Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 165–176

A MODIFIED SWIFT–HOHENBERG EQUATION

Maria B. Kania

Abstract. We consider the initial-boundary value problem for a modified Swift–Hohenberg equation in space dimension $n \le 7$. Based on the semigroup theory, we formulate this problem as an abstract evolutionary equation with sectorial operator in the main part. We show that the semigroup generated by this problem admits a global attractor in the phase space $H^2(\Omega) \cap H^1_0(\Omega)$ and characterize the contents of the attractor.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of class C^4. In this paper we study the fourth order parabolic equation

$$u_t + (-\Delta)^2 u + \varepsilon\Delta u + \delta^2 u + g(u) = 0, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where parameters ε and δ are positive. This equation is considered with the initial-boundary conditions

$$u(0,x) = u_0(x) \qquad \text{for } x \in \Omega, \tag{1.2}$$

$$u(t,x) = \Delta u(t,x) = 0 \quad \text{for } x \in \partial\Omega,\ t > 0. \tag{1.3}$$

When the parameter $\varepsilon = 2$ and the nonlinear term $g(u)$ takes the form of $u^3 - \alpha u^2 - \beta u + \gamma|\nabla u|^2$, $\alpha, \beta, \gamma \in \mathbb{R}$, then the equation (1.1) can be written as

$$u_t + (I + \Delta)^2 u + u^3 - \alpha u^2 - \kappa u + \gamma|\nabla u|^2 = 0, \tag{1.4}$$

2010 *Mathematics Subject Classification.* 35B41, 35K90.
Key words and phrases. Swift–Hohenberg equation, global attractor.

where $\kappa = \beta - \delta^2 + 1$. The above equation is known in the literature as the Swift–Hohenberg equation when $\alpha = \gamma = 0$, and as the modified Swift–Hohenberg equation when $\alpha \neq 0$ or $\gamma \neq 0$.

The Swift–Hohenberg equation was introduced in 1977 by J. B. Swift and P. C. Hohenberg [18] in connection with Rayleigh–Bénard's convection. Later, it has been shown that this equation is also a useful tool in the studies of a variety of problems, such as the Taylor–Couette flow [8], [15] and in the study of lasers [11]. The Swift–Hohenberg equation plays a central role in studies of pattern formation.

The problem of existence of the global attractor for the Swift–Hohenberg equation has been considered in [12], [13] and for the modified Swift–Hohenberg equation in [9], [14], [16]. In [12] the Swift–Hohenberg equation was equipped with the initial condition (1.2) and the boundary conditions

$$u(t,x) = \frac{\partial}{\partial n} u(t,x) = 0 \quad \text{for } x \in \partial\Omega,\ t > 0,$$

where Ω is a bounded planar domain with the smooth boundary $\partial\Omega$. In [13] A. Mielke and G. Schneider proved the existence of the global attractor for the Swift–Hohenberg equation in a weighted Sobolev space on the whole real line. A. V. Ion in [9] studied two-dimensional modified Swift–Hohenberg equation (1.4) with $\gamma = 0$ both in the case of a bounded and an unbounded domains Ω. M. Polat in [14] showed that the problem (1.2)–(1.4), where Ω is an open connected bounded domain in $\mathbb{R}^2$, $\alpha = 0$ and $u_0 \in H_0^2(\Omega)$ has a global attractor in $H_0^2(\Omega)$. L. Song, Y. Zhang and T. Ma generalized this result in [16]. They proved that for any $k \geq 0$ the Polat's problem has a global attractor in H_k. The fractional order spaces H_k, $k \geq 0$, are defined as follows

$$H_0 := L^2(\Omega), \quad H_{1/2} := H^2(\Omega) \cap H_0^1(\Omega), \quad H_{1/4} := \text{closure of } H_{\frac{1}{2}} \text{ in } H^1(\Omega),$$
$$H_1 := H^4_{\{I,\Delta\}}(\Omega), \quad H_k := H^{4k}(\Omega) \cap H_1 \quad \text{for } k \geq 1.$$

