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Non-uniqueness in a free boundary problem

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Abstract

We show that a result of Lewis and Vogel on uniqueness in a free boundary problem for the p-Laplace operator is sharp in two dimensions.

1. Introduction

Denote points in Euclidean 2 space $\mathbb{R}^2$ by $x = (x_1, x_2)$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^2$ and let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x. Set $B(x, r) = \{y \in \mathbb{R}^2 : |x - y| < r\}$ whenever $x \in \mathbb{R}^2$ and $r > 0$. Let dx denote Lebesgue measure on $\mathbb{R}^2$ and define k dimensional Hausdorff measure, in $\mathbb{R}^2$, $0 < k \leq 2$, as follows: For fixed $\delta > 0$ and $E \subset \mathbb{R}^2$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subset \bigcup B(x_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \ldots$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left(\sum \alpha(k) r_i^k \right)$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 0 < k \leq 2.$$

If O is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$ be the space of equivalence classes of functions u with distributional gradient $\nabla u = (u_{x_1}, u_{x_2})$, both of which are q th power integrable on O. Let

$$\|u\|_{1,q} = \|u\|_q + \|\nabla u\|_q$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Let $C_0^\infty(O)$ be the space of infinitely differentiable functions with

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compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let Ω be a domain (i. e. an open connected set) and suppose that the boundary of Ω (denoted $\partial\Omega$) is bounded and non empty. Let N be a neighborhood of $\partial\Omega$, p fixed, $1 < p < \infty$ and u a positive weak solution to the p Laplace differential equation in $\Omega \cap N$. That is $u \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Observe that if u is smooth and $\nabla u \neq 0$ in $\Omega \cap N$, then $\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0$ where $\nabla \cdot$ denotes divergence. We assume that u has zero boundary values on $\partial\Omega$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(N)$, then $u\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend u to $N \setminus \Omega$ by putting $u \equiv 0$ on $N \setminus \Omega$. Then $u \in W^{1,p}(N)$ and it follows from (1.1) as in [10] that there exists a positive finite Borel measure μ on $\mathbb{R}^2$ with support contained in $\partial\Omega$ and the property that

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = -\int \phi \, d\mu \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. We give a proof that μ exists provided u has a continuous extension to N. It suffices to show

$$F(\phi) = -\int_N \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx \geq 0.$$

for $\phi \geq 0$. Then the existence follows from the Riesz representation theorem and the basic estimates listed in section 2. To see this let $\phi = \big((\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta\big)\psi$ where $\psi \in C_0^\infty\big(B(z, r)\big)$ and $\psi = 1$ on $B(z, r/2)$ and supp $\psi \subset B(z, r)$ for some $z \in \partial\Omega$. Then supp $\phi \subset \Omega$ so we get

$$\begin{aligned} 0 &= \int_N \big\langle |\nabla u|^{p-2} \nabla u, \nabla \phi \big\rangle \, dx \\ &= \int_{u \geq \epsilon} \eta\big(\epsilon + \max(u - \epsilon, 0)\big)^{\eta - 1} |\nabla u|^p \psi \, dx \\ &\quad + \int_N \big((\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta\big) |\nabla u|^{p-2} \langle \nabla \psi, \nabla u \rangle \, dx \end{aligned} \tag{1.3}$$

Note that

$$\eta \int_{u \geq \epsilon} \big(\epsilon + \max(u - \epsilon, 0)\big)^{\eta - 1} |\nabla u|^p \psi \, dx \geq 0$$

so

$$0 \geq \int_N \big((\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta\big) |\nabla u|^{p-2} \langle \nabla \psi, \nabla u \rangle \, dx$$

Suppose r is so small that $u < 1$ in $B(z, r)$. Then

$$\big|(\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta)|\nabla u|^{p-2}\langle \nabla\psi, \nabla u\rangle\big| \leq \|\nabla\psi\|_\infty |\nabla u|^{p-1}.$$

Now $|\nabla u| \in L^{p-1}(\Omega)$ so we can use the dominated convergence theorem to take the limits under the integral sign as ϵ and η go to zero and get $F(\psi) \geq 0$. We can use a partition of unity to reduce the problem to such small r's. Note that if $\partial\Omega$ is smooth enough then

$$d\mu = |\nabla u|^{p-1} dH^{n-1} \tag{1.4}$$

Let E be a compact set and G an open set containing E. For fixed p, $1 < p < \infty$ set

$$K_p(E, G) = \inf\Big\{\int |\nabla\theta|^p\,dx\Big\}$$

where the infimum is taken over all $\theta \in C_0^\infty(G)$ with $\theta = 1$ on E. $K_p(E, G)$ is called the p-capacity of E relative to G.

In [17] Lewis and Vogel consider the following free boundary problem. Given $F \subset \mathbb{R}^n$ a compact convex set, $a > 0$, and $1 < p < \infty$, find a function u defined on a domain $D = D(a, p) \supset F$ with

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0 \text{ weakly in } D \setminus F, \tag{1.5a}$$

$$u(x) \to 1 \text{ whenever } x \to y \in F \tag{1.5b}$$

$$\text{and } u(x) \to 0 \text{ as } x \to y \in \partial D,$$

$$\mu = a^{p-1} H^{n-1} \text{ on } \partial D. \tag{1.5c}$$

They prove

Theorem A. *Suppose $K_p(F, G) > 0$ for some open $G \supset F$ and let D, u, p, a be as in* (1.5a), (1.5b) *and let μ be the measure corresponding to u as in* (1.2). *If μ satisfies* (1.5c) *and in addition there exists β, $0 < \beta < \infty$ and $r_0 > 0$, for which*

$$\mu[B(x, r)] \leq \beta r^{n-1}, \quad 0 < r < r_0 \tag{1.6}$$

then u and D are uniquely determined.

Previously Henrot and Shahgholian had considered the classical version of this problem that is the problem obtained by replacing (1.5c) by the condition $|\nabla u(x)| \to a$ whenever $x \to y \in \partial D$. In [11] they proved

Theorem B. *If $K_p(F, G) > 0$ for some open $G \supset F$ then there exists a unique $\hat{u}$, $\hat{D} = \hat{D}(a, p)$ such that* (1.5a), (1.5b) *are satisfied and $|\nabla u(x)| \to a$. Moreover $\hat{D}$ is convex with a smooth (C^∞) boundary.*

In this paper we show that theorem A is sharp in two dimensions, namely

Theorem 1. *Suppose* $n = 2$ *and* $K_p(F, G) > 0$ *for some open* $G \supset F$. *If* $a > 0$ *and* $1 < p < \infty$ *there exists a bounded domain* D *which is not convex, a* p *harmonic function* u *and a corresponding measure* μ *which satisfy* (1.5) *but* μ *does not satisfy* (1.6).

