The (L^1, L^1) bilinear Hardy-Littlewood function and Furstenberg averages

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Abstract

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on a non-atomic finite measure space. Consider the maximal function

$$R^* : (f, g) \in L^1 \times L^1 \to R^*(f, g)(x) = \sup_n \frac{f(T^n x) g(T^{2n} x)}{n}.$$

We show that there exist f and g such that $R^*(f, g)(x)$ is not finite almost everywhere. Two consequences are derived. The bilinear Hardy–Littlewood maximal function fails to be a.e. finite for all functions $(f, g) \in L^1 \times L^1$. The Furstenberg averages do not converge for all pairs of (L^1, L^1) functions, while by a result of J. Bourgain these averages converge for all pairs of (L^p, L^q) functions with $\frac{1}{p} + \frac{1}{q} \le 1$.

1. Introduction

The bilinear Hardy–Littlewood maximal function was introduced by Alberto Calderón in the 1960's. It is defined for f, g measurable functions as

$$M^*(f, g)(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x + s) g(x + 2s) ds.$$

Our purpose is to prove that M^* is not always a.e finite when the functions f and g are in L^1.

Theorem 1. *There exist functions f, g both in L^1 for which the bilinear Hardy-Littlewood maximal function*

$$M^*(f, g)(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x + s) g(x + 2s) ds$$

is not a.e. finite.

To prove this theorem we use Ergodic Theory. The Ergodic Theory version of the bilinear Hardy–Littlewood maximal function is defined for $f, g \geq 0$ as

$$\mathcal{M}(f,g)(x) = \sup_N \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^n x) g(T^{2n} x),$$

where T is an ergodic measure preserving transformation of a non-atomic probability measure space, $f \in L^p$, and $g \in L^q$. They are called Furstenberg averages. They appear in H. Furstenberg's paper in 1977 [5].

A transference argument shows that the class of functions in $L^p \times L^q$, for which M^* and $\mathcal{M}$ are a.e. finite is the same. For $f, g \geq 0$ we have

$$\mathcal{M}(f,g)(x) \geq \sup_N \frac{f(T^N x) g(T^{2N} x)}{2N+1},$$

the tail of the averages $\mathcal{M}(f,g)(x)$. We consider the maximal function

$$R^*(f,g)(x) = \sup_n \frac{f(T^n x) g(T^{2n}(x))}{n}.$$

In [1] we showed that for all $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} < 2$, R^* maps $L^p \times L^q$ into L^r as soon as $0 < r < 1/2$. This implies that $R^*(f,g)$ is finite almost everywhere and $\frac{f(T^n x) g(T^{2n} x)}{n} \to 0$ for a.e. x as $n \to \infty$.

In this paper we show that for $p = q = 1$, $R^*(f,g)(x)$ is not finite almost everywhere for all f and g.

Theorem 2. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving transformation on a finite non-atomic measure space. Then there exist functions f, g both in $L^1_+(X)$ for which the maximal function*

$$R^*(f,g)(x) = \sup_n \frac{f(T^n x) g(T^{2n} x)}{n}$$

is not finite a.e.

The example of the identity map shows that Theorem 2 is false without the ergodicity assumption.

Theorem 2 gives us three conclusions. First it solves an open problem in Ergodic Theory. Indeed, a deep result of J. Bourgain, [2], showed that the Furstenberg averages converge a.e. as soon as the Hölderian duality is respected, (i.e. $\frac{1}{p} + \frac{1}{q} \leq 1$). Theorem 2 shows that these averages do not converge for all pairs of (L^1, L^1) functions as the tail of these averages does not converge a.e. to zero for some functions $f, g \in L^1$. This is the content of the following result

Theorem 3. *Given an ergodic measure preserving transformation T, on a nonatomic probabilty measure space, we can find functions $f, g \in L^1$ for which the Furstenberg averages*

$$\frac{1}{N} \sum_{n=0}^{N} f(T^n x) g(T^{2n} x)$$

do not converge a.e.

Secondly, by transference the unboundedness of $\mathcal{M}(f, g)(x)$ implies the same result for the bilinear Hardy Littlewood maximal function in $(L^1 \times L^1)$ and gives a proof of Theorem 1.

A third consequence of Theorem 2 is that $1/2$ is an optimal bound for R^*.

In view of all these three consequences we just need to focus on proving Theorem 2.

For some further results and recent progress related to the bi- and trilinear Hardy–Littlewood maximal function we also refer to [7], [8], [3] and [4].

Let us fix some notation. Given $\phi : \mathbb{R} \to \mathbb{R}$, periodic by p we put

$$\overline{\int} \phi = \frac{1}{p} \int_0^p \phi(x) dx.$$

Given a Lebesgue measurable set A, periodic by p we put

$$\overline{\lambda}(A) = \frac{1}{p} \lambda(A \cap [0, p)).$$

For a function $\phi : \mathbb{R} \to \mathbb{R}$, we will denote by $\operatorname{spt}(\phi)$ the set of those x's for which $\phi(x) \neq 0$. This notation differs slightly from the support of a function. The functions for which we will apply it will be constant on intervals of the form $[k, k+1)$, $k \in \mathbb{Z}$ and hence $\operatorname{spt} \phi$ differs only by some endpoints of these intervals from the set which is usually considered to be the support of a function.

2. Main Results

We want to prove Theorem 2. To this end we introduce the following definition.

Suppose that $\mathcal{P} \subset \mathbb{N}$ is an infinite set, $\frac{1}{2} < \rho < 1$, $0 < \epsilon < \epsilon_0 \stackrel{\text{def}}{=} 1/10$, $s, \alpha \in \mathbb{N}$. We also suppose that $M \geq 2, M \in \mathbb{N}$

We will define the functions ϕ_i, $\psi_k : \mathbb{R} \to \mathbb{R}$ so that if Δ is any of these functions then $\Delta(x) = \Delta(\lfloor x \rfloor)$.

We fix the parameters ρ, ϵ and M.

Definition 1. An $\alpha - \mathcal{P} - s$-family with ψ-interval $[\alpha, \omega]$ consists of an integer n_s and of non-negative functions ϕ_i, $i = 1, ..., n_s$, ψ_k, $k = \alpha, ..., \omega$ which are periodic by an integer $p_s \in \mathcal{P}$, $p_s \geq \omega$ and

$$(2.1) \qquad \overline{\lambda} \left\{ x : \max_{k \in [\alpha, \omega]} \max_{l \leq p_s} \frac{\sum_{i=1}^{n_s} \phi_i(x+l)\psi_k(x+2l)}{l} \geq 1 \right\}$$
$$> \min \left\{ \epsilon, \frac{s(M-1)\epsilon 2^{-M}}{2048} \right\},$$

$$(2.2) \qquad s \cdot 2^{-M-1} < \sum_{i=1}^{n_s} \overline{\int} \phi_i < s \cdot 2^{-M+1},$$

$$(2.3) \qquad \rho\epsilon < \sum_{k=\alpha}^{\omega} \overline{\int} \psi_k < \epsilon.$$

Note that in this definition α is predetermined but ω is not. The number ω depends on α and on other objects of our construction like the functions ϕ and ψ.

The next lemma shows that the notion of $\alpha - \mathcal{P} - s$-family is "generic" among infinite sets $\mathcal{P}$ of positive integers in the following sense: if we can find one $\alpha - \mathcal{P}' - s$-family then for any other infinite subset $\mathcal{P}$ we will be able to find an $\alpha - \mathcal{P} - s$-family. This lemma allows us to restrict the proof of the existence of such families to infinite set of integers consisting only of powers of two.

Lemma 4. *Suppose that $\mathcal{P}' \subset \mathbb{N}$ is an arbitrary infinite set and there exists an $\alpha - \mathcal{P}' - s$-family with ψ'-interval $[\alpha, \omega]$. Then for an arbitrary other infinite set $\mathcal{P} \subset \mathbb{N}$ there exists an $\alpha - \mathcal{P} - s$-family with ψ-interval $[\alpha, \omega]$ as well. Moreover,*

$$(2.4) \qquad \overline{\int} \phi'_i \approx \overline{\int} \phi_i, \text{ in fact } \left(1 - \frac{p'_s}{p_s}\right) \overline{\int} \phi'_i \leq \overline{\int} \phi_i \leq \overline{\int} \phi'_i,$$

and

$$(2.5) \qquad \overline{\int} \psi'_k \approx \overline{\int} \psi_k, \text{ in fact } \left(1 - \frac{p'_s}{p_s}\right) \overline{\int} \psi'_k \leq \overline{\int} \psi_k \leq \overline{\int} \psi'_k$$

where ϕ'_i, ψ'_k belongs to the "old" $\alpha - \mathcal{P}' - s$-family and ϕ_i, ψ_k to the "new" $\alpha - \mathcal{P} - s$-family and these families are periodic by p'_s and p_s, respectively.

Proof. Suppose we have ϕ'_i, $i = 1, ..., n_s$, ψ'_k, $k = \alpha, ..., \omega$ periodic by $p'_s \in \mathcal{P}'$, $p'_s \geq \omega$ satisfying (2.1-2.3). Since $\mathcal{P}$ contains infinitely many terms there are arbitrarily large elements $p_s \in \mathcal{P}$. We will select a sufficiently large $p_s \gg p'_s$.

To define ϕ_i, ψ_k periodic by p_s it is sufficient to define them on $[0, p_s)$ and then extend their definition onto $\mathbb{R}$ by periodicity. If $x \in [0, \lfloor p_s/p'_s \rfloor \cdot p'_s)$ then set $\phi_i(x) = \phi'_i(x)$, $i = 1, ..., n_s$ and $\psi_k(x) = \psi'_k(x)$, $k = \alpha, ..., \omega$. If $x \in [\lfloor p_s/p'_s \rfloor \cdot p'_s, p_s)$ then set $\phi_i(x) = 0 = \psi_k(x)$ for $i = 1, ..., n_s$ and $k = \alpha, ..., \omega$. Since in (2.1–2.3) concerning $\overline{\lambda}$ and $\overline{\int}$ there are strict inequalities it is not difficult to see that (2.1–2.3) hold for ϕ_i and ψ_k if p_s is sufficiently large. For example, we show that (2.3) and (2.5) hold. We have

$$(2.6) \qquad \overline{\int} \psi_k = \frac{1}{p_s} \int_{[0,p_s)} \psi_k = \frac{1}{p_s} \int_{[0, \lfloor p_s/p'_s \rfloor p'_s)} \psi'_k$$

$$= \frac{\lfloor p_s/p'_s \rfloor p'_s}{p_s} \frac{1}{\lfloor p_s/p'_s \rfloor p'_s} \int_{[0, \lfloor p_s/p'_s \rfloor p'_s)} \psi'_k = \frac{\lfloor p_s/p'_s \rfloor p'_s}{p_s} \overline{\int} \psi'_k.$$

From this and

$$(2.7) \qquad \frac{p_s - p'_s}{p_s} < \frac{\lfloor p_s/p'_s \rfloor p'_s}{p_s} \leq 1$$

it follows (2.5) and adding these inequalities for $k = \alpha$ to ω we obtain (2.3) when p_s is sufficiently large. ∎

The next simple "independence lemma" will be useful later.

Lemma 5. *Suppose $t, n, \pi_1, \pi_2, \theta_1, \theta_2 \in \mathbb{Z}$, $\theta_1, \theta_2 \geq 0$. Consider two sets $\mathcal{X}_1, \mathcal{X}_2 \subset [t2^n, (t+1)2^n)$ with the following properties:*

a) They are "periodic within" $[t2^n + \theta_1, (t+1)2^n - \theta_2)$ by π_1 and π_2, respectively. This means that for $i = 1, 2$ if $x, x + \pi_i \in [t2^n + \theta_1, (t+1)2^n - \theta_2)$ then $x \in \mathcal{X}_i$ iff $x + \pi_i \in \mathcal{X}_i$.

b) They consist of integer intervals. This means that $x \in \mathcal{X}_i$ iff $\lfloor x \rfloor \in \mathcal{X}_i$ for $i = 1, 2$.

Then for relatively prime π_1 and π_2 we have

$$(2.8) \qquad \lambda(\mathcal{X}_1 \cap \mathcal{X}_2) < 2 \frac{\lambda(\mathcal{X}_1)\lambda(\mathcal{X}_2)}{2^n}$$

if 2^n is much larger than $\max\{\pi_1, \pi_2, \theta_1, \theta_2\}$.

