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POSITIVE SOLUTIONS FOR GENERALIZED NONLINEAR LOGISTIC EQUATIONS OF SUPERDIFFUSIVE TYPE

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Abstract. We consider a generalized version of the p-logistic equation. Using variational methods based on the critical point theory and truncation techniques, we prove a bifurcation-type theorem for the equation. So, we show that there is a critical value $\lambda_* > 0$ of the parameter $\lambda > 0$ such that the following holds: if $\lambda > \lambda_*$, then the problem has two positive solutions; if $\lambda = \lambda_*$, then there is a positive solution; and finally, if $0 < \lambda < \lambda_*$, then there are no positive solutions.

1. Introduction

In this paper we study the following nonlinear elliptic Dirichlet problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda f(z,u) - g(z,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a C^2 boundary $\partial\Omega$, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(\|Du\|^{p-2}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega)\ (1 < p < \infty)$$

and $\lambda > 0$ is a real parameter.

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When $f(z,x) = x^{q-1}$ and $g(z,x) = x^{r-1}$ with $q > 1$ and $p < r < p^*$ ($p^* = Np/(N-p)$ if $p < N$, $p^* = \infty$ if $p \geq N$), we have the so-called *p-logistic equation* (simply *logistic equation* when $p = 2$). Such equations are important, among other things, in biological models (see M. E. Gurtin and R. C Mac Camy [7]).

There are three different types of p-logistic equations, depending on the value of the exponent q with respect to p. Namely, we have:

(a) the *subdiffusive case*, when $1 < q < p < r$;
(b) the *equidiffusive case*, when $1 < q = p < r$;
(c) the *superdiffusive case*, when $1 < p < q < r$.

The first two cases (subdiffusive and equidiffusive) are similar and differ essentially from the superdiffusive case, which exhibits bifurcation-type phenomena for large values of the parameter $\lambda > 0$.

In this paper, we are interested in the superdiffusive case, which was studied by W. Dong and J. T. Chen [3] and S. Takeuchi [14], [15] (for $p \geq 2$). However, our formulation here is more general, since we do not restrict ourselves to the particular reaction term of the form $\lambda x^{q-1} - x^{r-1}$ with $1 < p < q < r$, as is the case in the aforementioned works. We also recall the recent work of F. Brock, L. Itturiaga and P. Ubilla [1], where the reaction term has the form $\lambda f(z,x)$, being $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a continuous function. However, their solution method limits their considerations to the case of a $(p-1)$-sublinear nonlinearity (see hypothesis (H_3) and the proofs of Lemmata 3.1 and 3.2). Related are also the works of Z. Guo [6] and D. Motreanu, V. V. Motreanu and N. S. Papageorgiou [11], [12].

In Z. Guo [6], $f(z,x) = f(x)$ is C^1, $g(z,x) = 0$ and $1 < p \leq 2$. Assuming that f is strictly increasing and $(p-1)$-sublinear, the author shows that there is a critical value $\lambda(p)$ of the parameter such that for all $\lambda > \lambda(p)$ the problem has at least two positive solutions. In [11], the authors examine nonlinear eigenvalue problems with the p-Laplacian and nonlinearities exhibiting a general polynomial growth. They prove a multiplicity result (three solutions, of which two of constant sign, one positive and the other negative) which is local in λ (for small values of $\lambda > 0$). In [12], the authors consider equations with a nonsmooth potential (hemivariational inequalities) and study the near resonant, resonant and nonresonant cases.

In this paper, using minimax methods based on the critical point theory together with suitable truncation techniques, we prove a bifurcation-type theorem for large values of the parameter $\lambda > 0$. Namely, under suitable assumptions on the nonlinearities f and g, we will show that there is a critical value $\lambda_* > 0$ such that the following holds: if $\lambda > \lambda_*$, then the problem (P_λ) has two positive solutions; if $\lambda = \lambda_*$, then there is a positive solution; and finally, if $0 < \lambda < \lambda_*$, then there are no positive solutions.

2. Mathematical background

Let X be a Banach space and X^* be its topological dual. By $\langle \cdot\, , \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$ be a real functional. A point $x_0 \in X$ is called a *critical point* of φ, if $\varphi'(x_0) = 0$. A number $c \in \mathbb{R}$ is called a *critical value* of φ, if there exists a critical point $x_0 \in X$ such that $\varphi(x_0) = c$. We say that φ satisfies the *Palais–Smale condition* if the following holds:

(PS) every sequence (u_n) in X, such that $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \to 0$ in X^*, admits a convergent subsequence.

The next result is known in the literature as the *mountain pass theorem*:

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies* (PS)*, $x_0, x_1 \in X$, $0 < r < \|x_1 - x_0\|$ are such that*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{\|u-x_0\|=r} \varphi(u) = \eta_r,$$
$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$$

where $\Gamma = \{\gamma \in C([0,1], \mathbb{R}) : \gamma(i) = x_i,\ i = 0, 1\}$, then, $c \ge \eta_r$ and c is a critical value of φ.

In the analysis of problem (P_λ) we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\Omega) : u(z) = 0 \text{ for all } z \in \partial\Omega\}.$$

Note that $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}(C_+) = \left\{u \in C_0^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here n denotes the outward unit normal to $\partial\Omega$.

Let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (the topological dual of $W_0^{1,p}(\Omega)$, with $p' = p/(p-1)$) be the nonlinear mapping defined by

$$\langle A(u), v\rangle = \int_\Omega \|Du\|^{p-2}(Du, Dv)dz \text{ for all } u, v \in W_0^{1,p}(\Omega). \tag{2.1}$$

Proposition 2.2. *The mapping A defined by* (2.1) *is continuous, strictly monotone* (*hence maximal monotone too*)*, bounded and of type* $(\mathrm{S})_+$*, that is, for every sequence (u_n) in $W_0^{1,p}(\Omega)$,*

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_n \langle A(u_n), u_n - u\rangle \le 0$$

imply that $u_n \to u$ *in* $W_0^{1,p}(\Omega)$.

In what follows, by $\widehat{\lambda}_1$ we denote the first eigenvalue of the negative p-Laplacian in $W_0^{1,p}(\Omega)$. It admits the following variational characterization:

$$\widehat{\lambda}_1 = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}. \tag{2.2}$$

We know that $\widehat{\lambda}_1 > 0$ and is simple and isolated. Moreover, the infimum in (2.2) is attained in the eigenspace of $\widehat{\lambda}_1$. It is clear from (2.2) that the elements of the eigenspace of $\widehat{\lambda}_1$ do not change sign. By $\widehat{u}_1$ we denote the L^p-normalized positive eigenfunction corresponding to $\widehat{\lambda}_1$. From nonlinear regularity theory (see, for example, N. S. Papageorgiou and S. Th. Kyritsi [13, p. 311–312]) we have that $\widehat{u}_1 \in C_+$. Then, invoking the nonlinear maximum principle of J. L. Vázquez [16], we conclude that $\widehat{u}_1 \in \operatorname{int}(C_+)$.