The proof uses the same method as the construction of pseudospheres in [16] to construct a domain which satisfies (1.5) but is not convex and thus is not the same as the domain in [11]. To outline this method let Ω be a domain and let u be a function which satisfies (1.5a), (1.5b) with D replaced by Ω and suppose $a = 1$. If $p < 2$ suppose that $|\nabla u| > 1$ on $\partial\Omega$ but if $p > 2$ suppose $|\nabla u| < 1$ on $\partial\Omega$. For a given small ϵ we add smooth bumps to $\partial\Omega$ by "pushing out" or "pushing in" along certain surface elements of $\{x \in \partial\Omega : |\nabla u(x)| > 1 + \epsilon\}$ or $\{x \in \partial\Omega : |\nabla u(x)| < 1 - \epsilon\}$ depending on whether $p > 2$ or $p < 2$. In this way we obtain a new domain $\Omega' \supset \Omega$ if $p < 2$ but $\Omega' \subset \Omega$ if $p > 2$ and we choose the bumps so that for $\epsilon \le t \le 1$

$$H^1(\partial\Omega') \ge H^1(\partial\Omega) + \eta(t) H^1\{x : |\nabla u(x)| > 1 + t\} \tag{1.7}$$

if $p < 2$ but

$$H^1(\partial\Omega') \ge H^1(\partial\Omega) + \eta(t) H^1\{x : |\nabla u(x)| < 1 - t\} \tag{1.8}$$

if $p > 2$. Here η is a positive function on $]0, \infty[$. Let u' be a function in Ω' which satisfies (1.5a), (1.5b) with D replaced by Ω'. If $p < 2$ then $\Omega \subset \Omega'$ and it follows that $u \le u'$ in Ω and by the maximum principle $|\nabla u'| > 1$ on $\partial\Omega \cap \partial\Omega'$. In section 3 we prove that $|\nabla u'| > 1$ on the bumps. If $p > 2$ we get $|\nabla u'| < 1$ in the same way. In section 4 we will show that there exists a certain elliptic partial differential equation for which u' is a solution and $\log|\nabla u'|$ is a supersolution if $1 < p < 2$ and a subsolution if $p > 2$. Then we use the divergence theorem as in [2] to prove that if $1 < p < 2$ then

$$\int_{\partial\Omega'} |\nabla u'|^{p-1} \log|\nabla u'| \, dH^1 \le C \tag{1.9}$$

and if $p > 2$ then

$$\int_{\partial\Omega'} |\nabla u'|^{p-1} \log|\nabla u'| \, dH^1 \ge C \tag{1.10}$$

where the constant C depends only on F. If $1 < p < 2$ this allows us to control the size of the set where $|\nabla u'|$ is large so that by pushing out and keeping $|\nabla u'| > 1$ we in fact keep $|\nabla u'|$ close to 1 for the most part. Likewise if $p > 2$ we are able to control the size of the set where $|\nabla u'|$ is close to zero.

Finally we use (1.7)-(1.10) and induction to construct D. We describe the case $p < 2$ in detail, the case $p > 2$ is similar. Let D_0 be a domain such that u_0 satisfies (1.5a) and (1.5b) with D replaced by D_0 and let $\Omega = D_0$. Modify Ω as above to get $\Omega' = D_1$ and $u' = u_1$. If D_k has been constructed for $0 \le k \le m$ we put $\epsilon_m = 2^{-m}\epsilon_0$ and modify D_m to obtain D_{m+1}. Set $D = \bigcup_0^\infty D_k$. The construction can be arranged so that D is not convex (see Section 4) which shows that it is not the domain in [17]. To prove (1.5c) we first note

$$C \ge \int d\mu_k = \int_{\partial D_k} |\nabla u_k|^{p-1}\, dH^1 \ge H^1(\partial D_k) \tag{1.11}$$

for $k = 0, 1, \ldots$ because $\mu_k(\partial D_k) \le C$ for some C independent of k (see Section 4). Second, for each $\delta > 0$ we have

$$\lim_{k\to\infty} H^1\big\{x \in \partial D_k : |\nabla u_k(x)| > 1 + \delta\big\} = 0 \tag{1.12}$$

since otherwise (1.7) and iteration would lead to a contradiction to (1.11). Next from (1.9) and the fact that $|\nabla u_k| > 1$ on ∂D_k we see that for $M > 1$ and $k = 0, 1, \ldots$

$$\log M \int_{\{|\nabla u_k| > M\}} |\nabla u_k|^{p-1}\, dH^1 \le \int_{\partial D_k} |\nabla u_k|^{p-1} \log |\nabla u_k|\, dH^1 \le C < \infty. \tag{1.13}$$

We also show that as $k \to \infty$

$$H^1|_{\partial D_k} \to H^1|_{\partial D} \text{ and } \mu_k \to \mu \tag{1.14}$$

weakly as measures on $\mathbb{R}^2$ in section 4. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ and $\phi \ge 0$. Then we get

$$\int \phi\, d\mu_k = \int_{\partial D_k} \phi |\nabla u_k|^{p-1}\, dH^1 \ge \int_{\partial D_k} \phi\, dH^1\,. \tag{1.15}$$

To obtain the reverse inequality let δ be a fixed small number and M be a fixed large number and put

$$E_k = \{x \in \partial D_k : 1 \le |\nabla u_k(x)| \le 1 + \delta\} \tag{1.16}$$

$$F_k = \{x \in \partial D_k : 1 + \delta < |\nabla u_k(x)| \le M\} \tag{1.17}$$

$$L_k = \{x \in \partial D_k : |\nabla u_k(x)| > M\} \tag{1.18}$$

for $k = 0, 1, \ldots$. Then

$$\int \phi\, d\mu_k = \int_{\partial D_k} \phi |\nabla u_k|^{p-1}\, dH^1 = \int_{E_k} \cdots + \int_{F_k} \cdots + \int_{L_k} \cdots = I_1 + I_2 + I_3.$$

It is clear that

$$|I_1| \le (1 + \delta)^{p-1} \int_{\partial D_k} \phi\, dH^1.$$

Also from (1.12) we have

$$|I_2| \le M^{p-1}\|\phi\|_\infty H^1\{x \in \partial D_k : 1+\delta < |\nabla u_k|\} \to 0$$

as $k \to \infty$. Using (1.13) we get

$$|I_3| \le \|\phi\|_\infty \int_{|\nabla u_k|>M} |\nabla u_k|^{p-1}\, dH^1 \le \frac{C}{\log M}\|\phi\|_\infty$$

Letting $k \to \infty$ we obtain from the above and (1.14)

$$\int_{\partial D} \phi\, dH^1 \le \int \phi\, d\mu \le (1+\delta)^{p-1} \int_{\partial D} \phi\, dH^1 + \frac{C}{\log M}\|\phi\|_\infty.$$

Finally letting $\delta \to 0$ and $M \to \infty$ we obtain

$$\int \phi\, d\mu = \int_{\partial D} \phi\, dH^1$$

which is what we wanted to prove. Finally the author would like to thank J. Lewis for pointing out this problem and helpful discussions.

2. Basic estimates

A Jordan curve J is said to be a k quasicircle $0 < k < 1$ if $J = f(\partial B(0,1))$ where $f \in W^{1,2}(\mathbb{R}^2)$ is a homeomorphism of $\mathbb{R}^2$ and

(2.1) $$|f_{\bar z}| \le k|f_z|, \qquad H^2 \text{ a. e. in } \mathbb{R}^2.$$

Here we use complex notation, $i = \sqrt{-1}$, $z = x_1 + ix_2$, $2f_{\bar z} = f_{x_1} + if_{x_2}$, $2f_z = f_{x_1} - if_{x_2}$. We call J a quasicircle if J is a k quasicircle for some $0 < k < 1$. Let w_1, w_2 be distinct points on the Jordan curve J and J_1, J_2 the arcs with endpoints w_1, w_2. Then J is said to satisfy the Ahlfors three point condition if there exists an $1 \le M < \infty$ such that for all $w_1, w_2 \in J$ we have

$$\min\{\text{diam} J_1, \text{diam} J_2\} \le M|w_1 - w_2|.$$

A Jordan curve J is a quasicircle if and only if it satisfies the Ahlfors three point condition. A domain Ω is said to be uniform provided there exists $M, 1 \le M < \infty$ such that if $w_1, w_2 \in \Omega$, then there is a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = w_1$, $\gamma(1) = w_2$, and

(2.2a) $$H^1(\gamma) \le M|w_1 - w_2|$$

(2.2b) $$\min\{H^1(\gamma([0,t])), H^1(\gamma([t,1]))\} \le Md(\gamma(t), \partial\Omega)$$

where $d(E,F)$ denotes the distance between two non-empty sets E and F. If $1 \le \tilde M < \infty$ and Ω is a domain a ball $B(w,r) \subset \Omega$ is said to be $\tilde M$ non-tangential if

$$\tilde M r > d(B(w,r), \partial\Omega) > \tilde M^{-1} r$$

If $w_1, w_2 \in \Omega$ a Harnack chain from w_1 to w_2 in Ω is a sequence of $\tilde{M}$ non-tangential balls such that the first ball contains w_1 the last ball contains w_2 and consecutive balls intersect. The conditions (2.2) are equivalent to

(2.3a) For any $w \in \partial\Omega, 0 < r \leq$ diam Ω, there exists $a = a_r(w) \in \Omega$ such that $M^{-1}r < |a - r| < r$ and $d(a, \partial\Omega) > M^{-1}r$

(2.3b) Given $\epsilon > 0, w_1, w_2 \in \Omega, d(w_j, \partial\Omega) > \epsilon$ and $|w_1 - w_2| < C\epsilon$, there is a Harnack chain from w_1 to w_2 whose length depends on C but not on ϵ.