Proof. Set $X_1 = \mathcal{X}_1 \cap \mathbb{Z}$, $X_2 = \mathcal{X}_2 \cap \mathbb{Z}$. Since the sets $\mathcal{X}_1$ and $\mathcal{X}_2$ consist of integer intervals $\lambda(\mathcal{X}_i) = \#X_i$, $(i = 1, 2)$ and $\lambda(\mathcal{X}_1 \cap \mathcal{X}_2) = \#(X_1 \cap X_2)$.

Suppose $i \in \{1, 2\}$ and

$$(2.9) \qquad t2^n + \theta_1 \leq k\pi_i < (k+1)\pi_i - 1 < (t+1)2^n - \theta_2.$$

Then

$$L_i \overset{\text{def}}{=} \frac{\#(X_i \cap [k\pi_i, (k+1)\pi_i))}{\pi_i}$$

does not depend on the choice of k as long as (2.9) is satisfied. If $n \to \infty$ with π_1, π_2, θ_1 and θ_2 fixed then

$$\frac{\lambda(\mathcal{X}_i)}{2^n} = \frac{\#X_i}{2^n} \to L_i.$$

Therefore, if 2^n is much larger than π_1, π_2, θ_1 and θ_2 then

$$(2.10) \qquad\qquad L_i < \sqrt[3]{2}\,\frac{\lambda(\mathcal{X}_i)}{2^n} \quad \text{holds for } i = 1, 2.$$

Suppose

$$t2^n + \theta_1 \leq k\pi_1\pi_2 < (k+1)\pi_1\pi_2 - 1 < (t+1)2^n - \theta_2.$$

Since π_1 and π_2 are relatively prime

$$L_1 L_2 = \frac{\#(X_1 \cap X_2 \cap [k\pi_1\pi_2, (k+1)\pi_1\pi_2))}{\pi_1\pi_2}.$$

If $n \to \infty$ with π_1, π_2, θ_1 and θ_2 fixed then

$$\frac{\lambda(\mathcal{X}_1 \cap \mathcal{X}_2)}{2^n} = \frac{\#(X_1 \cap X_2)}{2^n} \to L_1 L_2.$$

Therefore, if 2^n is much larger than π_1, π_2, θ_1 and θ_2 then

$$\frac{\lambda(\mathcal{X}_1 \cap \mathcal{X}_2)}{2^n} < L_1 L_2 \sqrt[3]{2}.$$

By using (2.10) we have

$$\frac{\lambda(\mathcal{X}_1 \cap \mathcal{X}_2)}{2^n} < L_1 L_2 \sqrt[3]{2} < \left(\sqrt[3]{2}\,\frac{\lambda(\mathcal{X}_1)}{2^n} \right) \left(\sqrt[3]{2}\,\frac{\lambda(\mathcal{X}_2)}{2^n} \right) \sqrt[3]{2}$$

and after multiplying by 2^n this implies (2.8). ∎

Theorem 2 is a consequence of Theorem 6. The proof is quite long. Therefore we have divided it into two steps and several substeps (14 SUB-STEPS). STEP 1 contains six substeps that correspond to the first induction step for $s = 1$. In STEP 2 we finish the induction argument on s. In Section 2.15 we show how this result allows us to derive a proof of Theorem 2. We recall that the parameters $\frac{1}{2} < \rho < 1$, $0 < \epsilon < \frac{1}{10} = \epsilon_0$, and $M \in \mathbb{N}$, $M \geq 2$, are fixed in the definition of an $\alpha - \mathcal{P} - s$-family.

Theorem 6. *Suppose that $\mathcal{P} \subset \mathbb{N}$ is an infinite set. Then for every $\alpha, s \in \mathbb{N}$ there exists an $\alpha - \mathcal{P} - s$-family.*

Proof. We do mathematical induction on s. By Lemma 4 we can suppose that $\mathcal{P}$ consists of powers of 2.

2.1. Step 1: The $s = 1$ case
Substep 1a: Interval supports

First we show that for any possible choice of α, $\mathcal{P}$, $\frac{1}{2} < \rho < 1$, $0 < \epsilon < \frac{1}{10}$, and $M \in \mathbb{N}, M \geq 2$, one can find $\alpha - \mathcal{P} - 1$-families.

We will select later a suitably large n_1 and for $i < n_1$ we set $\phi_i(x) = 0$ for all $x \in \mathbb{R}$. To define ϕ_{n_1} and the functions ψ_k, $k = \alpha, ..., \omega$ one could come up with a somewhat simpler definition but to help the reading of the more technical later steps of our induction we introduce already at the first step some of the features used later.

We choose integers $0 = k_0 < k_1 < \cdots < k_M$ so that $k_1 \gg \alpha$.

The interval support of ϕ_{n_1} at level k is defined as

$$(2.11) \quad \mathrm{ispt}_k(\phi_{n_1}) = \bigcup \{[(t-1)2^{n_1+k}, t2^{n_1+k}) :$$
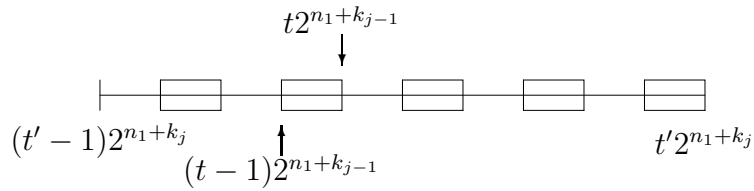$$t \in \mathbb{Z},\ \mathrm{spt}\,(\phi_{n_1}) \cap [(t-1)2^{n_1+k}, t2^{n_1+k}) \neq \emptyset\}.$$

Figure 1: One component of $\mathrm{ispt}_{k_j}(\phi_{n_1})$

Outside its support ϕ_{n_1} will vanish, so by giving its interval support at different levels we can define it. We zoom in during this definition. (This means that we define our interval supports as nested sets consisting of intervals and first we define the larger sets and then we define the smaller ones. This is similar to the procedure of defining the triadic Cantor set as the intersection of closed sets at level k consisting of 2^k many intervals of length 3^{-k}. Though in our construction we will use only finitely many steps of zooming in.) We set

$$\mathrm{ispt}_{k_M}(\phi_{n_1}) = \mathbb{R}.$$

(2.12)   For $j \in \{1, ..., M\}$ an interval $[(t-1)2^{n_1+k_{j-1}}, t2^{n_1+k_{j-1}}) \subset \mathrm{ispt}_{k_j}(\phi_{n_1})$ belongs to $\mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$ if and only if t is even.

(See Figure 1.) This implies

$$(2.13) \qquad \overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_1})) = \left(\frac{1}{2}\right)^{M-j},$$

and

$$(2.14) \quad \overline{\lambda}(\mathrm{ispt}_{k_{j-1}}(\phi_{n_1})) = \overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})) = \frac{1}{2}\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_1})).$$

It will be useful to keep in mind for further reference that $t - 1$, which corresponds to the left endpoint of a support interval is odd.

2.2. Substep 1b: Definitions of ϕ_{n_1} and $\psi_{k'_j}$

We will define ϕ_{n_1} so that it will be periodic by $2^{n_1+k_M} \in \mathcal{P}$.

If $[(t-1)2^{n_1}, t2^{n_1}) \subset \mathrm{ispt}_0(\phi_{n_1}) = \mathrm{ispt}_{k_0}(\phi_{n_1})$ then

$$(2.15) \qquad \phi_{n_1}(x) = 2^{n_1} \text{ if } x \in [(t-1)2^{n_1}, (t-1)2^{n_1}+1), \text{ and}$$
$$(2.16) \qquad \phi_{n_1}(x) = 0 \text{ if } x \in [(t-1)2^{n_1}+1, t \cdot 2^{n_1}).$$

From (2.13) and (2.14) used with $j = 1$ it follows that

$$(2.17) \qquad \overline{\int} \sum_{i=1}^{n_1} \phi_i = \overline{\int} \phi_{n_1} = 2^{-M}.$$

Set $k'_1 = \alpha$.

For any $k \neq k'_1$, $k \in \mathbb{N}$ and $x \in \mathrm{ispt}_{k_1}(\phi_{n_1})$ we set $\psi_k(x) = 0$.

For $x \notin \mathrm{ispt}_{k_1}(\phi_{n_1})$ we set $\psi_{k'_1}(x) = 0$, that is,

$$(2.18) \qquad \psi_{k'_1} \text{ is supported in } \mathrm{ispt}_{k_1}(\phi_{n_1}).$$

We will choose $\psi_{k'_1}$ so that it is constant on intervals of the form $[n, n+1)$ for all $n \in \mathbb{Z}$, it is periodic by $2^{n_1+k_M}$, its range is $\{2^{k'_1}, 0\}$ and

$$(2.19) \qquad \frac{1+\rho}{2}\epsilon \cdot 2^{n_1} < \int_{[(t-1)2^{n_1}+1, t2^{n_1}+1)} \psi_{k'_1}(x)dx < \epsilon 2^{n_1},$$

provided $[(t-1)2^{n_1}, t2^{n_1}) \subset \mathrm{ispt}_{k_1}(\phi_{n_1})$. Inequality (2.19) can be achieved if 2^{n_1} is sufficiently large.

Suppose $j \geq 2$ and we can also suppose that $\alpha \ll k_1 \ll ... \ll k_M$.

We will choose n_1 so that $n_1 \gg k_M$ and k'_j such that $k_{j-1} < k'_j \overset{\mathrm{def}}{=} k_{j-1} + 10 \ll k_j$. Our constants will be selected by induction in the following order: $k_1, k'_1, k_2, ... \; k_{j-1}, k'_j, k_j, ..., k_M, n_1$. Assumptions about k_j's and k'_j's are

given in Section 2.3, we emphasize that in (2.26) the value of the fraction does not depend on n_1. While we make assumptions about n_1 at (2.19), (2.22), (2.24) and (2.43).

The function

(2.20) $\psi_{k'_j}$ will be supported in $\mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$.

At the $s = 1$ case one nonzero $\psi_{k'_j}$ is sufficient, that is, we set $\psi_k(x) = 0$ for all $x \in \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$, $k \in \mathbb{N}$, $k \neq k'_j$.

To each $t \in \mathbb{Z}$ we associate a period $\pi(t) \gg 2^{k_M+1}$ so that

(2.21)   if $t \not\equiv t' \bmod 2^{k_M+1}$ then $(\pi(t), \pi(t')) = 1$.

These periods can be different primes. We suppose that if $\pi^* = \max_t \pi(t)$ then

(2.22) $2^{n_1} \gg (\pi^*)^{k_M}$ and

(2.23) if $x, x + \pi(t) \in [(t-1)2^{n_1} + 1, t2^{n_1} + 1)$ then $\psi_{k'_j}(x) = \psi_{k'_j}(x + \pi(t))$.

This means that $\psi_{k'_j}$ is "periodic" by $\pi(t)$ within $[(t-1)2^{n_1} + 1, t2^{n_1} + 1)$. We will choose $\psi_{k'_j}$ so that it is constant on intervals of the form $[n, n+1)$ for all $n \in \mathbb{N}$, it is periodic by $2^{n_1+k_M}$, its range is $\{2^{k'_j}, 0\}$ and

(2.24) $\dfrac{1+\rho}{2}\epsilon \cdot 2^{n_1} < \displaystyle\int_{[(t-1)2^{n_1}+1, t2^{n_1}+1)} \psi_{k'_j}(x)dx < \epsilon 2^{n_1}$,

provided $[(t-1)2^{n_1}, t2^{n_1}) \subset \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$. Inequality (2.24) can be achieved if 2^{n_1} is sufficiently large.

We need to say something about the points where

(2.25) $\dfrac{\phi_{n_1}(x+l)\psi_{k'_j}(x+2l)}{l} \geq 1$.

Since $\phi_{n_1}(x+l) = 2^{n_1}$ if $x + l \in \mathrm{spt}\,(\phi_{n_1})$ and $\psi_{k'_j}(x+2l) = 2^{k'_j}$ if $x + 2l \in \mathrm{spt}\,(\psi_{k'_j})$ we need to consider $l \leq 2^{n_1+k'_j}$.

