R}^N$. Forced problem*

$$\nabla[\phi(\Delta x_m)] = l_m \quad (1 \le m \le n-1), \quad \phi(\Delta x_0) = A, \quad \phi(\Delta x_{n-1}) = B \tag{3.1}$$

is solvable if and only if

$$\sum_{m=1}^{n-1} l_m = B - A. \tag{3.2}$$

In this case the general solution $(x_0, \dots, x_n)$ is given by

$$\begin{aligned} x_0 &\in \mathbb{R}^N, \\ x_1 &= x_0 + \phi^{-1}(A), \\ x_m &= x_0 + \phi^{-1}(A) + \sum_{j=1}^{m-1} \phi^{-1}\bigg(A + \sum_{k=1}^{j} l_k\bigg) \quad (2 \le m \le n-1), \\ x_n &= x_0 + \phi^{-1}(A) + \phi^{-1}(B) + \sum_{j=1}^{n-2} \phi^{-1}\bigg(A + \sum_{k=1}^{j} l_k\bigg). \end{aligned}$$

Proof. The proof follows by s simple computation and is left to the reader. □

Let $g_m: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ $(1 \le m \le n-1)$ be continuous and consider nonhomogeneous Neumann boundary value problem

$$\begin{aligned} \nabla[\phi(\Delta x_m)] &= g_m(x_m, \Delta x_m) \quad (1 \le m \le n-1), \\ \phi(\Delta x_0) &= A, \quad \phi(\Delta x_{n-1}) = B. \end{aligned} \tag{3.3}$$

Let $Q\colon [\mathbb{R}^N]^{n-1} \to \mathbb{R}^N$ be defined by

$$Q(l) = \frac{1}{n-1} \sum_{m=1}^{n-1} l_m.$$

It is clear that (3.3) can be written in the equivalent form

$$\begin{aligned} \nabla[\phi(\Delta x_m)] &= g_m(x_m, \Delta x_m) - Q[s(x_m, \Delta x_m)] \quad (1 \le m \le n-1), \\ \phi(\Delta x_0) &= A, \quad \phi(\Delta x_{n-1}) = B, \end{aligned} \tag{3.4}$$

and

$$Q[s(x_m, \Delta x_m)] = 0, \tag{3.5}$$

where

$$s_m(x_m, \Delta x_m) = g_m(x_m, \Delta x_m) - \frac{1}{n-1}(B - A) \quad (1 \le m \le n-1). \tag{3.6}$$

Now, we reformulate (3.4), (3.5) as a fixed point problem. Consider the operator

$$\begin{aligned} &M\colon [\mathbb{R}^N]^{n+1} \to [\mathbb{R}^N]^{n+1}, \quad M(x) = y, \\ y_0 &= x_0 + Q(s), \\ y_1 &= x_0 + Q(s) + \phi^{-1}(A), \\ y_m &= x_0 + Q(s) + \phi^{-1}(A) + \sum_{j=1}^{m-1} \phi^{-1}\left(A + \sum_{k=1}^{j} l_k\right) \quad (2 \le m \le n-1), \\ y_n &= x_0 + Q(s) + \phi^{-1}(A) + \phi^{-1}(B) + \sum_{j=1}^{n-2} \phi^{-1}\left(A + \sum_{k=1}^{j} l_k\right), \end{aligned}$$

where s is given in (3.6) and

$$l_m = g_m(x_m, \Delta x_m) - Q(s) \quad (1 \le m \le n-1). \tag{3.7}$$

Then, using Lemma 3.1 one has the following

Lemma 3.2. *The vector $x \in [\mathbb{R}^N]^{n+1}$ is a solution of* (3.3) *if and only if $M(x) = x$.*

In order to prove that M has at least one fixed point, we define the homotopy

$$\begin{aligned} &\mathcal{M}\colon [0,1] \times [\mathbb{R}^N]^{n+1} \to [\mathbb{R}^N]^{n+1}, \quad \mathcal{M}(\lambda, x) = y, \\ y_0 &= x_0 + Q(s), \\ y_1 &= x_0 + Q(s) + \phi^{-1}(\lambda A), \\ y_m &= x_0 + Q(s) + \phi^{-1}(\lambda A) + \sum_{j=1}^{m-1} \phi^{-1}\left(\lambda\left[A + \sum_{k=1}^{j} l_k\right]\right) \quad (2 \le m \le n-1), \\ y_n &= x_0 + Q(s) + \phi^{-1}(\lambda A) + \phi^{-1}(\lambda B) + \sum_{j=1}^{n-2} \phi^{-1}\left(\lambda\left[A + \sum_{k=1}^{j} l_k\right]\right), \end{aligned}$$

where s and l are defined in (3.6), (3.7) respectively. Note that $\mathcal{M}(1,\,\cdot\,) = M$.

