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On the interplay between Lorentzian Causality and Finsler metrics of Randers type

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Abstract

We obtain some results in both Lorentz and Finsler geometries, by using a correspondence between the conformal structure (Causality) of standard stationary spacetimes on $M = \mathbb{R} \times S$ and Randers metrics on S. In particular:

(1) For stationary spacetimes: we give a simple characterization of when $\mathbb{R} \times S$ is causally continuous or globally hyperbolic (including in the latter case, when S is a Cauchy hypersurface), in terms of an associated Randers metric. Consequences for the computability of Cauchy developments are also derived.

(2) For Finsler geometry: Causality suggests that the role of completeness in many results of Riemannian Geometry (geodesic connectedness by minimizing geodesics, Bonnet-Myers, Synge theorems) is played by the compactness of symmetrized closed balls in Finslerian Geometry. Moreover, under this condition we show that for any Randers metric R there exists another Randers metric $\tilde{R}$ with the same pregeodesics and geodesically complete.

Even more, results on the differentiability of Cauchy horizons in spacetimes yield consequences for the differentiability of the Randers distance to a subset, and vice versa.

1. Introduction

Randers metrics constitute the most typical class of non-reversible Finsler metrics, and the differences between their properties and those of Riemannian metrics become apparent. For example, they include compact Katok manifolds, which admit only finitely many closed geodesics (see [28, 46]); in

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particular, there are Katok metrics on the sphere S^2 with only two distinct closed geodesics, whereas any Riemannian metric on S^2 admits infinitely many (see [1, 22]). There are several ways to express a given Randers metric; for instance, when considered as a Zermelo metric, the characterization of those with constant flag curvature becomes more natural (see [3]). Moreover, there is an interesting relation between standard stationary spacetimes $(\mathbb{R}\times S, g)$ (see Eq. (4.1)) and Randers metrics (see Eq. (4.3)). This was pointed out in [14], where the associated Randers metric is called *Fermat metric*, and was used in this reference and others [4, 12, 15] to prove some properties of geodesics in stationary spacetimes. Such Fermat metrics are referred to as Optical Zermelo-Randers-Finsler metrics in [24], where some interpretations (concerning, for example, the case of constant flag curvature) are provided.

The purpose of the present paper is to develop in full the correspondence between the global conformal properties (Causality) of the stationary metric and the global geometric properties of the associated Fermat/optical metric. This will be useful for both geometries, Lorentzian and Finslerian, and it is summarized now.

1.1. Previous notions on stationary spacetimes

For the convenience of the reader, we recall first some basic elements on stationary spacetimes which are necessary in order to understand the announced correspondence with Randers metrics. Such elements are spread in physics and mathematics literature (see for example [13, 20, 31], [36, pp. 37–38], [45, Ch. 18]) and some of them will be developed in more detail in Section 4 (see also Section 3 for general background on Lorentzian Geometry).

A (normalized, standard) stationary spacetime is a smooth, connected, product manifold $M=\mathbb{R}\times S$ endowed with a Lorentzian metric g (with signature $(-,+,\dots,+)$) which can be written as:

$$g=-dt^2+\pi^*\omega\otimes dt+dt\otimes\pi^*\omega+\pi^*g_0, \tag{1.1}$$

where $\pi:\mathbb{R}\times S\to S$, $t:\mathbb{R}\times S\to\mathbb{R}$ are the natural projections, * denotes pullback and ω and g_0 are resp. a 1-form and a Riemannian metric on S. Here, the vector field K induced from the natural lifting to M of the canonical vector field on $\mathbb{R}$ is a (normalized, standard) timelike Killing vector field.

Let $\mathrm{Stat}(\mathbb{R}\times S)$ be the set of all the standard stationary metrics on $\mathbb{R}\times S$ as in (1.1). Notice that any such metric is determined by the pair (g_0,ω) composed by a Riemannian metric g_0 and a 1-form ω on S; such a pair will be called a *stationary data pair*. Conversely, any stationary data pair (g_0, ω) determines a Lorentzian metric through the expression (1.1) and, so, it defines an element of $\mathrm{Stat}(\mathbb{R} \times S)$. Therefore, $\mathrm{Stat}(\mathbb{R} \times S)$ can be also regarded as the set of all the stationary data pairs on S. We emphasize that two distinct stationary data pairs may yield isometric spacetimes. This is straightforward because an expression such as (1.1) can be derived by making any spacelike section S' to play the role of the initial section S. Nevertheless, we detail this assertion for clarity in future reference. Start with the metric (1.1) obtained for some stationary data pair (g_0, ω). On any embedded hypersurface $i : S' \hookrightarrow M$, one can induce a (possibly signature-changing) metric $g_0' = i^*g$, and S' is called *spacelike* if g_0' is a Riemannian (positive-definite) metric. Moreover, a 1-form ω' can be induced as $\omega'(v) = g(K, v)$ for all v tangent to S'. Assume that S' is spacelike and also a section, that is, S' can be written as the graph $S^f = \{(f(x), x) : x \in S\}$ of some smooth function f on S. The restriction $\pi|_{S^f} : S^f \to S$ is a diffeomorphism, and we write $\tilde{f} = (\pi|_{S^f})^{-1}$, $g_0^f = \tilde{f}^* g_0'$ and $\omega^f = \tilde{f}^*\omega'$. One can check that the 1-forms vary always in the same cohomology class as $\omega - \omega^f = df$, and the metrics satisfy $g_0 + \omega \otimes \omega = g_0^f + \omega^f \otimes \omega^f$ (Prop. 5.9). Now, the metric g^f in $\mathrm{Stat}(\mathbb{R} \times S)$ determined by the stationary data pair (g_0^f, ω^f) is isometric to the original one. In fact, the *change of initial section* $f_M : M \to M$

$$f_M(t, x) = (t + f(x), x) \qquad \forall (t, x) \in \mathbb{R} \times S \tag{1.2}$$

satisfies $f_M^* g = g^f$.

1.2. Stationary-Randers Correspondence (SRC)

Notice that any stationary data pair (g_0, ω) also determines a Finsler metric of Randers type on S, namely:

$$R(v) = \sqrt{h(v, v)} + \omega(v), \qquad \text{for } \ h = g_0 + \omega \otimes \omega, \ v \in TS.$$

Any Randers metric on S can be obtained in such a way. Therefore, the set of all such pairs (g_0, ω) can be also identified with the set $\mathrm{Rand}(S)$ of all the Randers metrics on S, and one has the natural bijective map

$$\mathrm{Stat}(\mathbb{R} \times S) \to \mathrm{Rand}(S), \qquad g \mapsto F_g, \tag{1.3}$$

where F_g is determined by the same stationary data pair (g_0, ω_0) which determines g. We call F_g the *Fermat metric* associated to g.

Now, some first relations between the geometric properties of the spacetime (M, g) and the Randers manifold (S, F_g) appear, for example:

(A1) The future-pointing (respectively, past-pointing) lightlike pregeodesics of (M, g) project onto the pregeodesics (respectively, reverse pregeodesics) of (S, F_g) (Prop. 4.1).

(A2) A slice $S_t = \{t\} \times S$ is a Cauchy hypersurface for (M, g) (and, then, (M, g) is globally hyperbolic) if and only if the corresponding Fermat metric F_g is forward and backward complete (Th. 4.4).

We can go further in this correspondence by noticing the following claim. Assume that two stationary data pairs $(g_0, \omega), (g_0', \omega')$ yield stationary metrics g, g' which are isometric by means of a change of the initial section as in (1.2). *Then, the associated Fermat metrics F_g, $F_{g'}$ must share the geometric properties which correspond to the intrinsic properties of g.* More precisely, define the following relations of equivalence in Rand(S) and Stat($\mathbb{R} \times S$), resp.:

$$\begin{array}{rcll} R \sim R' & \iff & R - R' = df & \text{for some smooth function } f \text{ on } S, \\ g \sim g' & \iff & g' = f_M^* g & \text{for some change of the initial section } f_M \\ & & & \text{as in (1.2)}, \end{array}$$

and let Rand(S)/ $\sim$, Stat($\mathbb{R} \times S$)/ $\sim$ be the corresponding quotient sets, resp. The bijection (1.3) induces a well-defined bijective map between the quotients

$$(\mathrm{Stat}(\mathbb{R} \times S)/\sim) \to (\mathrm{Rand}(S)/\sim)$$

(see Prop. 4.1 and 5.9). Then, the claim above yields, for example (see Theorem 4.3):

(B1) One (and then all) representative of the *class* $[g]$ ($\in \mathrm{Stat}(\mathbb{R} \times S)/\sim$) is globally hyperbolic if and only if one (and then all) representative of $[F_g]$($\in \mathrm{Rand}(S)/\sim$) satisfies that its symmetrized closed balls (i.e. the closed balls defined by the symmetrized distance associated to F_g) are compact. In particular, this happens if the Randers distance is forward or backward complete for some representative $F_g \in [F_g]$.

(B2) One (and then all) representative of $[g]$ is causally simple if and only if one (and then all) representative of $[F_g]$ is convex, i.e. any two points in S can be joined by a geodesic whose length is equal to the distance between the two points. As a consequence, g satisfies that $J^+(p)$ is closed for all p if and only if $J^-(p)$ is closed for all p.

1.3. Applications to standard stationary spacetimes

SRC yields explicit applications for the study of the so-called "causal ladder of spacetimes", and this can be extended to other causal properties. Recall that standard stationary spacetimes are always causally continuous, and the next conditions in the causal ladder are causal simplicity and global hyperbolicity (see Section 3). In general, these conditions may be difficult to check, but items (B1) and (B2) yield a complete characterization for standard stationary spacetimes[1]. We emphasize that, in (B1), global hyperbolicity is characterized even when $S_0 = \{0\} \times S$ is not a Cauchy hypersurface of $\mathbb{R} \times S$ (typically, global hyperbolicity is proved by checking that some candidate hypersurface is Cauchy but, in principle, the only candidate in our case would be S_0 or, equivalently, any slice $S_t = \{t\} \times S, t \in \mathbb{R}$). However, one can check directly when S_0 is a Cauchy hypersurface through the characterization in item (A2). This sharpens the natural rough estimates for Cauchy hypersurfaces in splitting type spacetimes, see [42].

An example of other causal properties which can be studied via Finslerian ones will be developed in Subsection 4.3. Here a simple application for the computation of Cauchy developments (Prop. 4.7), including the problem of the (non-)smoothness of the Cauchy horizon (Th. 4.10), is obtained.

1.4. Applications to Finsler geometry

They appear in several directions. First, the plain applications to Causality work also the other way round; so, results on, say, differentiability of Cauchy horizons, are translated in results on the differentiability of the distance function to a set for a Randers metric (Th. 5.12 and Cor. 5.13).

Causality also suggests the appropriate hypotheses to study the geometry of Randers metrics and, eventually, for any Finsler metric. Indeed, from SRC an analogy between Riemannian and Randers metrics for the problem of convexity becomes clear: *the role of the compactness of the symmetrized closed balls for a Randers metric is similar to the role of metric completeness for a Riemannian metric.* In fact, as globally hyperbolic spacetimes are causally continuous, the items (B1) and (B2) of SRC imply directly that any Randers metric with compact symmetrized closed balls is convex (see Lemma 5.1 for details). Moreover, this property can be also generalized to any Finsler metric, as it is carried out by using variational arguments in Theorem 5.2. Analogously, one can extend the Finslerian theorem of Bonnet-Myers (see [2, Theorems 7.7.1]) as well as other theorems where this is used (for example, Synge or the sphere theorem). Namely, forward or backward completeness can be replaced by the weaker condition of compactness for the symmetrized closed balls.