A few words about our general plan. We will be interested in certain intervals $[t_0 2^{n_1}, (t_0+1)2^{n_1}) \subset \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$, the exact assumption about these intervals will be given in (2.29). In these intervals $[t_0 2^{n_1}, (t_0+1)2^{n_1})$ we consider sets $\mathcal{X}_{t_0}$ satisfying (2.48). These sets are unions of the subsets $\mathcal{X}_{t_0,t_1}$ see (2.46). For the sets $\mathcal{X}_{t_0,t_1}$ we have (2.40). For fixed t_0 but different t_1's the sets $\mathcal{X}_{t_0,t_1}$ are sufficiently independent, so we have (2.43). Based on this we can obtain a lower estimate of the measure of $\mathcal{X}_{t_0}$ see (2.44), (2.45) and (2.47).

2.3. Substep 1c: The auxiliary sets $X'(j, n_1 + k'_j)$, $X(1/2, 3/4, j)$, $X(1/2, 3/4, j, e)$ and $X(1/2, 3/4, j, o)$

We can suppose that $k_j \gg k'_j = k_{j-1} + 10$ is so large that for most points $x \in \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$ we have $[x, x + 2 \cdot 2^{n_1+k'_j} + 2^{n_1}) \subset \mathrm{ispt}_{k_j}(\phi_{n_1})$. We denote the set of these x's by $X'(j, n_1 + k'_j)$. Hence, if k_j is sufficiently larger than k'_j then

$$(2.26) \qquad 1 \approx \frac{\overline{\lambda}(X'(j, n_1 + k'_j))}{\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1}))} > \frac{1}{2} \,.$$

We remark that our construction implies that in (2.26) the choice of k_j does not depend on the choice of n_1, which means that one can choose a $K_{j,0}$ such that for all $k_j \geq K_{j,0}$ we have (2.26) for any n_1. Denote by $X(1/2, 3/4, j)$ the set of those x for which there exists $t \in \mathbb{Z}$ such that

$$(2.27) \qquad x \in [(t + \tfrac{1}{2})2^{n_1+k_{j-1}}, (t + \tfrac{3}{4})2^{n_1+k_{j-1}}).$$

We split $X(1/2, 3/4, j)$ into two subsets depending on the parity of t.

If (2.27) holds and t is even then $x \in X(1/2, 3/4, j, e)$, while for odd t's $x \in X(1/2, 3/4, j, o)$.

Suppose

$$x, y \in X(1/2, 3/4, j), \; x < y, \; x \in X(1/2, 3/4, j, e), \; y \in X(1/2, 3/4, j, o),$$

and $l = \lfloor y \rfloor - \lfloor x \rfloor > 0$. Then, as the reader can verify, $\lfloor x \rfloor \in X(1/2, 3/4, j, e)$, $y' \overset{\mathrm{def}}{=} x + l \in X(1/2, 3/4, j, o)$ and $x + 2l \in [2t' \cdot 2^{n_1+k_{j-1}}, (2t'+1)2^{n_1+k_{j-1}})$ with a $t' \in \mathbb{Z}$. Hence,

$$(2.28) \quad \begin{aligned} &\text{if } x \in X'(j, n_1 + k'_j) \cap X(1/2, 3/4, j, e) \text{ and} \\ &y \in X(1/2, 3/4, j, o), \; 0 < l = \lfloor y \rfloor - \lfloor x \rfloor \leq 2^{n_1+k'_j} \text{ then} \\ &y' = x + l \in \mathrm{ispt}_{k_{j-1}}(\phi_{n_1}) \text{ and } x + 2l \in \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1}). \end{aligned}$$

2.4. Substep 1d: Estimate of the measure of those points where (2.25) holds in one 2^{n_1} grid interval, definition of the sets $\mathcal{X}_{t_0, t_1}$

Suppose

$$(2.29) \qquad [t_0 2^{n_1}, (t_0 + 1)2^{n_1}) \subset X'(j, n_1 + k'_j) \cap X(1/2, 3/4, j, e).$$

We want to obtain an estimate of the measure of those x's for which (2.25) holds. By (2.29) there exists $t'_0 \in \mathbb{Z}$ such that

$$(2.30) \qquad [t_0 2^{n_1}, (t_0 + 1)2^{n_1}) \subset \left[\left(2t'_0 + \tfrac{1}{2}\right)2^{n_1+k_{j-1}}, \left(2t'_0 + \tfrac{3}{4}\right)2^{n_1+k_{j-1}} \right).$$

Suppose

(2.31) $y \in X(1/2, 3/4, j, o) \cap \operatorname{spt}(\phi_{n_1})$ with $0 < l = \lfloor y \rfloor - \lfloor x \rfloor \le 2^{n_1 + k_j'}$.

Then $\phi_{n_1}(y) = 2^{n_1}$ and by (2.15)

(2.32)   there exists $t_1 \in \mathbb{Z}$ such that $y \in [t_1 2^{n_1}, t_1 2^{n_1} + 1)$.

Moreover, $l = t_1 2^{n_1} - \lfloor x \rfloor$ and $y' = x + l \in [t_1 2^{n_1}, t_1 2^{n_1} + 1)$ as well which implies $\phi_{n_1}(x + l) = \phi_{n_1}(y') = 2^{n_1}$ and $|y - y'| < 1$.
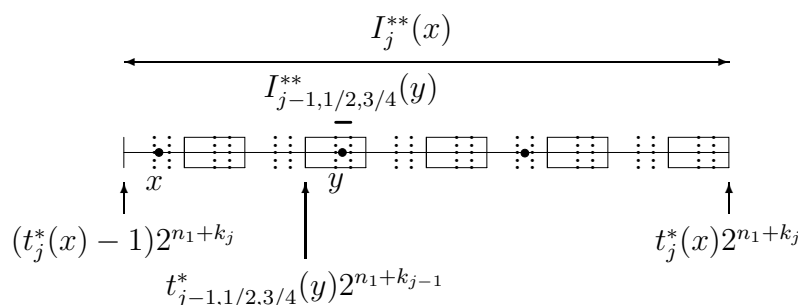

Figure 2: Notation related to $I_j^{**}(x)$

Denote by

(2.33) $I_j^{**}(x) = [(t_j^*(x) - 1)2^{n_1 + k_j}, t_j^*(x)2^{n_1 + k_j})$

$\qquad$ the component of $\operatorname{ispt}_{k_j}(\phi_{n_1})$ containing x.

By the definition of $X'(j, n_1 + k_j')$ the points $y' = x + l$, and $x + 2l$ belong to $I_j^{**}(x)$, that is, x, $y' = x + l$, and $x + 2l$ belong to the same component of $\operatorname{ispt}_{k_j}(\phi_{n_1})$. Moreover, by (2.28), $y' = x + l \in \operatorname{ispt}_{k_{j-1}}(\phi_{n_1})$ and $x + 2l \in \operatorname{ispt}_{k_j}(\phi_{n_1}) \setminus \operatorname{ispt}_{k_{j-1}}(\phi_{n_1})$. We introduce the notation

$$I_{j-1,1/2,3/4}^{**}(y) = \left[\left(t_{j-1,1/2,3/4}^*(y) + \frac{1}{2} \right) 2^{n_1 + k_{j-1}}, \left(t_{j-1,1/2,3/4}^*(y) + \frac{3}{4} \right) 2^{n_1 + k_{j-1}} \right)$$

for the interval containing $y \in X(1/2, 3/4, j)$. Then $y' \in I_{j-1,1/2,3/4}^{**}(y)$ holds as well. On Figure 2, $I_{j-1,1/2,3/4}^{**}(y)$ is the tiny interval containing y, its length is marked by a short line segment above y. Keep in mind that we supposed that $j \ge 2$. By our construction if

(2.34) $I^* = \left[\left(t^* + \frac{1}{2} \right) 2^{n_1 + k_{j-1}}, \left(t^* + \frac{3}{4} \right) 2^{n_1 + k_{j-1}} \right)$ is a component of

$\qquad \operatorname{ispt}_{k_{j-1}}(\phi_{n_1}) \cap X(1/2, 3/4, j)$ then

(2.35) $\lambda(\operatorname{ispt}_0(\phi_{n_1}) \cap I^*) = \left(\frac{1}{2} \right)^{j-1} \lambda(I^*) = \left(\frac{1}{2} \right)^{j-1} 2^{n_1 + k_{j-1} - 2}.$

This implies that there exist exactly

$$(2.36) \quad \left(\frac{1}{2}\right)^{j-1} 2^{k_{j-1}-2} \text{ many } t_1\text{'s such that } [t_1 2^{n_1}, t_1 2^{n_1} + 1) \subset \mathrm{spt}\,(\phi_{n_1}) \cap I^*.$$

We denote the set of these t_1's by $\mathcal{T}_1(I^*)$.

We still assume that t_0 satisfies (2.29). By (2.12) in the interval $[t_0 2^{n_1} + 2^{n_1+k'_j-1}, t_0 2^{n_1} + 2^{n_1+k'_j})$ there are $2^{k'_j-k_{j-1}-2}$ many I^*'s satisfying (2.34). Denote the set of the corresponding t^*'s by $\mathcal{T}^*(t_0)$. Finally, denote by $\mathcal{T}_1(t_0)$ the set of those t_1 which belong to a $\mathcal{T}_1(I^*)$ with I^* of the form in (2.34) and $t^* \in \mathcal{T}^*(t_0)$. Then by (2.36)

$$(2.37) \quad \#\mathcal{T}_1(t_0) = \left(\frac{1}{2}\right)^{j-1} 2^{k_{j-1}-2} 2^{k'_j-k_{j-1}-2} = \left(\frac{1}{2}\right)^j 2^{k'_j-3}.$$

Next suppose $t_1 \in \mathcal{T}_1(t_0)$ is fixed. If $y \in [t_1 2^{n_1}, t_1 2^{n_1} + 1)$ then by (2.36) $\phi_{n_1}(y) = 2^{n_1}$. For $x \in [t_0 2^{n_1}, (t_0+1)2^{n_1})$ set $l_{x,t_1} = t_1 2^{n_1} - \lfloor x \rfloor = \lfloor y \rfloor - \lfloor x \rfloor$ and $y' = x + l_{x,t_1}$. Then $\phi_{n_1}(x + l_{x,t_1}) = \phi_{n_1}(y') = \phi_{n_1}(y) = 2^{n_1}$. From $t_1 \in \mathcal{T}_1(t_0)$ it follows that $[t_1 2^{n_1}, t_1 2^{n_1} + 1) \subset [t_0 2^{n_1} + 2^{n_1+k'_j-1}, t_0 2^{n_1} + 2^{n_1+k'_j})$. This and $x \in [t_0 2^{n_1}, (t_0 + 1)2^{n_1})$ implies

$$(2.38) \quad 0 < l_{x,t_1} = t_1 2^{n_1} - \lfloor x \rfloor \leq (t_1 - t_0)2^{n_1} < 2^{n_1+k'_j} < 2^{n_1+k_M}.$$

We have $\phi_{n_1}(x + l_{x,t_1}) = 2^{n_1}$ and

$$(2.39) \quad \frac{\phi_{n_1}(x + l_{x,t_1})\psi_{k'_j}(x + 2l_{x,t_1})}{l_{x,t_1}} \geq 1$$

if $x + 2l_{x,t_1} \in \mathrm{spt}\,(\psi_{k'_j})$. For $t_1 \in \mathcal{T}_1(t_0)$ denote by $\mathcal{X}_{t_0,t_1}$ the set of those $x \in [t_0 2^{n_1}, (t_0 + 1)2^{n_1})$ for which $x + 2l_{x,t_1} \in \mathrm{spt}\,(\psi_{k'_j})$. This means that

$$(2.40) \quad \begin{array}{l} \text{for } x \in \mathcal{X}_{t_0,t_1} \subset [t_0 2^{n_1}, (t_0 + 1)2^{n_1}) \text{ there exists } l_{x,t_1} \text{ such that} \\ (2.39) \text{ holds and } y' = x + l_{x,t_1} \in [t_1 2^{n_1}, t_1 2^{n_1} + 1). \end{array}$$

