

# Overdetermined problems in unbounded domains with Lipschitz singularities

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## Abstract

We study the overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma, \end{cases}$$

where  $\Omega$  is a locally Lipschitz epigraph, that is  $C^3$  on  $\Gamma \subseteq \partial\Omega$ , with  $\partial\Omega \setminus \Gamma$  consisting in nonaccumulating, countably many points.

We provide a geometric inequality that allows us to deduce geometric properties of the sets  $\Omega$  for which monotone solutions exist.

In particular, if  $\mathcal{C} \in \mathbb{R}^n$  is a cone and either  $n = 2$  or  $n = 3$  and  $f \geq 0$ , then there exists no solution of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathcal{C}, \\ u > 0 & \text{in } \mathcal{C}, \\ u = 0 & \text{on } \partial\mathcal{C}, \\ \partial_\nu u = c & \text{on } \partial\mathcal{C} \setminus \{0\}. \end{cases}$$

This answers a question raised by Juan Luis Vázquez.

## Introduction

Let  $n \geq 2$ . We consider an epigraph in  $\mathbb{R}^n$ , that is

$$\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \Phi(x')\}.$$

We suppose that  $\Omega$  is locally Lipschitz and that it is  $C^3$  except, at most, at a countable family of points that do not accumulate.

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Explicitly, we suppose that there exists  $\mathcal{J} \subseteq \mathbb{N}$  and a family

$$\mathcal{F} := \{p'_j \in \mathbb{R}^{n-1}, j \in \mathcal{J}\}$$

such that

$$(0.1) \quad \inf_{j,k \in \mathcal{J}} |p'_j - p'_k| > 0$$

and

$$\Phi \in C^3\left(\mathbb{R}^{n-1} \setminus \left(\bigcup_{j \in \mathcal{J}} p'_j\right)\right).$$

Notice that  $\nabla\Phi$  exists a.e.: we suppose that

$$\|\nabla\Phi\|_{L^\infty(K)} < +\infty,$$

for any bounded set  $K$  in  $\mathbb{R}^{n-1}$ .

We denote  $p_j := (p'_j, \Phi(p'_j))$  and

$$\Gamma := \partial\Omega \setminus \left(\bigcup_{j \in \mathcal{J}} p_j\right).$$

We remark that, by construction, the exterior derivative  $\nu$  is always well defined at points of  $\Gamma$ .

Given  $c \in \mathbb{R}$ , we will study the following overdetermined elliptic problem:

$$(0.2) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma. \end{cases}$$

We will prove a geometric inequality for solutions of (0.2) and some rigidity results in low dimension.

For this, we introduce some notation.

Given a smooth function  $v$ , one may consider the level sets of  $v$ : in the vicinity of  $\{\nabla v \neq 0\}$ , these level sets are smooth manifolds, so one can consider the principal curvatures

$$\kappa_1, \dots, \kappa_{n-1}$$

at any point of such manifolds.

We set

$$\mathcal{K} := \sqrt{\kappa_1^2 + \dots + \kappa_{n-1}^2}.$$

Also, it is customary to consider the tangential gradient along level sets of  $v$  at these points, that is

$$\nabla_T g := \nabla g - \left(\nabla g \cdot \frac{\nabla v}{|\nabla v|}\right) \frac{\nabla v}{|\nabla v|}.$$

Thus, we may state the main results of this paper as follows:

**Theorem 1.** *Let  $u \in C^2(\Omega \cup \Gamma) \cap C(\bar{\Omega}) \cap W_{loc}^{1,\infty}(\Omega)$  be a solution of (0.2), with  $\partial_n u > 0$  in  $\Omega$ .*

*Then, for any  $\xi \in C_0^\infty(\mathbb{R}^n)$ ,*

$$(0.3) \quad \int_{\Omega} \left( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \xi^2 \leq \int_{\Omega} |\nabla u|^2 |\nabla \xi|^2.$$

**Theorem 2.** *Let  $u \in C^2(\Omega \cup \Gamma) \cap W^{1,\infty}(\Omega)$  be a solution of (0.2), with  $u > 0$  in  $\Omega$  and  $\Omega$  globally Lipschitz.*