In this paper we study another modification (1.1) of the equation

$$u_t + (I + \Delta)^2 u + u^3 - \kappa u = 0.$$

Notice that instead of the terms $2\Delta u$, u and $(u^3 - \kappa u)$ we consider the terms $\varepsilon \Delta u$, $\delta^2 u$ and $g(u)$, $(\varepsilon, \delta > 0)$, respectively. The first two exchanges imply that the equation (1.1) changes its properties depending on the value of the parameters ε and δ. This equation has 3 dissipative terms $((-\Delta)^2 u$, $\varepsilon\Delta u$, $\delta^2 u)$ and one of them $(\varepsilon \Delta u)$ has a bad sing. Therefore we can expect that the equation (1.1) will have nice properties if the term $\varepsilon \Delta u$ is subordinated to $(-\Delta)^2 u$ and $\delta^2 u$. Our main goal here is to show that if the parameter ε is sufficiently small compare to δ and μ^D (the least positive eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition), then the semigroup generated by the problem (1.1)–(1.3) admits a global attractor $\mathcal{A}$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, we show that $\mathcal{A} = \mathcal{M}(E_0)$, where $\mathcal{M}(E_0)$ is an unstable manifold of the set E_0 of the equilibrium points for the semigroup $\{T(t)\}$.

In this article we assume that $g\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying the following main assumptions:

(1.5) $$g \in C^1(\mathbb{R};\mathbb{R}),$$

(1.6) $$g(0) = 0,$$

(1.7) there exists $c_2 > 0$ such that for all $s_1, s_2 \in \mathbb{R}$

$$|g(s_1) - g(s_2)| \le c_2|s_1 - s_2|(1 + |s_1|^q + |s_2|^q),$$

where $q \ge 0$ can be arbitrarily large if $n \le 2$ and $0 \le q < 4/(n-2)$ if $n \ge 3$,

(1.8) there exist $0 < c_4 < M_1$ and $c_5 > 0$ such that for all $s \in \mathbb{R}$

$$-g(s)s \le c_4 s^2 + c_5,$$

where the constant M_1 is specified in the condition (2.3) below,

(1.9) there exist $c_6 > 0$ and $0 < c_7 < \delta^2$ such that for all $s \in \mathbb{R}$

$$-g'(s) \le c_6 s + c_7,$$

(1.10) there exists $M > 0$ such that for all $s \in \mathbb{R}$

$$-G(s) = -\int_0^s g(z)\,dz \le M.$$

Note that if $\beta < \delta^2$ and α is sufficiently large (i.e. $(c_6 - 2\alpha)^2 \le 12(c_7 - \beta)$), then the function $g(u) = u^3 - \alpha u^2 - \beta u$ satisfies the stated above assumptions for $n \le 3$. When $\beta \ge \delta^2$, regardless of space dimension, the assumption (1.9) is not satisfied. Moreover, the function $g(u) = u^3 - \alpha u^2 - \beta u$ grows too fast, when $n \ge 4$, i.e. the assumption (1.7) is not satisfied.

Notations. The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|$ and the scalar product on this space by $\langle\,\cdot\,,\,\cdot\,\rangle$. We reserve the letter C to denote arbitrary positive constants, which may vary from line to line. $|\Omega|$ denotes the measure of Ω.

We denote by $(-\Delta)$ the negative Laplacian in $L^2(\Omega)$ with the domain

$$D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega).$$

Since $(-\Delta)$ is a self-adjoint and positive definite operator (see [3, p. 41]), we can define for each $\alpha \ge 0$ its fractional powers $(-\Delta)^\alpha$. The domain $D((-\Delta)^\alpha)$ of $(-\Delta)^\alpha$ endowed with the norm

$$\|\phi\|_{D((-\Delta)^\alpha)} = \|(-\Delta)^\alpha \phi)\| \quad \text{for } \phi \in D((-\Delta)^\alpha)$$

is a Hilbert space (see [7, p. 29]). In particular

$$\begin{aligned} D((-\Delta)^{3/2})) &= \{\phi \in H_0^1(\Omega) : (-\Delta)\phi \in H_0^1(\Omega)\} \\ &= \{\phi \in H^3(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\} =: H^3_{\{I,\Delta\}}. \end{aligned}$$

and

$$\begin{aligned} D((-\Delta)^2)) &= \{\phi \in H^2(\Omega) \cap H_0^1(\Omega) : (-\Delta)\phi \in H^2(\Omega) \cap H_0^1(\Omega)\} \\ &= \{\phi \in H^4(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\} =: H^4_{\{I,\Delta\}}. \end{aligned}$$

Moreover, we infer from [5, Theorem 5.1.3] that the operator

$$(-\Delta)^\alpha : D((-\Delta)^\alpha) \to L^2(\Omega)$$

is also positive definite and self-adjoint for each $\alpha > 0$.