2.5. Substep 1e: "Periodicity and independence" of the sets $\mathcal{X}_{t_0,t_1}$

Observe that if $x \in [t_0 2^{n_1}, (t_0 + 1)2^{n_1})$ then $\lfloor x \rfloor \in [t_0 2^{n_1}, (t_0 + 1)2^{n_1} - 1]$ and $x + 2l_{x,t_1} \in [(2t_1 - t_0 - 1)2^{n_1} + 1, (2t_1 - t_0)2^{n_1} + 1)$, moreover by (2.23) $\mathrm{spt}\,(\psi_{k'_j})$ is "periodic" by $\pi(2t_1 - t_0)$, this implies that

$$(2.41) \quad \begin{array}{l} \text{if } x, x + \pi(2t_1 - t_0) \in [t_0 2^{n_1}, (t_0 + 1)2^{n_1}) \text{ then} \\ x \in \mathcal{X}_{t_0,t_1} \text{ iff } x + \pi(2t_1 - t_0) \in \mathcal{X}_{t_0,t_1}. \end{array}$$
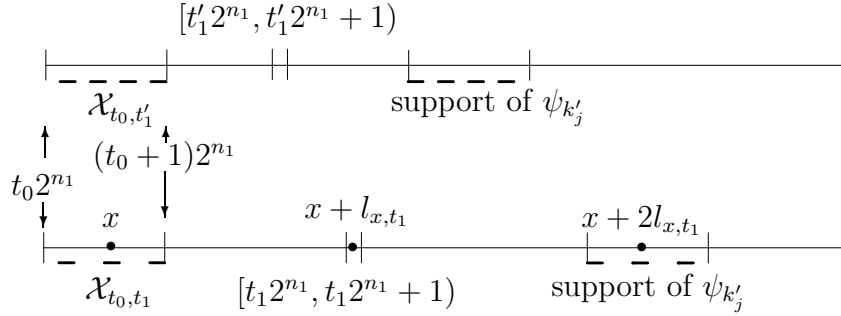
Figure 3: Sets $\mathcal{X}_{t_0,t_1}$ and $\mathcal{X}_{t_0,t_1'}$

From (2.24) it follows that

$$(2.42) \qquad \frac{1+\rho}{2}\epsilon 2^{-k_j'} \cdot 2^{n_1} < \lambda(\mathcal{X}_{t_0,t_1}) < \epsilon 2^{-k_j'} \cdot 2^{n_1}.$$

In (2.43) below it will be useful to keep in mind that the "density" of $\mathcal{X}_{t_0,t_1'}$ in $[t_0 2^{n_1}, (t_0+1)2^{n_1})$ is $\lambda(\mathcal{X}_{t_0,t_1'})/2^{n_1}$ and (2.42) holds for t_1' as well. By (2.21), (2.38) and (2.41) for $t_1, t_1' \in \mathcal{T}_1(t_0)$, $t_1 \neq t_1'$ the "periods" of $\mathcal{X}_{t_0,t_1}$ and $\mathcal{X}_{t_0,t_1'}$ are relatively prime, see Figure 3. By Lemma 5 if n_1 is sufficiently large these sets are independent in the sense that

$$(2.43) \qquad \lambda(\mathcal{X}_{t_0,t_1} \cap \mathcal{X}_{t_0,t_1'}) < \lambda(\mathcal{X}_{t_0,t_1})2\frac{\lambda(\mathcal{X}_{t_0,t_1'})}{2^{n_1}} < 2\epsilon 2^{-k_j'}\lambda(\mathcal{X}_{t_0,t_1}).$$

Hence, using (2.37) and (2.43) we infer

$$(2.44) \quad \lambda(\mathcal{X}_{t_0,t_1} \setminus \bigcup_{t_1' \neq t_1,\ t_1' \in \mathcal{T}_1(t_0)} \mathcal{X}_{t_0,t_1'}) > \left(1 - 2\epsilon \cdot 2^{-k_j'}\left(\frac{1}{2}\right)^j 2^{k_j'-3}\right)\lambda(\mathcal{X}_{t_0,t_1})$$

$$> \frac{1}{2}\lambda(\mathcal{X}_{t_0,t_1}).$$

This, (2.37), and (2.42) imply that

$$(2.45) \qquad \lambda(\bigcup_{t_1 \in \mathcal{T}_1(t_0)} \mathcal{X}_{t_0,t_1}) > \frac{1}{2}\sum_{t_1 \in \mathcal{T}(t_0)} \lambda(\mathcal{X}_{t_0,t_1})$$

$$> \frac{1}{4}\epsilon 2^{-k_j'} \cdot 2^{n_1}\left(\frac{1}{2}\right)^j \cdot 2^{k_j'-3} = \frac{\epsilon}{32}2^{n_1}\left(\frac{1}{2}\right)^j.$$

This means that in each interval $I(t_0) = [t_0 2^{n_1}, (t_0+1)2^{n_1})$ satisfying (2.29) we could find a set

$$(2.46) \qquad \mathcal{X}_{t_0} = \bigcup_{t_1 \in \mathcal{T}_1(t_0)} \mathcal{X}_{t_0,t_1},$$

such that

$$(2.47) \qquad \lambda(\mathcal{X}_{t_0}) > \frac{\epsilon}{32}\left(\frac{1}{2}\right)^j \lambda(I(t_0))$$

and

$$(2.48) \qquad \text{for } x \in \mathcal{X}_{t_0}, \ \exists l \leq 2^{n_1+k_M} \text{ such that } \frac{\phi_{n_1}(x+l)\psi_{k_j'}(x+2l)}{l} \geq 1.$$

2.6. Substep 1f: Conclusion of the $s=1$ case

Denote by $\mathcal{I}(j)$ the union of all intervals in $\mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$ which satisfy (2.29). Using (2.13), (2.14), (2.26) and (2.27) we obtain

$$(2.49) \qquad \overline{\lambda}(\mathcal{I}(j)) > \frac{1}{16}\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_1})) = \frac{1}{16}2^{-M+j}.$$

Denote by $\overline{\mathcal{X}}(j)$ the set of those $x \in \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$ for which there exists l such that (2.48) holds. Then (2.47) and (2.49) imply that

$$(2.50) \qquad \overline{\lambda}(\overline{\mathcal{X}}(j)) > \frac{\epsilon}{512}\left(\frac{1}{2}\right)^j \cdot 2^{-M+j} = \frac{\epsilon}{512}2^{-M}.$$

The sets $\overline{\mathcal{X}}(j) \subset \mathrm{ispt}_{k_j}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_1})$ and $\overline{\mathcal{X}}(j') \subset \mathrm{ispt}_{k_{j'}}(\phi_{n_1}) \setminus \mathrm{ispt}_{k_{j'-1}}(\phi_{n_1})$ are disjoint when $j \neq j'$ and hence using (2.50) for $j = 2, ..., M$ one can see that (2.1) holds with $s = 1$. We set $\omega = k_M'$. From (2.17) it follows (2.2) when $s = 1$. From (2.18), (2.19), (2.20) and (2.24) it follows (2.3).

2.7. Step 2: The general step of the induction
Substep 2a: usage of the functions from step $s-1$ of the induction

Next we turn to the general step of our induction. Suppose that for any possible choice of $\widetilde{\alpha}$, $\widetilde{\mathcal{P}}$, $\widetilde{\rho}$, $0 < \widetilde{\epsilon} < \epsilon_0$, and $\widetilde{M}$ one can find $\widetilde{\alpha} - \widetilde{\mathcal{P}} - (s-1)$-families. In case in (2.1), $\overline{\lambda}\{.\} > \epsilon$ holds for an $\alpha - \mathcal{P} - (s-1)$-family then one can define an $\alpha - \mathcal{P} - s$-family by choosing an almost arbitrary $\phi_{n_{s-1}+1} = \phi_{n_s}$ so that (2.2) holds. So we can assume that we work with $\alpha - \mathcal{P} - (s-1)$-families for which in (2.1), $\overline{\lambda}\{.\} \leq \epsilon$ holds. We define ϕ_{n_s} analogously to ϕ_{n_1}.

The interval supports are defined by

$$(2.51) \qquad \mathrm{ispt}_k(\phi_{n_s}) = \bigcup \left\{ [(t-1)2^{n_s+k}, t2^{n_s+k}) : \right.$$
$$\left. t \in \mathbb{Z}, \ \mathrm{spt}(\phi_{n_s}) \cap [(t-1)2^{n_s+k}, t2^{n_s+k}) \neq \emptyset \right\}.$$

We set

$$\mathrm{ispt}_{k_M}(\phi_{n_s}) = \mathbb{R}.$$

(2.52)   For $j \in \{1, ..., M\}$ an interval $[(t-1)2^{n_s+k_{j-1}}, t2^{n_s+k_{j-1}}) \subset \mathrm{ispt}_{k_j}(\phi_{n_s})$ belongs to $\mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$ if and only if t is even.

This implies

(2.53) $$\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_s})) = \left(\frac{1}{2}\right)^{M-j},$$

and

(2.54) $\overline{\lambda}(\mathrm{ispt}_{k_{j-1}}(\phi_{n_s})) = \overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})) = \dfrac{1}{2}\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_s})).$

It will be useful to keep in mind for further reference that $t-1$, which corresponds to the left endpoint of the support intervals is odd.

We will define ϕ_{n_s} so that it will be periodic by $2^{n_s+k_M} \in \mathcal{P}$.

If $[(t-1)2^{n_s}, t2^{n_s}) \subset \mathrm{ispt}_0(\phi_{n_s}) = \mathrm{ispt}_{k_0}(\phi_{n_s})$ then

(2.55) $\phi_{n_s}(x) = 2^{n_s}$ if $x \in [(t-1)2^{n_s}, (t-1)2^{n_s}+1)$, and
(2.56) $\phi_{n_s}(x) = 0$ if $x \in [(t-1)2^{n_s}+1, t \cdot 2^{n_s})$.

From (2.53) and (2.54) used with $j = 1$ it follows that

(2.57) $$\overline{\int} \phi_{n_s} = 2^{-M}.$$

We need more assumptions about $k_0 \ll \cdots \ll k_M$. Set $\alpha_1 = \alpha$. Choose an $\alpha_1 - \mathcal{P} - (s-1)$-family with ψ-interval $[\alpha_1, \omega_1]$ periodic by $p_{1,s-1} \in \mathcal{P}$. Recall that $\mathcal{P}$ consists of powers of 2 and hence $p_{1,s-1}$ is also a power of 2. As we remarked earlier we can suppose that for this family in (2.1), $\overline{\lambda}\{.\} \leq \epsilon$ holds. We denote the functions corresponding to this family by $\phi_{1,1}, ..., \phi_{1,n_{1,s-1}}$, $\psi_{1,\alpha_1}, ..., \psi_{1,\omega_1}$. For other parameters belonging to this family we adopt a similar subscript notation. We choose k_1 so that $k_1 \gg p_{1,s-1} \geq \omega_1$. If $x \in \mathrm{ispt}_{k_1}(\phi_{n_s})$ then we set $\phi_i(x) = \phi_{1,i}(x)$ for $i = 1, ..., n_{1,s-1}$ and $\psi_k(x) = \psi_{1,k}(x)$ for $k = \alpha_1, ..., \omega_1$, moreover, for other k's we set $\psi_k(x) = 0$ and for $i = n_{1,s-1}+1, ..., n_s - 1$ we set $\phi_i(x) = 0$. If k_1 is sufficiently larger than $p_{1,s-1}$ then the length of the components of $\mathrm{ispt}_{k_1}(\phi_{n_s})$ will be a multiple of $p_{1,s-1}$ and hence (2.2) and (2.3) will stay valid "relative to" $\mathrm{ispt}_{k_1}(\phi_{n_s})$.

By this we mean the following:

$$(2.58) \qquad (s-1) \cdot 2^{-M-1}\overline{\lambda}(\mathrm{ispt}_{k_1}(\phi_{n_s})) < \sum_{i=1}^{n_s-1} \overline{\int} \phi_i\big|_{\mathrm{ispt}_{k_1}(\phi_{n_s})}$$
$$< (s-1) \cdot 2^{-M+1}\overline{\lambda}(\mathrm{ispt}_{k_1}(\phi_{n_s})),$$

and

$$(2.59) \qquad \rho\epsilon\overline{\lambda}(\mathrm{ispt}_{k_1}(\phi_{n_s})) < \sum_{k=\alpha_1}^{\omega_1} \overline{\int} \psi_k\big|_{\mathrm{ispt}_{k_1}(\phi_{n_s})}$$
$$= \sum_{k=\alpha}^{\omega} \overline{\int} \psi_k\big|_{\mathrm{ispt}_{k_1}(\phi_{n_s})} < \epsilon\overline{\lambda}(\mathrm{ispt}_{k_1}(\phi_{n_s})).$$

Suppose $j \geq 2$ and k_{j-1} is defined. Let $\alpha_j = k_{j-1} + 10$ and choose an $\alpha_j - \mathcal{P} - (s-1)$-family with ψ-interval $[\alpha_j, \omega_j]$. Again, as we remarked earlier we can suppose that for this family $\overline{\lambda}\{.\} \leq \epsilon$ holds in (2.1). Denote the corresponding functions by $\phi_{j,1}, ..., \phi_{j,n_{j,s-1}}, \psi_{j,\alpha_j}, ..., \psi_{j,\omega_j}$ periodic by $p_{j,s-1}$. We will choose $k_j \gg p_{j,s-1} \geq \omega_j$.