To produce a pair of positive solutions of (P_λ), we will need the following strong comparison result which extends a similar result of M. Guedda and L. Veron [5] (where $\sigma = 0$ and the hypotheses are more restrictive).

Proposition 2.3. *If* $u_1 \in C_+$, $u_2 \in \operatorname{int}(C_+)$ *with* $u_1 \leq u_2$, $\beta_1, \beta_2 \in L^\infty(\Omega)$, $\sigma \geq 0$, $p < \theta < \infty$ *are such that*

$$-\Delta_p u_i + \sigma u_i^{\theta-1} = \beta_i \quad \text{in } \Omega\ (i = 1, 2)$$

and for every nonempty, compact subset K *of* Ω *there is* $\gamma_K > 0$ *such that*

$$\beta_2(z) - \beta_1(z) \geq \gamma_K \quad \text{for a.a. } z \in K, \tag{2.3}$$

then $u_2 - u_1 \in \operatorname{int}(C_+)$.

Proof. Let us define the coincidence set of u_1 and u_2 by

$$D = \{z \in \Omega : u_1(z) = u_2(z)\}$$

and the common critical set by

$$E = \{z \in \Omega : Du_1(z) = Du_2(z) = 0\}.$$

Claim. $D \subseteq E$.

Suppose that $z_0 \in D$. Because $u_1 \leq u_2$, we see that the function $(u_1 - u_2)$ attains its maximum at z_0, therefore

$$Du_1(z_0) = Du_2(z_0). \tag{2.4}$$

Reasoning by contradiction, suppose that $Du_2(z_0) \neq 0$. We can find an open ball B_0 centered at z_0 such that $B_0 \subseteq \Omega$ and

$$\|Du_2(z)\| > 0 \quad \text{for all } z \in B_0.$$

We set $y = u_2 - u_1$, so $y \in C_+$. We are going to prove that y solves a convenient elliptic equation with a positive reaction, so that we can apply the nonlinear strong maximum principle of J. L. Vázquez [16]. From our assumptions we know that

$$-\sum_{j=1}^{N} \frac{\partial}{\partial z_j}\left[\|Du_2\|^{p-2}\frac{\partial u_2}{\partial z_j} - \|Du_1\|^{p-2}\frac{\partial u_1}{\partial z_j}\right] = \beta_2 - \beta_1 - \sigma(u_2^{\theta-1} - u_1^{\theta-1}) \quad \text{in } \Omega.$$

An application of the mean value theorem yields the existence of $t_j \in [0,1]$ (depending on $z \in \Omega$) such that

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_j}\left[a_{ij}\left(\frac{\partial u_2}{\partial z_i} - \frac{\partial u_1}{\partial z_i}\right)\right] = \beta_2 - \beta_1 - \sigma(u_2^{\theta-1} - u_1^{\theta-1}) \quad \text{in } \Omega,$$

where

$$\begin{aligned} a_{ij} = {} & \|t_j Du_2 + (1-t_j)Du_1\|^{p-4} \\ & \cdot \left((p-2)\left(t_j\frac{\partial u_2}{\partial z_j} + (1-t_j)\frac{\partial u_1}{\partial z_j}\right) + \delta_{ij}\|t_j Du_2 + (1-t_j)Du_1\|^2\right). \end{aligned}$$

If we define a quasilinear second-order differential operator in divergence form by

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial z_j}\left[a_{ij}\frac{\partial u}{\partial z_i}\right], \tag{2.5}$$

we realize that y solves the auxiliary equation

$$-Ly = \beta_2 - \beta_1 - \sigma(u_2^{\theta-1} - u_1^{\theta-1}) \quad \text{in } B_0. \tag{2.6}$$

By (2.4) we have

$$a_{ij}(z_0) = \|Du_2(z_0)\|^{p-4}\left((p-2)\frac{\partial u_2}{\partial z_j}(z_0) + \delta_{ij}\|Du_2(z_0)\|^2\right)$$

and it is easily seen that there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(z_0)\xi_i\xi_j \geq \nu\|\xi\|^2 \quad \text{for all } \xi \in \mathbb{R}^N.$$

By choosing the ball B_0 even smaller if necessary, we may assume that the operator L is strictly elliptic. Besides, the right-hand side of (2.6) is positive (see (2.3)). Then, invoking Theorem 4 of J. L. Vázquez [16], we infer that

$$u_1(z) < u_2(z) \quad \text{for all } z \in B_0.$$

But, recall that $z_0 \in D \cap B_0$ and so $u_1(z_0) = u_2(z_0)$, a contradiction. This proves the Claim.

Since by hypothesis $u_2 \in \operatorname{int}(C_+)$, we see that E is a compact subset of Ω and D is a closed subset of E (see the Claim). It follows that D is compact. So, we can find a domain $\Omega_0 \subset \Omega$ such that

$$D \subset \Omega_0 \quad \text{and} \quad \overline{\Omega}_0 \subset \Omega.$$

There exists $\varepsilon \in]0,1[$ such that

$$\begin{aligned} u_1(z) + \varepsilon &< u_2(z) \quad \text{for all } z \in \partial\Omega_0, \\ \beta_1(z) + \varepsilon &< \beta_2(z) \quad \text{for a.a. } z \in \Omega_0. \end{aligned} \tag{2.7}$$

Also, we can choose $\delta \in]0,\varepsilon[$ such that

$$\sigma|t^{\theta-1} - s^{\theta-1}| \le \varepsilon \quad \text{for all } t, s \in \Big[\min_{\overline{\Omega}_0} u_1, \max_{\overline{\Omega}_0} u_1 + 1\Big] \text{ with } |t-s| \le \delta. \tag{2.8}$$

Then we have

$$\begin{aligned} -\Delta_p(u_1+\delta)(z) + \sigma[u_1(z)+\delta]^{\theta-1} &= -\Delta_p u_1(z) + \sigma\,[u_1(z)+\delta]^{\theta-1} \\ &\le -\Delta_p u_1(z) + \sigma u_1(z)^{\theta-1} + \varepsilon \quad \text{(see (2.8))} \\ &= \beta_1(z) + \varepsilon < \beta_2(z) \\ &= -\Delta_p u_2(z) + \sigma u_2(z)^{\theta-1} \qquad \text{for all } z \in \Omega_0. \end{aligned}$$