See [9] for references.

In the sequel c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence) which may depend only on p unless otherwise stated. In general $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1 which may only depend on $p, a_1, \dots, a_n$, not necessarily the same at each occurrence. We begin by stating some interior and boundary estimates for u a positive weak solution to the p Laplacian in $B(w, 4r) \cap \Omega$ with $u = 0$ on $\partial\Omega \cap B(w, 4r)$ when this set is nonempty. In this case we extend u to $B(w, 4r)$ by putting $u = 0$ on $B(w, 4r) \setminus \Omega$. Let $\max_{B(z,s)} u$, $\min_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z, s)$ whenever $B(z, s) \subset B(w, 4r)$.

Lemma 1. *Let u be as above. Then*

$$c^{-1} r^{p-2} \int_{B(w,r/2)} |\nabla u|^p \, dx \leq \max_{B(w,r)} u^p \leq c r^{-2} \int_{B(w,2r)} u^p \, dx.$$

If $B(w, 2r) \subset \Omega$, then

$$\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Proof. The first display in Lemma 1 is a standard subsolution estimate while the second display is a standard weak Harnack estimate for positive weak solutions to nonlinear partial differential equations of p Laplacian type (see [20]). ■

Lemma 2. *Let u be as in Lemma 1. Then u has a representative in $W^{1,p}\big(B(w, 4r) \cap \Omega\big)$ with Hölder continuous partial derivatives in $B(w, 4r) \cap \Omega$. That is for some $\sigma = \sigma(p) \in]0, 1[$ we have*

$$\begin{aligned} c^{-1} \big|\nabla u(w_1) - \nabla u(w_2)\big| &\leq \big(|w_1 - w_2|/s\big)^\sigma \max_{B(z,s)} |\nabla u| \\ &\leq c s^{-1} \big(|w_1 - w_2|/s\big)^\sigma \max_{B(z,2s)} u \end{aligned}$$

whenever $w_1, w_2 \in B(z, s)$ and $B(z, 4s) \subset B(w, 4r) \cap \Omega$.

Proof. The proof of Lemma 2 can be found in [4], [14] or [21] and in fact is true when $B(w,4r)\cap\Omega\subset\mathbb{R}^n$. In $\mathbb{R}^2$ the best Hölder exponent in Lemma 2 is known when $p>2$ while for $1<p\le 2$ a solution has continuous second partials (see [12]). ■

A mapping $h: B(w,4r)\cap\Omega\to\mathbb{R}^2$ is said to be quasiregular in $B(w,4r)\cap\Omega$ if $h\in W^{1,2}\big(B(w,4r)\cap\Omega\big)$ and (2.1) holds with f replaced by h in $B(w,4r)\cap\Omega$. From a factorization theorem for quasiregular mappings it follows that $h=\tau\circ f$ where f is quasiconformal in $\mathbb{R}^2$ and τ is an analytic function on $f\big(B(w,4r)\cap\Omega\big)$.

Lemma 3. *If u is as in Lemma 1 and $z=x_1+ix_2$ then u_z is quasiregular in $B(w,4r)\cap\Omega$ for some $0<k<1$ (depending only on p) and consequently ∇u has only isolated zeros in $B(w,4r)\cap\Omega$.*

Proof. For a proof of quasiregularity see [1], [15]. Since the zeros of an analytic function are isolated it follows from the factorization theorem that the zeros of ∇u are isolated. ■

Lemma 4. *If $B(w,4r)\subset\Omega$, $\nabla u\neq 0$ in $B(w,4r)$ and $\max_{B(w,2r)}|\nabla u|\le\lambda\max_{B(w,r)}|\nabla u|$ then*

$$\max_{B(w,2r)}|\nabla u|\le c(\lambda)\min_{B(w,r)}|\nabla u|$$

Proof. Note that $v=\log|\nabla u|$ is a weak solution in $B(w,4r)$ to the divergence form partial differential equation (see [19])

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}(A_{ij}(x)v_{x_j})=0$$

where the (A_{ij}) are bounded and uniformly elliptic (with constants depending only on p). Using Harnacks inequality for positive solutions to partial differential equations of this type (see [20]) applied to $\max_{B(w,2r)}v-v$ in $B(w,r)$ we obtain the lemma. ■

Lemma 5. *Let u be as in Lemma 1 and $w\in\partial\Omega$. If $p>2$ there exists $\alpha=\alpha(p)\in]0,1[$ such that u has a Hölder α continuous representative in $B(w,r)$ (also denoted u). Moreover if $x,y\in B(w,r)$ then*

$$\big|u(x)-u(y)\big|\le c\big(|x-y|/r\big)^{\alpha}\max_{B(w,2r)}u.$$

If $1<p\le 2$ and Ω is simply connected, then this inequality is also valid when $1<p\le 2$ with $\alpha=\alpha(p)$.

Proof. For $p > 2$, Lemma 5 is a consequence of Lemma 1 and Morreys inequality (see [6]). If $1 < p \leq 2$ and Ω is simply connected we deduce from the interior estimates in Lemma 2 that it suffices to consider only the case when $y \in B(w,r) \cap \partial\Omega$. We then show for some $\theta = \theta(p,k), 0 < \theta < 1$ that

$$(2.4) \quad \max_{B(z,\rho/4)} u \leq \theta \max_{B(z,\rho/2)} u \quad \text{whenever } 0 < \rho < r \text{ and } z \in \partial\Omega \cap B(w,r).$$

This inequality can then be iterated to get Lemma 5 for x, y as above. To prove (2.4) we use the fact that $B(z,\rho/4) \cap \partial\Omega$ and $B(z,\rho/4)$ have comparable p capacities (see [10]) and estimates for subsolutions to elliptic partial differential equations of p Laplacian type (see [8], [15]). ■

Lemma 6. *Let u, Ω, w be as in Lemma 5. Assume also that Ω is a uniform domain. Then there exist $c = c(M)$ and $\hat{c} = \hat{c}(M)$ with*

$$\max_{B(w,r/\hat{c})} u \leq c u(a_{r/\hat{c}}(w))$$

where M is as in (2.2) and $a_r(w)$ is as in (2.3). Hence

$$|u(x) - u(y)| \leq c(|x-y|/r)^\alpha u(a_{r/\hat{c}}(w))$$

for $x, y \in B(w, r/2\hat{c})$.