We repeat the above steps for $j = 2, ..., M$ and obtain

$$(2.60) \quad k_M \gg p_{M,s-1} \geq \omega_M \geq \alpha_M \gg k_{M-1} \gg \cdots$$
$$\gg k_j \gg p_{j,s-1} \geq \omega_j \geq \alpha_j \gg k_{j-1} \gg \cdots \gg k_1 \gg p_{1,s-1} \geq \omega_1.$$

The ψ-interval of our $\alpha - \mathcal{P} - s$-family will be defined by $\alpha = \alpha_1$ and $\omega = \omega_M$.

In the end we choose n_s so large that

$$(2.61) \qquad\qquad n_s > \max\{n_{j,s-1} : j = 1, ..., M\}.$$

Similarly to the $s = 1$ case our parameters are chosen in the order $\alpha = \alpha_1$, ω_1, $p_{1,s-1}$, k_1, ..., k_{j-1}, α_j, ω_j, $p_{j,s-1}$, k_j, ..., k_M, and then we choose some new parameters, the $\pi(t,j)$'s and finally we fix a large n_s. At the j'th step we have a k_{j-1} which determines $\alpha_j = k_{j-1}+10$. The $\alpha_j - \mathcal{P} - (s-1)$ family provides ω_j and $p_{j,s-1}$. If we have the value of $p_{j,s-1}$ then we choose k_j so that (2.83) holds. We emphasize again that, similarly to (2.26), (2.83) does not depend on the choice of n_s.

Since $\mathcal{P}$ consists of powers of 2 all $p_{j,s-1}$ are powers of 2. To each $j = 1, ..., M$ and each interval of the form $[(t-1)2^{n_s}, t2^{n_s})$, $(t \in \mathbb{Z})$ we assign a set $\mathcal{P}(t,j)$ consisting of infinitely many odd numbers such that if $\pi(t,j) \in \mathcal{P}(t,j)$ then

$$(2.62) \qquad\qquad \pi(t,j) \gg 2^{k_M} p_{M,s-1}$$

and

$$(2.63) \qquad \begin{aligned} &\text{if } t \not\equiv t' \bmod 2^{k_M+1}, \ \pi(t,j) \in \mathcal{P}(t,j), \pi(t',j) \in \mathcal{P}(t',j) \\ &\text{then } (\pi(t,j), \pi(t',j)) = 1. \end{aligned}$$

On the other hand,

(2.64)   if $t \equiv t' \bmod 2^{k_M+1}$ then $\mathcal{P}(t, j) = \mathcal{P}(t', j)$,

moreover

(2.65) $(\pi(t, j), \pi(t', j')) = 1$ if $j \neq j'$, $\pi(t, j) \in \mathcal{P}(t, j)$, $\pi(t', j') \in \mathcal{P}(t', j')$.

We have already defined for $x \in \operatorname{ispt}_{k_1}(\phi_{n_s})$ the values $\phi_i(x)$ for $i = 1, ..., n_s$ and $\psi_k(x)$ for all k.

We also want to define these functions when $x \notin \operatorname{ispt}_{k_1}(\phi_{n_s})$, that is, there exists a $j \geq 2$ such that $x \in \operatorname{ispt}_{k_j}(\phi_{n_s}) \setminus \operatorname{ispt}_{k_{j-1}}(\phi_{n_s})$.

Suppose $j \geq 2$. For any $t \in \mathbb{Z}$ from $\mathcal{P}(t, j)$ choose and fix a number $\pi(t, j)$ such that by Lemma 4 applied to the $\alpha_j - \mathcal{P} - (s-1)$-family with ψ-interval $[\alpha_j, \omega_j]$ we can select an $\alpha_j - \mathcal{P}(t, j) - (s-1)$-family with ψ-interval $[\alpha_j, \omega_j]$ periodic by $\pi(t, j)$. Denote the corresponding functions by $\phi_{t,j,1}, ..., \phi_{t,j,n_{j,s-1}}$, $\psi_{t,j,\alpha_j}, ..., \psi_{t,j,\omega_j}$. By (2.64) we can suppose that

(2.66)
$$\begin{aligned}
&\text{if } t \equiv t' \bmod 2^{k_M+1} \text{ then } \pi(t, j) = \pi(t', j),\\
&\phi_{t,j,i} = \phi_{t',j,i} \text{ for } i = 1, ..., n_{j,s-1},\\
&\text{and } \psi_{t,j,k} = \psi_{t',j,k} \text{ for } k = \alpha_j, ..., \omega_j.
\end{aligned}$$

We also have (2.62). By (2.63) and (2.65)

(2.67)
$$\begin{aligned}
&\text{if } t \not\equiv t' \bmod 2^{k_M+1} \text{ then } (\pi(t, j), \pi(t', j)) = 1, \text{ moreover}\\
&(\pi(t, j), \pi(t', j')) = 1 \text{ if } j \neq j', \text{ for any choice of } t \text{ and } t'.
\end{aligned}$$

Based on (2.5) we can suppose that the $\pi(t, j)$'s are chosen so large that

(2.68) $$\overline{\int} \psi_{t,j,k} \approx \overline{\int} \psi_{j,k}, \text{ for all } t, j, k.$$

By using $\rho\epsilon < \sum_{k=\alpha_j}^{\omega_j} \overline{\int} \psi_{j,k} < \epsilon$ and (2.68) we can obtain for large $\pi(t, j)$'s

(2.69) $$\rho\epsilon < \sum_{k=\alpha_j}^{\omega_j} \min_t \overline{\int} \psi_{t,j,k} \leq \sum_{k=\alpha_j}^{\omega_j} \max_t \overline{\int} \psi_{t,j,k} < \epsilon.$$

Similarly, based on (2.4) we can suppose that the $\pi(t, j)$'s are chosen so large that

(2.70) $$\overline{\int} \phi_{t,j,i} \approx \overline{\int} \phi_{j,i}, \text{ for all } t, j, i.$$

By using $(s-1)2^{-M-1} < \sum_{i=1}^{n_{s-1}} \overline{\int} \phi_{j,i} < (s-1)2^{-M+1}$ and (2.70) we can obtain for large $\pi(t,j)$'s

$$(2.71) \quad (s-1)2^{-M-1} < \sum_{i=1}^{n_{j,s-1}} \min_t \overline{\int} \phi_{t,j,i} \le \sum_{i=1}^{n_{j,s-1}} \max_t \overline{\int} \phi_{t,j,i} < (s-1)2^{-M+1}.$$

We suppose that

$$(2.72) \qquad \text{if } \pi^* = \max_{t,j} \pi(t,j) \text{ then } 2^{n_s} \gg (\pi^*)^{k_M}$$

(this means that n_s should be sufficiently large).

2.8. Substep 2b: Definitions of ϕ_i, $i = 1, ..., n_{j,s-1}$ and ψ_k, $k = \alpha_j, ..., \omega_j$

Suppose

$$(2.73) \qquad x \in [(t-1)2^{n_s} + 1, t2^{n_s} + 1) \cap (\mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})).$$

Set

$$(2.74) \qquad \begin{aligned} \phi_i(x) &= \phi_{t,j,i}(x) \text{ for } i = 1, ..., n_{j,s-1}, \text{ and} \\ \psi_k(x) &= \psi_{t,j,k}(x) \text{ for } k = \alpha_j, ..., \omega_j. \end{aligned}$$

For other k's set $\psi_k(x) = 0$.

$$(2.75) \qquad \text{We also put } \phi_i(x) = 0 \text{ for } n_{j,s-1} < i < n_s.$$

See Figure 4. It is also clear that by the definition of interval supports we have

$$(2.76) \qquad \phi_{n_s}(x) = 0 \text{ for any } x \in \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}).$$

By the periodicity assumptions about our corresponding $s-1$ families we have

$$(2.77) \qquad \begin{aligned} &\text{if } x, x + \pi(t,j) \in [(t-1)2^{n_s} + 1, t2^{n_s} + 1) \cap \\ &\quad (\mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})) \text{ then } \psi_k(x) = \psi_k(x + \pi(t,j)). \end{aligned}$$

If n_s is sufficiently large then by our induction step about $\alpha_j - \mathcal{P}(t,j) - (s-1)$-families we have

$$(2.78) \quad \overline{\lambda}\left\{ x : \max_{k \in [\alpha_1, \omega_M]} \max_{l \le \pi^*} \frac{\sum_{i=1}^{n_{s-1}} \phi_i(x+l)\psi_k(x+2l)}{l} \ge 1 \right\}$$
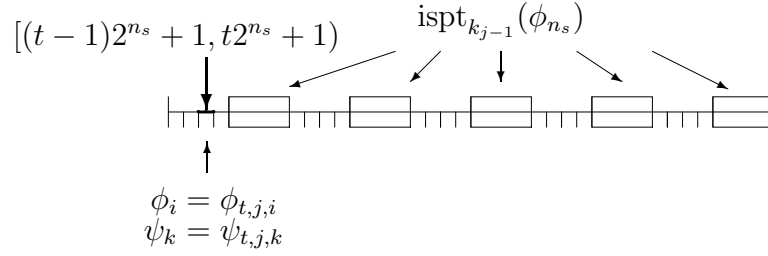$$> \frac{(s-1)(M-1)\epsilon 2^{-M}}{2048}.$$

$$[(t-1)2^{n_s}+1, t2^{n_s}+1) \qquad \text{ispt}_{k_{j-1}}(\phi_{n_s})$$

$$\phi_i = \phi_{t,j,i}$$
$$\psi_k = \psi_{t,j,k}$$

Figure 4: Definition of ϕ_i and ψ_k in one component of $\text{ispt}_{k_j}(\phi_{n_s})$

By choosing n_s sufficiently large and keeping in mind (2.75), (2.76) we have a version of (2.70) and (2.71) with respect to $\text{ispt}_{k_j}(\phi_{n_s}) \setminus \text{ispt}_{k_{j-1}}(\phi_{n_s})$

$$(2.79) \quad (s-1) \cdot 2^{-M-1}\overline{\lambda}(\text{ispt}_{k_j}(\phi_{n_s}) \setminus \text{ispt}_{k_{j-1}}(\phi_{n_s})) <$$

$$< \sum_{i=1}^{n_s-1} \overline{\int} \phi_i \big|_{\text{ispt}_{k_j}(\phi_{n_s})\setminus\text{ispt}_{k_{j-1}}(\phi_{n_s})}$$

$$< (s-1) \cdot 2^{-M+1}\overline{\lambda}(\text{ispt}_{k_j}(\phi_{n_s}) \setminus \text{ispt}_{k_{j-1}}(\phi_{n_s})),$$

and

$$(2.80) \quad \rho\epsilon\overline{\lambda}(\text{ispt}_{k_j}(\phi_{n_s}) \setminus \text{ispt}_{k_{j-1}}(\phi_{n_s})) < \sum_{k=\alpha_1}^{\omega_1} \overline{\int} \psi_k \big|_{\text{ispt}_{k_j}(\phi_{n_s})\setminus\text{ispt}_{k_{j-1}}(\phi_{n_s})}$$

$$= \sum_{k=\alpha}^{\omega} \overline{\int} \psi_k \big|_{\text{ispt}_{k_j}(\phi_{n_s})\setminus\text{ispt}_{k_{j-1}}(\phi_{n_s})} < \epsilon\overline{\lambda}(\text{ispt}_{k_j}(\phi_{n_s}) \setminus \text{ispt}_{k_{j-1}}(\phi_{n_s})).$$

2.9. Substep 2c: Definitions related to ϕ_{n_s}, the sets $X'(j, n_s + \omega_j)$, $X(1/2, 3/4, j)$, $X(1/2, 3/4, j, e)$ and $X(1/2, 3/4, j, o)$

We need to say something about the set of those x's where

$$(2.81) \qquad \max_{k' \in [\alpha_j, \omega_j]} \frac{\phi_{n_s}(x+l)\psi_{k'}(x+2l)}{l} \geq 1.$$

We split the above set of x's into subsets of those x's where

$$(2.82) \qquad \frac{\phi_{n_s}(x+l)\psi_{k'}(x+2l)}{l} \geq 1, \text{ with } \omega_j \geq k' \geq \alpha_j > k_{j-1}.$$

Since $\phi_{n_s}(x+l) = 2^{n_s}$ if $x + l \in \text{spt}(\phi_{n_s})$ and $\psi_{k'}(x+2l) = 2^{k'}$ if $x + 2l \in \text{spt}(\psi_{k'})$ we need to consider $l \leq 2^{n_s+k'}$.