From what stated above, the first inequality in (2.7) (recall that $\delta < \varepsilon$) and the weak comparison principle (see L. Damascelli [2]), it follows that

$$u_1(z) + \delta \le u_2(z) \quad \text{for all } z \in \Omega_0. \tag{2.9}$$

Now recall that $D \subset \Omega_0$. This fact, combined with (2.9), implies that $D = \emptyset$ and so

$$u_1(z) < u_2(z) \quad \text{for all } z \in \Omega.$$

Consider any $z_1 \in \partial\Omega$. Since $u_2 \in \operatorname{int}(C_+)$, we can find an open connected subset Ω_1 of Ω such that $z_1 \in \partial\Omega_1$ and $\delta > 0$ such that

$$\|Du_2(z)\| > \delta \text{for all } z \in \Omega_1.$$

As before, by choosing Ω_1 smaller if necessary, we achieve that the operator L (defined by (2.5)) is strictly elliptic and

$$Ly \ge 0 \quad \text{in } \Omega_1 \tag{2.10}$$

(recall $y = u_2 - u_1$). Since $y(z) > 0$ for all $z \in \Omega_1$ and $y(z_1) = 0$, then (2.10) and the results of J. L. Vázquez [16] (see Theorem 2 and the comments concerning its generalization) imply

$$\frac{\partial y}{\partial n}(z_1) < 0.$$

Since $z_1 \in \partial\Omega$ was arbitrary, we conclude that $y \in \operatorname{int}(C_+)$. $\square$

We conclude this section by introducing some notation: in what follows $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$ and by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Moreover, for every $y \in \mathbb{R}$, we set

$$y^{\pm} = \max\{\pm y, 0\}.$$

Finally, for every $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $u \in W_0^{1,p}(\Omega)$ we set

$$N_h(u)(z) = h(z, u(z)) \quad \text{for all } z \in \Omega.$$

3. Bifurcation theorem

In order to get a bifurcation result for problem (P_λ), we will assume the following hypotheses on the reaction nonlinearities f and g:

(H$_f$) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. measurable in $z \in \Omega$ and continuous in $x \in \mathbb{R}$) such that $f(z,0) = 0$ for almost all $z \in \Omega$ and $f(z,x) > 0$ for almost all $z \in \Omega$ and all $x > 0$, and

- (i) $|f(z,x)| \le a(z) + c|x|^{r-1}$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$ ($a \in L^\infty(\Omega)_+$, $c > 0$, $p < r < p^*$);
- (ii) $\lim_{x\to\infty} f(z,x)/x^{p-1} = \infty$, $\lim_{x\to\infty} f(z,x)/x^{\theta-1} = 0$ uniformly for almost all $z \in \Omega$ and $x \mapsto f(z,x)/x^{\theta-1}$ is nonincreasing in $]0,\infty[$ for almost all $z \in \Omega$ ($\theta > p$);
- (iii) $\lim_{x\to 0^+} f(z,x)/x^{p-1} = 0$ uniformly for almost all $z \in \Omega$;
- (iv) for every $\rho > 0$, there exists $\gamma_\rho > 0$ such that $f(z,x) \ge \gamma_\rho$ for almost all $z \in \Omega$ and all $x \ge \rho$.

Remark 3.1. Hypothesis (H$_f$)(ii) implies that f is $(p-1)$-superlinear, but need not satisfy the Ambrosetti–Rabinowitz condition.

(H$_g$) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. measurable in $z \in \Omega$ and continuous in $x \in \mathbb{R}$) such that $g(z,0) = 0$ for almost all $z \in \Omega$ and $g(z,x) \ge 0$ for almost all $z \in \Omega$ and all $x > 0$, and

- (i) $|g(z,x)| \le a(z) + c|x|^{r-1}$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$ (a, c, r as in (H$_f$)(i));
- (ii) $\lim_{x\to\infty} g(z,x)/x^{\theta-1} > 0$ uniformly for almost all $z \in \Omega$ and $x \mapsto g(z,x)/x^{p-1}$ is nondecreasing for almost all $z \in \Omega$ (θ as in (H$_f$)(ii));
- (iii) $\lim_{x\to 0^+} g(z,x)/x^{p-1} \le \widehat{\eta}$ uniformly for almost all $z \in \Omega$ ($\widehat{\eta} > 0$);

(H$_0$) for every $\xi > 0$ and every bounded interval $I \subset \mathbb{R}^+$ there exists $\sigma_{\xi,I} > 0$ such that $x \mapsto \lambda f(z,x) - g(z,x) + \sigma_{\xi,I} x^{\theta-1}$ is nondecreasing in $[0,\xi]$ for all $\lambda \in I$ and almost all $z \in \Omega$ (θ as in (H$_f$)(ii)).

Remark 3.2. Since we are interested in *positive* solutions of (P$_\lambda$) and hypotheses (H$_f$)(ii)–(iv) and (H$_g$)(ii), (iii) concern the positive semiaxis $[0,\infty[$, without any loss of generality we may (and will) assume that $f(z,x) = g(z,x) = 0$ for almost all $z \in \Omega$ and all $x \le 0$.

Our hypotheses are modeled on the superdiffusive p-logistic equation, but incorporate this case in a more general setting, as the following examples show:

Example 3.3. The superdiffusive p-logistic equation corresponds to the following choices: set for all $x \geq 0$

$$f(x) = x^{q-1}, \quad g(x) = x^{r-1} \quad \text{with } p < q < r < p^*.$$

These functions satisfy hypotheses (H_f), (H_g) and (H_0) with $\theta = r$.

Example 3.4. Set for all $x \in \mathbb{R}^+$

$$f(x) = x^{q-1}\bigg(\ln(1+x) + \frac{1}{p}\frac{x}{1+x}\bigg), \quad g(x) = x^{r-1} + \eta x^{\tau-1}$$

with $p < q$, $\tau < r < p^*$ and besides $q + 1 < r$ and $\eta > 0$. The above functions satisfy hypotheses (H_f), (H_g) and (H_0) with $\theta = r$.

By a *positive solution* for problem (P_λ) we mean a function $u \in \operatorname{int}(C_+)$ such that

$$\int_\Omega (\|Du\|^{p-2}Du, Dv)\,dz - \lambda\int_\Omega f(z,u)v\,dz + \int_\Omega g(z,u)v\,dz \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Let

$$\mathcal{S} = \{\lambda > 0 : (\mathrm{P}_\lambda) \text{ has a positive solution}\}.$$

In the following propositions, we will investigate some properties of the set $\mathcal{S}$.

Proposition 3.5. *If hypotheses* (H_f) *and* (H_g) *hold, then* $\inf(\mathcal{S}) > 0$ (*we set* $\inf(\emptyset) = \infty$).