2. Operator $A_{\varepsilon\delta}$ and its properties

Let $\varepsilon, \delta > 0$. We denote by $A_{\varepsilon\delta}$ the operator $(-\Delta)^2 + \varepsilon\Delta + \delta^2 I$ in $L^2(\Omega)$ with the domain $D(A_{\varepsilon\delta}) = H^4_{\{I,\Delta\}}$. We will show that $A_{\varepsilon\delta}$ is bounded from below, self-adjoint and has compact resolvent. Let the constant c_1 be such that the interpolation estimate

$$\|\nabla\phi\|^2 \le c_1 \|\Delta\phi\| \|\phi\|, \quad \text{for all } \phi \in H^2(\Omega) \cap H_0^1(\Omega) \tag{2.1}$$

holds. Using the Cauchy inequality we can write it in a more suitable for us form

$$\varepsilon\|\nabla\phi\|^2 \le \|\Delta\phi\|^2 + \left(\frac{c_1\varepsilon}{2}\right)^2 \|\phi\|^2. \tag{2.2}$$

Proposition 2.1. *The operator $A_{\varepsilon\delta}$ is bounded from below. Moreover, if $\varepsilon \in (0, 2\delta/c_1)$, then $A_{\varepsilon\delta}$ is positive definite (the constant c_1 is as above).*

Proof. Integrating by parts we obtain

$$\langle A_{\varepsilon\delta}\phi, \phi\rangle = \|\Delta\phi\|^2 - \varepsilon\|\nabla\phi\|^2 + \delta^2\|\phi\|^2.$$

The estimate (2.2) implies that

$$\langle A_{\varepsilon\delta}\phi, \phi\rangle \ge \left(\delta^2 - \left(\frac{c_1\varepsilon}{2}\right)^2\right)\|\phi\|^2 =: M_1\|\phi\|^2. \tag{2.3}$$

□

Proposition 2.2. *The operator $A_{\varepsilon\delta}$ is self-adjoint in $L^2(\Omega)$.*

Proof. Using the Cauchy inequality and the Nirenberg–Gagliardo inequality:

$$\|\Delta\phi\| \le c\|(-\Delta)^2\phi\|^{1/2}\|\phi\|^{1/2} \quad \text{for all } \phi \in H^4_{\{I,\Delta\}},$$

we obtain

$$\varepsilon\|\Delta\phi\| \le \frac{(c\varepsilon)^2}{2}\|\phi\| + \frac{1}{2}\|(-\Delta)^2\phi\|.$$

By [10, Theorem 4.3] we infer that the operator $(-\Delta)^2 + \varepsilon\Delta$ is self-adjoint, and hence $A_{\varepsilon\delta}$ is self-adjoint as well. $\square$

3. Setting of the problem and its local solvability

Consider the Cauchy problem in Ω for the modified Swift–Hohenberg equation

$$\begin{cases} u_t + (-\Delta)^2 u + \varepsilon\Delta u + \delta^2 u + g(u) = 0 & \text{for } x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \\ u(t,x) = \Delta u(t,x) = 0 & \text{for } x \in \partial\Omega,\ t > 0, \end{cases} \tag{3.1}$$

where $\delta > 0$, $0 < \varepsilon < 2\delta/c_1$ (the constant c_1 was defined in (2.1)), Ω is a nonempty, bounded, open subset of $\mathbb{R}^n$ and $\partial\Omega \in C^4$. In the study of local solvability of (3.1) we need that $g\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying the following assumptions

$$g \in C(\mathbb{R};\mathbb{R}), \tag{3.2}$$

(3.3) there exists $c_2' > 0$ such that for all $s_1, s_2 \in \mathbb{R}$

$$|g(s_1) - g(s_2)| \le c_2'|s_1 - s_2|(1 + |s_1|^{q'} + |s_2|^{q'}),$$

where the exponent $q' \ge 0$ can be arbitrarily large if $n \le 4$ and $0 \le q' \le 4/(n-4)$ if $n > 4$.

Remark 3.1. Note that the conditions (3.2) and (3.3) are weaker than (1.5) and (1.7), respectively.

Remark 3.2. To simplify the presentation we formulate explicitly a direct consequence of the conditions (3.2) and (3.3)

(3.4) there exists $c_3 > 0$ such that for all $s \in \mathbb{R}$

$$|g(s)| \le c_3(1 + |s|^{q'+1}),$$

q' as above.