Proof. The first display in Lemma 6 follows from Harnacks principle in Lemma 1, Hölder continuity of u in Lemma 5 and the fact that Ω is a uniform domain and a general argument which can be found in [3]. The second display follows from the first display and Lemma 5 ■

To proceed we consider the following scenario. Let Ω be a domain such that $\partial\Omega$ is C^4. Let $w \in \partial\Omega$ and let u be a positive p harmonic function in $\Omega \cap B(w,2r)$ and assume that $\Omega \cap B(w,2r)$ has only one component. We further assume that $\nabla u \neq 0$ in $\Omega \cap B(w,r)$. We have

Lemma 7. *Let u be as above. If $x \in \Omega \cap B(w,r)$ there exists a $c \geq 1$ depending only on k and p such that*

$$c^{-1} d(x,\partial\Omega)^{-1} u(x) \leq |\nabla u(x)| \leq c d(x,\partial\Omega)^{-1} u(x)$$

where $d(x,\partial\Omega)$ denotes the distance from x to $\partial\Omega$

Proof. Choose $y \in B(x, d(x,\partial\Omega))$ with $u(y) = u(x)/2$. Apply the mean value theorem of calculus to u restricted to the line segment with endpoints x, y. From this and Lemma 6 it follows that there exists a constant $c \geq 4$

and z such that $y \in B(x, (1-c^{-1})d(x, \partial\Omega))$ and z is on the line segment between x and y and

$$u(x)/2 = |u(x) - u(y)| \leq |\nabla u(z)||x - y|.$$

Using this inequality and Lemma 2 we see for some positive $\tilde{c}$ that if $t_1 = (1-c^{-1})d(x, \partial\Omega), t_2 = (1-(2c)^{-1})d(x, \partial\Omega)$ then

$$\tilde{c}^{-1}u(x)/d(x, \partial\Omega) \leq \max_{B(x,t_1)} |\nabla u| \leq \max_{B(x,t_2)} |\nabla u| \leq \tilde{c}u(x)/d(x, \partial\Omega). \tag{2.5}$$

From (2.5) and Lemma 4 we conclude that Lemma 7 is valid for u at x. ■

Let θ be a function whose graph is after a rotation and translation $\Omega \cap B(w, r/2)$ and suppose that the C^4-norm of θ is bounded by c/r. The condition (3.1) stated in the next section is clearly sufficient. At each point $x \in \partial\Omega \cap B(w, r/2)$ we can find a tangential ball $B(z, \rho) \subset \Omega \cap B(w, r)$ with $x \in \partial B(z, \rho)$ and radius $\rho > 0$ depending only on λ and r. Let v be the p harmonic function which is zero on $\partial B(z, \rho)$ and $\inf_{\partial B(z,\rho/2)} u$ on $\partial B(z, \rho/2)$. Then $v \leq u$ in the annulus $B(z, \rho) \setminus B(z, \rho/2)$. Therefore

$$|\nabla u(t)| \geq c^{-1}u(t)d(t, \partial\Omega)^{-1} \geq c^{-1}v(t)d(t, \partial\Omega)^{-1} \geq c^{-1} \inf_{\partial B(z,\rho/2)} u/\rho$$

for t in the annulus where we used the fact that $v(x) = A|x - z|^{\frac{p-2}{p-1}} + B$ to compute ∇v. Then by Harnack's inequality we get a lower bound in terms of $\max_{B(w,r)} u$. We can argue in the same way to get an upper bound so that we have

$$c^{-1} \max_{B(w,r)} u/r \leq |\nabla u|(t) \leq c \max_{B(w,r)} u/r \tag{2.6}$$

for t in $B(w, r/2) \cap \Omega$ and thus $u \in W^{1,2}(B(w, r/2))$.

Let $\tilde{u}(x) = u(rx + w)/r$. Then $\tilde{u}$ is a solution to the p Laplace equation in $B(0, 2) \cap \tilde{\Omega}$ where $\tilde{\Omega} = \{x \in \mathbb{R}^2 : rx + w \in \Omega\}$. Let Φ be a differentiable mapping from $B(0, 2)$ to $B(0, 2)$ such that 0 is mapped to 0 and $]-1, 1[$ is mapped to $\partial\tilde{\Omega} \cap B(0, 2)$ and $\{(x, y) \in B(0, 2) : y > 0\}$ is mapped to $B(0, 2) \cap \tilde{\Omega}$. Define $v = \tilde{u} \circ \Phi$ in $\{(x, y) \in B(0, 2) : y > 0\}$ and let $v(x, y) = 0$ in $\{(x, y) \in B(0, 2) : y < 0\}$. Then v satisfies an equation of the form

$$\nabla \cdot \left(\langle A\nabla v, \nabla v\rangle^{p/2-1} A\nabla v\right) = 0 \tag{2.7}$$

in $B(0, 2)^+ = B(0, 2) \cap \{(x, y) \in B(0, 2) : y > 0\}$ where $A = [A_{ij}]$ is a symmetric matrix whose coefficients are in C^1. From our work above it follows that

$$c^{-1} \max_{B(0,2)} v \leq |\nabla v|(x) \leq c \max_{B(0,2)} v \tag{2.8}$$

for $x \in B(0,1)^+$ where the constant may depend on Φ. If we let $A(x,\xi) = \langle A\xi, \xi\rangle^{p/2-1} A\xi$ then we have

$$(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \le c\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle$$

and

$$|\nabla_x A(x,\eta)| \le c|\eta|^{p-1} \tag{2.9}$$

where ∇_x denotes the gradient with respect to the x variable.

Lemma 8. *Let v be as above. Then v has weak derivatives of second order and $v_{x_1} \in W^{1,2}(B(0,1/2))$ and we have*

$$\int_{B(z,\rho/2)} \sum_{i=1}^{2} |v_{x_j x_i}|^2 \, dx \le \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_j} - a|^2 \, dx + c\rho^2 (\max_{B(0,1)} v)^2$$

if $B(z,\rho) \subset B(0,1/2)^+ = \{(x_1,x_2) \in B(0,1/2) : x_2 > 0\}$ and $a \in \mathbb{R}$. In addition we have

$$\int_{B(z,\rho/2)} \sum_{i=1}^{2} |v_{x_1 x_i}|^2 \, dx \le \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_1}|^2 \, dx + c\rho^2 (\max_{B(0,1)} v)^2$$

for any $z \in B(0,1/2)$ and $\rho \le 1/4$

Proof. Let

$$D_k^h v(x) = \frac{v(x + he_k) - v(x)}{h}$$

where e_k denotes the k-th unit vector. Let ζ be a smooth function such that $\zeta = 1$ on $B(z,\rho/2)$, $\operatorname{supp} \zeta \subset B(z,\rho)$ and $|\nabla\zeta| \le c/\rho$ for some constant c. Since $v \in W^{1,2}(B(0,1)^+)$ and $v = 0$ on $\{(x_1,x_2) : x_2 = 0\}$ the function $\phi = D_k^{-h}\big(\zeta^2(D_k^h v - a)\big)$ belongs to $W_0^{1,2}\big(B(0,1)^+\big)$ if $B(z,\rho) \subset B(0,1/2)^+$ and if $a = 0$ and $k = 1$ we have $\phi \in W_0^{1,2}\big(B(0,1)^+\big)$ for any $z \in B(0,1/2)$ and $\rho \le 1/4$. This function is therefore an admissible test function. We obtain

$$\begin{aligned}
0 &= \int \Big\langle A(x,\nabla v), \nabla\big(D_k^{-h}(\zeta^2(D_k^h v - a))\big)\Big\rangle \, dx \\
&= \int \Big\langle D_k^h A(x,\nabla v), \nabla\big(\zeta^2(D_k^h v - a)\big)\Big\rangle \, dx \\
&= \int \Big\langle \frac{A(x+he_k, \nabla v(x+he_k)) - A(x+he_k, \nabla v(x))}{h}, \nabla\big(\zeta^2(D_k^h v - a)\big)\Big\rangle \, dx \\
&\quad + \int \Big\langle \frac{A(x+he_k, \nabla v(x)) - A(x, \nabla v(x))}{h}, \nabla\big(\zeta^2(D_k^h v - a)\big)\Big\rangle \, dx = \mathrm{I} + \mathrm{II}
\end{aligned}$$

The first of these integrals is

$$\int \left\langle \frac{A(x+he_k, \nabla v(x+he_k)) - A(x+he_k, \nabla v(x))}{h}, \zeta^2 D_k^h \nabla v \right\rangle dx$$
$$+ \int \left\langle \frac{A(x+he_k, \nabla v(x+he_k)) - A(x+he_k, \nabla v(x))}{h}, 2\zeta \nabla \zeta (D_k^h v - a) \right\rangle dx$$