Proof. By virtue of hypotheses (H_f)(ii) and (H_g)(ii), we see that we can find $\lambda_0 > 0$ such that

$$\lambda_0 f(z,x) - g(z,x) \leq \widehat{\lambda}_1 x^{p-1} \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq 0. \tag{3.1}$$

We will show that $\lambda_0 \leq \inf(\mathcal{S})$, arguing by contradiction. Let $\lambda \in \,]0, \lambda_0[$ and suppose that (P_λ) has a positive solution $u_\lambda \in \operatorname{int}(C_+)$. Then

$$A(u_\lambda) = \lambda N_f(u_\lambda) - N_g(u_\lambda), \tag{3.2}$$

where A is given by (2.1) and N_f, N_g are the Nemytzki operators defined in Section 2. We act on (3.2) with u_λ and obtain

$$\begin{aligned}
\|Du_\lambda\|_p^p &= \lambda\int_\Omega f(z,u_\lambda)u_\lambda\,dz - \int_\Omega g(z,u_\lambda)u_\lambda\,dz \\
&< \lambda_0\int_\Omega f(z,u_\lambda)u_\lambda\,dz - \int_\Omega g(z,u_\lambda)u_\lambda\,dz \quad (\text{see } (\mathrm{H}_f)) \\
&\leq \widehat{\lambda}_1\|u_\lambda\|_p^p \quad (\text{see } (3.1)).
\end{aligned}$$

Besides, from (2.2) we have

$$\widehat{\lambda}_1 \|u_\lambda\|_p^p \leq \|Du_\lambda\|_p^p,$$

a contradiction. Hence, $\lambda \notin \mathcal{S}$ and we have $\lambda_0 \leq \inf(\mathcal{S})$. □

Proposition 3.6. *If hypotheses* (H_f), (H_g) *and* (H_0) *hold, then* $\mathcal{S} \neq \emptyset$. *Moreover, for all* $\lambda, \mu \in \mathbb{R}^+$, $\lambda \in \mathcal{S}$ *and* $\lambda < \mu$ *imply* $\mu \in \mathcal{S}$.

Proof. We set for all $(z, x) \in \Omega \times \mathbb{R}$

$$F(z,x) = \int_0^x f(z,s)\,ds, \quad G(z,x) = \int_0^x g(z,s)\,ds,$$

and, for a fixed $\lambda > 0$, we define the energy functional $\varphi_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ by putting

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \lambda \int_\Omega F(z,u)\,dz + \int_\Omega G(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Obviously we have $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$. Also, exploiting the compact embedding of $W_0^{1,p}(\Omega)$ into $L^r(\Omega)$, we can easily check that φ_λ is sequentially weakly lower semicontinuous.

From hypotheses (H_g)(i), (ii), we can find $\eta, c_\eta > 0$ such that

$$G(z,x) \geq \eta x^\theta - c_\eta \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq 0. \tag{3.3}$$

Fix $\varepsilon \in \,]0, \eta/\lambda[$. By virtue of hypotheses (H_f)(i), (ii) we can find $c_\varepsilon > 0$ such that

$$F(z,x) \leq \varepsilon x^\theta + c_\varepsilon \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq 0. \tag{3.4}$$

We have for all $u \in W_0^{1,p}(\Omega)$

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{p}\|Du\|_p^p - \lambda \int_\Omega F(z,u^+)\,dz + \int_\Omega G(z,u^+)\,dz \\ &\geq \frac{1}{p}\|Du\|_p^p + (\eta - \lambda\varepsilon)\|u^+\|_\theta^\theta - c_1\end{aligned}$$

for some $c_1 > 0$ (see (3.3) and (3.4)), hence we infer that the functional φ_λ is coercive. So, by the Weierstrass theorem, there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf_{W_0^{1,p}(\Omega)} \varphi_\lambda = m_\lambda. \tag{3.5}$$

We have, for $\lambda > 0$ big enough,

$$m_\lambda < 0. \tag{3.6}$$

Indeed, let $\overline{u} \in \mathrm{int}(C_+)$. We have $F(z, \overline{u}) > 0$ almost everywhere in Ω and so, for $\lambda > 0$ big enough,

$$\varphi_\lambda(\overline{u}) < 0.$$

From (3.5) and (3.6) it follows, for $\lambda > 0$ big enough, $u_\lambda \neq 0$.

Also, from (3.5) we get, for $\lambda > 0$ big enough, $\varphi'_\lambda(u_\lambda) = 0$, that is,

$$A(u_\lambda) = \lambda N_f(u_\lambda) - N_g(u_\lambda). \tag{3.7}$$

We act on (3.7) with $u_\lambda^- \in W_0^{1,p}(\Omega)$, obtaining $\|Du_\lambda^-\|_p = 0$ (recall that $f(z,x) = g(z,x) = 0$ for almost all $z \in \Omega$ and $x \leq 0$). Hence $u_\lambda \geq 0$ in Ω and $u_\lambda \neq 0$. From (3.7) we have

$$-\Delta_p u_\lambda = \lambda f(z,u_\lambda) - g(z,u_\lambda) \quad \text{a.e. in } \Omega \text{ (see [13])}$$

and nonlinear regularity theory implies that $u_\lambda \in C_+ \setminus \{0\}$. Now we apply hypothesis (H_0) with $I = \{\lambda\}$ and $\xi = \|u_\lambda\|_\infty$ and find $\widehat{\sigma} > 0$ such that

$$-\Delta_p u_\lambda + \widehat{\sigma} u_\lambda^{\theta-1} = \lambda f(z,u_\lambda) - g(z,u_\lambda) + \widehat{\sigma} u_\lambda^{\theta-1} \geq 0 \quad \text{a.e. in } \Omega,$$

which, by the results of J. L. Vázquez [16], implies $u_\lambda \in \mathrm{int}(C_+)$. Therefore $\mathcal{S} \neq \emptyset$.

Next, suppose that $\lambda \in \mathcal{S}$ and $\mu > \lambda$: we will show that $\mu \in \mathcal{S}$. Choose $\tau \in \,]0,1[$ such that $\lambda = \tau^{\theta-p}\mu$ (recall that $\theta > p$). Since $\lambda \in \mathcal{S}$, there exists $u_\lambda \in \mathrm{int}(C_+)$ solution of (P_λ). Set $\underline{u} = \tau u_\lambda \in \mathrm{int}(C_+)$. We have

$$-\Delta_p \underline{u} = \tau^{p-1}(-\Delta_p u_\lambda) = \tau^{p-1}[\lambda f(z,u_\lambda) - g(z,u_\lambda)] \quad \text{a.e. in } \Omega. \tag{3.8}$$

Here we invoke our monotonicity assumptions. By hypothesis (H_f)(ii) we have

$$\tau^{p-1}\lambda f(z,u_\lambda) = \mu\tau^{\theta-1} f(z,u_\lambda) \leq \mu f(z,\underline{u}) \quad \text{a.e. in } \Omega. \tag{3.9}$$

Besides, from (H_g)(ii) we have

$$\tau^{p-1} g(z,u_\lambda) \geq g(z,\underline{u}) \quad \text{a.e. in } \Omega. \tag{3.10}$$