With the use of the operator $A_{\varepsilon\delta}$ the problem (3.1) on $L^2(\Omega)$ will be rewritten in an abstract way as

$$\begin{cases} u_t + A_{\varepsilon\delta} u = -g(u) & \text{for } t > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{3.5}$$

Note that the Nemytskiĭ operator $g\colon H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ corresponding to the function $g\colon \mathbb{R} \to \mathbb{R}$ (which we denote also by g for simplicity) is well defined. Indeed, for $u \in H^2(\Omega) \cap H^1_0(\Omega)$, thanks to (3.4), we have

$$\|g(u)\| \le C\bigg(\int_\Omega 1 + |u|^{2(q'+1)}\,dx\bigg)^{1/2} \le C(|\Omega|^{1/2} + \|u\|^{q'+1}_{L^{2(q'+1)}(\Omega)}). \tag{3.6}$$

Then as a consequence of the Sobolev type inclusion

$$H^2(\Omega) \subset L^p(\Omega), \tag{3.7}$$

where p is arbitrarily large if $n \le 4$ and $p \le 2n/(n-4)$ if $n > 4$, we obtain

$$\|g(u)\| \le C(1 + \|u\|^{q'+1}_{H^2(\Omega)\cap H^1_0(\Omega)}).$$

Theorem 3.3. *Under the assumptions* (3.2) *and* (3.3) *for each* $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$ *there exists a unique local solution* u *of the problem* (3.5) *in* $L^2(\Omega)$, *defined on its maximal interval of existence* $(0, \tau_{\max})$ *and satisfying*

$$u \in C([0,\tau_{\max}), H^2(\Omega)\cap H^1_0(\Omega)) \cap C^1((0,\tau_{\max}), L^2(\Omega)) \cap C((0,\tau_{\max}), D(A_{\varepsilon\delta})).$$

Proof. Since $A_{\varepsilon\delta}$ is a sectorial operator, it suffices to show (see [3, Chapter 2], [7, Chapter 3]) that the nonlinearity $g\colon H^2(\Omega)\cap H^1_0(\Omega) \to L^2(\Omega)$ is Lipschitz continuous on each bounded subset of $H^2(\Omega)\cap H^1_0(\Omega)$. Fix such a bounded set G and let $u, v \in G$. From the assumption (3.3) we obtain

$$\|g(u) - g(v)\| \le C\bigg(\int_\Omega |u-v|^2(1 + |u|^{2q'} + |v|^{2q'})\,dx\bigg)^{1/2}.$$

If $q' = 0$, then the proof is obvious, so we assume that $q' \neq 0$. Using the Hölder inequality we get

$$\|g(u) - g(v)\| \le C(\|u-v\| + \|u-v\|_{L^{2r/(r-1)}(\Omega)}(\|u\|^{q'}_{L^{2q'r}(\Omega)} + \|v\|^{q'}_{L^{2q'r}(\Omega)})),$$

where $r > \max\{1, 1/(2q')\}$ and $r = n/(q'(n-4))$ for $n > 4$. Then, thanks to (3.7), we deduce that

$$\begin{aligned} \|g(u) - g(v)\| &\le C(\|u-v\|_{H^2(\Omega)\cap H^1_0(\Omega)}(1 + \|u\|^{q'}_{H^2(\Omega)\cap H^1_0(\Omega)} + \|v\|^{q'}_{H^2(\Omega)\cap H^1_0(\Omega)}) \\ &\le C(G)\|u-v\|_{H^2(\Omega)\cap H^1_0(\Omega)}, \end{aligned}$$

which proves the claim. □

4. Global solutions

In this section we study the global solvability of (3.5) in the case of space dimension $n \le 7$. We prove that when the parameter ε is sufficiently small (i.e. the condition (4.2) is satisfied), under the additional growth restrictions on the function g, local solution can be extended to the global ones.

As usual, to show global in time extendibility of the local solution to (3.5) obtained in Theorem 3.3, we need first to get suitable a priori estimates.