The first term in this expression can be bounded below by

$$c^{-1} \int \Big(|\nabla v(x+he_k)| + |\nabla v(x)| \Big)^{p-2} \zeta^2 |D_k^h \nabla v|^2 \, dx$$

and the second term can be bounded above by

$$\begin{aligned} \int \big(|\nabla v(x+he_k)| + |\nabla v(x)|\big)^{p-2} &|D_k^h \nabla v| |\nabla \zeta| |2\zeta (D_k^h v - a)| \, dx \\ &\leq \epsilon \int \big(|\nabla v(x+he_k)| + |\nabla v(x)|\big)^{p-2} |D_k^h \nabla v|^2 \zeta^2 \, dx \\ &\quad + \frac{c}{\epsilon} \int \big(|\nabla v(x+he_k)| + |\nabla v(x)|\big)^{p-2} |\nabla \zeta|^2 |D_k^h v - a|^2 \, dx \end{aligned}$$

by Youngs inequality. As for II we get

$$\begin{aligned} \mathrm{II} &\leq c \int |\nabla v|^{p-1} \zeta^2 |D_k^h \nabla v| \, dx + c \int |\nabla v|^{p-1} |\zeta| |\nabla \zeta| |D_k^h v - a| \, dx \\ &\leq \epsilon \int |\nabla v|^{p-2} \zeta^2 |D_k^h \nabla v|^2 dx + \frac{c}{\epsilon} \int |\nabla v|^p \zeta^2 dx + \frac{c}{\rho^2} \int |\nabla v|^{p-2} |D_k^h v - a|^2 dx. \end{aligned}$$

Choosing ϵ small enough and using (2.8) to estimate $|\nabla v|^{p-2}$ we get

$$\int_{B(z,\rho/2)} |D_k^h \nabla v|^2 \, dx \leq \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_k} - a|^2 \, dx + \rho^2 (\max_{B(0,1)} v)^2.$$

We conclude that $(D_k^h v)\zeta \in W_0^{1,2}(B(z,\rho))$ with a norm independent of h. It now follows from a weak compactness argument that $v_{x_k}\zeta \in W_0^{1,2}(B(z,\rho))$ and

$$\int_{B(z,\rho/2)} \sum_{i=1}^{2} |v_{x_k x_i}|^2 \, dx \leq \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_k} - a|^2 \, dx + \rho^2 (\max_{B(0,1)} v)^2.$$

This is what we wanted to prove. ■

Recall that if $\psi \in W^{1,2}(B(z,\rho))$ and $\psi_{B(z,\rho)} = \frac{1}{|B(z,\rho)|} \int_{B(z,\rho)} \psi \, dx$ then

$$|\psi(x) - \psi_{B(z,\rho)}| \leq C \int_{B(z,\rho)} \frac{|\nabla \psi(y)|}{|x-y|} \, dy$$

Let $\psi(x) = v_{x_1}(x)$ and $1/4 > \rho$. It follows if $x \in B(0, 1/4)$

$$|\psi(x) - \psi_{B(z,\rho)}| \leq \int_{B(z,\rho)} \frac{|\nabla\psi(y)|}{|x-y|}\,dy = \int_{B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\,dy + \int_{B(z,\rho)\setminus B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\,dy$$

and by Hölders inequality

$$\begin{aligned}(2.10)\quad & \int_{B(z,2\rho)\setminus B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\,dy \\ &\leq c\left(\int_{B(z,2\rho)} |\nabla\psi(y)|^q\,dy\right)^{1/q}\left(\int_{B(z,2\rho)\setminus B(x,\delta)} \frac{1}{|x-y|^{\frac{q}{q-1}}}\,dy\right)^{\frac{q-1}{q}} \\ &\leq c\|\nabla\psi\|_q \delta^{(q-2)/q}\end{aligned}$$

and for the other integral we have the estimate

$$\begin{aligned}(2.11)\quad \int_{B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\,dy &\leq \sum_{n=0}^{\infty} 2\left(\frac{2^k}{\delta}\right)\int_{\{2^{-(k+1)}\delta\leq|x-y|\leq 2^{-k}\delta\}} |\nabla\psi(y)|\,dy \\ &\leq 2\sum_{k=0}^{\infty} \frac{\delta}{2^k} M(|\nabla\psi|) \leq 2\delta M(|\nabla\psi|)\end{aligned}$$

Here $M(f)$ denotes the maximal function of f. We conclude

$$|\psi(x) - \psi_{B(z,\rho)}| \leq 2\delta M(|\nabla\psi|) + \delta^{(q-2)/q}\|\nabla\psi\|_q$$

and if we choose

$$\delta = \left(\frac{\|\nabla\psi\|_q}{2M(|\nabla\psi|)}\right)^{q/2}$$

we get

$$|\psi(x) - \psi_{B(z,\rho)}|^2 \leq cM(|\nabla\psi|)^{2-q}\|\nabla\psi\|_q^q$$

Integrating and applying Hölder's inequality yields for $1 < q < \frac{3}{2}$

$$\begin{aligned}(2.12)\quad \|\psi(x) - \psi_{B(z,\rho)}\|_2^2 &\leq \rho\left(\int_{B(z,2\rho)} M(|\nabla\psi|)^{4-2q}\,dx\right)^{1/2}\|\nabla\psi\|_q^q \\ &\leq \rho\left(\int_{B(z,2\rho)} |\nabla\psi|^{4-2q}\,dx\right)^{1/2}\|\nabla\psi\|_q^q\end{aligned}$$

If $B(z,\rho) \subset B(0,1/2)^+$ Lemma 8 and (2.12) yield with $\psi = v_{x_1}$, $\psi_{B(z,\rho)} = a$

(2.13)

$$\int_{B(z,\rho/2)} \sum_i |v_{x_1x_i}|^2\,dx$$
$$\leq \frac{c}{\rho}\left(\int_{B(z,\rho)}\left(\sum_{i,j}|v_{x_jx_i}|\right)^{4-2q} dx\right)^{1/2}\int_{B(z,\rho)}\left(\sum_{i,j}|v_{x_jx_i}|\right)^q dx + c\rho^2(\max_{B(0,1)} v)^2$$

If $B(z,\rho) \cap B(0,1/2)^- \neq \emptyset$ then we take $x = (x_1,x_2) \in B(z,\rho) \cap B(0,1/2)^+$ and let $x^* = (x_1,-x_2)$. Note that if $x,y \in B(0,1)^+$ then $|x-y| \leq |x^*-y|$. Since $\psi = 0$ in $B(0,1)^-$ we get

$$|\psi_{B(z,\rho)}| \leq \int_{B(z,\rho)} \frac{|\nabla\psi(y)|}{|x^*-y|}\,dy \leq c\int_{B(z,\rho)} \frac{|\nabla\psi(y)|}{|x-y|}\,dy$$

since $\psi(x^*) = 0$. This allows us to get rid of $\psi_{B(z,\rho)}$ in our work above and we see that (2.13) holds in this case as well.

Lemma 9. *Let u be defined as above Lemma 7 and v be defined as above Lemma 8. Then $v \in C^4(\overline{B(0,1/4)}^+)$ and we have*

$$|D^2v|(x) \leq c\max_{B(0,1)} v \tag{2.14}$$

for x in $B(0,1/4)^+$. For the function u we have $u \in C^4(\overline{\Omega \cap B(w,r/8)})$ and

$$|\nabla u|(x) \leq \frac{c}{r}\max_{B(w,r)} u \tag{2.15}$$

$$|D^2u|(x) \leq \frac{c}{r^2}\max_{B(w,r)} u \tag{2.16}$$

for $x \in B(w,r/8) \cap \Omega$.