Using (3.9) and (3.10) in (3.8), we obtain

$$-\Delta_p \underline{u} \leq \mu f(z,\underline{u}) - g(z,\underline{u}) \quad \text{a.e. in } \Omega. \tag{3.11}$$

We consider problem (P_μ) and we truncate its reaction term as follows:

$$h_\mu(z,x) = \begin{cases} \mu f(z,\underline{u}(z)) - g(z,\underline{u}(z)) & \text{if } x \leq \underline{u}(z), \\ \mu f(z,x) - g(z,x) & \text{if } x > \underline{u}(z). \end{cases}$$

This is a Carathéodory function. We set

$$H_\mu(z,x) = \int_0^x h_\mu(z,s)\,ds \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}$$

and consider the functional $\widehat{\varphi}_\mu \in C^1(W_0^{1,p}(\Omega))$ defined by

$$\widehat{\varphi}_\mu(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega H_\mu(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Recalling (3.3) and (3.4) with $\varepsilon \in]0, \eta/\mu[$, we easily get

$$H_\mu(z,x) \le (\mu\varepsilon - \eta)x^\theta + c_2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0, \text{ for some } c_2 > 0. \tag{3.12}$$

Applying (3.12), we have for all $u \in W_0^{1,p}(\Omega)$

$$\widehat{\varphi}_\mu(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega H_\mu(z,u^+)\,dz \ge \frac{1}{p}\|Du\|_p^p + (\eta - \mu\varepsilon)\|u^+\|_\theta^\theta - c_3$$

(for some $c_3 > 0$). Thus, we may conclude that $\widehat{\varphi}_\mu$ is coercive. Clearly, it is also sequentially weakly lower semicontinuous. So, by virtue of the Weierstrass theorem, there exists $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\mu(u_\mu) = \inf_{W_0^{1,p}(\Omega)} \widehat{\varphi}_\mu,$$

which implies $\widehat{\varphi}'_\mu(u_\mu) = 0$, that is,

$$A(u_\mu) = N_{h_\mu}(u_\mu). \tag{3.13}$$

On (3.13) we act with $(\underline{u} - u_\mu)^+ \in W_0^{1,p}(\Omega)$, obtaining

$$\begin{aligned}\langle A(u_\mu), (\underline{u} - u_\mu)^+\rangle &= \int_\Omega h_\mu(z,u_\mu)(\underline{u} - u_\mu)^+\,dz \\ &= \int_\Omega [\mu f(z,\underline{u}) - g(z,\underline{u})](\underline{u} - u_\mu)^+\,dz \ge \langle A(\underline{u}), (\underline{u} - u_\mu)^+\rangle\end{aligned}$$

(see (3.11)). This implies

$$\langle A(u_\mu) - A(\underline{u}), (\underline{u} - u_\mu)^+\rangle \ge 0. \tag{3.14}$$

On the other hand, from the well-known inequalities

$$(\|x\|^{p-2}x - \|y\|^{p-2}y, x - y) \ge \begin{cases} \gamma\|x-y\|^p & \text{if } p > 2, \\ \gamma\|x-y\|^2(1 + \|x\| + \|y\|)^{p-2} & \text{if } 1 < p < 2, \end{cases}$$

which hold for some $\gamma > 0$ and for all $x, y \in \mathbb{R}^N$, we easily get

$$\begin{aligned}&\langle A(\underline{u}) - A(u_\mu), (\underline{u} - u_\mu)^+\rangle \\ &\qquad = \int_{\{\underline{u} > u_\mu\}} (\|D\underline{u}\|^{p-2}\underline{u} - \|Du_\mu\|^{p-2}Du_\mu, D\underline{u} - Du_\mu)\,dz \ge 0.\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we deduce that $|\{\underline{u} > u_\mu\}|_N = 0$, that is, $u_\mu \ge \underline{u}$ almost every in Ω. Recalling the definition of $h_\mu(z,x)$, (3.13) becomes

$$A(u_\mu) = \mu N_f(u_\mu) - N_g(u_\mu),$$

so $u_\mu \in \operatorname{int}(C_+)$ solves (P_μ) and $\mu \in \mathcal{S}$. $\square$

Set $\lambda_* = \inf(\mathcal{S})$.

Proposition 3.7. *If hypotheses* (H_f), (H_g) *and* (H_0) *hold and* $\lambda > \lambda_*$, *then problem* (P_λ) *has at least two positive solutions.*

Proof. Choose $\mu \in]\lambda_*, \lambda[\cap \mathcal{S}$. Then, there exists $u_\mu \in \mathrm{int}(C_+)$ a positive solution of problem (P_μ), that is,

$$-\Delta_p u_\mu = \mu f(z, u_\mu) - g(z, u_\mu) \quad \text{in } \Omega. \tag{3.16}$$

Arguing as in the proof of Proposition 3.6, we define

$$h_\lambda(z,x) = \begin{cases} \lambda f(z, u_\mu(z)) - g(z, u_\mu(z)) & \text{if } x \le u_\mu(z), \\ \lambda f(z,x) - g(z,x) & \text{if } x > u_\mu(z), \end{cases}$$

the potential function

$$H_\lambda(z,x) = \int_0^x h_\lambda(z,s)\,ds \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}$$

and the truncated functional

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega H_\lambda(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly, $\widehat{\varphi}_\lambda \in C^1(W_0^{1,p}(\Omega))$ and is sequentially weakly lower semicontinuous. Moreover, from (H_f)(ii) and (H_g)(ii), we deduce the existence of $\eta > 0$, $\varepsilon \in]0, \eta/\lambda[$ and $c_4 > 0$ such that

$$H_\lambda(z,x) \le (\varepsilon\lambda - \eta)x^\theta + c_4 \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge 0. \tag{3.17}$$

From (3.17) we get

$$\widehat{\varphi}_\lambda(u) \ge \frac{1}{p}\|Du\|_p^p + (\eta - \varepsilon\lambda)\|u^+\|_\theta^\theta - c_5 \quad \text{for all } u \in W_0^{1,p}(\Omega) \ (c_5 > 0),$$

hence $\widehat{\varphi}_\lambda$ is coercive. By the Weierstrass theorem, there exists $u_\lambda^0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda(u_\lambda^0) = \inf_{W_0^{1,p}(\Omega)} \widehat{\varphi}_\lambda. \tag{3.18}$$

In particular,

$$A(u_\lambda^0) = N_{h_\lambda}(u_\lambda^0). \tag{3.19}$$

Acting on (3.19) with $(u_\mu - u_\lambda^0)^+ \in W_0^{1,p}(\Omega)$, we obtain as above that $u_\lambda^0 \ge u_\mu$ almost everywhere in Ω. Thus, $u_\lambda^0 > 0$ and (3.19) becomes

$$-\Delta_p u_\lambda^0 = \lambda f(z, u_\lambda^0) - g(z, u_\lambda^0) \quad \text{in } \Omega. \tag{3.20}$$

Then, nonlinear regularity theory and the maximum principle of Vázquez imply $u_\lambda^0 \in \mathrm{int}(C_+)$, and so we have obtained a positive solution of (P_λ) (see (3.20)).