First a priori estimate. To get a priori estimate in $L^2(\Omega)$ we assume that the condition (1.8) holds. Multiplying (3.1) by u and integrating by parts we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 - \varepsilon\|\nabla u\|^2 + \delta^2\|u\|^2 + \int_\Omega g(u)u\,dx = 0.$$

Then, thanks to (2.2) and (1.8), we get an estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + M_2\|u\|^2 \le c_5|\Omega|,$$

where $M_2 = (M_1 - c_4) > 0$. Consequently,

$$\|u(t)\|^2 \le \left(\|u_0\|^2 + \frac{c_5|\Omega|}{M_2}\right)e^{-2M_2 t} + \frac{c_5|\Omega|}{M_2}. \tag{4.1}$$

Second a priori estimate. Let the parameter ε be such that

$$0 < \varepsilon < \min\left\{\mu_1^D, \frac{2\delta}{c_1}\right\}, \tag{4.2}$$

where μ_1^D denotes the least positive eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition and the constant c_1 was defined in (2.1). To obtain a priori estimate in $H_0^1(\Omega)$ we need the following extra assumptions on the nonlinear term g:

$$g \in C^1(\mathbb{R};\mathbb{R}), \tag{1.5}$$

$$g(0) = 0, \tag{1.6}$$

(1.9) there exists $c_6 > 0$ and $0 < c_7 < \delta^2$ such that, for all $s \in \mathbb{R}$,

$$-g'(s) \le c_6 s + c_7.$$

Multiplying (3.1) by $-\Delta u$ and integrating by parts, due to (1.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\nabla\Delta u\|^2 - \varepsilon\|\Delta u\|^2 + \delta^2\|\nabla u\|^2 = -\int_\Omega g'(u)|\nabla u|^2\,dx.$$

Thanks to (1.9) and the Hölder inequality we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\nabla\Delta u\|^2 - \varepsilon\|\Delta u\|^2 + (\delta^2 - c_7)\|\nabla u\|^2 &\le c_6\int_\Omega u|\nabla u|^2\,dx \\ &\le C\|u\|\|u\|^2_{W^{1,4}(\Omega)}.\end{aligned}$$

Then using the Nirenberg–Gagliardo inequality

$$\|\phi\|_{W^{1,4}(\Omega)} \le c\|\phi\|^{1-\eta}\|\phi\|^{\eta}_{H^3(\Omega)}$$

with some $\eta \in ((4+n)/12, 1)$, $n < 8$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\nabla\Delta u\|^2 - \varepsilon\|\Delta u\|^2 + (\delta^2 - c_7)\|\nabla u\|^2 \\ &\qquad\le C\|u\|^{3-2\eta}\|u\|^{2\eta}_{H^3(\Omega)} \le C\|u\|^{3-2\eta}\|\nabla\Delta u\|^{2\eta}.\end{aligned}$$

By the Young inequality we get next

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + (1-\zeta)\|\nabla\Delta u\|^2 - \varepsilon\|\Delta u\|^2 + (\delta^2 - c_7)\|\nabla u\|^2 \le C\|u\|^{(3-2\eta)/(1-\eta)},$$

where ζ is sufficiently small, such that $((1-\zeta)\mu_1^D - \varepsilon) > 0$. Using Poicaré's inequality

$$\|\nabla\phi\|^2 \ge \mu_1^D\|\phi\|^2, \quad \text{for all } \phi \in H_0^1(\Omega),$$

we can write

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + ((1-\zeta)\mu_1^D - \varepsilon)\|\Delta u\|^2 + (\delta^2 - c_7)\|\nabla u\|^2 \le C\|u\|^{(3-2\eta)/(1-\eta)}.$$

Finally, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + (\delta^2 - c_7)\|\nabla u\|^2 \le C\|u\|^{(3-2\eta)/(1-\eta)}, \tag{4.3}$$

which gives the required $H_0^1(\Omega)$ estimate.

From now on we assume additionally that the condition (1.7), stronger than (3.3), holds.

Theorem 4.1. *Let $n \le 7$ and the parameter ε be such that the condition* (4.2) *holds. Under the assumptions* (1.5)–(1.9) *the local solution u to* (3.5) *exists globally in time.*

Proof. Note that for every $p \in [1,\infty)$ if $n = 1,2$, and for every $p \in [1,(n+2)/(n-2))$ if $n \ge 3$, we have the following Nirenberg–Gagliardo type inequality

$$\|\phi\|_{L^{2p}(\Omega)} \le C\|\phi\|^{\eta}_{H^2(\Omega)\cap H_0^1(\Omega)}\|\phi\|^{1-\eta}_{H_0^1(\Omega)} \tag{4.4}$$

with some $\eta \in [0, 1/p)$.