A different type of applications of SRC appears for those Randers metrics $R(v) = \sqrt{h(v,v)} + \omega(v)$ with equal h but different ω in the same cohomology class. As claimed in Subsection 1.2, all of them are Fermat metrics for

[1] As these properties are conformally invariant, the normalization $g(K,K) = -1$ for g in SRC is just a non-restrictive choice. Through the remainder of the paper, the results will be written without this normalization in order to make the expressions valid directly for all stationary spacetimes (and also trivially for all conformastationary spacetimes).

canonically isometric standard stationary spacetimes and, therefore, they share the elements which correspond to the (intrinsic) geometry of these spacetimes. The first of these elements is the set of pregeodesics (which corresponds to the lightlike pregeodesics of the spacetime, item (A1)). Another shared property is the possible compactness of symmetrized closed balls (as this corresponds to global hyperbolicity, item (B1)) as well as convexity (as this corresponds to causal simplicity, item (B2)).

More subtly, recall that the fact that each slice $S_t = \{t\} \times S$ is a Cauchy hypersurface, is not an intrinsic property for the spacetime (say, it depends on the choice of the initial spacelike hypersurface S_0). In spite of this, such a property was also characterized in terms of the completeness of the corresponding Fermat metric (item (A2)). Therefore, one obtains a surprising consequence for Randers metrics: *the class of a Randers metric R (according to* Subsection 1.2) *contains a forward and backward complete representative (necessarily with the same pregeodesics as R) if and only if its symmetrized closed balls are compact*, see[2] Th. 5.10. We recall that, even though the symmetrized distance d_s appears sometimes in the literature (see for example [39, 40]), there are no similar results to previous ones, as far as we know. A difficulty of d_s is that it is not constructed as the distance associated to a length structure (see Appendix).

The interaction is also symbiotic in other intermediate aspects. For example, under the Finslerian viewpoint, the completeness of the symmetrized distance d_s of the Randers metric $R(v) = \sqrt{h(v,v)}+\omega(v)$ is easily a sufficient (but not necessary) condition for the completeness of the associated Riemannian metric h. This yields that, if a standard stationary spacetime (M, g) is globally hyperbolic, then the Riemannian metric $g_0 + \omega \otimes \omega$ on S, (the "quotient metric by the flow of K", under the normalization $g(K,K) = -1$), which is the Riemannian metric h of the Fermat metric F_g, must be complete (see Eq. (4.1) and Cor. 5.6), and counterexamples to the converse follow easily, Example 5.5.

1.5. Plan of work

Due to the quite interdisciplinary nature of this work, a special effort has been carried out in order to keep it self-contained. The paper is organized as follows. In Section 2 we introduce the basic notions and results in Finsler manifolds necessary for the remainder of the paper. Moreover, we give an

[2]In comparison, notice that a well-known result by Nomizu and Ozeki [34] states that any Riemannian manifold is globally conformal to a complete one –with different pregeodesics, in general. If a Riemannian metric is incomplete then, regarded as a Finsler metric, it contains non-compact (symmetrized) closed balls and, so, no Randers metric in its class will be complete by SRC (B1).

extension of the Hopf-Rinow theorem for Finsler manifolds which considers the symmetrized balls (see Prop. 2.2). In Section 3, we recall the basic notions on Causality in spacetimes, including the causal ladder. In Section 4, the Fermat metric associated to any (non-necessarily normalized) standard stationary spacetime is introduced (see (4.3) and (4.1)). Then, the applications to Causality of standard stationary spacetimes are obtained (Th. 4.3, 4.4). Moreover, in Prop. 4.7 we characterize the future/past Cauchy development of a subset A contained in a Cauchy hypersurface, by using the distance function from A in the Fermat metric. As a consequence, we give a result about the measure of the subset of non-differentiable points in the Cauchy horizons $H^{\pm}(A)$ when A is a domain with enough regular edge (Th. 4.10). In Section 5, we exploit the expression of Randers metrics as Fermat ones in order to obtain the applications to Finsler Geometry. First, we prove the existence and multiplicity of connecting geodesics for general Finsler manifolds (Th. 5.2) taking into account the hypothesis on Randers metrics suggested by Causality (Lemma 5.1). Second, we retrieve a necessary condition for a standard stationary spacetime to be globally hyperbolic (Cor. 5.6), and obtain a result on geodesic completeness for Randers metrics with compact symmetrized closed balls (Th. 5.10). We finish the section by applying some results on differentiability of Cauchy horizons in [16] to the differentiability of the distance function to a subset in a Randers metric (Th. 5.12 and Cor. 5.13). In the Appendix, we discuss the relation between the different metrics which appear in Randers manifolds, introducing the length metric associated to the symmetrized distance.

As possible further developments, notice that most of the results in this paper apply only to Randers metrics. It is natural to wonder which of them can be extended to arbitrary (non-reversible) Finsler manifolds.

2. Finsler metrics

Let M be a C^∞, paracompact, connected manifold of dimension n and $F : TM \to [0, +\infty]$ be a continuous function. We say that (M, F) is a Finsler manifold if

1. F is C^∞ in $TM \setminus 0$, i. e. away from the zero section,
2. F is fiberwise positively homogeneous of degree one, i. e. $F(x, \lambda y) = \lambda F(x, y)$ for every $(x, y) \in TM$ and $\lambda > 0$,
3. F^2 is fiberwise strongly convex, i. e. the matrix

$$g_{ij}(x,y) = \Big[\frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x,y)\Big] \tag{2.1}$$

is positive definite for every $(x, y) \in TM \setminus 0$.

Here we are using the notation of [2], that is, (x, y) denotes the natural coordinates in TM associated to a chart in M. In what follows, v will denote directly a vector in TM and the reference to the base point (which eventually will be denoted with a different letter, $p, q \in M$) will be omitted when there is no possibility of confusion.

Given a Finsler manifold (M, F), it can be proven that F must be in fact positive away from the zero section, and it satisfies the *triangle inequality*, that is,

$$F(v_1 + v_2) \leq F(v_1) + F(v_2), \tag{2.2}$$

where equality holds iff $v_2 = \alpha v_1$ or $v_1 = \alpha v_2$ for some $\alpha \geq 0$, and the *fundamental inequality*,

$$\sum_{ij} g_{ij}(v) w^i v^j \leq F(w) F(v),$$

where $v \neq 0$, and equality holds iff $w = \alpha v$ for some $\alpha \geq 0$.

A remarkable property of Finsler metrics is that they may be non reversible, that is, in general $F(-v) \neq F(v)$. So, one defines the *reverse Finsler metric* $\tilde{F}$ by $\tilde{F}(v) = F(-v)$ for every $v \in TM$, which is again a Finsler metric. The most typical non-reversible examples are Randers metrics. Let (M, h) be a Riemannian manifold and ω be a 1-form on M such that $|\omega(v)| < \sqrt{h(v, v)}$ for any $v \in TM$. The Randers metric R on (M, h) associated to ω is then

$$R(v) = \sqrt{h(v, v)} + \omega(v), \tag{2.3}$$

(see [2, Chapter 11]). The reverse metric $\tilde{R}$ is obtained just replacing ω by $-\omega$.

2.1. Distance function and length

For any Finsler metric, one can define naturally the (Finslerian) distance as

$$\mathrm{d}(p, q) = \inf_{\gamma \in C(p,q)} \ell_F(\gamma), \tag{2.4}$$

where $C(p, q)$ is the set of piecewise smooth curves from p to q and $\ell_F(\gamma)$ is the Finslerian length of $\gamma : [a, b] \to \mathbb{R}$, that is

$$\ell_F(\gamma) = \int_a^b F(\dot{\gamma}(s)) \mathrm{d}s.$$

Because of (2.2), d satisfies the triangle inequality. As all the properties of a distance but symmetry are fulfilled, the pair (M, d) is referred sometimes as a *generalized metric space* (see [2, Section 6.2] or [19] for a detailed study).

When the Finsler metric is non-reversible, d is not symmetric, because the length of a curve γ may not coincide with the length of its reverse curve $\tilde{\gamma}(s) = \gamma(b+a-s) \in M$. This also translates to Cauchy sequences, that is, we say that a sequence $\{x_i\}_{i\in\mathbb{N}}$ in M is *forward* (resp. *backward*) Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $i, j > N$, then $\mathrm{d}(x_i, x_j) < \varepsilon$ whenever $i \leq j$ (resp. $i \geq j$). Moreover, there are two kinds of (open) balls, *forward* balls, that is,

$$B^+(p, r) = \{x \in M : \mathrm{d}(p, x) < r\},$$

where $p \in M$ and $r \geq 0$, and *backward* balls,

$$B^-(p, r) = \{x \in M : \mathrm{d}(x, p) < r\}.$$

As usual, a bar will denote closure, as in the closed balls $\bar{B}^+(p, r)$ or $\bar{B}^-(p, r)$. The topologies generated by the forward and the backward balls agree with the underlying manifold topology (see [2, Section 6.2 C]).

2.2. Geodesics

There are several ways to define geodesics of Finsler manifolds. We can use any of the connections associated to a Finsler manifold, for example: Chern, Cartan, Berwald or Hashiguchi connections (see [2]). Another possibility is to define a geodesic as a smooth critical curve of the length functional (see [2, Prop. 5.1.1]); nevertheless, as in the Riemannian case, such a functional is invariant by reparametrizations of the curves, and its critical points will be called *pregeodesics* here[3]. Geodesics affinely parametrized by arc length are the critical points of the energy functional

$$E_F(\gamma) = \int_a^b F^2(\dot{\gamma}(s))\mathrm{d}s, \tag{2.5}$$

defined in the space of H^1-curves $\gamma : [a, b] \to M$ with fixed endpoints (see for example [14, Prop. 2.3]). Another consequence of non-reversibility is that geodesics are non-reversible, that is, the reverse curve of a geodesic may not be a geodesic. This leads us to define two exponential maps at every point $p \in M$; the first one will be taken as the natural exponential, since it is analogous to the Riemannian exponential for geodesics departing from p, say, $\exp_p(v) = \gamma_v(1)$, where γ_v is the unique geodesic, such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. The second one, which we call *reverse exponential map*, $\tilde{\exp}$, can be defined as the exponential map associated to the reverse Finsler metric $\tilde{F}$ (see [2, Chapter 6]).

[3] We observe that we will use a different convention from [2], where pregeodesics are called geodesics. So, our geodesics are the speed constant geodesics in [2].

2.3. Geodesic completeness and Hopf-Rinow theorem

There are two types of geodesic completeness, *forward* when the domain of the geodesics can be always extended to $(a, +\infty)$, for some $a \in \mathbb{R}$, and *backward*, when it can be extended to $(-\infty, b)$ for some $b \in \mathbb{R}$. In these circumstances, the classical Hopf-Rinow theorem splits into a forward and a backward version (see [2, Th. 6.6.1] or [18]).