*Suppose*

- *either that  $n = 2$*
- *or that  $n = 3$  and  $f \geq 0$ .*

*Then,  $\Omega$  cannot be coercive, that is, it cannot be that*

$$\lim_{|x'| \rightarrow +\infty} \Phi(x') = +\infty.$$

The result in Theorem 1 may be seen as a weighted Poincaré inequality. Similar inequalities have been used first in [5, 6], where no boundary term was present, and in [2, 3] to deduce symmetry results for PDEs. In [4] related inequalities have been used for problems like (0.2) in smooth domains. Differently than [4], in this paper we take into account also domains with Lipschitz singularities: indeed, when the domains are smooth, Theorems 1 and 2 here boil down to Theorems 1.1 and 1.6 in [4].

As a side remark, we also notice that the left hand side of (0.3) is well-defined, since  $\nabla u \neq 0$  in  $\Omega$ .

We now use Theorem 2 in order to answer a question posed to us by Juan Luis Vázquez [7]. For this, let  $\mathcal{C}$  be a cone.

More precisely, if  $n = 2$ , we write

$$t^+ := \begin{cases} |t| & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \quad \text{and} \quad t^- := \begin{cases} |t| & \text{if } t < 0, \\ 0 & \text{if } t \geq 0, \end{cases}$$

and, given  $\alpha^+, \alpha^- \in (0, +\infty)$ , we define the cone

$$\mathcal{C} := \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } x_2 > \alpha^+ x_1^+ + \alpha^- x_1^-\}.$$

When  $n \geq 3$ , given  $\alpha \in (0, +\infty)$ , we write the cone as

$$\mathcal{C} := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \alpha|x|\}.$$

With this notation, we obtain the following result:

**Corollary 3.** *If*

- either  $n = 2$
- or  $n = 3$  and  $f \geq 0$ ,

*then there exists no solution  $u \in C^2(\overline{\mathcal{C}} \setminus \{0\}) \cap W^{1,\infty}(\mathcal{C})$  of*

$$(0.4) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \mathcal{C}, \\ u > 0 & \text{in } \mathcal{C}, \\ u = 0 & \text{on } \partial\mathcal{C}, \\ \partial_\nu u = c & \text{on } \partial\mathcal{C} \setminus \{0\}. \end{cases}$$

Corollary 3 is a simple consequence of Theorem 2. We also recall that solutions of (0.4) satisfy

$$\partial_n u(x) > 0 \text{ for any } x \in \Omega,$$

thanks to Theorem 1.3 in [1].

We prove Theorems 1 and 2 in the forthcoming Sections 1 and 2, respectively.

### 1. Proof of Theorem 1

Let now  $\xi \in C_0^\infty(\mathbb{R}^n)$ .

We define

$$\rho := \inf_{j,k \in \mathcal{J}} |p'_j - p'_k|.$$

We recall that  $\rho > 0$  because of (0.1).

We fix  $K > 2$  and  $\eta > 0$  such that

$$(1.1) \quad \eta \leq \min \left\{ 1, \frac{\rho}{2}, \frac{1}{\log K} \right\}.$$

We define

$$\tau_{\eta,K}(x) := \begin{cases} 0 & \text{if } |x| \leq \eta/K, \\ \frac{\log|x| - \log(\eta/K)}{\log K} & \text{if } \eta/K < |x| \leq \eta, \\ 1 & \text{if } |x| > \eta. \end{cases}$$

Notice that  $\tau_{\eta,K}$  is Lipschitz continuous and

$$(1.2) \quad |\nabla \tau_{\eta,K}(x)| \leq \frac{\chi_{(B_\eta \setminus B_{\eta/K})}(x)}{|x| \log K}.$$

Here above, as customary, we have denoted by  $\chi_A$  the characteristic function of the set  $A$ .

We also set

$$\xi_{\eta,K}(x) := \xi(x) \cdot \prod_{j \in \mathcal{J}} \tau_{\eta,K}(x - p_j).$$

This function is well-defined, since  $\tau_{\eta,K}(x - p_j) = 1$  for  $x \notin B_\eta(p_j)$  and these balls are disjoint.

