Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 383–389

FIXED POINTS OF HEMI-CONVEX MULTIFUNCTIONS

Seyed M. A. Aleomraninejad — Shahram Rezapour
Naseer Shahzad

Abstract. The notion of hemi-convex multifunctions is introduced. It is shown that each convex multifunction is hemi-convex, but the converse is not true. Some fixed point results for hemi-convex multifunctions are also proved.

1. Introduction

Throughout this paper we suppose that X and Y are Banach spaces and M is a nonempty convex subset of X. We denote the family of all nonempty subsets of X by 2^X and the family of all nonempty closed and bounded subsets of X by $\mathrm{CB}(X)$. Also, we denote the Hausdorff metric on $\mathrm{CB}(X)$ by H, i.e.

$$H(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\}$$

for all $A,B \in \mathrm{CB}(X)$, where $d(x,A) = \inf_{a\in A}\|x-a\|$.

Let $T\colon X \to 2^Y$ be a multifunction. The graph of T is defined by

$$\mathrm{Gr}(T) = \{(x,y) : x \in X,\ y \in T(x)\}.$$

The multifunction T is called closed (resp. convex) whenever $\mathrm{Gr}\, T$ is closed (resp. convex). Also, T is called upper semi-continuous (resp. lower semi-continuous) whenever $\{x \in X : T(x) \subset A\}$ (resp. $\{x \in X : T(x) \cap A \neq \emptyset\}$) is open for

2010 *Mathematics Subject Classification.* 47H04, 47H10.
Key words and phrases. Convex multifunction, fixed point, hemi-convex multifunction.

all open subsets A of Y. Some authors work on convex multifunctions (see for example; [4]–[6] and [10]), whereas some authors work on nonconvex multifunctions (see for example [2]). In 1980, Yanagi defined the notion of semi-convex multifunctions ([9]). Later on, Bae and Park reviewed some fixed point theorems for multivalued mappings in Banach spaces by using the notion of semi-convex type multifunctions ([3]). The aim of this paper is to give the notion of hemi-convexity of multifunctions which is weaker than convexity of multifunctions. We show that this notion is independent of the notion of semi-convex multifunctions. We also prove some fixed point results for hemi-convex multifunctions.

2. Main results

Definition 2.1. Let M be a convex subset of a Banach space X and $r > 0$. We say that the multifunction $T: M \to 2^M$ is r-hemi-convex whenever

$$d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) \leq r$$

for all $\lambda \in [0,1]$ and $x, y \in M$ with $d(x, T(x)) < r$ and $d(y, T(y)) < r$. We say that T is hemi-convex whenever T is r-hemi-convex for all $r > 0$.

It is clear that each convex multifunction on a Banach space is a hemi-convex multifunction. Now, by providing the following example we show that the converse is not true.

Example 2.2. Define the multifunction $T: \mathbb{R} \to 2^{\mathbb{R}}$ by $T(x) = [2x, 3x]$ if $x \geq 0$ and $T(x) = [3x, 2x]$ if $x < 0$. Then T is not convex whereas T is hemi-convex. In fact, $(1,2), (-1,-3) \in \mathrm{Gr}(T)$, but for $\lambda = 1/2$ we have

$$\lambda(1,2) + (1-\lambda)(-1,-3) \notin \mathrm{Gr}(T).$$

Since $d(x, T(x)) = |x|$ for all $x \in \mathbb{R}$, T is a hemi-convex multifunction.

Let M be a convex subset of a Banach space X. We say that the multifunction $T: M \to \mathrm{CB}(X)$ is semi-convex whenever for each $x, y \in M$, $z = \lambda x + (1-\lambda)y$, where $\lambda \in [0,1]$, and any $x_1 \in T(x)$, $y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$ (see [9]). Now, by providing next examples, we show that the notions semi-convexity and hemi-convexity are independent, although both extend the notion of convexity of multifunctions.

Example 2.3. Define the multifunction $T: \mathbb{R} \to 2^{\mathbb{R}}$ by $T(x) = \{-x+1\}$ if $x \geq 0$ and $T(x) = [x+1, x+2]$ if $x < 0$. Then T is hemi-convex whereas T is not semi-convex.