Theorem 2.1. *Let (M, F) be a Finsler manifold, then the following properties are equivalent:*

(*a*) *The generalized metric space (M, d) is forward (resp. backward) complete.*

(*b*) *The Finsler manifold (M, F) is forward (resp. backward) geodesically complete.*

(*c*) *At every point $p \in M$, $\exp_p$ (resp. $\tilde{\exp}_p$) is defined on all of T_pM.*

(*d*) *At some point $p \in M$, $\exp_p$ (resp. $\tilde{\exp}_p$) is defined on all of T_pM.*

(*e*) *Heine-Borel property: every closed and forward (resp. backward) bounded subset of (M, d) is compact.*

Moreover, if any of the above conditions holds, (M, F) is convex[4]*, i.e., every pair of points $p, q \in M$ can be joined by a minimizing geodesic from p to q.*

In order to overcome the lack of symmetry of the distance d in (2.4), we can define the symmetrized distance as

$$\mathrm{d}_s(p, q) = \frac{1}{2}(\mathrm{d}(p, q) + \mathrm{d}(q, p)). \tag{2.6}$$

It is easy to see that d_s is a distance in the classical sense, even though it is not constructed as a length metric (see Appendix). We will denote its associated balls as $B_s(x, r)$, for $x \in M$ and $r \geq 0$. We can wonder if a Hopf-Rinow theorem holds also for d_s. As a first answer, we obtain the following result.

Proposition 2.2. *Let (M, F) be a Finsler manifold. The following properties are equivalent:*

(*a*) *Heine-Borel property: the* symmetrized closed balls *$\bar{B}_s(x, r)$ are compact for all $x \in M$ and $r > 0$.*

(*b*) *$\bar{B}^+(x, r) \cap \bar{B}^-(x, r)$ is compact for any $x \in M$ and $r > 0$.*

[4]Notice that there is no "forward and backward" convexity, as the former would be equivalent to the latter.

(c) $\bar{B}^+(x, r_1) \cap \bar{B}^-(y, r_2)$ *is compact for any* $x, y \in M$ *and* $r_1, r_2 > 0$.

Moreover, if any of the above conditions hold, then the metric space (M, d_s) *is complete.*

Proof. $(a) \Rightarrow (b)$. It follows from the obvious inclusion $\bar{B}^+(x,r)\cap\bar{B}^-(x,r)\subset \bar{B}_s(x,r)$, as the latter ball is compact.

$(b) \Rightarrow (c)$. As $\bar{B}^-(y, r_2) \subset \bar{B}^-(x, r_2 + \mathrm{d}(y, x))$, the result follows from $\bar{B}^+(x,r_1)\cap\bar{B}^-(y,r_2)\subset\bar{B}^+(x,r_3)\cap\bar{B}^-(x,r_3)$, where $r_3=\max\{r_1, r_2+\mathrm{d}(y,x)\}$.

$(c) \Rightarrow (a)$. Straightforward from $\bar{B}_s(x,r) \subset \bar{B}^+(x, 2r) \cap \bar{B}^-(x, 2r)$.

Finally, (a) implies that the metric d_s is complete because a d_s-Cauchy sequence is always contained in a ball $\bar{B}_s(x,r)$ for some $x \in M$ and $r>0$. ■

Theorem 2.1 cannot be claimed to prove convexity under the Heine-Borel property in the last proposition, as hypotheses (a)–(c) above are weaker than those in the theorem. However, we will see in Section 5 that these hypotheses do imply convexity (see Th. 5.2). Nevertheless, there is no relation between the hypotheses and completeness. In fact, Example 4.6 shows a Randers metric that satisfies (a)-(c) in Prop. 2.2, but with forward and backward incomplete geodesics. Moreover, in the following example we exhibit a Randers metric with complete symmetrized distance that does not satisfy the equivalent conditions (a) to (c) of Prop. 2.2 (in fact, the manifold is d_s-bounded but non-compact), see also Subsection 5.3.

Example 2.3. Consider $\mathbb{R}^2$ and there two smooth bump functions μ_+, μ_- such that $0 \leq \mu_\pm \leq 1$ and satisfying :

$$\mu_\pm(x,y) \equiv \mu_\pm(x) = \begin{cases} 0 & \text{if } |x \mp 3| \geq 2 \\ 1 & \text{if } |x \mp 3| \leq 1 \end{cases}$$

and the metric $h = \mathrm{d}x^2 + \mathrm{d}y^2$. Now, consider the 1-form ω:

$$\omega = (\mu_+(x) - \mu_-(x))\frac{y^2}{1+y^2}\mathrm{d}y$$

and the corresponding Randers metric: $F(v) = \sqrt{h(v,v)}+\omega(v)$ for $v \in T\mathbb{R}^2$, with associated distance d and symmetrized distance d_s. Finally, consider the strip $-6 \leq x \leq 6$, construct a quotient M by identifying each two $(-6, y), (6, y)$, and regard F as a Randers metric on M.

Easily, the lines $s \mapsto (3, s)$ and $s \mapsto (-3, -s)$ have infinite length for F, whereas the curves $s \mapsto (3, -s)$ and $s \mapsto (-3, s)$ have finite length equal to π. Thus, the distance d, and therefore the symmetrized one d_s, are finite (say, obviously bounded by $12+\pi$). As a consequence, neither d nor d_s can satisfy the property of Heine-Borel (M is non-compact but included in a ball), and d is incomplete. Nevertheless, d_s is complete, as no non-converging sequence $\{p_n\}_n$ can be Cauchy; in fact, either the limsup of $\{d(p_m, p_{m+k})\}_k$ or the one of $\{d(p_{m+k}, p_m)\}_k$ is bounded away from zero.

3. Causality of spacetimes

3.1. Lorentzian manifolds and spacetimes

Our notation and conventions on Causality will be standard, as in [6, 26, 33, 35, 41]. So, for a Lorentzian manifold (M, g), $\dim M \geq 2$, the metric g on M has index $(-, +, \ldots, +)$, and a tangent vector $v \in TM$, is *timelike* when $g(v, v) < 0$, *spacelike* when $g(v, v) > 0$, *lightlike* if $g(v, v) = 0$ but $v \neq 0$ and *causal* if it is timelike or lightlike; following [33] vector 0 will be regarded as non-spacelike and non-causal –even though this is not by any means the unique convention in the literature. At every point $p \in M$ the *causal cone* is the subset of causal vectors in T_pM, which has exactly two connected components. A *time-orientation* is a smooth choice of a causal cone at every point, which will be called the *future* causal cone –in opposition to the non-chosen one or *past* causal cone. A *spacetime* is a connected C^∞ Lorentzian manifold (M, g) endowed with a time-orientation (which is not written explicitly in the notation). The latter can be determined by a timelike vector field T which defines the future orientation and, so, a causal vector $v \in TM$ is future-pointing (resp. past-pointing) if $g(v, T) < 0$ (resp. $g(v, T) > 0$). A piecewise smooth curve $\gamma : [a, b] \to M$ will be said timelike (and analogously spacelike, lightlike, causal, or future/past-pointing) if so is its velocity $\dot\gamma(s)$ at every $s \in [a, b]$. Spacetimes are used in *General Relativity* as models of (regions of the) Universe. The points of M are also called *events* (they represent all possible "here-now") and massive (resp. massless) particles are described by future-pointing timelike (resp. lightlike) curves.

Causality studies the properties associated to the causal cones, as, for example, if two events can be connected by means of a causal curve. As two Lorentzian metrics on the same manifold are (pointwise) conformal iff they have equal causal cones, Causality is essentially the same thing as conformal geometry in Lorentzian Geometry (even though usually the former refers to the global viewpoint of the latter). Given two events p and q in a spacetime, we say that they are chronologically related, and write $p \ll q$ (resp. strictly causally related $p < q$) if there exists a future-pointing timelike (resp. causal) curve γ from p to q; p is causally related to q if either $p < q$ or $p = q$, denoted $p \leq q$. Relations such as $p \leq q \ll r \Rightarrow p \ll r$ are well-known. The *chronological future* (resp. *causal future*) of $p \in M$ is defined as $I^+(p) = \{q \in M : p \ll q\}$ (resp. $J^+(p) = \{q \in M : p \leq q\}$). Analogous notions appear reversing the word "future" by "past" and, so, one writes $I^-(p), J^-(p)$.

3.2. Causal properties of spacetimes

The *causal ladder* groups spacetimes in the following families, ordered by strictly increasingly better causal properties:

$$\begin{aligned}\text{chronological} \Leftarrow \text{causal} \Leftarrow \text{distinguishing} \Leftarrow \text{strongly causal}\\ \Leftarrow \text{stably causal} \Leftarrow \text{causally continuous}\\ \Leftarrow \text{causally simple} \Leftarrow \text{globally hyperbolic}\end{aligned}$$

In the following we will give a brief account of these spacetime classes; for further information see [6, 26, 41], or the survey [33].

A spacetime is *chronological* (resp. *causal*) if $p \notin I^+(p)$ (resp. $p \notin J^+(p)$) for every $p \in M$; this comprises the nonexistence of timelike or causal closed curves. A spacetime is future (resp. past) *distinguishing* if $I^+(p) = I^+(q)$ (resp. $I^-(p) = I^-(q)$) implies $p = q$, and *distinguishing* if it is both future and past distinguishing. It is easy to prove that distinguishing spacetimes are causal. Intuitively, *strong causality* means the in existence of almost closed timelike curves, and this is equivalent to obtaining a basis of the manifold topology with the subsets of the type $I^+(p) \cap I^-(q)$. A spacetime is *stably causal* if it is causal and it remains causal when we open slightly the light cones. This is equivalent (see [9] or [43]) to the existence of a temporal function on (M, g), that is, a smooth function $t : M \to \mathbb{R}$ with a past-pointing timelike gradient (thus, t is also a *time function*, i.e., continuous and strictly increasing on every future-pointing causal curve). A spacetime is said causally continuous when the maps $I^\pm : M \to \mathcal{P}(M)$ are one to one (i.e., the spacetime is distinguishing) and continuous (here $\mathcal{P}(M)$ is the set of parts of M endowed with the topology which admits as a basis the collection $\{\mathcal{O}_K\}_{K \subset M}$, where each open $\mathcal{O}_K$ contains all the subsets of M which do not intersect the compact set $K \subset M$, see [33, Defn. 3.59, Prop. 3.38]). A spacetime is *causally simple* when it is causal and all causal futures and pasts $J^\pm(p)$ are closed for every $p \in M$ [11]. Finally, a spacetime is *globally hyperbolic* when it admits a Cauchy hypersurface, that is, a subset S which meets exactly once every inextensible timelike curve –which can be chosen as a smooth spacelike hypersurface, necessarily crossed once by inextensible causal curves. This is equivalent to be causal with $J^+(p) \cap J^-(q)$ compact for every pair $p, q \in M$ (see [8, 9, 10, 11, 23] or [33, Sections 3.11.2, 3.11.3] for details).

4. Fermat metrics applied to stationary spacetimes

A spacetime (M, g) is called *stationary* if it admits a *stationary vector field*, i.e., a timelike Killing vector field. Let $\mathbb{R} \times S$ be a product manifold with natural projection on the first factor $t : \mathbb{R} \times S \to \mathbb{R}$, called *standard time*, and on the second factor $\pi : \mathbb{R} \times S \to S$. $(\mathbb{R} \times S, g)$ is a *standard stationary* spacetime if there exists a Riemannian metric g_0 on S, a positive function $\beta : S \to \mathbb{R}$ and a smooth 1-form ω on S, such that:

$$g = -(\beta \circ \pi)dt^2 + \pi^*\omega \otimes dt + dt \otimes \pi^*\omega + \pi^* g_0, \tag{4.1}$$

where $\pi^*\omega$ and $\pi^* g_0$ are the pullback of ω and g_0 on $\mathbb{R} \times S$ through π. The (future) time-orientation is determined by the *standard stationary vector field* K obtained as the lift to $\mathbb{R} \times S$ through t of the natural vector field on $\mathbb{R}$.