We now take  $\Omega_{\eta,K}$  to be an open set with  $C^3$  boundary such that

$$\Omega \setminus \left( \bigcup_{j \in \mathcal{J}} B_{\eta/(2K)}(p_j) \right) \subset \Omega_{\eta,K} \subset \Omega \setminus \left( \bigcup_{j \in \mathcal{J}} B_{\eta/(4K)}(p_j) \right).$$

We make use of (1.4) of [4] to obtain that

$$(1.3) \quad \begin{aligned} & \int_{\Omega_{\eta,K}} \left( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \varphi^2 \\ & + \limsup_{\epsilon \rightarrow 0^+} \int_{\partial \Omega_{\eta,K}} \frac{\varphi^2}{\epsilon + \partial_n u} \left( |\nabla u|^2 \partial_{n,\nu}^2 u - \partial_{i,\nu}^2 u \partial_i u \partial_n u \right) \leq \\ & \leq \int_{\Omega_{\eta,K}} |\nabla u|^2 |\nabla \varphi|^2 \end{aligned}$$

for any  $\varphi \in W_0^{1,\infty}(\mathbb{R}^n)$ .

On the other hand, from (3.15) in [4] and (0.2) here, we know that

$$(1.4) \quad |\nabla u|^2 \partial_{n,\nu}^2 u - \partial_{i,\nu}^2 u \partial_i u \partial_n u = 0 \quad \text{on } \Gamma.$$

Also, by construction,

$$(1.5) \quad \xi_{\eta,K} = 0 \quad \text{in } \bigcup_{j \in \mathcal{J}} B_{\eta/(2K)}(p_j).$$

From (1.4) and (1.5) we thus obtain

$$\begin{aligned} & \int_{\partial \Omega_{\eta,K}} \frac{\xi_{\eta,K}^2}{\epsilon + \partial_n u} \left( |\nabla u|^2 \partial_{n,\nu}^2 u - \partial_{i,\nu}^2 u \partial_i u \partial_n u \right) = \\ & = \sum_{j \in \mathcal{J}} \int_{\partial \Omega_{\eta,K} \cap B_{\eta/(2K)}(p_j)} \frac{\xi_{\eta,K}^2}{\epsilon + \partial_n u} \left( |\nabla u|^2 \partial_{n,\nu}^2 u - \partial_{i,\nu}^2 u \partial_i u \partial_n u \right) = 0. \end{aligned}$$

Consequently, by taking  $\varphi := \xi_{\eta,K}$  in (1.3), we obtain

$$(1.6) \quad \int_{\Omega_{\eta,K}} \left( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \xi_{\eta,K}^2 \leq \int_{\Omega_{\eta,K}} |\nabla u|^2 |\nabla \xi_{\eta,K}|^2.$$

Also, recalling (1.1) and (1.2), a straightforward computation gives that

$$(1.7) \quad \begin{aligned} |\nabla \xi_{\eta,K}(x)| - |\nabla \xi(x)| &\leq |\nabla \xi_{\eta,K}(x) - \nabla \xi(x)| \leq \\ &\leq \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)} \sum_{j \in \mathcal{J}} \left( 1 + \frac{1}{|x - p_j| \log K} \right) \chi_{(B_\eta(p_j) \setminus B_{\eta/K}(p_j))}(x) \\ &\leq 2\|\xi\|_{W^{1,\infty}(\mathbb{R}^n)} \sum_{j \in \mathcal{J}} \frac{\chi_{(B_\eta(p_j) \setminus B_{\eta/K}(p_j))}(x)}{|x - p_j| \log K}. \end{aligned}$$

We now fix an auxiliary parameter  $\delta > 0$  and we use a scaled Cauchy Inequality to deduce from (1.7) that

$$\begin{aligned} |\nabla \xi_{\eta,K}(x)|^2 &\leq (1 + \delta) |\nabla \xi(x)|^2 \\ &\quad + C_\delta \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \left[ \sum_{j \in \mathcal{J}} \frac{\chi_{(B_\eta(p_j) \setminus B_{\eta/K}(p_j))}(x)}{|x - p_j| \log K} \right]^2. \end{aligned}$$