Estimating $\|g(u)\|$ as in (3.6) we get

$$\|g(u)\| \le C(1 + \|u\|^{q+1}_{L^{2(q+1)}(\Omega)}).$$

Then, thanks to (4.4), we obtain

$$\begin{aligned} \|g(u)\| &\le C(1 + \|u\|^{\eta(q+1)}_{H^2(\Omega)\cap H_0^1(\Omega)}\|u\|^{(1-\eta)(q+1)}_{H_0^1(\Omega)}) \\ &\le C\max\{1; \|u\|^{(1-\eta)(q+1)}_{H_0^1(\Omega)}\}(1 + \|u\|^{\eta(q+1)}_{H^2(\Omega)\cap H_0^1(\Omega)}) \end{aligned} \tag{4.5}$$

with some $\eta \in [0, 1/(q+1))$. Finally, it follows from [3, Theorem 3.1.1] and the estimates (4.1), (4.3) and (4.5) that any local solution to (3.5) exists globally in time. $\square$

Denote by $\{T(t)\}$ the C^0 semigrup of global solutions to (3.5), which is defined on $H^2(\Omega) \cap H_0^1(\Omega)$ via the relation

$$T(t)u_0 = u(t), \quad t \ge 0.$$

5. Existence and the structure of the global attractor for (3.5)

Following [3, Chapter 4] and [4, Section 1.6] we will study now existence and structure of the global attractor for the semigroup $\{T(t)\}$. We know that the resolvent of $A_{\varepsilon\delta}$ is compact. If we prove that the set of $E_0 := \{v \in H^2(\Omega) \cap H^1_0(\Omega): T(t)v = v \text{ for all } t \geq 0\}$ is bounded in $H^2(\Omega)\cap H^1_0(\Omega)$ and that there exists a "nice" Lyapunov type functional L for $\{T(t)\}$, then $\{T(t)\}$ will have a global attractor $\mathcal{A}$ coinciding with the unstable manifold of E_0 in $H^2(\Omega)\cap H^1_0(\Omega)$ (see [3, Theorem 4.2.3] and [4, Theorem 6.1]). We first show that the set E_0 is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$.

5.1. Stationary solutions of the problem (3.1). We present here some simple estimates of the stationary solutions $v \in H^2(\Omega) \cap H^1_0(\Omega)$ of the problem (3.1). Note that the stationary solution v solves the problem

$$\begin{cases} (-\Delta)^2 v + \varepsilon\Delta v + \delta^2 v + g(v) = 0 & \text{for } x \in \Omega, \\ v(x) = v_0(x) & \text{for } x \in \Omega, \\ v(x) = \Delta v(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{5.1}$$

Multiplying the equation (5.1) first by v then by $-\Delta v$ and estimating as in the first and the second a priori estimates above it is easy to show that

$$\|v\|^2 \leq \frac{c_5|\Omega|}{M_2}, \qquad \|\nabla v\|^2 \leq \frac{C}{(\delta^2 - c_7)} \|v\|^{(3-2\eta)/(1-\eta)},$$

and

$$\|\Delta v\|^2 \leq \frac{C}{((1-\zeta)\mu_1^D - \varepsilon)} \|v\|^{(3-2\eta)/(1-\eta)}.$$

It follows from the above estimates that the set E_0 of stationary solutions is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$.

5.2. Lyapunov functional. In this subsection we discuss properties of a Lyapunov type functional $L: H^2(\Omega)\cap H^1_0(\Omega) \to \mathbb{R}$ (connected with the problem (3.1)) defined by

$$L(u) = \|\Delta u\|^2 - \varepsilon\|\nabla u\|^2 + \delta^2\|u\|^2 + 2\int_\Omega G(u)\,dx,$$

where the function G is a primitive of g and satisfies the assumption (1.10).

Remark 5.1. Notice that as a direct consequence of the condition (1.7) we obtain:

(5.2) there exists $c_8 > 0$ such that, for all $s_1, s_2 \in \mathbb{R}$,

$$|G(s_1) - G(s_2)| \leq c_8|s_1 - s_2|(1 + |s_1|^{q+1} + |s_2|^{q+1}),$$

where $q \geq 0$ can be arbitrarily large if $n \leq 2$ and $q \in [0, 4/(n-2))$ if $n \geq 3$.