A stationary spacetime with a prescribed stationary vector field X admits an isometry with a standard stationary spacetime whose differential maps X in K, if and only if the flow of X is complete and M admits a spacelike section i.e., a spacelike hypersurface S which is crossed exactly once by any inextensible integral curve of X. In this case, an isometry with (4.1) is obtained by moving S by means of the flow of X, and all the possible isometries correspond with all the possible spacelike sections. Remarkably, the existence of such a section is determined by the level where the spacetime is positioned in the causal ladder. Concretely, a stationary spacetime is isometric to a standard stationary one if and only if it is distinguishing and admits a complete stationary vector field (see [27] for this result and other details).

4.1. Fermat principle

According to the relativistic Fermat principle, any lightlike pregeodesics is a critical point of the arrival time function corresponding to an *observer* (defined as a future-pointing timelike curve, up to re-parametrization), see [29, 38]. In a standard stationary spacetime $(\mathbb{R} \times S, g)$, this implies that future-pointing lightlike geodesics project onto pregeodesics of a Finsler metric in S. More precisely, when a vertical line $\mathbb{R} \ni s \mapsto (s, x_1) \in \mathbb{R} \times S$ is regarded as an observer, the arrival time $\mathrm{AT}(\gamma)$ of a (future-pointing) lightlike curve $\gamma : [a, b] \to \mathbb{R} \times S$, $\gamma = (t_\gamma, x_\gamma)$ (i.e. $\gamma(s) = (t_\gamma(s), x_\gamma(s))$, $s \in [a, b]$), joining a point $(t_\gamma(a), x_\gamma(a))$ with the vertical line, can be expressed as $\mathrm{AT}(\gamma) = t_\gamma(b)$, where $t_\gamma(b) - t_\gamma(a)$ is equal to:

$$\int_a^b \left(\frac{1}{\beta(x_\gamma)}\omega(\dot{x}_\gamma) + \sqrt{\frac{1}{\beta(x_\gamma)} g_0(\dot{x}_\gamma, \dot{x}_\gamma) + \frac{1}{\beta(x_\gamma)^2}\omega(\dot{x}_\gamma)^2} \right) \mathrm{d}s. \tag{4.2}$$

As a result, future-pointing lightlike geodesics project onto the pregeodesics of the non-reversible Finsler metric of Randers type on S

$$F(v) = \frac{1}{\beta}\,\omega(v) + \sqrt{\frac{1}{\beta}\,g_0(v,v) + \frac{1}{\beta^2}\,\omega(v)^2}, \quad \forall v \in TS \tag{4.3}$$

which we call *Fermat metric*. Prescribing x_γ, the t_γ-component can be recovered as $t_\gamma(s) = t_\gamma(a) + \int_a^s F(\dot{x}_\gamma(\mu))\mathrm{d}\mu$. By a similar reasoning, past-pointing lightlike geodesics project onto pregeodesics of the reverse Finsler metric

$$\tilde{F}(v) = -\frac{1}{\beta}\,\omega(v) + \sqrt{\frac{1}{\beta}g_0(v,v) + \frac{1}{\beta^2}\,\omega(v)^2}.$$

Such Fermat metrics were introduced in [14] to obtain some multiplicity results for lightlike geodesics and timelike geodesics with fixed proper time from an event to a vertical line in globally hyperbolic standard stationary spacetimes as a consequence of multiplicity results for Finsler metrics.

Prescribing a piecewise smooth future-pointing lightlike curve $\gamma : [a,b] \to \mathbb{R}\times S, \gamma = (t_\gamma, x_\gamma)$, its projection on S is a curve with Fermat length $\ell_F(x_\gamma) = t_\gamma(b) - t_\gamma(a)$. Conversely, a piecewise smooth curve $x_\gamma : [a,b] \to S$ can be lifted to a future-pointing lightlike curve $[a,b] \ni s \to \gamma(s) = (t_\gamma(s), x_\gamma(s)) \in \mathbb{R} \times S$ by choosing

$$t_\gamma(s) = \int_a^s F(\dot{x}_\gamma(\mu))\mathrm{d}\mu$$

and therefore, $t_\gamma(b) - t_\gamma(a) = \ell_F(x_\gamma)$. Easily then:

Proposition 4.1. *Let $z_0 = (t_0, x_0)$, $L_{x_1} = \{(t, x_1) : t \in \mathbb{R}\}$ be, respectively, a point and a line in a standard stationary spacetime. Then z_0 can be joined with L_{x_1} by means of a future-pointing (resp. past-pointing) lightlike pregeodesic $t \mapsto \gamma(t) = (t, x_\gamma(t))$ (resp. $t \mapsto \gamma(t) = (2t_0 - t, x_\gamma(t))$) starting at z_0, if and only if x_γ is a unit speed geodesic of the Fermat metric F (resp. $\tilde{F}$) which joins x_0 with x_1. In this case, the interval of time $t_1 - t_0$ (resp. $t_0 - t_1$) such that $\gamma(t_1) \in L_{x_1}$ is equal to the length of the curve $x_\gamma(t), t \in [t_0, t_1]$ (resp. $t \in [t_0, 2t_0 - t_1]$) computed with F (resp. $\tilde{F}$).*

In fact, a (future-pointing) curve parametrized with the standard time $t \mapsto \gamma(t) = (t, x_\gamma(t))$ is lightlike iff x_γ is parametrized with Fermat speed $F(\dot{x}_\gamma) = 1$ and, among these curves, Fermat's principle states that the critical curves of the arrival time (4.2) are lightlike pregeodesics. As the arrival time coincides (up to an initial additive constant) with the Fermat length of x_γ, it follows that x_γ must be a geodesic for the Fermat metric (see also [14, Th. 4.1]).

4.2. Causality via Fermat metrics

The essential point about Fermat metrics is that they contain all the causal information of a standard stationary spacetime. Let $\gamma = (t_\gamma, x_\gamma) : [0,1] \to \mathbb{R} \times S$, be a future-pointing differentiable causal curve, then

$$g_0(\dot{x}_\gamma, \dot{x}_\gamma) + 2\omega(\dot{x}_\gamma)\dot{t}_\gamma - \beta(x_\gamma)\dot{t}_\gamma^2 \leq 0.$$

Analyzing the quadratic equation in $\dot{t}_\gamma$ and using that $\beta > 0$ we deduce that either

$$\dot{t}_\gamma \geq \frac{1}{\beta(x_\gamma)}\,\omega(\dot{x}_\gamma) + \sqrt{\frac{1}{\beta(x_\gamma)}\,\omega(\dot{x}_\gamma, \dot{x}_\gamma) + \frac{1}{\beta(x_\gamma)^2}\,\omega(\dot{x}_\gamma)^2} = F(\dot{x}_\gamma)$$

or

$$\dot{t}_\gamma \leq \frac{1}{\beta(x_\gamma)}\,\omega(\dot{x}_\gamma) - \sqrt{\frac{1}{\beta(x_\gamma)}\,g_0(\dot{x}_\gamma, \dot{x}_\gamma) + \frac{1}{\beta(x_\gamma)^2}\,\omega(\dot{x}_\gamma)^2} = -\tilde{F}(\dot{x}_\gamma). \tag{4.4}$$

As γ has been chosen to be future-pointing, we have

$$g(K, (\dot{t}_\gamma, \dot{x}_\gamma)) = \omega(\dot{x}_\gamma) - \beta(x_\gamma)\dot{t}_\gamma < 0.$$

This implies that (4.4) cannot be satisfied and then $\dot{t}_\gamma \geq F(\dot{x}_\gamma) \geq 0$. Moreover, $\dot{t}_\gamma > 0$ because, otherwise, $\dot{t}_\gamma(s) = 0 = \dot{x}_\gamma(s)$ at some s and γ would not be causal there (in particular, this proves that the standard time $t : \mathbb{R} \times S \to \mathbb{R}$ is a temporal function and the spacetime is stably causal). In [14, Th. 4.2] it is proven that if the Fermat metric is forward or backward complete, then the spacetime is globally hyperbolic and if the slice $S_0 = \{0\} \times S$ is a Cauchy hypersurface then the Fermat metric is forward and backward complete. In the following we will give a complete characterization of the causal properties in terms of the Fermat metric. As a first step, let us obtain a description of the chronological past and future. From now on, the balls $B^+(x,r)$ and $B^-(x,r)$ will correspond to the forward and the backward balls in the generelized metric space (S, F), F defined in (4.3).

Proposition 4.2. *Let* $(\mathbb{R} \times S, g)$ *be a standard stationary spacetime as in* (4.1). *Then*

$$I^+(t_0, x_0) = \bigcup_{s>0} \{t_0 + s\} \times B^+(x_0, s),$$

$$I^-(t_0, x_0) = \bigcup_{s<0} \{t_0 - s\} \times B^-(x_0, s).$$

Proof. (We will consider just the first equality.) Let $(t_1, x_1) \in I^+(t_0, x_0)$. As $I^+(t_0, x_0)$ is open, one finds easily a lightlike piecewise geodesic $\gamma = (t_\gamma, x_\gamma)$ joining (t_0, x_0) and $(t_1 - \varepsilon, x_1)$ for some small $\varepsilon > 0$ such that $(t_1 - \varepsilon, x_1)$

remains in $I^+(t_0,x_0)$ (see for example [21, Prop. 2]). Then x_γ is a curve joining x_0 and x_1 with Fermat length equal to $t_1-t_0-\varepsilon$, so that $x_1\in B^+(x_0,t_1-t_0)$ and $(t_1,x_1)\in\{t_0+(t_1-t_0)\}\times B^+(x_0,t_1-t_0)$.

Conversely, let $x_1\in B^+(x_0,s)$, and take an arc length parametrized curve $x_\gamma:[0,b]\to S$ joining x_0 and x_1 with Fermat length $b<s$. The future-pointing lightlike curve $\gamma(r)=(t_0+r,x_\gamma(r))$ for $r\in[0,b]$ yields $(t_0,x_0)\leq(t_0+b,x_1)$. As, trivially, $(t_0+b,x_1)\ll(t_0+s,x_1)$ we have $(t_0+s,x_1)\in I^+(t_0,x_0)$. ∎

Theorem 4.3. *Let $(\mathbb{R}\times S,g)$ be a standard stationary spacetime. Then $(\mathbb{R}\times S,g)$ is causally continuous and*

- (a) *the following assertions become equivalent:*
  - (i) *$(\mathbb{R}\times S,g)$ is causally simple,*
  - (ii) *$J^+(p)$ is closed for all p,*
  - (iii) *$J^-(p)$ is closed for all p and*
  - (iv) *the associated Finsler manifold (S,F) is convex,*
- (b) *it is globally hyperbolic if and only if the symmetrized closed balls $\bar{B}_s(x,r)$ are compact for every $x\in S$ and $r>0$.*

Proof. The first assertion is known (see [27]) but we can give a simple proof in terms of Fermat metrics. It is enough to prove that $(\mathbb{R}\times S,g)$ is future and past reflecting (see for example [33, Def. 3.59, Lemma 3.46] or [6, Th. 3.25, Prop. 3.2]), and we will focus on the latter, that is, $I^+(p)\supset I^+(q)$ implies $I^-(p)\subset I^-(q)$. Let $p=(t_0,x_0)$ and $q=(t_1,x_1)$. Then $I^+(p)\supset I^+(q)$ implies that $\mathrm{d}(x_0,x_1)\leq t_1-t_0$ by the first equality in Prop. 4.2. Therefore, using the second equality, $I^-(p)\subset I^-(q)$.