Since the balls  $B_\eta(p_j)$  are disjoint, we can write the above inequality as

$$(1.8) \quad \begin{aligned} |\nabla \xi_{\eta,K}(x)|^2 &\leq (1 + \delta) |\nabla \xi(x)|^2 \\ &\quad + C_\delta \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \sum_{j \in \mathcal{J}} \frac{\chi_{(B_\eta(p_j) \setminus B_{\eta/K}(p_j))}(x)}{|x - p_j|^2 (\log K)^2}. \end{aligned}$$

Now, we denote by  $\mathcal{S} \subset \mathbb{R}^n$  the support of  $\xi$ , and we define

$$\mathcal{J}_\mathcal{S} := \{j \in \mathcal{J} \text{ s.t. } B_\eta(p_j) \cap \mathcal{S} \neq \emptyset\}.$$

We remark that  $\mathcal{J}_\mathcal{S}$  is a finite set, so we denote by  $C_\mathcal{S} \in \mathbb{N}$  its cardinality.

Then, by (1.8),

$$\begin{aligned}
 & \int_{\Omega_{\eta,K}} |\nabla u(x)|^2 |\nabla \xi_{\eta,K}(x)|^2 dx \leq \\
 & \leq \int_{\mathcal{J}} |\nabla u(x)|^2 \left[ (1+\delta) |\nabla \xi(x)|^2 + C_\delta \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \sum_{j \in \mathcal{J}} \frac{\chi_{(B_\eta(p_j) \setminus B_{\eta/K}(p_j))}(x)}{|x - p_j|^2 (\log K)^2} \right] dx \\
 & \leq (1+\delta) \int_{\Omega} |\nabla u(x)|^2 |\nabla \xi(x)|^2 dx \\
 & \quad + C_\delta \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \|u\|_{W^{1,\infty}(\mathcal{J})}^2 \sum_{j \in \mathcal{J}} \int_{(B_\eta(p_j) \setminus B_{\eta/K}(p_j))} \frac{1}{|x - p_j|^2 (\log K)^2} dx \\
 & = (1+\delta) \int_{\Omega} |\nabla u(x)|^2 |\nabla \xi(x)|^2 dx \\
 & \quad + \frac{C_\delta \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \|u\|_{W^{1,\infty}(\mathcal{J})}^2}{(\log K)^2} \sum_{j \in \mathcal{J}} \int_{\eta/K}^\eta \frac{r^{n-1}}{r^2} dr \\
 & \leq (1+\delta) \int_{\Omega} |\nabla u(x)|^2 |\nabla \xi(x)|^2 dx \\
 & \quad + \frac{C_\delta C_{\mathcal{J}} \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \|u\|_{W^{1,\infty}(\mathcal{J})}^2}{(\log K)^2} \int_{\eta/K}^\eta \frac{1}{r} dr \\
 & = (1+\delta) \int_{\Omega} |\nabla u(x)|^2 |\nabla \xi(x)|^2 dx + \frac{C_\delta C_{\mathcal{J}} \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \|u\|_{W^{1,\infty}(\mathcal{J})}^2}{\log K}.
 \end{aligned}$$

This and (1.6) give that

$$\begin{aligned}
 & \int_{\Omega_{\eta,K}} \left( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \xi_{\eta,K}^2 \leq \\
 & \leq (1+\delta) \int_{\Omega} |\nabla u(x)|^2 |\nabla \xi(x)|^2 dx + \frac{C_\delta C_{\mathcal{J}} \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)}^2 \|u\|_{W^{1,\infty}(\mathcal{J})}^2}{\log K}.
 \end{aligned}$$

We now take  $\eta = 1/\log K$  and we send  $K \rightarrow +\infty$  (notice that (1.1) allows us to do so), so that we obtain

$$\int_{\Omega} \left( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \xi^2 \leq (1+\delta) \int_{\Omega} |\nabla u|^2 |\nabla \xi|^2.$$

By taking  $\delta$  as small as we wish, we obtain (0.3), thus completing the proof of Theorem 1.