In fact, let $x = -1$, $y = 1$, $z = x/2 + y/2 = 0$, $x_1 = 0$, $y_1 = 0 \in T(y) = \{0\}$ and $z_1 = 1 \in T(z) = \{1\}$. Then, the relation $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$ does not hold. Hence, T is not semi-convex.

On the other hand, $d(x, T(x)) = |2x-1|$ if $x \geq 0$ and $d(x, T(x)) = 1$ if $x < 0$. Without loss of generality, suppose that $x < y$ and $r > 0$.

If $x, y \geq 0$, $d(x, T(x)) < r$ and $d(y, T(y)) < r$, then $d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) \leq r$.

If $x, y < 0$, then $d(x, T(x)) = 1$, $d(y, T(y)) = 1$ and $d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) = 1$.

If $x < 0$, $y \geq 0$ and $\lambda x + (1-\lambda)y < 0$, then $d(x, T(x)) = 1$ and $d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) = 1$.

If $d(y, T(y)) < r$ and $r \geq 1$, then $d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) \leq \min\{1, r\}$.

If $x < 0$, $y \geq 0$ and $\lambda x + (1-\lambda)y \geq 0$, then $d(x, T(x)) = 1$, $d(y, T(y)) = |2y-1|$ and $-1 \leq 2(\lambda x + (1-\lambda)y) - 1 \leq 2y - 1$. Thus, the relation

$$d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) = |2(\lambda x + (1-\lambda)y) - 1| \leq \max\{1, |2y-1|\}$$

implies that T is hemi-convex.

EXAMPLE 2.4. Define the multifunction $T\colon \mathbb{R} \to 2^{\mathbb{R}}$ by $T(x) = [x, x+1]$ if $x > 0$ and $T(x) = \{\sqrt[3]{x}\}$ if $x \leq 0$. Then T is semi-convex whereas T is not hemi-convex.

In fact, let $x = 1$, $y = -1$ and $z = x/4 + 3y/4 = -1/2$. Then, $d(x, T(x)) = d(y, T(y)) = 0$ while $d(z, T(z)) = d(-1/2, -\sqrt[3]{1/2}) > 0$. Hence, T is not hemi-convex.

Now, without loss of generality suppose that $x < y$.

If $x, y > 0$ or $x, y < 0$ and $z = (\lambda x + (1-\lambda)y)$, it is easy to see that for each $x_1 \in T(x)$ and $y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$.

If $x \leq 0$, $y > 0$ and $z = \lambda x + (1-\lambda)y \leq 0$, then for each $x_1 = \sqrt[3]{x} \in T(x)$ and $y_1 \in T(y)$ we have $\sqrt[3]{x} = x_1 \leq z_1 = \{\sqrt[3]{\lambda x + (1-\lambda)y}\} \leq 0 < y_1$. Hence, $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$.

If $x \leq 0$, $y > 0$ and $z = \lambda x + (1-\lambda)y > 0$, then for each $x_1 = \sqrt[3]{x} \in T(x)$ and $y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\sqrt[3]{x} = x_1 \leq 0 < z_1 \leq y_1$. Hence, $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$. Therefore, T is semi-convex.

THEOREM 2.5. *Let $T, T_n\colon M \to \mathrm{CB}(M)$ be given. If T_n is a hemi-convex multifunction for all $n \geq 1$ and $H(T_n(x), T(x)) \to 0$ for all $x \in M$, then T is a hemi-convex multifunction.*

PROOF. Fix $\varepsilon > 0$, $r > 0$, $0 \leq \lambda \leq 1$ and $x, y \in M$ with $d(x, T(x)) < r$, $d(y, T(y)) < r$. Choose a natural number N such that

$$H(T_n(x), T(x)) < \varepsilon, \quad H(T_n(y), T(y)) < \varepsilon,$$
$$H(T_n(\lambda x + (1-\lambda)y), T(\lambda x + (1-\lambda)y)) < \varepsilon$$

for all $n \geq N$. Then, for each $n \geq N$ we have

$$\begin{aligned} d(x, T_n(x)) &\leq d(x, T(x)) + H(T_n(x), T(x)) < r + \varepsilon \\ d(y, T_n(y)) &\leq d(y, T(y)) + H(T_n(y), T(y)) < r + \varepsilon. \end{aligned}$$