For the remainder, put $p=(t_0,x_0)$ and recall that, by using Prop. 4.2 and [35, Lemma 14.6] resp.:

$$\overline{I}^+(t_0,x_0)=\bigcup_{s\geq 0}\{t_0+s\}\times\bar{B}^+(x_0,s)\quad\text{and}\quad J^+(p)\subset\overline{I}^+(p)=\overline{J}^+(p). \tag{4.5}$$

Notice also that the condition of *causality* in the definitions of causal simplicity and global hyperbolicity are automatically satisfied.

For the proof of (a), it is enough to check that (ii) and (iv) are equivalent.

Implication $(ii)\Rightarrow(iv)$. Take any pair of points x_0,x_1 in S. From (4.5),

$$(\mathrm{d}(x_0,x_1),x_1)\in J^+(0,x_0)\setminus I^+(0,x_0),$$

and there exists a future-pointing lightlike geodesic joining the points $(0,x_0)$ and $(\mathrm{d}(x_0,x_1),x_1)$ (see for example [35, Prop. 10.46]). Clearly, the projection in S of this geodesic is the required minimizing Fermat geodesic.

Implication $(ii) \Leftarrow (iv)$. We have to prove that the inclusion in (4.5) is an equality. Any $(t_1, x_1) \in \partial J^+(p)$ satisfies $t_1 = t_0 + \mathrm{d}(x_0, x_1)$. So, take the minimal Fermat geodesic x starting at x_0 and ending at x_1. The associated lightlike geodesic starting at p in $(\mathbb{R}\times S, g)$ connects p and (t_1, x_1) as required.

Proof of (b). $(\Rightarrow)$ Consider the points (r, x) and $(-r, x)$. Global hyperbolicity implies that $J^{\pm}(p)$ is closed; thus, Prop. 4.2 and (4.5) yield:

$$\{0\} \times \left(\bar{B}^+(x, r) \cap \bar{B}^-(x, r)\right) = (\{0\} \times S) \cap J^+(-r, x) \cap J^-(r, x),$$

and the right-hand side is compact, also by global hyperbolicity. Then by Prop. 2.2 we conclude.

$(\Leftarrow)$ Given two points (t_0, x_0) and (t_1, x_1) in $\mathbb{R} \times S$,

$$J^+(t_0, x_0) \cap J^-(t_1, x_1) \subset \bigcup_{s\in[0,t_1-t_0]} \{t_0+s\} \times (\bar{B}^+(x_0, s) \cap \bar{B}^-(x_1, t_1-t_0-s)). \tag{4.6}$$

Moreover, the subset in the right-hand side is compact. Indeed, any sequence $\{(s_k, y_k)\}_k$ in it has $\{s_k\} \subset [0, t_1 - t_0]$ and, thus, $\{s_k\} \to \bar{s}$, up to a subsequence. Moreover, $\{y_k\}_k \subset \bar{B}^+(x_0, t_1 - t_0) \cap \bar{B}^-(x_1, t_1 - t_0)$, which is compact (see part (c) of Prop. 2.2). Thus, again, $\{y_k\} \to \bar{y}$, up to a subsequence, and by the continuity of the distance $(\bar{s}, \bar{y}) \in \{t_0 + \bar{s}\} \times \bar{B}^+(x_0, \bar{s}) \cap \bar{B}^-(x_1, t_1 - t_0 - \bar{s})$, which concludes the compactness. By Eq. (4.6), the closure of $J^+(t_0, x_0) \cap J^-(t_1, x_1)$ is compact. As the spacetime is strongly causal, this implies that $J^+(t_0, x_0) \cap J^-(t_1, x_1)$ is compact (see [6, Lemma 4.29]), and global hyperbolicity follows. ∎

Theorem 4.3 allows us to determine easily examples of standard stationary spacetimes which are not causally simple, even in the static case ($\omega = 0$), extending [44, Rem. 3.2]. Notice also that we have characterized global hyperbolicity, which is a property intrinsic to the spacetime, independently of how it is written as a standard stationary one. Nevertheless, the fact that S is a Cauchy hypersurface (more precisely, a slice $\{t_0\} \times S$, and trivially then any slice, is Cauchy) will depend on the concrete choice, and it is characterized next.

Theorem 4.4. *Let* $(\mathbb{R} \times S, g)$ *be a standard stationary spacetime. A slice* $S_{t_0} = \{t_0\} \times S$, $t_0 \in \mathbb{R}$, *is a Cauchy hypersurface if and only if the Fermat metric* F *on* S *is forward and backward complete.*

Proof. As the slice is spacelike and acausal, it is Cauchy iff any future-pointing inextensible null pregeodesic $\gamma\colon (a, b) \to \mathbb{R} \times S$ meets S_{t_0} once (see [37, Prop. 5.14] or [35, Cor. 14.54]) and, in this case, γ crosses all the slices. As γ can be parametrized with the standard time, $\gamma(t) = (t, x_\gamma(t))$, this curve will cross all the slices iff the inextensible domain (a, b) of its

x_γ-component is equal to $\mathbb{R}$. But the possible x_γ-components are all the (unit speed) Fermat geodesics, so, their domains are equal to $\mathbb{R}$ if and only if (S, F) is forward and backward complete. ■

Remark 4.5. From the proof of Th. 4.4, a more precise result follows:

(S, F) is forward (resp. backward) complete iff any future-pointing (resp. past-pointing) inextensible lightlike geodesic –and, then, also any timelike curve– starting at $S_0 = \{0\} \times S$ crosses all the slices S_t for $t > 0$ (resp. $t < 0$).

Informally, this means that forward/backward completeness is equivalent to the property that the slices behave as Cauchy hypersurfaces for future/past-pointing causal curves. As a consequence, one can construct a Fermat metric, with compact symmetrized closed balls, which is forward and (or) backward incomplete, by taking a globally hyperbolic stationary spacetime and splitting it as a standard one with respect to a non-Cauchy spacelike hypersurface S. The following example illustrates this situation.

Example 4.6. Consider Lorentz-Minkowski spacetime $\mathbb{L}^2$, i.e $\mathbb{R}^2$ endowed with the metric $g = -\mathrm{d}t^2 + \mathrm{d}x^2$ and the spacelike section given by the curve

$$\alpha(\theta) = \begin{cases} (-\cosh\theta + 1, \sinh\theta) & \text{if} \quad \theta \in (-\infty, 0], \\ (\cosh\theta - 1, \sinh\theta) & \text{if} \quad \theta \in [0, +\infty). \end{cases}$$

As emphasized at the beginning of the present section, we can express the Minkowski metric as a standard stationary spacetime by using the "spacelike hypersurface" α (one can easily smooth α in a neighborhood of 0, this will not affect the discussion below). Putting $v_\theta = \alpha'(\theta)$, a new standard stationary splitting of $\mathbb{L}^2$ is determined by $\beta \equiv 1$, $g_0(\mu v_\theta, \mu v_\theta) = \mu^2$ and

$$\omega(\mu v_\theta) = g(\partial_t, \mu v_\theta)/\beta = \begin{cases} \mu \sinh\theta & \text{if} \quad \theta \le 0, \\ -\mu\sinh\theta & \text{if} \quad \theta \ge 0. \end{cases}$$

The associated Fermat metric is

$$F(\mu v_\theta) = \begin{cases} \sqrt{\mu^2(1 + \sinh^2\theta)} + \mu\sinh\theta & \text{if} \quad \theta \in (-\infty, 0], \\ \sqrt{\mu^2(1 + \sinh^2\theta)} - \mu\sinh\theta & \text{if} \quad \theta \in [0, +\infty). \end{cases}$$

The length of $\mathbb{R}$ with this metric is

$$\int_{-\infty}^{+\infty} F(v_\theta)\,\mathrm{d}\theta = \int_{-\infty}^{0} (\cosh\theta + \sinh\theta)\,\mathrm{d}\theta + \int_{0}^{+\infty} (\cosh\theta - \sinh\theta)\,\mathrm{d}\theta = 2$$

and, thus $(\mathbb{R}, F)$ is neither forward nor backward complete, even though its symmetrized closed balls are compact. Obvious modifications in the branches of α yield only forward and backward completeness, as pointed out in Rem. 4.5.

4.3. Cauchy developments

As the final application in this section, we will construct Cauchy developments in terms of the Fermat metric. A subset A of a spacetime M is *achronal* if no $x, y \in A$ satisfies $x \ll y$; in this case, the *future* (*resp. past*) *Cauchy development* of A, denoted by $D^+(A)$ (resp. $D^-(A)$), is the subset of points $p \in M$ such that every past- (resp. future)- inextensible causal curve through p meets A. The union $D(A) = D^+(A) \cup D^-(A)$ is the *Cauchy development* of A. The *future* $H^+(A)$ (*resp. past* $H^-(A)$) *Cauchy horizon* is defined as

$$H^\pm(A) = \{p \in \bar{D}^\pm(A) : I^\pm(p) \cap D^\pm(A) = \emptyset\}.$$

Intuitively, $D(A)$ is the region of M a priori predictable from data in A, and its *horizon* $H(A) = H^+(A) \cup H^-(A)$, the boundary of this region.

Proposition 4.7. *Let* $(\mathbb{R} \times S, g)$ *be a standard stationary spacetime as in* (4.1)*, with* $S_0 = \{0\} \times S$ *a Cauchy hypersurface,* $A \subset S$*, and* $A_{t_0} = \{t_0\} \times A$ *the corresponding* (*necessarily achronal*) *subset of* $\{t_0\} \times S$*. Then*

$$D^+(A_{t_0}) = \{(t, y) : \mathrm{d}(x, y) > t - t_0 \text{ for every } x \notin A \text{ and } t \geq t_0\}, \tag{4.7}$$

$$D^-(A_{t_0}) = \{(t, y) : \mathrm{d}(y, x) > t_0 - t \text{ for every } x \notin A \text{ and } t \leq t_0\}, \tag{4.8}$$

where d *is the distance in* S *associated to the Fermat metric.*

Moreover, the Cauchy horizons can be described as

$$H^+(A_{t_0}) = \{(t, y) : \inf_{x \notin A} \mathrm{d}(x, y) = t - t_0\} \tag{4.9}$$

$$H^-(A_{t_0}) = \{(t, y) : \inf_{x \notin A} \mathrm{d}(y, x) = t_0 - t\}. \tag{4.10}$$

Proof. Clearly, $D^+(A_{t_0})$ is contained in the semi-space $t \geq t_0$. Given a point $(t, y) \in \mathbb{R} \times S$, $t \geq t_0$, every past-inextensible causal curve meets the Cauchy hypersurface $\{t_0\} \times S$ at some point. From the definition, $(t, y) \notin D^+(A_{t_0})$ iff there exists $x \in A^c$ (where A^c is the complementary subset of A in S) such that $(t, y) \in J^+(t_0, x)$. As $J^+(t_0, x)$ is closed, this is equivalent to $d(x, y) \leq t - t_0$ (recall (4.5)), and the conclusion on $D^+(A_{t_0})$ follows.