We introduce the following assumption:

($\mathrm{H}_{g,A,B}$) *There exists* $R > 0$ *such that*

$$\sum_{m=1}^{n-1} g_m(x_m, \Delta x_m) \neq B - A,$$

for all $x \in [\mathbb{R}^N]^{n+1}$ *satisfying*

$$\min_{0 \le m \le n} |x_m| \ge R \quad \textit{and} \qquad \max_{0 \le m \le n-1} |\Delta x_m| < a.$$

Lemma 3.3. *Assume that* ($\mathrm{H}_{g,A,B}$) *holds. If* $(\lambda, x) \in [0,1] \times [\mathbb{R}^N]^{n+1}$ *is such that* $x = \mathcal{M}(\lambda, x)$, *then*

$$||x||_\infty < R + na.$$

Proof. Let $(\lambda, x) \in [0,1] \times [\mathbb{R}^N]^{n+1}$ be such that $x = \mathcal{M}(\lambda, x)$. It follows that

$$\max_{0 \le m \le n-1} |\Delta x_m| < a,$$

and

$$\sum_{m=1}^{n-1} g_m(x_m, \Delta x_m) = B - A,$$

implying that

$$\min_{0 \le m \le n} |x_m| < R.$$

Hence, using that

$$|x_m| \le \min_{0 \le m \le n} |x_m| + \sum_{j=0}^{n-1} |\Delta x_j| \quad (0 \le m \le n),$$

we get the result. □

Consider the continuous function

$$\gamma\colon \mathbb{R}^N \to \mathbb{R}^N, \quad c \mapsto \sum_{m=1}^{n-1} g_m(c, 0).$$

Theorem 3.4. *If assumption* ($\mathrm{H}_{g,A,B}$) *holds, then for all sufficiently large* $\rho > 0$,

$$d_B[I - \mathcal{M}(1,\,\cdot\,), B(\rho), 0] = (-1)^N d_B[\gamma, B(R), B - A].$$

If furthermore

$$d_B[\gamma, B(R), B - A] \neq 0,$$

then (3.3) *has at least one solution.*

Proof. Note that, from assumption ($\mathrm{H}_{g,A,B}$) it follows that $\gamma(c) \neq B - A$ for all $|c| \ge R$, which implies that the Brouwer degree $d_B[\gamma, B(R), B - A]$ is well defined. On the other hand, taking $\rho > R + na$, using Lemma 3.3 and the homotopy invariance of Brouwer degree, we get that

$$d_B[I - \mathcal{M}(1, \cdot\,), B(\rho), 0] = d_B[I - \mathcal{M}(0, \cdot\,), B(\rho), 0].$$

But, the range of $\mathcal{M}(0, \cdot\,)$ is isomorphic to $\mathbb{R}^N$. Actually,

$$\mathcal{M}_m(0, x) = x_0 + Q(s)(x) \quad (0 \le m \le n)$$

where s is given in (3.6). Hence, using the reduction and excision properties of the Brouwer degree we have

$$\begin{aligned} d_B[I - \mathcal{M}(0, \cdot\,), B(\rho), 0] &= d_B[I - \mathcal{M}(0, \cdot\,)|_{\mathbb{R}^N}, B(\rho), 0] \\ &= (-1)^N d_B[\gamma, B(R), B - A]. \end{aligned}$$

Now, the result follows from the existence property of the Brouwer degree. $\square$

An immediate consequence of Theorem 3.4 is the following

Corollary 3.5. *Assume that there exists $\varepsilon \in \{-1, 1\}$ and $R > 0$ such that*

$$\varepsilon(g_m(x + y, z) - (n-1)^{-1}(B - A)|x) > 0$$

for all $1 \le m \le n-1$, $|x| \ge R$, $|y| < na$ and $|z| < a$, then (3.3) *has at least one solution.*

Remark 3.6. Similar considerations hold also for Dirichlet and periodic boundary value problems. In these cases, in order to construct the associated fixed point operators (see [2] for $N = 1$), ϕ must be of gradient type like in [1].

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