## 2. Proof of Theorem 2

We observe that, under the assumptions of Theorem 2,

$$u \in W^{1,\infty}(\Omega) \subset C(\overline{\Omega}).$$

We suppose, by contradiction, that  $\Omega$  is coercive.

Then,  $\partial_n u > 0$ , thanks to Theorem 1.3 in [1].

Thus, when  $n = 2$ , the claim of Theorem 2 follows from (0.3) here and Lemma 5.1 in [4].

Thus, we focus on the case in which  $n = 3$  and  $f \geq 0$ .

For any  $t \geq 0$  and any  $(x', x_3) \in \Omega$ , we define

$$u_t(x', x_3) := u(x', x_3 + t).$$

Due to standard elliptic regularity theory, we have that the following limit exists for any  $x' \in \mathbb{R}^2$ , with  $(x', x_3) \in \Omega$ , and it is attained in  $C^2(\mathbb{R}^2)$ :

$$(2.1) \quad u_\infty(x') := \lim_{t \rightarrow +\infty} u_t(x', x_3).$$

In particular,

$$(2.2) \quad \Delta u_\infty + f(u_\infty) = 0 \quad \text{in } \mathbb{R}^2.$$

We also set

$$F(r) := \int_0^r f(s) \, ds.$$

Note that  $F' = f \geq 0$  and so  $F$  is nondecreasing. Accordingly,

$$F(u(x)) \leq F(u_t(x)) \quad \text{for any } x \in \Omega$$

and so

$$(2.3) \quad F(u(x', x_3)) \leq F(u_\infty(x')) \quad \text{for any } (x', x_3) \in \Omega.$$

Now, we take  $\Omega_\epsilon \subseteq \Omega$  to be a  $C^3$  coercive epigraph that approaches  $\Omega$  when  $\epsilon \rightarrow 0^+$ .

We make use of Lemma 9.1 in [4] (applied here to  $u$  in the smooth domain  $\Omega_\epsilon$ ): we obtain, for any  $t \geq 0$ ,

$$\int_{B_R \cap \Omega_\epsilon} \frac{|\nabla u|^2}{2} - F(u) \, dx \leq CR^2 + \int_{B_R \cap \Omega_\epsilon} \frac{|\nabla u_t|^2}{2} - F(u_t) \, dx,$$

for a suitable constant  $C \geq 0$ .

Therefore, keeping  $t$  fixed and sending  $\epsilon \rightarrow 0^+$ ,

$$\int_{B_R \cap \Omega} \frac{|\nabla u|^2}{2} - F(u) \, dx \leq CR^2 + \int_{B_R \cap \Omega} \frac{|\nabla u_t|^2}{2} - F(u_t) \, dx.$$

We now send  $t \rightarrow +\infty$  and we conclude that

$$\int_{B_R \cap \Omega} \frac{|\nabla u(x)|^2}{2} - F(u(x)) \, dx \leq CR^2 + \int_{B_R \cap \Omega} \frac{|\nabla u_\infty(x')|^2}{2} - F(u_\infty(x')) \, d(x', x_3).$$

Therefore, from (2.3),

$$(2.4) \quad \int_{B_R \cap \Omega} \frac{|\nabla u(x)|^2}{2} \, dx \leq CR^2 + \int_{B_R \cap \Omega} \frac{|\nabla u_\infty(x')|^2}{2} \, d(x', x_3).$$

We now observe that, by (2.2), it holds that  $\Delta u_\infty \leq 0$  in  $\mathbb{R}^2$  and therefore, by a classical Liouville Theorem, we have that  $u_\infty$  is constant.

Hence, (2.4) becomes

$$(2.5) \quad \int_{B_R \cap \Omega} \frac{|\nabla u(x)|^2}{2} \, dx \leq CR^2.$$

Thus, in the light of (0.3) and (2.5), we may now apply Corollary 9.4 of [4]: we obtain that  $\partial\Omega$  is a hyperplane, in contradiction with the fact that  $\Omega$  is coercive.

This completes the proof of Theorem 2.

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