Indeed,

$$|G(s_1) - G(s_2)| = |g(\xi)||s_1 - s_2| \le c_3(1 + |\xi|^{q+1})|s_1 - s_2|,$$

but $|\xi| \le \max\{|s_1|; |s_2|\}$, hence

$$|G(s_1) - G(s_2)| \le c_3(1 + |s_1|^{q+1} + |s_2|^{q+1})|s_1 - s_2|.$$

REMARK 5.2. Since $G(0) = 0$, due to (5.2), it is easy to show that

$$|G(s)| \le 2c_8(1 + |s|^{q+2}), \quad s \in \mathbb{R}, \tag{5.3}$$

where q and the constant c_8 are as above.

Note that L is well defined. Indeed, for $u \in H^2(\Omega) \cap H^1_0(\Omega)$, we have

$$\begin{aligned}|L(u)| &\le C\|\Delta u\|^2 + 4c_8 \int_\Omega (1 + |u|^{q+2})\, dx \le C(1 + \|\Delta u\|^2 + \|u\|^{q+2}_{L^{q+2}(\Omega)}) \\ &\le C(1 + \|\Delta u\|^2 + \|u\|^{q+2}_{H^2(\Omega)\cap H^1_0(\Omega)}).\end{aligned}$$

We have the following properties of the functional L:

THEOREM 5.3.

(a) *L is bounded from below.*
(b) *L is continuous on $H^2(\Omega) \cap H^1_0(\Omega)$.*
(c) *For each $u \in H^2(\Omega) \cap H^1_0(\Omega)$ the function $(0,\infty) \ni t \mapsto L(T(t)u) \in \mathbb{R}$ is nonincreasing.*
(d) *If for some $t_0 > 0$ and $v \in H^2(\Omega) \cap H^1_0(\Omega)$ the equation $L(v) = L(T(t_0)v)$ holds, then $v = T(t)v$ for all $t \in [0, t_0]$.*

PROOF. (a) Since $\varepsilon < 2\delta/c_1$, thanks to (2.2) and (1.10), we obtain that L is bounded from below by $-2M|\Omega|$. Indeed,

$$L(u) \ge M_1\|u\|^2 + 2\int_\Omega G(u)\, dx \ge -2M|\Omega|$$

(the constant M_1 was defined in (2.3)).

(b) Let $u, u_n \in H^2(\Omega)\cap H^1_0(\Omega)$, $n \in \mathbb{N}$, be such that $\|u - u_n\|_{H^2(\Omega)\cap H^1_0(\Omega)} \to 0$ as $n \to \infty$. Since

$$\begin{aligned}|L(u_n) - L(u)| &\le \|\Delta u_n - \Delta u\|(\|\Delta u_n\| + \|\Delta u\|) + \int_\Omega |G(u) - G(u_n)|\, dx \\ &\quad + \varepsilon\|\nabla u_n - \nabla u\|(\|\nabla u_n\| + \|\nabla u\|) + \delta^2\|u_n - u\|(\|u_n\| + \|u\|)\end{aligned}$$

it suffices to show that $\int_\Omega |G(u) - G(u_n)|\, dx \to 0$ as $n \to \infty$. From (5.2) we have

$$\int_\Omega |G(u) - G(u_n)|\, dx \le c_8 \int_\Omega |u - u_n|(1 + |u|^{q+1} + |u_n|^{q+1})\, dx.$$

Using the Hölder inequality we get

$$\int_\Omega |G(u) - G(u_n)|\, dx \le C\|u - u_n\|_{L^{r/(r-1)}(\Omega)}(1 + \|u\|^{q+1}_{L^{r(q+1)}(\Omega)} + \|u_n\|^{q+1}_{L^{r(q+1)}(\Omega)}),$$

where $r > 1$ and $r = 2n/((q+1)(n-4))$ for $n > 4$. Thus from (3.7) it follows that

$$\int_\Omega |G(u) - G(u_n)|\,dx \le C\|u - u_n\|_{H^2(\Omega)\cap H^1_0(\Omega)} \times (1 + \|u\|^{q+1}_{H^2(\Omega)\cap H^1_0(\Omega)} + \|u_n\|^{q+1}_{H^2(\Omega)\cap H^1_0(\Omega)}).$$