Proof. It follows from lemma 8 that v is a strong solution of (2.7). Writing the equation in nondivergence form we obtain

$$|v_{x_2x_2}|^2 \leq c\left(\sum_{i=1}^2 |v_{x_1x_i}|\right)^2 + c(\max_{B(0,1)} v)^2$$

Let $g = \sum |v_{x_i x_j}|$. We obtain

(2.17)
$$\begin{aligned}\fint_{B(z,\rho)} g^2\,dx &\le \fint_{B(z,2\rho)} g^q\,dx \left(\fint_{B(z,2\rho)} g^{4-2q}\,dx\right)^{1/2} + \rho^2(\max_{B(0,1)} v)^2\\ &\le \epsilon\left(\fint_{B(z,2\rho)} g^q\,dx\right)^{2/q} + C\left(\fint_{B(z,2\rho)} g^{4-2q}\,dx\right)^{1/(2-q)} + \rho^2(\max_{B(0,1)} v)^2\\ &\le \epsilon\fint_{B(z,2\rho)} g^2\,dx + C\left(\fint_{B(z,2\rho)} g^{4-2q}\,dx\right)^{1/(2-q)} + \rho^2(\max_{B(0,1)} v)^2\end{aligned}$$

where we first used Youngs inequality and then Jensens inequality. In a ball $B(x,2t) \subset B(0,1)$ we define

$$f(y) = \frac{\delta(y)}{2t} g(y)$$

where $\delta(y)$ is the distance from y to $\partial B(x,2t)$ and note

$$2f(y) \ge g(y) \text{ for } y \in B(x,t) \text{ and } f(y) \le g(y) \text{ for } y \in B(x,2t) \tag{2.18}$$

If $z \in B(x,2t)$ then

$$\begin{aligned}\fint_{B(z,\delta(z)/2)} f^2(y)\,dy &\le \left(\frac{2}{\delta(z)}\right)^2 \int_{B(z,\delta(z)/2)} \left(\frac{\delta(y)}{2t}\right)^2 g^2(y)\,dy\\ &\le \frac{4}{t^2}\int_{B(x,2t)} g^2(y)\,dy = \lambda_0^2\end{aligned} \tag{2.19}$$

Let $\mu_0^2 = \lambda_0^2 + 2t^2(\max_{B(0,1)} v)^2$, take $\lambda \ge \lambda_0$, let $\mu^2 = \lambda^2 + 2t^2(\max_{B(0,1)} v)^2$ and $F(\mu) = \{z \in B(x,2t) : f(z) > \mu\}$. Then it follows from differentiation theory that for almost every $z \in F(\mu)$ there exists $\rho > 0$ such that

$$\fint_{B(z,\rho)} f^2\,dx > \mu^2$$

If $z \in F(\mu)$ and ρ is sufficiently small it follows from (2.19) that we can select ρ such that $10\rho < \delta(z)/2$ and

$$\fint_{B(z,10\rho)} f^2\,dx < \mu^2 \tag{2.20}$$

$$\fint_{B(z,\rho)} f^2\,dx > \mu^2 \tag{2.21}$$

Then we obtain

$$\text{(2.22)}\quad \int_{B(z,2\rho)} g^2\,dy \le \int_{B(z,10\rho)} g^2\,dy \le (10\rho)^2\left(\frac{2t}{\delta(z)}\right)^2 \fint_{B(z,10\rho)} f^2(y)\,dy$$
$$\le (10\rho)^2\left(\frac{2t}{\delta(z)}\right)^2 \fint_{B(z,\rho)} f^2\,dy \le c\int_{B(z,\rho)} g^2\,dy.$$

Along with (2.17) this gives the estimate

$$\text{(2.23)}\qquad \fint_{B(z,\rho)} g^2\,dx \le C\left(\fint_{B(z,2\rho)} g^{4-2q}\,dx\right)^{1/(2-q)} + 2t^2(\max_{B(0,1)} v)^2$$

Since $10\rho < \delta(z)/2$ we have $\delta(z)/4 < \delta(y)/2 < \delta(z)$ for all $y \in B(z,2\rho)$. Therefore

$$\fint_{B(z,\rho)} f^2\,dx \le C\left(\fint_{B(z,2\rho)} f^{4-2q}\,dx\right)^{1/(2-q)} + 2t^2(\max_{B(0,1)} v)^2$$

From (2.21) it now follows

$$\text{(2.24)}\qquad \lambda^{4-2q} \le C\fint_{B(z,2\rho)} f^{4-2q}\,dx$$

so

$$\text{(2.25)}\quad \fint_{B(z,10\rho)} f^2\,dx \le \mu^2 = \lambda^2 + 2t^2(\max_{B(0,1)} v)^2$$
$$\le C\lambda^{2q-2}\left(\fint_{B(z,2\rho)} f^{4-2q}\,dx\right) + 2t^2(\max_{B(0,1)} v)^2$$

Let $E(\mu) = \{y \in B(x,2t) : f(y) < \mu\}$ and note

$$\text{(2.26)}\qquad \int_{E(\delta\mu)\cap B(z,2\rho)} f^{4-2q}\,dx \le (\delta\mu)^{4-2q} m(B(z,2\rho))$$

where m denotes two dimensional Lebesgue measure. By a well known covering theorem we can find a sequence of balls $\{B(z_i,\rho_i)\}$ such that (2.21), (2.20) and (2.25) hold and

$$\text{(2.27)}\qquad m(F(\mu)\setminus \bigcup_i B(z_i,10\rho_i)) = 0$$

$$\text{(2.28)}\qquad B(z_i,2\rho_i)\cap B(z_j,2\rho_j) = \emptyset \quad i \ne j$$

Now we have

$$\begin{aligned}(2.29)\qquad \int_{F(\mu)} f^2\,dx &\le \sum_i \int_{B(z_i,10\rho_i)} f^2\,dx \\ &\le \lambda^{2q-2}\left(\sum_i \int_{B(z_i,2\rho_i)} f^{4-2q}\,dx\right) + 2t^2(\max_{B(0,1)} v)^2 \\ &\le C\lambda^{2q-2}\left(\int_{F(\delta\mu)} f^{4-2q}\,dx\right) + 2t^2(\max_{B(0,1)} v)^2.\end{aligned}$$

Let M be a large number and put

$$(2.30)\qquad \tilde{f} = \min\{f, M\}$$

$$(2.31)\qquad \tilde{F}(\mu) = \{z \in B(x,2t) : \tilde{f}(z) > \mu\}$$

Then it follows that

$$(2.32)\qquad \int_{\tilde{F}(\mu)} \tilde{f}^2\,dx \le C\lambda^{2q-2}\left(\int_{\tilde{F}(\delta\mu)} \tilde{f}^{4-2q}\,dx\right) + 2t^2(\max_{B(0,1)} v)^2$$

Now we get with integration by parts and Fubini's theorem

(2.33)

$$\begin{aligned}\int_{\tilde{F}(\mu_0)} \tilde{f}^{2+\gamma}\,dx &= \gamma \int_{\tilde{F}(\mu_0)} \tilde{f}^2 \int_0^{\tilde{f}} \mu^{\gamma-1}\,d\mu\,dx = \gamma\int_{\mu_0}^\infty \mu^{\gamma-1}\int_{\tilde{F}(\mu)} \tilde{f}^2\,dx\,d\mu \\ &\le \gamma\int_{\mu_0}^\infty \mu^{\gamma+2q-3}\left(\int_{\tilde{F}(\delta\mu)} \tilde{f}^{4-2q}\,dx\right) + 2t^2(\max_{B(0,1)} v)^2\,d\mu \\ &= \frac{(4-2q)\gamma\delta^{4-2q}}{\gamma+2q-2}\int_{\mu_0}^\infty \mu^{1+\gamma} m(\tilde{F}(\delta\mu))\,d\mu \\ &\quad + \frac{\gamma}{\gamma+2q-2}\mu_0^{\gamma+2q-2}\left(\int_{\tilde{F}(\delta\mu_0)} \tilde{f}^{4-2q}\,dx\right) + 2t^2(\max_{B(0,1)} v)^2\end{aligned}$$