More words about our general plan. We will be interested in certain intervals $[t_0 2^{n_s}, (t_0 + 1) 2^{n_s}) \subset \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$, the exact assumption about these intervals will be given in (2.86). In these intervals $[t_0 2^{n_s}, (t_0 + 1) 2^{n_s})$ we consider the sets $\mathcal{X}'_{t_0}$ satisfying (2.120). By (2.117) these sets will be the unions of the auxiliary sets $\mathcal{X}'_{t_0, k'}$ defined in (2.114) via the sets $\mathcal{X}_{t_0, t_1, k'}$ and $\mathcal{X}_{t_0, s-1}$. For the sets $\mathcal{X}_{t_0, t_1, k'}$ with $\alpha_j \leq k' \leq \omega_j$ we have (2.101). For fixed t_0 and t_1 by (2.100) there can be at most one k' for which an $\mathcal{X}_{t_0, t_1, k'}$ is defined. For fixed t_0 but different t_1's and k''s the sets $\mathcal{X}_{t_0, t_1, k'}$ are sufficiently independent, so we have (2.106) and (2.107). It is a new feature that we also need to consider the sets $\mathcal{X}_{t_0, s-1}$ defined in (2.109). These sets take care of points coming from the $(s-1)$-families of earlier steps of the induction. They will also be sufficiently independent of the sets $\mathcal{X}_{t_0, t_1, k'}$, see (2.112) and then (2.113-2.118). Hence, in (2.119) we obtain our lower estimate of the measure of the sets $\mathcal{X}'_{t_0}$. By (2.121) these sets are disjoint from $\mathcal{X}_{t_0, s-1}$.

Now we return to the details of the proof of Theorem 6. One can assume similarly to the one family case the following: Suppose that $k_j \gg \omega_j \geq k'$ is so large that for most points of $x \in \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$ we have $[x - 2^{n_s}, x + 2 \cdot 2^{n_s + \omega_j} + 2^{n_s}) \subset \mathrm{ispt}_{k_j}(\phi_{n_s})$. We denote the set of these x's by $X'(j, n_s + \omega_j)$. If $k_j \gg \omega_j$ then

$$(2.83) \qquad 1 \approx \frac{\overline{\lambda}(X'(j, n_s + \omega_j))}{\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}))} > \frac{1}{2}.$$

Similarly to (2.26) this estimate does not depend on the choice of n_s. Denote by $X(1/2, 3/4, j)$ the set of those x for which there exists $t \in \mathbb{Z}$ such that $x \in [(t + \frac{1}{2}) 2^{n_s + k_{j-1}}, (t + \frac{3}{4}) 2^{n_s + k_{j-1}})$. We split $X(1/2, 3/4, j)$ into two subsets depending on the parity of t. If t is even then $x \in X(1/2, 3/4, j, e)$ and if t is odd then $x \in X(1/2, 3/4, j, o)$. Suppose $x, y \in X(1/2, 3/4, j)$, $x < y$, $x \in X(1/2, 3/4, j, e)$, $y \in X(1/2, 3/4, j, o)$ and $0 < l = \lfloor y \rfloor - \lfloor x \rfloor$. Then, as the reader can verify, $\lfloor x \rfloor \in X(1/2, 3/4, j, e)$, $\lfloor y \rfloor \in X(1/2, 3/4, j, o)$, $y' \stackrel{\mathrm{def}}{=} x + l \in X(1/2, 3/4, j, o)$ and $x + 2l \in [2t' \cdot 2^{n_s + k_{j-1}}, (2t' + 1) \cdot 2^{n_s + k_{j-1}})$ with a $t' \in \mathbb{Z}$.

Hence,

$$(2.84) \qquad \begin{aligned} &\text{if } x \in X'(j, n_s + k') \cap X(1/2, 3/4, j, e), \ y \in X(1/2, 3/4, j, o), \text{ and} \\ &0 < l = \lfloor y \rfloor - \lfloor x \rfloor \leq 2^{n_s + k'} \leq 2^{n_s + \omega_j} \text{ then} \\ &x \in \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}), \ y' = x + l \in \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}), \text{ and} \\ &x + 2l \in \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k-1}(\phi_{n_s}). \end{aligned}$$

Moreover, from $x \in [(t + \frac{1}{2}) 2^{n_s + k_{j-1}}, (t + \frac{3}{4}) 2^{n_s + k_{j-1}})$ with t even it follows

that

$$(2.85) \qquad [x - 2^{n_s}, x + 2^{n_s} + 1) \subset [t2^{n_s+k_{j-1}}, (t+1)2^{n_s+k_{j-1}})$$
$$\subset \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}).$$

2.10. Substep 2d: Estimate of the measure of those points where (2.82) holds in one 2^{n_s} grid interval, definition of the sets $\mathcal{X}_{t_0, t_1, k'}$

Suppose

$$(2.86) \qquad [t_0 2^{n_s}, (t_0+1)2^{n_s}) \subset X'(j, n_s + \omega_j) \cap X(1/2, 3/4, j, e).$$

By (2.85) used with $x = t_0 2^{n_s}$ we have

$$(2.87) \qquad [(t_0-1)2^{n_s}, (t_0+1)2^{n_s}+1) \subset \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}).$$

We want to obtain an estimate of the measure of those x's for which (2.82) and hence (2.81) holds for a suitable l.

By (2.86) there exists $t_0' \in \mathbb{Z}$ such that

$$(2.88) \qquad [t_0 2^{n_s}, (t_0+1)2^{n_s}) \subset \left[\left(2t_0' + \frac{1}{2}\right)2^{n_s+k_{j-1}}, \left(2t_0' + \frac{3}{4}\right)2^{n_s+k_{j-1}} \right).$$

Suppose

$$(2.89) \qquad y \in X(1/2, 3/4, j, o), \ 0 < l = \lfloor y \rfloor - \lfloor x \rfloor \leq 2^{n_s+k'} \leq 2^{n_s+\omega_j} \text{ and}$$
$$y \in \mathrm{spt}\,(\phi_{n_s}).$$

By (2.89), $\phi_{n_s}(y) = 2^{n_s}$ and by (2.55)

$$(2.90) \qquad \exists\, t_1 \in \mathbb{Z} \text{ such that } y \in [t_1 2^{n_s}, t_1 2^{n_s} + 1).$$

Moreover, $l = t_1 2^{n_s} - \lfloor x \rfloor$ and $y' = x + l \in [t_1 2^{n_s}, t_1 2^{n_s} + 1)$ as well, which implies $\phi_{n_s}(x + l) = \phi_{n_s}(y') = 2^{n_s}$. Denote by

$$(2.91) \qquad I_j^{**}(x) = [(t_j^*(x) - 1)2^{n_s+k_j}, t_j^*(x)2^{n_s+k_j})$$

the component of $\mathrm{ispt}_{k_j}(\phi_{n_s})$ containing x.

By the definition of $X'(j, n_s + \omega_j)$, $y' = x + l$ and $x + 2l$ are in $I_j^{**}(x)$, that is, x, $y' = x + l$, and $x + 2l$ belong to the same component of $\mathrm{ispt}_{k_j}(\phi_{n_s})$. Moreover, by (2.84), $y' = x + l \in \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$ and $x + 2l \in \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$. We introduce the notation

$$I_{j-1, 1/2, 3/4}^{**}(y) = \left[\left(t_{j-1, 1/2, 3/4}^*(y) + \frac{1}{2}\right)2^{n_s+k_{j-1}}, \left(t_{j-1, 1/2, 3/4}^*(y) + \frac{3}{4}\right)2^{n_s+k_{j-1}} \right)$$

for the interval containing $y \in X(1/2, 3/4, j)$. Then $y' \in I^{**}_{j-1,1/2,3/4}(y)$ holds as well. Recall that we still suppose $j \geq 2$. By our construction, see Section 2.1 and the second paragraph of 2.7 we have

(2.92)   if $I^* = \left[\left(t^* + \dfrac{1}{2}\right) 2^{n_s+k_{j-1}}, \left(t^* + \dfrac{3}{4}\right) 2^{n_s+k_{j-1}} \right)$ is a component of

$\mathrm{ispt}_{k_{j-1}}(\phi_{n_s}) \cap X(1/2, 3/4, j)$ then

(2.93) $\lambda(\mathrm{ispt}_0(\phi_{n_s}) \cap I^*) = \left(\dfrac{1}{2}\right)^{j-1} \lambda(I^*) = \left(\dfrac{1}{2}\right)^{j-1} 2^{n_s+k_{j-1}-2}.$

This implies that there exist

(2.94) $\left(\dfrac{1}{2}\right)^{j-1} 2^{k_{j-1}-2}$ many t_1's such that $[t_1 2^{n_s}, t_1 2^{n_s} + 1) \subset \mathrm{spt}(\phi_{n_s}) \cap I^*.$

We denote the set of these t_1's by $\mathcal{T}_1(I^*)$. In the interval $[t_0 2^{n_s} + 2^{n_s+k'-1}, t_0 2^{n_s} + 2^{n_s+k'})$ there exist $2^{k'-k_{j-1}-2}$ many I^*'s satisfying (2.92). Denote the set of the corresponding t^*'s by $\mathcal{T}^*(t_0, k')$. Finally, denote by $\mathcal{T}_1(t_0, k')$ the set of those t_1 which belong to a $\mathcal{T}_1(I^*)$ with I^* of the form in (2.92) and $t^* \in \mathcal{T}^*(t_0, k')$. Then

(2.95) $\#\mathcal{T}_1(t_0, k') = \left(\dfrac{1}{2}\right)^{j-1} 2^{k_{j-1}-2} \cdot 2^{k'-k_{j-1}-2} = \left(\dfrac{1}{2}\right)^{j} 2^{k'-3}.$

Next, suppose $t_1 \in \mathcal{T}_1(t_0, k')$ is fixed. Then, from $y \in [t_1 2^{n_s}, t_1 2^{n_s} + 1)$ it follows that $\phi_{n_s}(y) = 2^{n_s}$. For $x \in [t_0 2^{n_s}, (t_0 + 1) 2^{n_s})$ set

(2.96) $l_{x,t_1} = t_1 2^{n_s} - \lfloor x \rfloor = \lfloor y \rfloor - \lfloor x \rfloor.$

Then $\phi_{n_s}(x + l_{x,t_1}) = \phi_{n_s}(y') = \phi_{n_s}(y) = 2^{n_s}$. From $t_1 \in \mathcal{T}_1(t_0, k')$ it follows that

(2.97) $[t_1 2^{n_s}, t_1 2^{n_s} + 1) \subset I^* \subset [t_0 2^{n_s} + 2^{n_s+k'-1}, t_0 2^{n_s} + 2^{n_s+k'})$

with a suitable I^*. This implies

(2.98) $0 < t_1 2^{n_s} - (t_0 + 1) 2^{n_s} + 1 \leq l_{x,t_1} \leq t_0 2^{n_s} + 2^{n_s+k'} - t_0 2^{n_s} = 2^{n_s+k'}.$

We have $\phi_{n_s}(x + l_{x,t_1}) = 2^{n_s}$ and

(2.99) $\dfrac{\phi_{n_s}(x + l_{x,t_1}) \psi_{k'}(x + 2l_{x,t_1})}{l_{x,t_1}} \geq 1$
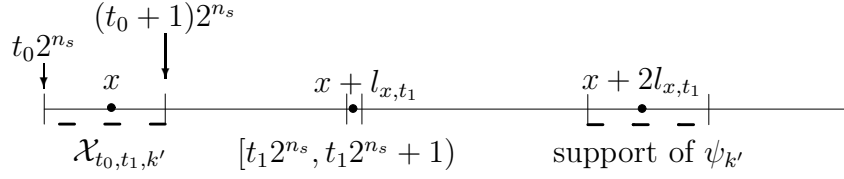
holds if $x + 2l_{x,t_1} \in \mathrm{spt}(\psi_{k'}).$

Figure 5: The set $\mathcal{X}_{t_0,t_1,k'}$

Denote by $\mathcal{X}_{t_0,t_1,k'}$ the set of those $x \in [t_0 2^{n_s}, (t_0 + 1)2^{n_s})$ for which $x + 2l_{x,t_1} \in \mathrm{spt}\,(\psi_{k'})$. By (2.97)

(2.100) for fixed t_0, t_1 there can be at most one k' for which $t_1 \in \mathcal{T}_1(t_0, k')$.