The characterization of the Cauchy horizon is obtained by taking into account that $\bar{D}^+(A_{t_0}) = \{(t, y) : \inf_{x \notin A} \mathrm{d}(x, y) \geq t - t_0;\ t \geq t_0\}$ and using the property $(s, x) \in I^+(t, y)$ iff $\mathrm{d}(y, x) < s - t$, which follows from Prop. 4.2. Assume that $\inf_{x \notin A} \mathrm{d}(x, y) = t - t_0$, and thus $(t, y) \in \bar{D}^+(A_{t_0})$. Then by the Finslerian Hopf-Rinow theorem this infimum is attained at some $\bar{x} \in \overline{A^c}$, i.e. $\mathrm{d}(\bar{x}, y) = t - t_0$. Let $(s_0, x_0) \in I^+(t, y)$, then $\mathrm{d}(\bar{x}, x_0) \leq \mathrm{d}(\bar{x}, y) + \mathrm{d}(y, x_0) < t - t_0 + s_0 - t = s_0 - t_0$. Moreover, as $\bar{x} \in \overline{A^c}$ there exists $\tilde{x} \in A^c$ such that $\mathrm{d}(\tilde{x}, x_0) < s_0 - t_0$. Therefore, (4.7) implies that $(s_0, x_0) \notin D^+(A_{t_0})$, and the inclusion $\supset$ in (4.9) is proved. The other inclusion follows easily. Indeed, if there is $(t,y) \in H^+(A_{t_0})$ such that $\inf_{x\notin A} \mathrm{d}(x,y) > t - t_0$, then we can choose $\varepsilon > 0$ small enough such that $(t+\varepsilon, y) \in D^+(A_{t_0}) \cap I(t,y)$. The pasts are obtained analogously. ■

Remark 4.8. Such a result can be extended in some different directions:

(A) If $\mathbb{R} \times S$ is globally hyperbolic, then any acausal compact spacelike submanifold A with boundary can be extended to a spacelike Cauchy hypersurface S_A [10]. Thus, $D(A)$ can be computed in terms of the Fermat metric associated to the standard stationary splitting for S_A.

(B) Even if $\mathbb{R} \times S$ is not globally hyperbolic (or S is not Cauchy) Cauchy developments could be studied by using the Cauchy boundary associated to the Finslerian metric (see [19], for properties of this boundary).

Next, the results in [30] on the regularity of the Finslerian distance function from the boundary will be used to obtain some extensions of the results on differentiability of horizons in [7] for the class of standard stationary spacetimes. We begin by describing the central result in [30]. Let (S,F) be a complete Finsler n-manifold, and $\Omega \subset S$ an open connected subset such that its boundary $\partial\Omega$ satisfies the Hölder condition $C^{2,1}_{\mathrm{loc}}$. Let G be the subset of Ω containing the points where the closest point from $\partial\Omega$ is unique. Then $\Sigma = \Omega \setminus G$ is a subset of the set of points where the inner "normal" geodesics from $\partial\Omega$ do not minimize anymore (i.e., the cut locus). Now denote by $\ell(y)$ the length of such a inner normal geodesic from $y \in \partial\Omega$ to the first hit in the cut locus $m(y) \in \Sigma$. In [30, Th. 1.5-Cor. 1.6] the authors proved the following (optimal) result:

Theorem 4.9. *Under the ambient hypotheses above, for any $N > 0$ the function*

$$\partial\Omega \ni y \mapsto \min(N, \ell(y)) \in \mathbb{R}^+$$

is Lipschitz-continuous on any compact subset of $\partial\Omega$. As a consequence

$$\mathfrak{h}^{n-1}(\Sigma \cap B) < +\infty$$

for any bounded subset B, where $\mathfrak{h}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Now, let A be a closed achronal hypersurface with boundary of a spacetime (M,g) of dimension $(n+1)$. It is known that any point p in $H^+(A)$ admits a *generator*, i.e., a lightlike geodesic through p entirely contained in $H^+(A)$ which is either past-inextensible or has a past endpoint in the boundary of A (see for example [26, Prop. 6.5.3]). Let us denote by $N(p)$ the number of generators through $p \in H^+(A) \setminus A$, and $H^+_{\mathrm{mul}}(A)$ the *crease set* [7, 17] i.e., the set of points $p \in H^+(A) \setminus A$ with $N(p) > 1$. It is known that $H^+(A) \setminus A$ is a topological hypersurface which satisfies a Lipschitz condition (see [26, Prop. 6.3.1]); therefore, its non-differentiable points constitute a set of zero $\mathfrak{h}^n$-measure (even though this set may be highly non-negligible [17]). Moreover, the set of points where $H^+(A) \setminus A$ is not differentiable coincides with the crease $H^+_{\rm mul}(A)$ (see [7]). Using Th. 4.9 and Prop. 4.7, the following more accurate estimate on the measure of this set is obtained.

Theorem 4.10. *Let $(\mathbb{R}\times S, g)$ be a standard stationary $(n+1)$-dimensional spacetime with $S_0 = \{0\}\times S$ Cauchy, and let $\Omega \subset S$ be an open connected subset with $C^{2,1}_{\rm loc}$ boundary $\partial\Omega$. Put $A = \bar{\Omega}$, $A_{t_0} = \{t_0\}\times A$, and let $H^+_{\rm mul}(A_{t_0})$ the crease set of $H^+(A_{t_0})$. Then, for any compact (or Fermat bounded) subset $B \subset S$, we have that*

$$\mathfrak{h}^{n-1}((B\times\mathbb{R})\cap H^+_{\rm mul}(A_{t_0})) < +\infty.$$

5. Causality applied to Randers metrics

Consider now any Randers manifold (S, R) as defined in (2.3). In [3], the authors use the expression of a Randers metric as a Zermelo metric in order to classify Randers metrics of constant flag curvature. Here we will study Randers metrics with compact symmetrized closed balls by exploiting their expression as Fermat metrics for a standard stationary spacetime as in (4.1). Concretely:

$$\begin{cases} \beta = 1, \\ g_0 = h - \omega\otimes\omega, \end{cases} \tag{5.1}$$

(see also [12] for a description of the equivalence between Randers, Zermelo and Fermat metrics).

5.1. Geodesic connectedness in Randers metrics

The first consequence, for Randers metrics, of the interplay with Fermat ones is the following.

Lemma 5.1. *Let (M, R) be a (connected) Randers manifold. If its symmetrized closed balls are compact then it is convex.*

Proof. Consider the expression of the Randers metric as a Fermat metric described in (5.1). We know from Th. 4.3 (*b*) that the associated standard stationary spacetime is globally hyperbolic and then causally simple. By part (*a*) of the same theorem, the Fermat metric is convex, *i. e.*, there exists a minimal geodesic between every two points $p, q \in M$. ■

This lemma is interesting because its proof provides a geometrical understanding of the compactness assumption on the simmetrized closed balls and it suggests the optimal Finslerian result. Indeed, Lemma 5.1 can be generalized to every non-reversible Finsler metric by using analytical techniques. The key point is to prove that, if the symmetrized closed balls are compact, the energy functional (2.5) of a Finsler metric satisfies the Palais-Smale condition on the manifold $\Omega_{p,q}$ of H^1-curves joining two given points p and q of M. By using variational arguments, in [14, Th. 3.1], it is proved that if (M, F) is forward or backward complete then the energy E satisfies the Palais-Smale condition on $\Omega_{p,q}$. But the forward or backward completeness is only used to show that, given a Palais-Smale sequence, there exists a uniformly convergent subsequence. To that end, it is enough to prove that the supports of the sequence are contained in a compact subset and, then, to apply the Ascoli-Arzelà theorem. This follows from the Finslerian Hopf-Rinow theorem, using that the forward (or backward) closed balls are compact. But we can show easily that the Palais-Smale sequence is contained in the intersection of two closed balls $\bar{B}^+(p, r_1) \cap \bar{B}^-(q, r_2)$ for some $r_1, r_2 \in \mathbb{R}$. Therefore, by Prop. 2.2, it is enough to assume that the symmetrized closed balls are compact. Once the Palais-Smale condition is satisfied, a standard minimization argument based on the Deformation Lemma (see for instance [32]) applies, giving the existence of a geodesic connecting p and q and with length equal to $d(p, q)$. Moreover, by using Ljusternik-Schnirelmann theory, we obtain also the existence of infinitely many connecting geodesics if M is not contractible. Summing up:

Theorem 5.2. *Any Finsler metric with compact symmetrized closed balls is convex, i.e., for any $p, q \in M$ there exists a geodesic joining p to q with length equal to the distance $d(p, q)$. Moreover, if M is not contractible then infinitely many connecting geodesics with divergent lengths exist.*

Further developments of this result for manifolds with boundary have been obtained in [5].

Remark 5.3. The Finslerian Theorem of Bonnet-Myers (see for example [2, Theorem 7.7]) can be also extended by assuming compactness of symmetrized closed balls rather than forward or backward completeness. The proof of the theorem under this hypothesis can be accomplished by following the same steps as in [2, Theorem 7.7]. As a consequence, this condition can also be considered in Synge's Theorem (see [2, Theorem 8.8.1]) and, then, in theorems which may be formulated by using it implicitly, as the sphere theorem (see Rademacher's version focused on non-reversible Finsler metrics, [40]).

5.2. Completeness of the symmetrized distance

The role of compact symmetrized closed balls in previous results suggests to discuss when a Randers metric has complete symmetrized distance. Along the way, we will obtain some necessary conditions for a standard stationary spacetime to be globally hyperbolic.

Proposition 5.4. *Let* (M, R) *be a Randers metric as in* (2.3) *with complete symmetrized distance. Then the Riemannian manifold* (M, h) *must be complete.*

Proof. It is enough to prove that $\mathrm{d}_s(p, q) \leq \mathrm{d}_h(p, q)$, which can be done as follows. Let ℓ_h, ℓ_R denote, resp., the length measured with h and R. For any smooth curve α joining p and q and its reverse curve $\tilde{\alpha}$, one has $\mathrm{d}_s(p, q) \leq \frac{1}{2}(\ell_R(\alpha) + \ell_R(\tilde{\alpha})) = \ell_h(\alpha)$. As for every $\varepsilon > 0$, α can be chosen such that $\ell_h(\alpha) < \mathrm{d}_h(p, q) + \varepsilon$, the required inequality follows. ∎

Completeness of the Riemannian metric h is only a necessary condition for the completeness of d_s, as the following counterexample shows.