Claim. $u_\lambda^0 - u_\mu \in \mathrm{int}(C_+)$.

First, we use hypothesis (H_0): there is $\widehat{\sigma} > 0$ such that, for all $\lambda' \in]\lambda_*, \lambda]$ and almost all $z \in \Omega$, the mapping $x \mapsto \lambda' f(z,x) - g(z,x) + \widehat{\sigma} x^{\theta-1}$ is nondecreasing in $[0, \|u_\lambda^0\|_\infty]$. We have

$$\begin{aligned} -\Delta_p u_\mu + \widehat{\sigma} u_\mu^{\theta-1} &= \mu f(z,u_\mu) - g(z,u_\mu) + \widehat{\sigma} u_\mu^{\theta-1} && \text{(see (3.16))} \\ &< \lambda f(z,u_\mu) - g(z,u_\mu) + \widehat{\sigma} u_\mu^{\theta-1} && \text{(recall } \mu < \lambda \text{ and see } (\mathrm{H}_f)) \\ &\le \lambda f(z,u_\lambda^0) - g(z,u_\lambda^0) + \widehat{\sigma}(u_\lambda^0)^{\theta-1} && \text{(recall } u_\mu \le u_\lambda^0) \\ &= -\Delta_p u_\lambda^0 + \widehat{\sigma}(u_\lambda^0)^{\theta-1} && \text{(see (3.20)).} \end{aligned}$$

We set for all $z \in \Omega$

$$\begin{aligned} \widehat{\beta}_1(z) &= \mu f(z,u_\mu) - g(z,u_\mu) + \widehat{\sigma} u_\mu^{\theta-1}, \\ \overline{\beta}(z) &= \lambda f(z,u_\mu) - g(z,u_\mu) + \widehat{\sigma} u_\mu^{\theta-1}, \\ \widehat{\beta}_2(z) &= \lambda f(z,u_\lambda^0) - g(z,u_\lambda^0) + \widehat{\sigma}(u_\lambda^0)^{\theta-1}. \end{aligned}$$

Summarizing, we have $u_\mu, u_\lambda^0 \in \operatorname{int}(C_+)$, $\widehat{\sigma} > 0$, $\widehat{\beta}_1, \widehat{\beta}_2 \in L^\infty(\Omega)$ and

$$-\Delta u_\mu + \widehat{\sigma} u_\mu^{\theta-1} = \widehat{\beta}_1(z), \quad -\Delta_p u_\lambda^0 + \widehat{\sigma}(u_\lambda^0)^{\theta-1} = \widehat{\beta}_2(z) \quad \text{in } \Omega.$$

Moreover, let $K \subset \Omega$ be a compact set and set $\rho_K = \min_K u_\lambda^0 > 0$. By hypothesis (H_f)(iv), there exists $\gamma_K > 0$ such that

$$f(z,x) \ge \gamma_K \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge \rho_K.$$

Thus, we have

$$\widehat{\beta}_2(z) - \widehat{\beta}_1(z) \ge \overline{\beta}(z) - \widehat{\beta}_1(z) = (\lambda - \mu) f(z,u_\mu) \ge (\lambda - \mu)\gamma_K$$

for almost all $z \in K$, so condition (2.3) is fulfilled. We apply Proposition 2.3 to prove the Claim.

We set

$$V = \{u \in W_0^{1,p}(\Omega) : u(z) \ge u_\mu(z) \text{ for a.a. } z \in \Omega\}.$$

Note that

$$\widehat{\varphi}_\lambda(u) = \varphi_\lambda(u) - \widehat{c} \quad \text{for all } u \in V, \tag{3.21}$$

with

$$\widehat{c} = \int_\Omega [\lambda f(z,u_\mu)u_\mu - g(z,u_\mu)u_\mu]\,dz - \int_\Omega [\lambda F(z,u_\mu) - G(z,u_\mu)]\,dz.$$

There exists $\delta > 0$ such that, for all $h \in C_0^1(\overline{\Omega})$ with $\|h\|_{C_0^1(\overline{\Omega})} < \delta$,

$$(u_\lambda^0 - u_\mu) + h \in C_+ \quad \text{(see the Claim).}$$

By (3.18) and (3.21), u_λ^0 is a $C_0^1(\overline{\Omega})$-local minimizer of φ_λ. We invoke Theorem 1.1 of Garcia Azorero, Manfredi and Peral Alonso [4] to conclude that u_λ^0 is a $W_0^{1,p}(\Omega)$-local minimizer of φ_λ.

By virtue of hypotheses (H_f)(iii) and (H_g)(iii), given $\varepsilon \in]0, \widehat{\lambda}_1/(\lambda+1)[$, we can find $\widehat{\delta} > 0$ such that

$$F(z,x) \le \frac{\varepsilon}{p}|x|^p \quad \text{and} \quad G(z,x) \ge -\frac{\varepsilon}{p}|x|^p \tag{3.22}$$

for almost all $z \in \Omega$ and all $|x| \le \widehat{\delta}$.

Let $u \in C_0^1(\overline{\Omega})$ be such that $\|u\|_{C_0^1(\overline{\Omega})} \le \widehat{\delta}$. Then,

$$\varphi_\lambda(u) \ge \frac{1}{p}\|Du\|_p^p - (\lambda+1)\frac{\varepsilon}{p}\|u\|_p^p \ge \frac{1}{p}\left[1 - \frac{\varepsilon(\lambda+1)}{\widehat{\lambda}_1}\right]\|Du\|_p^p.$$

Thus we have

$$\varphi_\lambda(u) \ge 0 = \varphi_\lambda(0) \quad \text{for all } u \in C_0^1(\overline{\Omega}),\ \|u\|_{C_0^1(\overline{\Omega})} \le \widehat{\delta},$$

that is 0 is a $C_0^1(\overline{\Omega})$-local minimizer of φ_λ, hence a $W_0^{1,p}(\Omega)$-local minimizer of φ_λ (see [4]).