Thus, $d(\lambda x + (1-\lambda)y, T_n(\lambda x + (1-\lambda)y)) \leq r + \varepsilon$. Hence, for each $n \geq N$ we have

$$\begin{aligned} d(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)) &\leq d(\lambda x + (1-\lambda)y, T_n(\lambda x + (1-\lambda)y)) \\ &\quad + H(T_n(\lambda x + (1-\lambda)y), T(\lambda x + (1-\lambda)y)) < r + 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, we obtain $d(\lambda x+(1-\lambda)y, T(\lambda x+(1-\lambda)y)) \leq r$. Therefore, T is a hemi-convex multifunction. □

Theorem 2.6. *Let $T\colon M \to \mathrm{CB}(M)$ be an upper semi-continuous hemi-convex multifunction. Then the set of fixed points of T is convex and closed.*

Proof. Set $F = \{x : x \in T(x)\}$. For each $x, y \in F$ we have $d(x, T(x)) = 0$ and $d(y, T(y)) = 0$. Thus, $d(T(\lambda x + (1-\lambda)y), \lambda x + (1-\lambda)y) = 0$ and so $\lambda x + (1-\lambda)y \in F$ for all $\lambda \in [0,1]$, because T is a closed-valued multifunction. Since T is upper semi-continuous and closed-valued, $\mathrm{Gr}(T)$ is closed.

Let $\{x_n\}_{n\geq 1}$ be a sequence in F with $x_n \to x$. Since $x_n \in T(x_n)$, $(x_n, x_n) \in \mathrm{Gr}(T)$. Hence, $(x, x) \in \mathrm{Gr}(T)$ and so $x \in F$. □

Definition 2.7. Let M be a convex subset of a Banach space X and $r > 0$. We say that the function $f\colon X \to \mathbb{R}$ is r-hemi-convex on M whenever

$$f(\lambda x + (1-\lambda)y) < r$$

for all $\lambda \in [0,1]$ and $x, y \in M$ with $f(x) < r$ and $f(y) < r$. We say that f is hemi-convex on M whenever f is r-hemi-convex on M for all $r > 0$.

Lemma 2.8. *Let M be a convex subset of a Banach space X, $\delta > 0$, $m \geq 2$ and $f\colon X \to \mathbb{R}$ a hemi-convex function on M. If $x_1, \ldots, x_m \in M$ with $f(x_i) < \delta$ for $i = 1, \ldots, m$ and $\lambda_1, \ldots, \lambda_m \in [0, \infty)$ with $\sum_{i=1}^m \lambda_i = 1$, then*

$$f\bigg(\sum_{i=1}^m \lambda_i x_i\bigg) < \delta.$$

Proof. We prove this by induction. For $m = 2$ we have nothing to prove. Suppose that this lemma holds for each $1 \leq k \leq m-1$. We have to prove it for m. Note that, one can assume $\lambda_1 \neq 0$ and so

$$f\bigg(\sum_{i=1}^m \lambda_i x_i\bigg) = f\bigg(\lambda_1 x_1 + \sum_{i=2}^m \lambda_i x_i\bigg) = f\bigg(\lambda_1 x_1 + (1-\lambda_1)\sum_{i=2}^m \frac{\lambda_i}{(1-\lambda_1)} x_i\bigg).$$

Put $y = \sum_{i=2}^m (\lambda_i/(1-\lambda_1)) x_i$. Since $\sum_{i=2}^m \lambda_i/(1-\lambda_1) = 1$, by assumption of the induction, we have $f(y) < \delta$.

Now, by the case of $m = 2$, we obtain

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) = f\left(\lambda_1 x_1 + (1-\lambda_1)\sum_{i=2}^{m} \frac{\lambda_i}{(1-\lambda_1)} x_i\right) = f(\lambda_1 x_1 + (1-\lambda_1)y) < \delta.$$

This completes the proof. $\square$

THEOREM 2.9. *Let M be a weakly compact subset of X, $T\colon M \to \mathrm{CB}(X)$ a multifunction and $\inf_{x\in M} d(x, T(x)) = 0$. If the function $f\colon M \to [0,\infty)$, defined by $f(x) = d(x, T(x))$, is lower semi-continuous and hemi-convex on M, then T has a fixed point in M.*