Example 5.5. Consider $\mathbb{R}^2$ with the Euclidean metric $\langle \cdot, \cdot \rangle$ and the sequence of points $\{p_n = (0, n)\}_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, choose a unit-speed injective curve $\gamma_n = (x_n, y_n) : [0, 2] \to \mathbb{R}^2$ from p_n to p_{n+1}, with $\gamma_n|_{(0,2)}$ contained in the set $0 < x$, $n < y < n + 1$. Consider also the y-symmetric curves $\tilde{\gamma}_n = (-x_n, y_n)$. Let $\varepsilon_n > 0$ small enough and $0 < \alpha_n < 1$, close to 1, such that $\varepsilon_n + 1 - \alpha_n < 2^{-n-1}$. Choose functions $\mu_n : [0, 2] \to \mathbb{R}$, $\mu_n = \alpha_n \hat{\mu}_n$, where $\hat{\mu}_n$ is a bump function equal to 1 in a neighborhood of $[\varepsilon_n, 2 - \varepsilon_n]$, and equal to 0 in a neighborhood of 0 and 2. Finally, define a 1-form along the curves γ_n, $\tilde{\gamma}_n$ as $\omega_{\gamma_n(s)}(v) = -\langle \mu_n(s)\dot{\gamma}_n(s), v\rangle$ and $\omega_{\tilde{\gamma}_n(s)}(v) = \langle \mu_n(s)\dot{\tilde{\gamma}}_n(s), v\rangle$, and extend it to a 1-form ω on all $\mathbb{R}^2$ with norm strictly less than 1 at every point.

Now consider the Randers metric R in $\mathbb{R}^2$ associated to the Euclidean metric and the 1-form ω. The (non-converging) sequence $\{p_n\}_n$ is Cauchy for the symmetrized distance d_s of R. In fact, both $\mathrm{d}(p_n, p_{n+1})$ and $\mathrm{d}(p_{n+1}, p_n)$ are smaller than 2^{-n}, as can be checked by computing the Randers length of the curves γ_n and $t \mapsto \tilde{\gamma}_n(2 - t)$. For example, for the first one:

$$\begin{aligned}
\ell_R(\gamma_n) &= \int_0^2 \Big(\omega\big(\dot{\gamma}_n(s)\big) + \sqrt{\langle \dot{\gamma}_n(s), \dot{\gamma}_n(s)\rangle}\Big)\, \mathrm{d}s \\
&\leq 2\varepsilon_n + \int_{\varepsilon_n}^{2-\varepsilon_n} \Big(-\alpha_n \langle \dot{\gamma}_n(s), \dot{\gamma}_n(s)\rangle + \sqrt{\langle \dot{\gamma}_n(s), \dot{\gamma}_n(s)\rangle}\Big)\, \mathrm{d}s \\
&< 2\varepsilon_n + 2(1 - \alpha_n) < 2^{-n},
\end{aligned}$$

as required.

Finally, as a straightforward consequence of Theorem 4.3 (b) and Propositions 2.2 and 5.4, we have:

Corollary 5.6. *Let* $(\mathbb{R} \times S, g)$ *be a globally hyperbolic standard stationary spacetime as in* (4.1). *Then the Riemannian metric on* S

$$h = \frac{1}{\beta} g_0 + \frac{1}{\beta^2} \omega \otimes \omega,$$

is complete.

Remark 5.7. In the static case ($\omega = 0$), the converse is also true. Nevertheless, this is not true in general (a counterexample would follow easily from Example 5.5).

5.3. Randers metrics with the same pregeodesics

Recall first that different Randers metrics with the same pregeodesics can be obtained by adding an exact 1-form $\mathrm{d}f$ with small enough norm, concretely, such that $\mathrm{d}f(v) < 1$ for all $v \in TS$ with $R(v) = 1$. We denote by R^f the Randers metric $R^f(v) = R(v) - \mathrm{d}f$. In order to check that the pregeodesics of R and R^f coincide, notice that those joining two fixed points $p, q \in M$ are the critical points of the Finslerian length. As for any curve $\alpha : [a, b] \to M$ joining p and q we have that

$$\int_a^b R^f(\dot\alpha(s))\,\mathrm{d}s = \int_a^b R(\dot\alpha(s))\,\mathrm{d}s + f(p) - f(q),$$

the critical curves of R and R^f coincide. Moreover, the symmetrized distance associated to R^f coincides trivially with the one associated to R.[5]

From the viewpoint of SRC in the Introduction, the Randers metric R can be seen as a Fermat one F_g for a spacetime $(M = \mathbb{R} \times S, g)$, and the function $f : S \to \mathbb{R}$ yields a section $S^f = \{(f(x), x) : x \in S\} \subset \mathbb{R} \times S$. As we show in the Introduction, if this section is spacelike then it induces a different Fermat metric on S (as well as a different splitting of (M, g) as a standard stationary spacetime with fixed K). The next results show that S^f is spacelike iff $R^f = R - \mathrm{d}f$ is Randers and the corresponding Fermat metric F_{g^f} on S coincides with R^f.

Proposition 5.8. *Let* (S, R) *be a Randers manifold and* $(\mathbb{R} \times S, g)$ *be the standard stationary spacetime associated to it via* (5.1). *Let* $f : S \to \mathbb{R}$ *be*

[5] This fact can be used to find an example of a Randers metric with complete symmetrized distance but with non-compact symmetrized closed balls as in Example 2.3, in a more elegant way.

a smooth function, then $S^f = \{(f(x), x) \in \mathbb{R} \times S \,:\, x \in S\}$ *is a spacelike hypersurface if and only if*

$$\sup_{v \in TS,\ R(v)=1} \mathrm{d}f(v) < 1 \tag{5.2}$$

and, in this case, $R^f = R - \mathrm{d}f$ *is also a Randers metric on* S.

Proof. Let $\tilde{f}\colon S \to S^f \subset \mathbb{R}\times S$ be the map defined as $\tilde{f}(x) = (f(x), x)$. The tangent space at $\tilde{f}(x)$ to S^f is given by $T_{\tilde{f}(x)}S^f = \{\tilde{\xi} = (\mathrm{d}f_x(\xi), \xi) : \xi \in T_xS\}$. Evaluating g on vector fields $\tilde{\xi} \in TS^f$, we get

$$g(\tilde{\xi}, \tilde{\xi}) = g_0(\xi, \xi) + 2\omega(\xi)\mathrm{d}f(\xi) - \mathrm{d}f(\xi)^2. \tag{5.3}$$

By definition, S^f is spacelike if and only if

$$g_0(\xi, \xi) + 2\omega(\xi)\mathrm{d}f(\xi) - \mathrm{d}f(\xi)^2 > 0,$$

for every $\xi \in TS,\ \xi \neq 0$. Since $g_0 = h - \omega \otimes \omega$, this condition is equivalent to

$$-R(-\xi) < \mathrm{d}f(\xi) < R(\xi),$$

and then to

$$\mathrm{d}f(\xi) < R(\xi),$$

for every $\xi \in TS,\ \xi \neq 0$. ∎

Proposition 5.9. *Let* (S, R) *be a Randers manifold and* $(\mathbb{R} \times S, g)$ *be the standard stationary spacetime associated to it via* (5.1). *Let* $f : S \to \mathbb{R}$ *be a smooth function and assume that* $S^f = \{(f(x), x) \in \mathbb{R} \times S \,:\, x \in S\}$ *is a spacelike hypersurface. Let* g_0' *be the Riemannian metric induced by* g *on* S^f *and* ω' *be the* 1*-form on* S^f *defined as* $\omega'(\tilde{\xi}) = g(K, \tilde{\xi})$, *for all* $\tilde{\xi} \in TS^f$. *Consider the standard stationary metric* g^f *on* $\mathbb{R} \times S$ *defined by* $g_0^f = \tilde{f}^* g_0'$, $\omega^f = \tilde{f}^*\omega'$, $\beta = 1$, *where* $\tilde{f}\colon S \to S^f$ *is the map* $\tilde{f}(x) = (f(x), x)$. *Then the Fermat metric* F_{g^f} *on* S *associated to* g^f *is equal to* $R^f = R - df$.

Proof. For any $\xi \in TS$ we have

$$g_0^f(\xi, \xi) = g_0'(\mathrm{d}\tilde{f}(\xi), \mathrm{d}\tilde{f}(\xi)) = g_0(\xi, \xi) + 2\omega(\xi)\mathrm{d}f(\xi) - \mathrm{d}f(\xi)^2.$$

Moreover

$$\omega^f(\xi) = \omega'(\mathrm{d}\tilde{f}(\xi)) = g(K, (\mathrm{d}f(\xi), \xi)) = \omega(\xi) - \mathrm{d}f(\xi)$$

The Fermat metric associated to g^f is equal to

$$\begin{aligned} F_{g^f}(\xi) &= \omega^f(\xi) + \sqrt{g_0^f(\xi, \xi) + \omega^f(\xi)^2} = \omega(\xi) - \mathrm{d}f(\xi) + \sqrt{g_0(\xi, \xi) + \omega(\xi)^2} \\ &= \omega(\xi) - \mathrm{d}f(\xi) + \sqrt{h(\xi, \xi)} = R^f(\xi). \end{aligned}$$

∎

Theorem 5.10. *Let (S, R) be a Randers metric. There exists $f : S \to \mathbb{R}$ such that the Randers metric given by*

$$R^f(v) = R(v) - \mathrm{d}f(v) \tag{5.4}$$

for $v \in TS$ is geodesically complete if and only if the symmetrized closed balls of (S, R) are compact.

Proof. Consider the standard stationary spacetime $(\mathbb{R} \times S, g)$ associated to the Randers metric R as described in (4.1) and (5.1). By Th. 4.3 (b), compactness of the symmetrized closed balls is equivalent to global hyperbolicity. This property is also equivalent to the existence of a smooth spacelike Cauchy hypersurface [9], which can be given as the graph of some smooth function $f : S \to \mathbb{R}$. Moreover, recall that a spacelike hypersurface obtained as a graph S^f is Cauchy iff the Fermat metric F^f associated to it is forward and backward complete (Th. 4.4). This is equivalent to the completeness of the pullback of F^f on S through the map $\tilde{f}$. From Prop. 5.9, the pullback metric is equal to R^f. ■

5.4. Cut loci of Randers metrics via Cauchy horizons

In Subsection 4.3, the properties of the Fermat distance from some subset A yield consequences on the horizon corresponding to A. Next, we will see that the correspondence is also fruitful in the converse way. In fact, the applications of general results on Cauchy horizons for Riemannian Geometry were already pointed out in [16]. Here, this will be extended to Finsler Geometry.

Let (S, R) be a connected Randers manifold, not necessarily forward or backward complete. Given any closed subset $C \subset S$, the distance function $\rho_C \colon S \to [0, +\infty)$ is the infimum of the lengths of the smooth curves in S from[6] C to p. The function ρ_C is Lipschitz, more precisely $|\rho_C(p) - \rho_C(q)| \leq 2\,\mathrm{d}_s(p, q)$. Let $I \subset [0, +\infty)$ be a (non-empty) interval. We say that $\gamma : I \to S$ is a *C-minimizing segment* if it is a unit speed geodesic such that $\rho_C(\gamma(s)) = s$ for all $s \in I$. We emphasize that the interval I (which may be open, half open or closed) may not contain 0. Reasoning in the Finsler case as in [16, Prop. 9] for the Riemannian one, we have:

Proposition 5.11. *Every $p \in S \setminus C$ lies on at least one C-minimizing segment.*

[6] All the results will be obtained for the distance ρ_C from $C \subset M$ to a point $p \in M$. Analogous results hold for the distance from a point $p \in M$ to the subset $C \subset M$ –they are reduced to the former case by considering the reverse Finsler metric.