Without any loss of generality, we may assume that $\varphi_\lambda(u_\lambda^0) \ge 0$ (the argument is similar if the reverse inequality holds). Moreover, we assume that u_λ^0 is an isolated critical point of φ_λ (otherwise we have a whole sequence of pairwise distinct positive solutions of (P_λ) and so we are done). Reasoning as in D. Motreanu, V. V. Motreanu and N. S. Papageorgiou [10] (see the proof of Proposition 6), we can find $\rho \in]0, \|u_\lambda^0\|[$ such that

$$\varphi(u_\lambda^0) < \inf_{\partial B_\rho(u_\lambda^0)} \varphi_\lambda = \eta_\rho, \tag{3.23}$$

where $\partial B_\rho(u_\lambda^0) = \{u \in W_0^{1,p}(\Omega) : \|u - u_\lambda^0\| = \rho\}$.

Recall that φ_λ is coercive (see the proof of Proposition 3.6). This implies that φ_λ satisfies (PS). Indeed, if (u_n) is a sequence in $W_0^{1,p}(\Omega)$ such that $(\varphi_\lambda(u_n))$ is bounded in $\mathbb{R}$ and $\varphi_\lambda'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$, coercivity of φ_λ implies that (u_n) is bounded. So, we may assume that there is $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega). \tag{3.24}$$

Set $\varepsilon_n = \|\varphi_\lambda'(u_n)\|_{W^{-1,p'}(\Omega)}$. We have for all $n \in \mathbb{N}$

$$\left|\langle A(u_n), u_n - u\rangle - \lambda\int_\Omega f(z,u_n)(u_n-u)\,dz + \int_\Omega g(z,u_n)(u_n-u)\,dz\right| \le \varepsilon_n\|u_n - u\|,$$

which implies

$$\lim_n \langle A(u_n), u_n - u\rangle = 0 \quad \text{(see (3.24)}$$

which in turn gives $u_n \to u$ in $W_0^{1,p}(\Omega)$ by the $(\mathrm{S})_+$ property of the mapping A (see Proposition 2.2).

This, together with (3.23), allows us to apply Theorem 2.1. So, we find $u_\lambda^1 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda^1) \ge \eta_\rho \quad \text{and} \quad \varphi_\lambda'(u_\lambda^1) = 0. \tag{3.25}$$

From the inequality in (3.25) and (3.23) we see that $u^1_\lambda \notin \{0, u^0_\lambda\}$. Also, from the equality in (3.25) we have

$$A(u^1_\lambda) = \lambda N_f(u^1_\lambda) - N_g(u^1_\lambda).$$

From this, reasoning as before, we deduce that $u^1_\lambda \geq 0$ in Ω and solves (P_λ). Moreover, by nonlinear regularity theory we have $u^1_\lambda \in C_+ \setminus \{0\}$. Finally, using hypothesis $(\mathrm{H})_0$, we can find $\sigma > 0$ such that the mapping $x \mapsto \lambda' f(z,x) - g(z,x) + \sigma x^{\theta-1}$ is nondecreasing in $[0, \|u^1_\lambda\|_\infty]$ for almost all $z \in \Omega$ and all $\lambda' \in \,]\lambda_*, \lambda]$. Then, we easily get

$$\Delta_p u^1_\lambda \leq \sigma (u^1_\lambda)^{\theta-1} \quad \text{in } \Omega.$$

By the results of J. L. Vázquez [16], recalling that $\theta > p$, we finally get $u^1_\lambda \in \operatorname{int}(C_+)$. Thus, (P_λ) admits at least two positive solutions u^0_λ and u^1_λ. □

Next, we show that $\lambda_* \in \mathcal{S}$:

Proposition 3.8. *If hypotheses* (H_f), (H_g) *and* (H_0) *hold, then problem* (P_{λ_*}) *has at least one positive solution.*

Proof. Let (λ_n) be a decreasing sequence in $\mathcal{S}$ such that $\lambda_n \to \lambda_*$. Let $u_n \in \operatorname{int}(C_+)$ be the corresponding positive solutions for problems (P_{λ_n}), for all $n \geq 1$. We have

$$A(u_n) = \lambda_n N_f(u_n) - N_g(u_n) \text{ for all } n \geq 1. \tag{3.26}$$

Hypotheses (H_g)(i), (ii) imply that we can find $\eta > 0$, $c_6 > 0$ such that

$$g(z,x)x \geq \eta x^\theta - c_6 \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq 0. \tag{3.27}$$

Similarly, chosen $\varepsilon \in \,]0, \eta/\lambda_1[$, hypotheses (H_f)(i), (ii) imply that we can find $c_7 > 0$ such that

$$f(z,x)x \leq \eta x^\theta + c_7 \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq 0. \tag{3.28}$$

Acting on (3.26) with u_n and applying (3.27) and (3.28), we have

$$\begin{aligned} \|Du_n\|_p^p &= \lambda_n \int_\Omega f(z,u_n)u_n\,dz - \int_\Omega g(z,u_n)u_n\,dz \\ &\leq (\lambda_n\varepsilon - \eta)\|u_n\|_\theta^\theta + c_8 \quad (c_8 > 0) \\ &\leq c_8 \qquad\qquad\qquad\qquad \text{(recall that } (\lambda_n) \text{ is decreasing).} \end{aligned}$$

Thus, (u_n) is bounded in $W^{1,p}_0(\Omega)$. We may assume that there exists $u_* \in W^{1,p}_0(\Omega)$ such that

$$u_n \rightharpoonup u_* \quad \text{in } W^{1,p}_0(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega). \tag{3.29}$$

Now we act on (3.26) with $u_n - u_* \in W_0^{1,p}(\Omega)$:

$$\begin{aligned}\langle A(u_n), u_n - u_*\rangle \le& (\lambda_n + 1)\int_\Omega [a(z) + cu_n^{r-1}]|u_n - u_*|\,dz \quad \text{(see (H}_f\text{)(i), (H}_g\text{)(i))}\\ \le& (\lambda_1 + 1)[\|a\|_\infty + c\|u_n\|_r^{r-1}]\|u_n - u_*\|_r.\end{aligned}$$

From (3.29) it follows $\limsup_n \langle A(u_n), u_n - u_*\rangle \le 0$, which, by $(\mathrm{S})_+$ property (see Proposition 2.2), implies that $u_n \to u_*$ in $W_0^{1,p}(\Omega)$.

Passing to the limit in (3.26) and using the above convergence, we have

$$A(u_*) = \lambda_* N_f(u_*) - N_g(u_*),$$

hence $u_* \in C_+$ and it solves (P_{λ_*}) (as before, via nonlinear regularity theory).

Claim. $u_* \neq 0$.