Moreover,

(2.101)   if $x \in \mathcal{X}_{t_0,t_1,k'} \subset [t_0 2^{n_s}, (t_0 + 1)2^{n_s})$ then (2.99) holds
$$\text{and } x + l_{x,t_1} \in [t_1 2^{n_s}, t_1 2^{n_s} + 1).$$

Dividing (2.98) by 2^{n_s} and rearranging we obtain that for $t_1 \in \mathcal{T}_1(t_0, k')$

$$0 < t_1 - t_0 \le 2^{k'} + 1 - \frac{1}{2^{n_s}},$$

since t_1 and t_0 are integers recalling (2.60) and (2.82) we have

(2.102) $0 < t_1 - t_0 \le 2^{k'} < 2^{\omega_j} < 2^{k_M}.$

2.11. Substep 2e: "Periodicity and independence" of the sets $\mathcal{X}_{t_0,t_1,k'}$

Observe that if $x \in [t_0 2^{n_s}, (t_0 + 1)2^{n_s})$ then $\lfloor x \rfloor \in [t_0 2^{n_s}, (t_0 + 1)2^{n_s} - 1]$ and by (2.96), $x + 2l_{x,t_1} \in [(2t_1 - t_0 - 1)2^{n_s} + 1, (2t_1 - t_0)2^{n_s} + 1)$, moreover by (2.77) in this interval

(2.103) $\psi_{k'}$ is "periodic" by $\pi(2t_1 - t_0, j).$

This implies that

(2.104)   if $x, x + \pi(2t_1 - t_0, j) \in [t_0 2^{n_s}, (t_0 + 1)2^{n_s})$ then
$x \in \mathcal{X}_{t_0,t_1,k'}$ iff $x + \pi(2t_1 - t_0, j) \in \mathcal{X}_{t_0,t_1,k'}.$

The "periodic density" of the measure of the support of $\psi_{2t_1-t_0,j,k'}$ in $[(2t_1-t_0-1)2^{n_s}+1, (2t_1-t_0)2^{n_s}+1)$ equals $2^{-k'}\overline{\int}\psi_{2t_1-t_0,j,k'}$. So the measure of

this support in an interval of length 2^{n_s} is approximately $(\overline{\int}\psi_{2t_1-t_0,j,k'})2^{-k'}2^{n_s}$ provided 2^{n_s} is much larger than $\pi(2t_1-t_0,j)$. Hence, we have by (2.5) that

$$(2.105) \quad \frac{1}{4}\left(\overline{\int}\psi_{2t_1-t_0,j,k'}\right)2^{-k'}2^{n_s} < \left(\overline{\int}\psi_{2t_1-t_0,j,k'}\right)2^{-k'}2^{n_s}$$

$$\approx \lambda(\mathcal{X}_{t_0,t_1,k'}) < 2\left(\overline{\int}\psi_{2t_1-t_0,j,k'}\right)2^{-k'}2^{n_s}.$$

Suppose $t_1 \in \mathcal{T}_1(t_0,k')$. Denote by $\mathcal{K}(t_0,t_1,k')$ the set of those $(t_1',k'') \neq (t_1,k')$ for which $t_1' \in \mathcal{T}_1(t_0,k'')$. Observe that if $t_1 = t_1'$ then from (2.100) it follows that for $k'' \neq k'$ we have $t_1 \notin \mathcal{T}_1(t_0,k'')$. Hence for $(t_1',k'') \in \mathcal{K}(t_0,t_1,k')$ we have $t_1' \neq t_1$. By (2.63), (2.102) and (2.104) if $(t_1',k'') \in \mathcal{K}(t_0,t_1,k')$ then the "periods" of $\mathcal{X}_{t_0,t_1,k'}$ and $\mathcal{X}_{t_0,t_1',k''}$ are relatively prime and these sets are "independent" in the sense that, using (2.105) and Lemma 5 for large n_s as well, we have

$$(2.106) \quad \lambda(\mathcal{X}_{t_0,t_1,k'} \cap \mathcal{X}_{t_0,t_1',k''}) < \lambda(\mathcal{X}_{t_0,t_1,k'}) \cdot 2 \cdot \frac{\lambda(\mathcal{X}_{t_0,t_1',k''})}{2^{n_s}}$$

$$< 4 \cdot 2^{-k''}\left(\overline{\int}\psi_{2t_1'-t_0,j,k''}\right)\lambda(\mathcal{X}_{t_0,t_1,k'}).$$

Using (2.69), (2.95), (2.100) and (2.106) we obtain

(2.107)

$$\lambda\Big(\mathcal{X}_{t_0,t_1,k'} \setminus \bigcup_{(t_1',k'')\in\mathcal{K}(t_0,t_1,k')} \mathcal{X}_{t_0,t_1',k''}\Big) >$$

$$> \left(1 - \sum_{k''=\alpha_j}^{\omega_j} 4\cdot 2^{-k''}\left(\max_t \overline{\int}\psi_{t,j,k''}\right)\left(\frac{1}{2}\right)^j 2^{k''-3}\right)\lambda(\mathcal{X}_{t_0,t_1,k'}) > \frac{3}{4}\lambda(\mathcal{X}_{t_0,t_1,k'}).$$

2.12. Substep 2f: Estimates of the measure of points where the maximal operator is large for the $(s-1)$-functions, the sets $\mathcal{X}_{t_0,s-1}$

We recall that by (2.84), (2.86) and (2.87)

$$[t_0 2^{n_s}, (t_0+1)2^{n_s}+1) \subset \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s}).$$

By (2.73), (2.74) and (2.75)

$$(2.108) \quad \text{on } [t_0 2^{n_s}+1, (t_0+1)2^{n_s}+1) \text{ the functions}$$
$$\phi_i, \psi_k, i = 1,...,n_{j,s-1}, k = \alpha_j,...,\omega_j \text{ are the restrictions of an}$$
$$\alpha_j - \mathcal{P}(t_0+1,j) - (s-1) - \text{family periodic by } \pi(t_0+1,j).$$

Set

$$\mathcal{X}_{t_0,s-1} = \{x \in [t_0 2^{n_s}, (t_0 + 1)2^{n_s}) :$$

(2.109)
$$\max_{k \in [\alpha_j,\omega_j]} \max_{l \le \pi^*} \frac{\sum_{i=1}^{n_{j,s-1}} \phi_i(x + l)\psi_k(x + 2l)}{l} \ge 1\}.$$

These are the "old" points in $[t_0 2^{n_s}, (t_0 + 1)2^{n_s})$ where the inequality from (2.109) holds. We will need those "new" points from the sets $\mathcal{X}_{t_0,t_1,k'}$ which do not belong to these "old" sets $\mathcal{X}_{t_0,s-1}$. By (2.108)

(2.110)
$$\text{if } x, x + \pi(t_0 + 1, j) \in [t_0 2^{n_s} + 1, (t_0 + 1)2^{n_s} - 2\pi^*) \text{ then}$$
$$x \in \mathcal{X}_{t_0,s-1} \text{ iff } x + \pi(t_0 + 1, j) \in \mathcal{X}_{t_0,s-1}.$$

In (2.72) we supposed that $\pi(t_0 + 1, j) \le \pi^* \ll 2^{n_s}$, so for most part of $[t_0 2^{n_s}, (t_0 + 1)2^{n_s})$ the set $\mathcal{X}_{t_0,s-1}$ is "periodic" by $\pi(t_0 + 1, j)$.

By our assumptions for our $\alpha_j - \mathcal{P}(t_0 + 1, j) - (s - 1)$-family $\overline{\lambda}\{.\} \le \epsilon < \epsilon_0 = \frac{1}{10}$ holds in (2.1) therefore, for sufficiently large n_s

(2.111)
$$\lambda(\mathcal{X}_{t_0,s-1}) = \lambda(\mathcal{X}_{t_0,s-1} \cap [t_0 2^{n_s}, (t_0 + 1)2^{n_s})) < \frac{1}{8}2^{n_s}.$$

2.13. Substep 2g: Independence estimates of "old" and "new" sets, that is, of $\mathcal{X}_{t_0,s-1}$ and $\mathcal{X}_{t_0,t_1,k'}$. Estimates of the "new contribution" set $\mathcal{X}'_{t_0}$

By (2.110), $\mathcal{X}_{t_0,s-1}$ is "periodic" by $\pi(t_0 + 1, j)$ while by (2.104), $\mathcal{X}_{t_0,t_1,k'}$ is "periodic" by $\pi(2t_1 - t_0, j)$. By (2.60) and (2.102)

$$0 < 2t_1 - t_0 - (t_0 + 1) = 2t_1 - 2t_0 - 1 < 2^{k'+1} \le 2^{\omega_j+1} \ll 2^{k_M+1}.$$

Hence by (2.63), $\pi(t_0+1, j)$ and $\pi(2t_1-t_0, j)$ are relatively prime and the sets $\mathcal{X}_{t_0,t_1,k'}$ are sufficiently "independent" of $\mathcal{X}_{t_0,s-1}$. By (2.111) and Lemma 5 for $2^{n_s} \gg \pi(t_0 + 1, j)$ we have

(2.112)
$$\lambda(\mathcal{X}_{t_0,t_1,k'} \cap \mathcal{X}_{t_0,s-1}) < \frac{1}{4}\lambda(\mathcal{X}_{t_0,t_1,k'}).$$

Using (2.105), (2.107) and (2.112) we infer

(2.113)
$$\lambda\left(\mathcal{X}_{t_0,t_1,k'} \setminus \left(\mathcal{X}_{t_0,s-1} \cup \bigcup_{(t'_1,k'') \in \mathcal{K}(t_0,t_1,k')} \mathcal{X}_{t_0,t'_1,k''}\right)\right)$$
$$> \frac{1}{2}\lambda(\mathcal{X}_{t_0,t_1,k'}) > \frac{1}{8}\left(\min_t \overline{\int} \psi_{t,j,k'}\right) 2^{-k'}2^{n_s}.$$

This estimates the "new contribution" of $\mathcal{X}_{t_0,t_1,k'}$ to the set of points where (2.82) holds. Set

$$(2.114) \quad \mathcal{X}'_{t_0,k'} = \bigcup_{t_1 \in \mathcal{T}_1(t_0,k')} \left(\mathcal{X}_{t_0,t_1,k'} \setminus \left(\bigcup_{(t'_1,k'') \in \mathcal{K}(t_0,t_1,k')} \mathcal{X}_{t_0,t'_1,k''} \cup \mathcal{X}_{t_0,s-1} \right) \right).$$

From $k' \neq k''$ and $x \in \mathcal{X}'_{t_0,k'} \cap \mathcal{X}'_{t_0,k''}$ it follows that $x \in \mathcal{X}_{t_0,t_1,k'}$ with $t_1 \in \mathcal{T}_1(t_0,k')$ and $x \in \mathcal{X}_{t_0,t'_1,k''}$ with $t'_1 \in \mathcal{T}_1(t_0,k'')$, that is $(t'_1,k'') \in \mathcal{K}(t_0,t_1,k')$, but this contradicts (2.114). Hence

$$(2.115) \qquad\qquad \mathcal{X}'_{t_0,k'} \cap \mathcal{X}'_{t_0,k''} = \emptyset \text{ for } k' \neq k''.$$