PROOF. Choose a sequence $\{x_n\}_{n\geq 1}$ in M such that $d(x_n, T(x_n)) \to 0$. Since M is weakly compact, there exists a subsequence $\{z_n\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $z_n \xrightarrow{w} x_0$ for some $x_0 \in M$. Since f is a lower semi-continuous function, for each $\varepsilon > 0$ choose $\delta > 0$ such that $f(x_0) < f(y) + \varepsilon/2$ for all $y \in M$ with $\|y - x_0\| < \delta$ ([7]). Since $f(z_n) \to 0$, there exists a natural number N such that $f(z_n) < \varepsilon/2$ for all $n \geq N$.

We denote again the sequence $\{z_n\}_{n\geq N}$ by $\{z_n\}_{n\geq 1}$. Since $z_n \xrightarrow{w} x_0$, there exist a sequence $\{y_i\}_{i\geq 1}$ in M and a sequence $\{\alpha_{in}\}_{i,n\geq 1}$ in $[0,\infty)$ such that for each i we have $y_i = \sum_{n=1}^{\infty} \alpha_{in} z_n$, where $\sum_{n=1}^{\infty} \alpha_{in} = 1$ and only finitely many $\{\alpha_{in}\}$ are not zero, and $y_i \to x_0$ originally ([8; Theorem 3.13]). But, by Lemma 2.8, we have $f(y_i) < \varepsilon/2$ for all $i \geq 1$. Thus, for sufficiently large i, we obtain

$$f(x_0) < f(y_i) + \frac{\varepsilon}{2} < \varepsilon.$$

Hence, $f(x_0) = 0$ and so $x_0 \in T(x_0)$. $\square$

If $T\colon M \to \mathrm{CB}(M)$ is an upper semi-continuous multifunction, then the function $f(x) = d(x, T(x))$ is lower semi-continuous ([1, Proposition 4.2.6]). Also, note that the function $f(x) = d(x, T(x))$ is hemi-convex whenever T so is. We say that the function $f(x) = d(x, T(x))$ has the property (B) whenever $f(x_n) \to \infty$ for all sequences $\{x_n\}$ with $\|x_n\| \to \infty$. The following example shows that weak compactness of M is a necessary condition in Theorem 2.9.

EXAMPLE 2.10. Consider the multifunction $T\colon (0,\infty) \to 2^{(0,\infty)}$ given by

$$T(x) = \left\{x + \frac{1}{x}\right\}.$$

It is clear that T is a hemi-convex multifunction, $\inf_{x\in(0,\infty)} d(x, T(x)) = 0$ and the function $f(x) = d(x, T(x))$ is lower semi-continuous and hemi-convex. But it is clear that T has no fixed point.

The following example shows that there are many multifunctions which satisfy the condition $\inf_{x\in M} d(x, T(x)) = 0$.

Example 2.11. Let M be a convex and bounded subset of a Banach space X, $u \in M$ a fixed element and $T\colon M \to \mathrm{CB}(M)$ a nonexpansive multifunction. For each $n \geq 2$ define $T_n\colon M \to \mathrm{CB}(M)$ by $T_n(x) = u/n + (1 - 1/n)T(x)$. Since $H(T_n(x), T_n(y)) \leq (1-1/n)\|x-y\|$ for all $x, y \in M$ and $n \geq 2$, T_n is a contraction multifunction and so for each $n \geq 2$ there exists $x_n \in M$ such that $x_n \in T_n(x_n)$. Note that $d(x_n, T(x_n)) \to 0$ and so $\inf_{x\in M} d(x, T(x)) = 0$.

Definition 2.12. Let M be a convex subset of a Banach space X and $T_n, T\colon M \to \mathrm{CB}(M)$ a sequence of multifunctions. We say that $\{T_n\}$ strongly converges to T whenever for each $\varepsilon > 0$ there exists a natural number n_0 such that $H(T_n(x), T(x)) < \varepsilon$ for all $n \geq n_0$ and $x \in M$. In this case, we write $T_n \twoheadrightarrow T$.