By choosing δ small enough this gives

$$(2.34)\qquad \int_{\tilde{F}(\mu_0)} \tilde{f}^{2+\gamma}\,dx \le C\mu_0^{\gamma+2q-2}\left(\int_{B(x,2t)} \tilde{f}^{4-2q}\,dx\right) + t^2(\max_{B(0,1)} v)^2$$

By the monotone convergence theorem we see that this inequality holds for f and by (2.18), (2.19) and Jensen's inequality that

$$(2.35)\qquad \left(⨍_{B(x,t)} g^{2+\gamma}\,dx\right)^{1/(2+\gamma)} \le C\left(⨍_{B(x,2t)} g^2\,dx\right)^{1/2} + c(\max_{B(0,1)} v)^2$$

This implies that $v \in W^{2,2+\gamma}(B(0,1/4)^+)$ and from Morrey's inequality we see that $v \in C^{1,\alpha}(\overline{B(0,1/4)}^+)$ and the $C^{1,\alpha}$ norm of v is bounded by the $W^{2,2+\gamma}$ norm of v. If we write (2.7) in nondivergence form we obtain an equation

$$\sum_{i,j} a_{ij}(x,\nabla v)v_{x_i x_j} + b(x,\nabla v) = 0.$$

Since the matrix A in (2.7) is smooth and the function $v \in C^{1,\alpha}(\overline{B(0,1/4)}^+)$ it follows that $a_{ij} \in C^{\alpha}(\overline{B(0,1/4)}^+)$. Also (2.8) gives us that the equation is strictly elliptic. Then lemma 9 follows from boundary Schauder estimates (see [18, chapter 6]). ■

3. Preliminary reductions

Assume Ω is a bounded domain of class C^4. This means that for each $y \in \partial\Omega$ there exists $s > 0$ such that $B(y,s) \cap \partial\Omega$ is a part of the graph of a four times continuously differentiable function defined on a line in $\mathbb{R}^2$ and $B(y,s) \cap \Omega$ lies above the graph. We use compactness and a standard covering argument to obtain $y^1, \dots, y^N \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{N} B(y^i, 100r) \quad \text{and} \quad B(y^i,10r) \cap B(y^j,10r) = \emptyset, \quad i \neq j$$

If r is sufficiently small and $y = y^i$ then it follows from the implicit function theorem that there exists a function $\theta = \theta(\cdot, y)$ four times continuously differentiable on $\mathbb{R}$ with $\theta(0) = 0$ and $\theta_x(0) = 0$ such that after a rotation of the axes, if necessary:

$$\begin{aligned} \partial\Omega \cap B(y, 1000r^{1/2}) &\subset \big\{(x_1 + y_1, \theta(x_1) + y_2) : x_1 \in \mathbb{R}\big\} \\ \Omega \cap B(y, 1000r^{1/2}) &\subset \big\{(x_1 + y_1, x_2) : x_2 - y_2 > \theta(x_1), x_1 \in \mathbb{R}\big\} \end{aligned}$$

Let

$$K_1 = \max_{y \in \{y^i\}_1^N} \left(\max_{x \in \partial\Omega \cap B(y,100r^{1/2})} \sum_{k=1}^{4} |\theta^{(k)}(\cdot, y)| \right)$$

and for $0 < \epsilon < \sigma_0 \leq 10^{-3}$ choose $r_0 > 0$ so small that for $0 < r \leq r_0$

$$K_1 r^{1/2} \leq 10^{-3} r^{1/4} \leq 10^{-9}\epsilon^4 \tag{3.1}$$

which is possible since $K_1 < +\infty$ by compactness of $\partial\Omega$. Let u be a function satisfying (1.5a)–(1.5b) with D replaced by Ω and assume that $u \in C^4(\overline{\Omega})$

and $|\nabla u| > 1$ on $\partial\Omega$. Let

$$K_2 = \max_{y\in\{y^i\}_1^N}\left(\max_{x\in\bar\Omega\cap B(y,100r^{1/2})}\sum|\partial_\alpha u(x)|\right)$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multiindex and $0 \le |\alpha| \le 4$. Choose r_0 even smaller so that if $0 < r \le r_0$ then

$$K_2 r^{1/2} \le 10^{-3} r^{1/4} \le 10^{-9}\epsilon^4 \tag{3.2}$$

Let l be the largest nonnegative integer such that $2^{-l}\sigma_0 > \epsilon$ and let $\sigma_k = 2^{-k}\sigma_0$ for $k = 0, 1, \ldots$. Put

$$E_k = \big\{x \in \partial\Omega : 1 + \sigma_k < |\nabla u(x)| \le 1 + \sigma_{k-1}\big\}, \tag{3.3}$$

for $1 \le k \le l+1$ and

$$E_0 = \big\{x \in \partial\Omega : |\nabla u(x)| > 1 + \sigma_0\big\} \tag{3.4}$$

Let $\psi \ge 0$ be a C^∞ function on $\mathbb{R}$ with $\max\psi = 1$ and support in the unit interval. Let L be the set of all $y \in \{y^i\}_1^N$ for which

$$B(y, 100r) \cap \bigcup_{k=0}^{l+1} E_k \ne \emptyset$$

For a fixed $y = (y_1, y_2) \in L$ let j be the smallest nonnegative integer with

$$B(y, 100r) \cap E_j \ne \emptyset \tag{3.5}$$

Put

$$\xi(x_1) = \theta(x_1) - \sigma_j^4 r\psi(x_1/r\sigma_j^2) + y_2 \quad x_1 \in \mathbb{R}$$

Now we define Ω' as follows

(i) $\Omega \setminus \bigcup_{y\in L} B(y, 10r) = \Omega' \setminus \bigcup_{y\in L} B(y, 10r)$

(ii) $\partial\Omega' \cap B(y, 10r) = \{(x_1 + y_1, \xi(x_1)) : x_1 \in \mathbb{R}\} \cap B(y, 10r)$

(iii) $\Omega' \cap B(y, 10r) = \{(x_1 + y_1, x_2) : x_2 > \xi(x_1)\} \cap B(y, 10r)$.

Clearly Ω' is of class C^4.

Lemma 10. *Let u' be defined by* (1.5a)–(1.5b) *with D replaced by Ω'. Then $u' \in C^4(\bar\Omega')$ and if r_0 is small enough*

$$|\nabla u'(x)| > 1, \quad x \in \partial\Omega'. \tag{3.6}$$

Proof. First $u' \in C^4(\overline{\Omega})$ follows from lemma 9 since $\nabla u' \neq 0$ in Ω' (see [15]). If $x \in \partial\Omega' \cap \partial\Omega$ then it follows from the maximum principle that (3.6) is true. Let $Z(y,t) = \{(x_1,x_2) : |x_i - y_i| < t, i = 1,2\}$. If $x \in \partial\Omega' \setminus \partial\Omega$ we first note that since ψ has support in the unit interval

$$(\partial\Omega' \setminus \partial\Omega) \cap B(y,10r) \subset Z(y,r) \tag{3.7}$$

whenever $y \in L$. From the maximum principle and (3.7) it follows that to prove (3.6) it suffices to show that

$$|\nabla u^*(x)| > 1 \quad x \in Z(y,r) \cap \partial\Omega^* \tag{3.8}$$

where Ω^* is obtained by adding just one bump to Ω at the point y and u^* satisfies (1.5a)-(1.5b) with D replaced by Ω^*.