By (2.95) and (2.113)

$$(2.116) \qquad \lambda(\mathcal{X}'_{t_0,k'}) > \left(\frac{1}{8} \left(\min_t \overline{\int} \psi_{t,j,k'} \right) 2^{-k'} 2^{n_s} \right) \left(\frac{1}{2} \right)^j 2^{k'-3}$$

$$= \frac{1}{64} \left(\min_t \overline{\int} \psi_{t,j,k'} \right) 2^{n_s} \left(\frac{1}{2} \right)^j.$$

Set

$$(2.117) \qquad\qquad \mathcal{X}'_{t_0} = \cup_{k'=\alpha_j}^{\omega_j} \mathcal{X}'_{t_0,k'}.$$

Then by $\rho > 1/2$, (2.69), (2.115) and (2.116)

$$(2.118) \qquad \lambda(\mathcal{X}'_{t_0}) > \sum_{k'=\alpha_j}^{\omega_j} \frac{1}{64} \left(\min_t \overline{\int} \psi_{t,j,k'} \right) 2^{n_s} \left(\frac{1}{2} \right)^j$$

$$= \frac{1}{64} \left(\sum_{k'=\alpha_j}^{\omega_j} \min_t \overline{\int} \psi_{t,j,k'} \right) 2^{n_s} \left(\frac{1}{2} \right)^j > \frac{1}{128} \epsilon 2^{n_s} \left(\frac{1}{2} \right)^j.$$

Hence in each interval $I(t_0) \subset [t_0 2^{n_s}, (t_0+1)2^{n_s})$ satisfying (2.86) we have found a set $\mathcal{X}'_{t_0}$ such that

$$(2.119) \qquad\qquad \lambda(\mathcal{X}'_{t_0}) > \frac{\epsilon}{128} \left(\frac{1}{2} \right)^j \lambda(I(t_0))$$

and for $x \in \mathcal{X}'_{t_0}$

$$(2.120) \qquad \exists l \leq 2^{n_s+\omega_j} \text{ such that } \frac{\phi_{n_s}(x+l) \sum_{k'=\alpha_j}^{\omega_j} \psi_{k'}(x+2l)}{l} \geq 1.$$

Moreover,

$$(2.121) \qquad\qquad \mathcal{X}'_{t_0} \cap \mathcal{X}_{t_0,s-1} = \emptyset.$$

2.14. Substep 2h: Conclusion

Denote by $\mathcal{I}(j)$ the union of all intervals in $\mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$ which satisfy (2.86).

Then by (2.53), (2.54), (2.83), and the definition of $X(1/2, 3/4, j, e)$ we have

$$(2.122) \qquad \overline{\lambda}(\mathcal{I}(j)) > \frac{1}{16}\overline{\lambda}(\mathrm{ispt}_{k_j}(\phi_{n_s})) = \frac{1}{16}2^{-M+j}.$$

Denote by $\overline{\mathcal{X}}(j)$ the set of those $x \in \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \big(\mathrm{ispt}_{k_{j-1}}(\phi_{n_s}) \cup \bigcup_{t_0} \mathcal{X}_{t_0,s-1}\big)$ for which there exists l such that (2.120) holds. Then (2.119) and (2.122) imply that

$$(2.123) \qquad \overline{\lambda}(\overline{\mathcal{X}}(j)) > \frac{\epsilon}{128 \cdot 16}\left(\frac{1}{2}\right)^j 2^{-M+j} = \frac{\epsilon}{2048}2^{-M}.$$

When $j \neq j'$ the sets $\overline{\mathcal{X}}(j) \subset \mathrm{ispt}_{k_j}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j-1}}(\phi_{n_s})$ and $\overline{\mathcal{X}}(j') \subset \mathrm{ispt}_{k_{j'}}(\phi_{n_s}) \setminus \mathrm{ispt}_{k_{j'-1}}(\phi_{n_s})$ are disjoint from each other and from the sets $\cup_{t_0} \mathcal{X}'_{t_0,s-1}$. Using the estimates (2.123) for $j = 2, ..., M$ and (2.78) we obtain that (2.1) holds with s as well. Inequality (2.2) follows from (2.57), (2.58), and (2.79). Inequality (2.3) follows from (2.59) and (2.80). ∎

2.15. How to derive Theorem 2 from Theorem 6

We define the dynamical system $(Y_s, \mathcal{B}_s, \overline{\lambda}_s, T_{p_s})$ where $Y_s = [0, p_s)$, $T_{p_s} : x \rightarrow x + 1 \bmod p_s$, $\mathcal{B}_s$ is the σ-field of measurable subsets on Y_s and $\overline{\lambda}_s$ is the normalized Lebesgue measure on $[0, p_s)$.

By Theorem 6 we have

$$\overline{\lambda}_s\left\{x : \max_{l \leq p_s} \sum_{k=\alpha}^{\omega} \sum_{i=1}^{n_s} \frac{\phi_i(T_{p_s}^l(x))\psi_k(T_{p_s}^{2l}(x))}{l} \geq 1\right\} > \min\left\{\epsilon, \frac{s\epsilon(M-1)2^{-M}}{2048}\right\}.$$

Dividing by the norms of the sums of the non-negative functions ψ_k and ϕ_i we derive the estimate

$$\overline{\lambda}_s\left\{x : \max_{l \leq p_s} \sum_{k=\alpha}^{\omega} \sum_{i=1}^{n_s} \frac{\phi_i(T_{p_s}^l(x))\psi_k(T_{p_s}^{2l}(x))}{l\|\sum_{i=1}^{n_s}\phi_i\|_1\|\sum_{k=\alpha}^{\omega}\psi_k\|_1} \geq \frac{1}{\|\sum_{i=1}^{n_s}\phi_i\|_1\|\sum_{k=\alpha}^{\omega}\psi_k\|_1}\right\}$$
$$> \min\left\{\epsilon, \frac{s\epsilon(M-1)2^{-M}}{2048}\right\}.$$

For $0 < \epsilon < \epsilon_0$ fixed, one can pick s and M such that $\frac{s(M-1)2^{-M}}{2048} \geq 1$. Actually we will choose s to be equal to $\lfloor \frac{2048 \cdot 2^M}{M-1} \rfloor + 1$. We can observe that

with this choice for $M > 2$ we have $\lfloor \frac{2048 \cdot 2^M}{M-1} \rfloor + 1 \le 2\frac{2048 \cdot 2^M}{M-1}$. Using the estimates (2.2) and (2.3) we have

$$\overline{\lambda}_s \left\{ x : \max_{l \le p_s} \sum_{k=\alpha}^{\omega} \sum_{i=1}^{n_s} \frac{\phi_i(T_{p_s}^l(x))\psi_k(T_{p_s}^{2l}(x))}{l\| \sum_{i=1}^{n_s} \phi_i \|_1 \| \sum_{k=\alpha}^{\omega} \psi_k \|_1} \ge \frac{1}{\epsilon s 2^{-M+1}} \right\} > \epsilon.$$

Therefore, if we set $F_s \overset{\text{def}}{=} \frac{\sum_{i=1}^{n_s} \phi_i}{\| \sum_{i=1}^{n_s} \phi_i \|_1}$ and $G_s \overset{\text{def}}{=} \frac{\sum_{k=\alpha}^{\omega} \psi_k}{\| \sum_{k=\alpha}^{\omega} \psi_k \|_1}$ then we have

$$\overline{\lambda}_s \left\{ x : \max_{l \le p_s} \frac{F_s(T_{p_s}^l(x))G_s(T_{p_s}^{2l}(x))}{l} \ge \frac{1}{\epsilon s 2^{-M+1}} \right\} > \epsilon.$$

As we have assumed that $s = \lfloor \frac{2048 \cdot 2^M}{M-1} \rfloor + 1$, the last inequality implies

(2.124) $$\overline{\lambda}_s \left\{ x : \max_{l \le p_s} \frac{F_s(T_{p_s}^l(x))G_s(T_{p_s}^{2l}(x))}{l} \ge \frac{(M-1)}{4 \cdot 2048\epsilon} \right\} > \epsilon.$$

To simplify further the notation we set $C = \frac{1}{8192\epsilon}$. This constant depends only on ϵ, a fixed positive number. We introduce different values of M with the sequence $M_j = (j+1)^5$. To these values the subsequence s_j of natural numbers is automatically associated, where $s_j = \lfloor \frac{2048 \cdot 2^{(j+1)^5}}{(j+1)^5 - 1} \rfloor + 1$. We define now the measure preserving system

$$\prod_{j=1}^{\infty} (Y_{s_j}, \mathcal{B}_{s_j}, \overline{\lambda}_{s_j}, T_{p_{s_j}})$$

and the functions

$$W(y_1, y_2, ..., y_n, ...) = \sum_{j=1}^{\infty} \sqrt{\beta_j} F_{s_j}(y_j), \quad Q(y_1, y_2, ..., y_n, ...) = \sum_{j=1}^{\infty} \sqrt{\beta_j} G_{s_j}(y_j)$$

where the constants β_j will be specified a bit later. The space is $Y = \prod_{j=1}^{\infty} Y_{s_j}$, and the measure preserving transformation acting on Y is the map U defined as $U(y_1, y_2, ..., y_n, ...) = (T_{p_{s_1}}(y_1), T_{p_{s_2}}(y_2), ..., T_{p_{s_n}}(y_n), ...)$. The invariant measure under U, is denoted by μ, and it is the product of the measures $\overline{\lambda}_{s_j}$. We have for any positive number z,

$$\mu \left\{ y \in Y : \sup_l \frac{W(U^l(y))Q(U^{2l}(y))}{l} \ge z \right\}$$

$$\ge \overline{\lambda}_{s_j} \left\{ y_j \in Y_{s_j} : \beta_j \sup_l \frac{F_{s_j}(T_{p_{s_j}}^l(y_j))G_{s_j}(T_{p_{s_j}}^{2l}(y_j))}{l} \ge z \right\}.$$

We can select now the terms β_j. We put $\beta_j = (\frac{\pi^2}{6} - 1)^{-2}(j + 1)^{-4}$. Observe that this guarantees that the functions W and Q have norm 1 in $L^1(\mu)$. We also have $\lim_j (M_j - 1)\beta_j = \infty$. With these conditions we can conclude without difficulty. For each j we have

$$\mu\left\{y \in Y : \sup_l \frac{W(U^l(y))Q(U^{2l}(y))}{l} \geq C\beta_j(M_j - 1)\right\}$$
$$\geq \overline{\lambda}_{s_j}\left\{y_j \in Y_{s_j} : \sup_l \frac{F_{s_j}(T^l_{p_{s_j}}(y_j))G_{s_j}(T^{2l}_{p_{s_j}}(y_j))}{l} \geq C(M_j - 1)\right\} > \epsilon.$$

Therefore, we have found a complete non-atomic measure space, a measure preserving system and functions W and Q with L^1 norm 1 such that

$$\mu\left\{y \in Y : \sup_l \frac{W(U^l(y))Q(U^{2l}(y))}{l} = \infty\right\} > \epsilon.$$

Thus Theorem 2 holds for this measure preserving system. By using the disintegration of μ into ergodic components i.e. $d\mu = d\mu_c dc$ and for a.e. c and U with respect to μ_c is ergodic we have

$$\int \mu_c\left\{y \in Y : \sup_l \frac{W(U^l(y))Q(U^{2l}(y))}{l} = \infty\right\}dc > \epsilon.$$

Therefore there exists at least one c^* such that

$$\mu_{c^*}\left\{y \in Y : \sup_l \frac{W(U^l(y))Q(U^{2l}(y))}{l} = \infty\right\} > \epsilon.$$

To duplicate this for any ergodic measure preserving system we can use Halmos's result [6] on the density of the conjugates of an ergodic measure preserving system for the weak topology. We have for any ergodic measure preserving system $(B, \mathcal{F}, m, D)$ on a complete non-atomic finite measure space, for any j

$$\sup_{\|f\|_1=1, \|g\|_1=1} m\left\{b : \sup_l \frac{f(D^l b)g(D^{2l}b)}{l} > M_j'\right\}$$
$$\geq \sup_{\|W\|_1=1, \|Q\|_1=1} \mu_{c^*}\left\{y \in Y : \sup_l \frac{W(U^l y)Q(U^{2l}y)}{l} > M_j'\right\} > \epsilon.$$

The details are omitted here. Similar details are given in the appendix of [3] in the case of the averages associated with the triple a.e. recurrence and these arguments apply to the situation considered here.

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