Theorem 2.13. *Let M be a weakly compact subset of X, $T\colon M \to \mathrm{CB}(M)$ a multifunction and $T_n\colon M \to \mathrm{CB}(M)$ an upper semi-continuous hemi-convex multifunction for all $n \geq 1$. If each T_n has at least one fixed point in M and $T_n \twoheadrightarrow T$, then T has a fixed point.*

Proof. Since each T_n has at least one fixed point in M, $\inf_{x\in M} d(x, T_n(x)) = 0$ for all $n \geq 1$. Let $\varepsilon > 0$ be given. Choose a natural number n_0 such that $H(T_n(x), T(x)) < \varepsilon$ for all $n \geq n_0$. Since

$$d(x, T(x)) \leq d(x, T_n(x)) + H(T_n(x), T(x)) \leq d(x, T_n(x)) + \varepsilon,$$

for all $n \geq n_0$, $\inf_{x\in M} d(x, T(x)) \leq \varepsilon$. Hence, $\inf_{x\in M} d(x, T(x)) = 0$. By Theorem 2.5, T is hemi-convex and so is the function $f(x) = d(x, T(x))$. Since T is upper semi-continuous, the function $f(x) = d(x, T(x))$ is lower semi-continuous. Now by using Theorem 2.9, T has a fixed point. $\square$

The next example shows that strong convergence of the sequence $\{T_n\}_{n\geq 1}$ is a necessary condition in Theorem 2.13.

Example 2.14. Let $X = \mathbb{R}$ and $M = [0, 2]$. Define $T\colon M \to \mathrm{CB}(M)$ by $T(x) = \{x + 1\}$ if $x < 1$, $T(x) = \{x - 1\}$ if $x > 1$ and $T(x) = \{0, 2\}$ if $x = 1$. Moreover, for each $n \geq 2$, let $T_n\colon M \to \mathrm{CB}(M)$ be defined by $T_n(x) = T(x)$ if $x \neq 1/n$ and $T_n(x) = [0, 2]$ if $x = 1/n$. It is easily seen that T_n is upper semi-continuous, $d(x, T_n(x)) = 1$ if $x \neq 1/n$, $d(x, T_n(x)) = 0$ if $x = 1/n$ and T_n has a fixed point for each $n \geq 2$. This implies that T_n is hemi-convex. Evidently $H(T_n(x), T(x)) \to 0$ for all $x \in M$, but T has no fixed point.

Theorem 2.15. *Let X be an uniformly convex Banach space, $T\colon X \to \mathrm{CB}(X)$ an upper semi-continuous hemi-convex multifunction, $\inf_{x\in M} d(x, T(x)) = 0$. If the function $f(x) = d(x, T(x))$ has the property* (B)*, then T has a fixed point.*

PROOF. Choose a sequence $\{x_n\}$ in X such that $f(x_{n+1}) \leq f(x_n)$ and $f(x_n) \to 0$. Now, for each $n \geq 1$ define $F_n = \{x \in X : f(x) \leq f(x_n)\}$. Since the function $f(x) = d(x, T(x))$ has the property (B), each F_n is a nonempty bounded subset of X. Since T is upper semi-continuous, the function $f(x) = d(x, T(x))$ is lower semi-continuous and so each F_n is a closed subset of X. Also, each F_n is convex because T is a hemi-convex multifunction. Now by using [1, Theorem 2.3.14], there exists $x_0 \in X$ such that $x_0 \in \bigcap_{n=1}^{\infty} F_n$. Thus, $f(x_0) \leq f(x_n)$ for all $n \geq 1$. Hence, $f(x_0) = 0$ and so $x_0 \in T(x_0)$. □

Acknowledgments. The authors express their gratitude to two anonymous referees for their helpful suggestions on a previous version of this paper and especially providing Example 2.14.

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Manuscript received May 7, 2010

SEYED M. A. ALEOMRANINEJAD AND SHAHRAM REZAPOUR
Department of Mathematics
Azarbaidjan University of Tarbiat Moallem
Azarshahr, Tabriz, IRAN

E-mail address: sh.rezapour@azaruniv.edu

NASEER SHAHZAD
Department of Mathematics
King AbdulAziz University
P.O. Box 80203
Jeddah 21859, SAUDI ARABIA

E-mail address: nshahzad@kau.edu.sa