We argue by contradiction. So, suppose that $u_* = 0$. Set $y_n = u_n/\|u_n\|$ for all $n \ge 1$. Then (y_n) is a bounded sequence in $W_0^{1,p}(\Omega)$ and we may assume that there exists $y \in W_0^{1,p}(\Omega)$ such that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega). \tag{3.30}$$

From (3.26) we have

$$A(y_n) = \lambda_n \frac{N_f(u_n)}{\|u_n\|^{p-1}} - \frac{N_g(u_n)}{\|u_n\|^{p-1}} \quad \text{for all } n \ge 1. \tag{3.31}$$

Hypotheses (H_f)(i), (iii) imply that, given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$|f(z,x)| \le \varepsilon(x^+)^{p-1} + c_\varepsilon(x^+)^{r-1} \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge 0. \tag{3.32}$$

Similarly, from hypotheses (H_g)(i), (iii) we see that we an find $\eta > \widehat{\eta}$ and $c_9 > 0$ such that

$$|g(z,x)| \le \eta(x^+)^{p-1} + c_9(x^+)^{r-1} \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge 0. \tag{3.33}$$

We return to (3.26). Invoking Theorem 7.1 of O. A. Ladyzhenskaya and N. N. Ural'tseva [8, p. 286] and Theorem 1 of G. M. Lieberman [9] (see also N. S. Papageorgiou and S. Th. Kyritsi [13, p. 311]), we can find $\alpha \in]0,1[$ and $c_{10} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_{10} \quad \text{for all } n \ge 1. \tag{3.34}$$

Since $C_0^{1,\alpha}(\overline{\Omega})$ is compactly embedded into $C_0^1(\overline{\Omega})$, we have

$$u_n \to 0 \quad \text{in } C_0^1(\overline{\Omega}) \text{ (see (3.29) and recall that we are assuming } u_* = 0). \tag{3.35}$$

We have for all $n \ge 1$ and almost all $z \in \Omega$

$$\begin{aligned}\frac{|f(z,u_n)|}{\|u_n\|^{p-1}} &\le \frac{\varepsilon u_n^{p-1} + c_\varepsilon u_n^{r-1}}{\|u_n\|^{p-1}} \quad &&\text{(see (3.32))}\\ &\le c_{11} y_n^{p-1} &&(c_{11} > 0),\end{aligned}$$

hence

$$\int_\Omega \left[\frac{|f(z,u_n)|}{\|u_n\|^{p-1}}\right]^{p'} dz \le c_{11}^{p'} \|y_n\|_p^{p-1} \quad \text{for all } n \ge 1.$$

So, the sequence $(N_f(u_n)/\|u_n\|^{p-1})$ is bounded in $L^{p'}(\Omega)$ (see (3.30)). We may assume that there exists $v \in L^{p'}(\Omega)$ such that

$$N_f(u_n)/\|u_n\|^{p-1} \rightharpoonup v \quad \text{in } L^{p'}(\Omega). \tag{3.36}$$

Reasoning as in D. Motreanu, V. V. Motreanu and N. S. Papageorgiou [10] (see the proof of Proposition 8), we show that $v = 0$.

Indeed, we know from (H_f) that $v \ge 0$ in Ω. Arguing by contradiction, we assume that there exists $\delta > 0$ such that $|E|_N > 0$, where

$$E = \{z \in \Omega : v(z) > \delta\}.$$

We choose $\varepsilon \in \,]0, \delta\widehat{\lambda}_1^{-1/p'} |E|_N^{1/p'}[$ in (3.32). We have $\chi_E \in L^p(\Omega)$, so by (3.36) we get

$$\lim_n \int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}} \chi_E(z)\,dz = \int_\Omega v(z)\chi_E(z)\,dz. \tag{3.37}$$

From the definition of E we get

$$\int_\Omega v(z)\chi_E(z)\,dz \ge \delta |E|_N. \tag{3.38}$$

For all $n \ge 1$ we have

$$\begin{aligned}
\int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}} \chi_E(z)\,dz &= \int_E \frac{f(z,u_n)}{\|u_n\|^{p-1}}\,dz \\
&\le \int_E \frac{\varepsilon u_n^{p-1} + c_\varepsilon u_n^{r-1}}{\|u_n\|^{p-1}}\,dz \qquad \text{(see (3.32))} \\
&\le \varepsilon \widehat{\lambda}_1^{1/p'} |E|_N^{1/p} + c_{12}\|u_n\|^{r-p} \quad (c_{12} > 0),
\end{aligned}$$

which implies

$$\lim_n \int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}} \chi_E(z)\,dz = \varepsilon \widehat{\lambda}_1^{1/p'} |E|_N^{1/p} < \delta |E|_N \quad \text{(see (3.29))}. \tag{3.39}$$

Clearly, (3.37)–(3.39) give a contradiction. So $v = 0$ in Ω.

By an analogous (though a little more sophisticated) argument, we deduce that the sequence $(N_g(u_n)/\|u_n\|^{p-1})$ is bounded in $L^{p'}(\Omega)$, and there exists $\gamma \in L^\infty(\Omega)$ such that

$$\frac{N_g(u_n)}{\|u_n\|^{p-1}} \rightharpoonup \gamma y \quad \text{in } L^{p'}(\Omega) \quad \text{and} \quad 0 \le \gamma(z) \le \widehat{\eta} \quad \text{for a.a. } z \in \Omega. \tag{3.40}$$

Acting on (3.31) with $y_n - y \in W_0^{1,p}(\Omega)$, and passing to the limit as $n \to \infty$, we obtain

$$\limsup_n \langle A(y_n), y_n - y\rangle \le 0 \quad \text{(see (3.36) and (3.40))}.$$

Then,

$$y_n \to y \quad \text{in } W_0^{1,p}(\Omega) \tag{3.41}$$

(see Proposition 2.2). Finally, from (3.31), (3.36) and (3.40) we have

$$A(y) = -\gamma y^{p-1},$$

which in turn implies

$$\|Dy\|_p^p = -\int_\Omega \gamma(z) y^{p-1}\, dz \le 0,$$

that is, $y = 0$, contradicting (3.41) as $\|y_n\| = 1$ for all $n \ge 1$. This proves the Claim.

So $u_* \in C_+ \setminus \{0\}$ and is a solution of (P_{λ_*}). With the aid of hypothesis (H_0) and the nonlinear maximum principle of J. L. Vázquez [16], we have $u_* \in \mathrm{int}(C_+)$, i.e. $\lambda_* \in \mathcal{S}$. □

Finally, summarizing the situation for problem (P_λ), we can state the following bifurcation-type theorem:

THEOREM 3.9. *If hypotheses* (H_f), (H_g) *and* (H_0) *hold, then there exists* $\lambda_* > 0$ *such that*

(a) *for all* $\lambda > \lambda_*$, *problem* (P_λ) *has at least two positive solutions;*
(b) *for* $\lambda = \lambda_*$, *problem* (P_λ) *has at least one positive solution;*
(c) *for all* $\lambda \in \,]0, \lambda_*[$, *problem* (P_λ) *has no positive solutions.*

PROOF. Follows from Propositions 3.5–3.8. □

REMARK 3.10. Theorem 3.9 extends the results of W. Dong, J. T. Chen [3] and S. Takeuchi [14], [15].

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