From now on we will assume that C-minimizing segments are defined in their maximal domain. We say that a C-minimizing segment has a *cut point* iff its interval of definition is of the form $[a, b]$ or $(a, b]$ with $b < +\infty$ being then $p = \gamma(b)$ the cut point. The set of all the cut points is called the *cut locus* of C in S, denoted Cut_C. For any $p \in S \setminus C$ let $N_C(p)$ be the number of C-minimizing segments passing through p. By Prop. 5.11, $N_C(p) \geq 1$ for every $p \in S \setminus C$, and it is easy to see that if $N_C(p) \geq 2$ then $p \in \mathrm{Cut}_C$.

Now (taking into account formulas (5.1)), consider the standard stationary metric constructed for the *reverse* Randers metric $\tilde{R}$ so that the past-pointing lightlike geodesics correspond to the geodesics of (S, R). We will focus in the "lower half" spacetime $(M = \big(-\infty, 0) \times S, g\big)$. These choices are convenient because we will use the general notion of horizon in [16], i.e, a *future horizon* $\mathcal{H}$ is[7] an achronal, closed, future null geodesically ruled topological hypersurface. Here *future null geodesically ruled* means that each point $p \in \mathcal{H}$ belongs to a future inextensible lightlike geodesic $\Gamma \subset \mathcal{H}$, i.e. a *null generator* Γ of $\mathcal{H}$. Let us call $\mathcal{H}$ the graph of $-\rho_C$ in M, that is

$$\mathcal{H} = \{(-\rho_C(x), x) : x \in S \setminus C\},$$

which is a future horizon in the sense above. Up to reparametrization, the null generators are precisely, the curves $s \mapsto (-\rho_C(\gamma(s)), \gamma(s)) = (s, \gamma(s))$, where γ is a C-minimizing segment of (S, R). Then, the number $N_C(x)$ of C-minimizing segments through x coincides with the number $N(-\rho_C(x), x)$ of null generators of $\mathcal{H}$ through the point $(-\rho_C(x), x)$. In addition, the set $\mathcal{H}_{\mathrm{end}}$ (given by the past endpoints of the null generators of $\mathcal{H}$) coincides with the set $\{(-\rho_C(p), p) : p \in \mathrm{Cut}_C\}$. After a result by Beem and Królak (see [7, Th. 3.5] and also [17, Prop. 3.4]) a point $p \in \mathcal{H}$ is differentiable iff $N(p) = 1$. Then, as a consequence we obtain:

Theorem 5.12. *Let (S, R) be a Randers manifold, and $C \subset S$ a closed subset. A point $p \in S \setminus C$ is a differentiable point of the distance function ρ_C from C if and only if it is crossed by exactly one minimizing segment, i.e., $N_C(p) = 1$.*

As a final remark, notice that, in the preceding discussion, the fact that $\mathcal{H}$ is a Lipschitz hypersurface (and, then by Rademacher theorem, almost everywhere differentiable) and the result in [16, Th. 1] about the zero $\mathfrak{h}^n$-measure of the set of smooth ends, yield directly:

Corollary 5.13. *If C is a closed set in an n-dimensional Randers manifold (S, R), then $\mathfrak{h}^n(\mathrm{Cut}_C) = 0$.*

[7] These requirements would be fulfilled by the horizons of Cauchy developments in Section 4.3, if one removes some parts of the spacetime; for example, for A closed, $H^-(A) \setminus A$ would be a future horizon of $M \setminus A$.

We observe that the result in [30] (see Th. 4.9 above) says that, when the subset C is regular enough, then the Hausdorff dimension of Cut_C is at most $n-1$. As far as we are aware it is not known if there exists a subset C such that the Hausdorff dimension of Cut_C is equal to n.

6. Appendix: the symmetrized distance and its path metric space

As we have commented in Section 2, the Finslerian distance d_F of a non-reversible Finsler manifold (M, F) is not a true distance, as it is non-symmetric. One can symmetrize it to obtain the so-called symmetrized distance in (2.6), but then the analog to Hopf-Rinow theorem does not hold (see the counterexample 2.3). Indeed, d_s is *not* constructed as length metric (see [25]). In fact, we can construct from the symmetrized distance its associated length metric as follows. Given a continuous curve $\alpha : I \to M$ with $I \subset \mathbb{R}$ an interval, an arbitrary subset of $\mathbb{R}$, we define the *dilatation* of α, $\mathrm{dil}(\alpha)$ as

$$\mathrm{dil}(\alpha) = \sup_{s,t\in I; s\neq t} \frac{\mathrm{d}_s(\alpha(s), \alpha(t))}{|s-t|},$$

and the *local dilatation at* $t_0 \in I$ as

$$\mathrm{dil}_{t_0}(\alpha) = \lim_{\varepsilon\to 0} \mathrm{dil}(\alpha|_{(t_0-\varepsilon, t_0+\varepsilon)}).$$

As M is a (connected) manifold, we can consider just the class of piecewise smooth curves, and define the length associated to d_s on $[a, b] \subset I$ as:

$$l(\alpha) = \int_a^b \mathrm{dil}_t(\alpha)\, \mathrm{d}t.$$

This length determines a length metric d_s^l and, then, the path-metric space (M, d_s^l) for which the Hopf-Rinow theorem does hold (see [25, pp. 2–9]). For a Randers metric, using that $\mathrm{d}_s(p, q) \leq \mathrm{d}_h(p, q)$ for every $p, q \in M$ (see the proof of Prop. 5.4), and the definition of length distance, we can easily deduce that $\mathrm{d}_s(p, q) \leq \mathrm{d}_s^l(p, q) \leq \mathrm{d}_h(p, q)$ for every $p, q \in M$. We will show that actually $\mathrm{d}_s^l = \mathrm{d}_h$.

Proposition 6.1. *Let* $R(v) = \sqrt{h(v, v)} + \omega(v)$ *be a Randers metric and* d_s *the symmetrized distance of* d_R. *Then the length distance* d_s^l *associated to* d_s *coincides with the distance* d_h.

Proof. It is enough to prove that given a curve α, parametrized by the h-length, the local dilatations $\mathrm{dil}^s_{t_0}(\alpha)$, $\mathrm{dil}^h_{t_0}(\alpha)$ for d_s and d_h satisfy:

$$\mathrm{dil}^s_{t_0}(\alpha) \geq \mathrm{dil}^h_{t_0}(\alpha)\, (= 1), \tag{6.1}$$

as the converse follows from $\mathrm{d}_s^l \leq \mathrm{d}_h$. Consider a convex neighborhood U of $\alpha(t_0)$ in the Randers metric $R(v) = \sqrt{h(v,v)} + \omega(v)$, $v \in TM$. We can assume that the closure of U is compact and contained in a chart $(\tilde{U}, x)$ such that $x(\tilde{U})$ is a Euclidean ball in $\mathbb{R}^n$. Moreover, we can take a constant $C > 0$ such that

$$\frac{1}{C}|\mathrm{d}x(v)| \leq \sqrt{h(v,v)} \leq CR(v) \quad \forall v \in T\tilde{U},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Consider an interval I such that $\alpha(I) \subset U$. Given $s, t \in I$, let γ_1 be the Randers pregeodesic in U from $\alpha(s)$ to $\alpha(t)$ and γ_2 the Randers pregeodesic in U from $\alpha(t)$ to $\alpha(s)$ both defined in $[0,1]$ and parametrized with constant h-Riemannian speed. Let γ be the closed curve defined in $[0,1]$ as γ_1 and in $[1,2]$ as $\gamma(t) = \gamma_2(t-1)$. Then

$$\frac{\mathrm{d}_s(\alpha(s),\alpha(t))}{|s-t|} = \frac{1}{2|s-t|}\int_0^2 R(\dot\gamma(\mu))\mathrm{d}\mu \geq \frac{\mathrm{d}_h(\alpha(s),\alpha(t))}{|s-t|} + \frac{1}{2|s-t|}\int_0^2 \omega(\dot\gamma(\mu))\mathrm{d}\mu,$$

and (6.1) will follow if

$$\lim_{s,t\to t_0} \frac{1}{2|s-t|}\int_0^2 \omega(\dot\gamma(\mu))\,\mathrm{d}\mu = 0. \tag{6.2}$$

Write in the chosen coordinates $\omega = \sum_i \omega_i \mathrm{d}x^i$ and $x \circ \gamma = (\gamma^1, \ldots, \gamma^n)$ so that $|\mathrm{d}x(\dot\gamma)|^2 = \sum_i (\dot\gamma^i)^2$. Recall that $h(\dot\gamma(\mu), \dot\gamma(\mu))$ is constant in $[0,1]$ as well as in $[1,2]$ and, so, for all $\mu \in [0,2]$:

$$\begin{aligned} |\mathrm{d}x(\dot\gamma(\mu))| &\leq C\sqrt{h(\dot\gamma(\mu),\dot\gamma(\mu))} < C\,\ell_h(\gamma) \leq C^2\,\ell_R(\gamma) \\ &= 2C^2\,\mathrm{d}_s(\alpha(s),\alpha(t)) \leq 2C^2\,\mathrm{d}_h(\alpha(s),\alpha(t)) \leq 2C^2|s-t|, \end{aligned} \tag{6.3}$$

where ℓ_h and ℓ_R are the lengths associated to h and R respectively. So, the mean value theorem and (6.3) imply the existence of $\mu_i \in (0, \mu)$ such that:

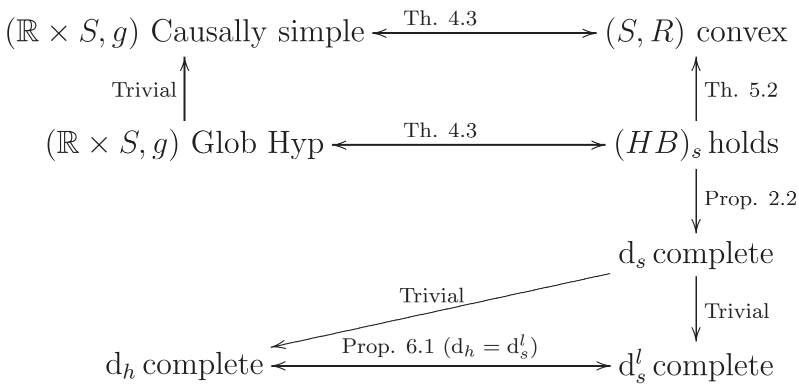
$$\begin{aligned} \omega(\dot\gamma(\mu)) &= \sum_{i=1}^n \omega_i(\gamma(\mu))\,\dot\gamma^i(\mu) \\ &= \sum_{i=1}^n \omega_i(\gamma(0))\,\dot\gamma^i(\mu) + \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j}(\gamma(\mu_i))\,\dot\gamma^j(\mu_i)\,\dot\gamma^i(\mu)\,\mu \\ &\leq \sum_{i=1}^n \omega_i(\gamma(0))\,\dot\gamma^i(\mu) + 2\,\tilde{C}(s-t)^2, \end{aligned}$$

for some constant $\tilde{C}$ independent of t, s. Integrating this expression:

$$\frac{1}{2|s-t|}\left|\int_0^2 \omega(\dot\gamma(\mu))\,\mathrm{d}\mu\right| \leq \tilde{C}|s-t|,$$

and (6.2) follows, as required. ■

Observe that Example 2.3 yields a Randers manifold (S, R) where d_s is complete with S d_s-bounded, non-compact and d_h-unbounded. The involved Hopf-Rinow type relations are summarized in the following diagram:



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