(c) Multiplying (3.1) by $2u_t$, we obtain

$$2\|u_t\|^2 + \frac{d}{dt}\bigg(\|\Delta u\|^2 - \varepsilon\|\nabla u\|^2 + \delta^2\|u\| + 2\int_\Omega G(u)\,dx\bigg) = 0,$$

hence

$$\frac{d}{dt}L(u(t)) = -2\|u_t\|^2 \le 0.$$

(d) Let $t_0 > 0$ and $v \in H^2(\Omega) \cap H^1_0(\Omega)$ be such that $L(v) = L(T(t_0)v)$. We know that

$$\frac{d}{dt}L(u(t)) = -2\|u_t\|^2, \quad t > 0.$$

Since the expression $L(T(t)v)$ is nonincreasing in time (see (c)) the equality $L(v) = L(T(t_0)v)$ implies that $L(v) = L(T(t_0)v)$ for all $t \in [0, t_0]$, so that

$$\frac{d}{dt}L(u(t)) = 0, \quad t \in (0, t_0].$$

Consequently, $u_t(t, x) = 0$ almost everywhere in Ω for $t \in (0, t_0]$. □

REMARK 5.4. Note that from the condition (d) of Theorem 5.3 it follows that

(d)' If for some $t_0 > 0$ and $v \in H^2(\Omega)\cap H^1_0(\Omega)$ the equation $L(v) = L(T(t_0)v)$ holds, then $v = T(t)v$ for all $t > 0$.

Indeed, let $t_0 > 0$ and $v \in H^2(\Omega) \cap H^1_0(\Omega)$ be such that $L(v) = L(T(t_0)v)$. The condition (d) of Theorem 5.3 implies that

$$v = T(t)v \quad \text{for all } t \in [0, t_0]. \tag{5.4}$$

Since $\{T(t)\}$ is the semigroup, thanks to (5.4), we have

$$L(T(2t_0)v) = L(T(t_0)T(t_0)v) = L(T(t_0)v) = L(v).$$

So that $v = T(t)v$ for all $t \in [0, 2t_0]$. By induction, we obtain that $v = T(t)v$ for $t \ge 0$.

THEOREM 5.5. *Let $n \le 7$ and the parameter ε be such that the condition* (4.2) *holds. Under the assumptions* (1.5)–(1.10) *the semigroup $\{T(t)\}$ has a global attractor $\mathcal{A}$ coinciding with the unstable manifold of the set of stationary solutions E_0 in $H^2(\Omega) \cap H^1_0(\Omega)$.*

The proof is a direct consequence of [3, Theorem 4.2.3] and [4, Theorem 6.1].

References

[1] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, 1992.

[2] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc., Providence, Rhode Island, 2002.

[3] J. W. Cholewa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, 2000.

[4] I. D. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, Acta, Kharkiv, 2002.

[5] R. Czaja, *Differential Equations with Sectorial Operator*, Wydawnictwo Uniwersytetu Śląskiego, Katowice, 2002.

[6] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer–Verlag, Berlin, 1981.

[8] P. C. Hohenberg and J. B. Swift, *Effects of additive noise at the onset of Rayleigh–Bénard's convection*, Phys. Rev. A **46** (1992), 4773–4785.

[9] A. V. Ion, *Atractori Globali Şi Varietăţi Inerţiale Pentru Două Probleme Din Mecanica Fluidelor*, Seria Matematică Aplicată Şi Industrială, Editura Universităţii din Piteşti, 2000.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer–Verlag, Berlin, 1980.

[11] J. Lega, J. V. Moloney and A. C. Newell, *Swift–Hohenberg equation for lasers*, Phys. Rev. Lett. **73** (1994), 2978–2981.

[12] G. Lin, H. Gao, J. Duan and V. J. Ervin, *Asymptotic dynamical difference between the nonlocal and local Swift–Hohenberg models*, J. Math. Phys. **41** (2000), 2077–2089.

[13] A. Mielke and G. Schneider, *Attractors for modulation equations on unbounded domains – existence and comparison*, Nonlinearity **8** (1995), 734–768.

[14] M. Polat, *Global attractor for a modified Swift–Hohenberg equation*, Comput. Math. Appl. **57** (2009), 62–66.

[15] Y. Pomeau and P. Manneville, *Wave length selection in cellular flows*, Phys. Lett. A **75** (1980), 296–298.

[16] L. Song, Y. Zhang and T. Ma, *Global attractor of a modified Swift–Hohenberg equation in H^k spaces*, Nonlinear Anal. **72** (2010), 183–191.

[17] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer–Verlag, New York, 1997.

[18] J. B. Swift and P. C. Hohenberg, *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A **15** (1977), 319–328.

Manuscript received February 10, 2010

Maria B. Kania
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice, Poland

E-mail address: mkania@math.us.edu.pl