We note that since $|\nabla u(x)| > 1$ on $\partial\Omega$ it follows from (3.2) that $u_{x_2} > 1/2$ when $x \in Z(y,r)$. Let $t_0 = \min_Y u$ where $Y = \{(x_1,x_2) \in \partial Z(y,r) \cap \Omega : |y_2 - x_2| = r\}$. Note that $ct_0 \geq \max_{\Omega \cap Z(y,r)} u$ by Harnack's inequality. Let $U = \Omega \cap Z(y,r) \cap \{u < t_0\}$ and note that u is increasing on $\partial U \cap \partial Z(y,r)$. Let $U^* = \Omega^* \cap Z(y,r) \cap \{u(x) < t_0\}$. Define v to be the p harmonic function in U^* such that $v = 0$ on $\partial\Omega^*$ and $v = u$ on $\partial U^* \setminus \partial\Omega^*$. Note that $v \leq u^*$ in U^* by the boundary maximum principle so it suffices to show $|\nabla v| > 1$ on $\partial\Omega^*$. In order to do this we need to apply the estimates in section 2 to the function v. This requires us to show that $\nabla v \neq 0$.

Consider the function v^ϵ in U^* which solves the equation

$$\nabla \cdot ((|\nabla v^\epsilon|^2 + \epsilon)^{p/2-1} \nabla v^\epsilon) \tag{3.9}$$

and satisfies $v^\epsilon = v$ on ∂U^*. This equation is strictly elliptic so it follows from Schauder estimates (see [13] or [18]) that v^ϵ is real analytic in the interior of U^* and continous in the closure of U^* (see [13]). If $t < t_0$ the set $\partial U \cap \{u = t\}$ contains exactly two points. Since $v^\epsilon = u$ on $\partial U^* \setminus \partial\Omega^*$ the set $\{v^\epsilon(x) > s\}$ is connected in U^* ($s < t_0$) since each component must intersect the boundary of U by the maximum principle for v^ϵ. We note that it follows from [15] that if $|\nabla v^\epsilon(x_0)| = 0$ then $\{v^\epsilon(x) > v^\epsilon(x_0)\}$ can not be connected. Since we have already concluded that these sets are connected we see that $\nabla v^\epsilon \neq 0$ in U^*. Now one can argue as in [15] to obtain $\nabla v \neq 0$ in U^*. Since u is Hölder continous there exists a λ which depends only on p so that $u < t_0$ in $Z(y, \lambda r)$ so $Z(y,\lambda r) \cap \Omega^* \subset U^*$. Thus we have $\nabla v \neq 0$ in $Z(y, \lambda r)$.

Now we can apply lemma 9 to v and obtain

$$\max_{Z(y,\lambda r/8)\cap\Omega^*} |D^2 v| \leq \frac{c}{r^2} \max_{Z(y,\lambda r)} v \leq \frac{c}{r^2} \max_{Z(y,r)} u \leq \frac{c}{r} |\nabla u|(t)$$

for $t \in Z(y,r)$.

Let σ_0 be so small that $\sigma_0 < \lambda/8$. By the maximum principle $|\nabla v| \geq |\nabla u|$ on $\partial\Omega \cap \partial\Omega^*$ and from our construction we know that there exists some point $x \in \partial\Omega \cap B(y, 100r)$ such that $1 + \sigma_j \leq |\nabla u|(x)$. From (3.2) it follows that $|\nabla u|(x) \geq 1 + \sigma_j/2$ for all $x \in \partial\Omega \cap B(y, 100r)$. Pick a point $z \in \partial\Omega \cap \partial\Omega^* \cap B(y, \sigma_j^2 r)$. By (3.2) we see that

$$|\nabla u(t)| \leq |\nabla u(z)| + 10^{-9}\epsilon^4 r^{1/2}$$

for $t \in Z(y, r)$. Choosing σ_0 smaller so that $C\sigma_0 < 10^{-3}$ and using the mean value theorem and (3.2) we obtain for $x \in \partial\Omega^* \cap B(y, \sigma_j^2 r)$

(3.10)
$$\begin{aligned} |\nabla v(z) - \nabla v(x)| &\leq \max_{Z(y,r)\cap\Omega^*} |D^2 v||z - x| \\ &\leq c|\nabla u|(t)\frac{|z-x|}{r} \leq 10^{-3}\sigma_j|\nabla u|(z) + 10^{-12}\epsilon^4 r^{1/2}\sigma_j \end{aligned}$$

and since $|\nabla v|(z) \geq |\nabla u|(z)$

$$\begin{aligned} |\nabla v|(x) &\geq (1 - 10^{-3}\sigma_j)|\nabla v|(z) - 10^{-12}\epsilon^4 r^{1/2}\sigma_j \\ &\geq (1 - 10^{-3}\sigma_j)(1 + \frac{1}{2}\sigma_j) - 10^{-12}\epsilon^4 r^{1/2}\sigma_j > 1. \end{aligned} \tag{3.11}$$

Which is what we needed to prove. ■

Lemma 11. *Let Ω, Ω' be as above. If $\epsilon \leq t \leq 1$*

$$H^1(\partial\Omega') \geq H^1(\partial\Omega) + \eta(t)H^1\{x : |\nabla u(x)| > 1 + t\} \tag{3.12}$$

if $p < 2$ but

$$H^1(\partial\Omega') \geq H^1(\partial\Omega) + \eta(t)H^1\{x : |\nabla u(x)| < 1 - t\} \tag{3.13}$$

if $p > 2$. Here η is a positive function on $]0, \infty[$.

Proof. To prove (3.12) let

$$c_2 = \int_{\mathbb{R}} |\psi'(x)|^2\, dx$$

and choose σ_0 even smaller so that

$$\sigma_0 \leq c_2 \leq 2(\max_{\mathbb{R}} |\psi'|)^2 \leq \sigma_0^{-1}10^{-6} \tag{3.14}$$

Then it follows from (3.1) and the definition of σ_j

$$
\begin{aligned}
(3.15)\qquad H^1(Z(y,r)\cap\partial\Omega') &= \int_{-r}^{r}\sqrt{1+|\xi'|^2}\,dx \\
&\geq \int_{-r}^{r}\sqrt{1+\sigma_j^4|\psi'(x/r)|^2}\,dx - 2\epsilon^8 r \\
&= r\int_{-1}^{1}\sqrt{1+\sigma_j^4|\psi'(x)|^2}\,dx - 2\epsilon^8 r \\
&\geq (1+\frac{1}{4}\sigma_j^4 c_2 - \epsilon^8)2r \geq \frac{1}{8}\sigma_j^4 c_2 2r + H^1(Z(y,r)\cap\partial\Omega).
\end{aligned}
$$

Take $t \geq \epsilon$ and let k be the least nonnegative integer such that $t \geq \sigma_k$, $0 \leq k \leq l+1$. Let $J = J(k)$ be the set of all i such that (3.5) holds with $y = y^i$ and $j \leq k$. From (3.1) it is clear that

$$
\begin{aligned}
(3.16)\qquad H^1\{x\in\partial\Omega : |\nabla u(x)| \geq 1+t\} &\leq H^1\Big(\bigcup_{i\in J} B(y^i,100r)\cap\partial\Omega\Big) \\
&\leq 2\sum_{i\in J} 200r
\end{aligned}
$$

and we conclude that

$$
(3.17)\qquad H^1(\partial\Omega') \geq H^1(\partial\Omega) + c_3\sigma_k^4 H^1\{x\in\partial\Omega : |\nabla u(x) > 1+t\}
$$

Let

$$
(3.18)\qquad \eta(t) = \begin{cases} c_3\sigma_0^4 & \text{if } \sigma_0 \leq t \\ c_3\sigma_k^4 & \text{if } \sigma_k \leq t < \sigma_{k-1}, k = 1,2,\dots \end{cases}
$$

Since η does not depend on Ω this proves (3.12). The case when $p > 2$ is similar. ∎

4. Proof of Theorem 1

Lemma 12. *Let u, Ω be as above. If $1 < p < 2$ then*

$$
(4.1)\qquad \int_{\partial\Omega} |\nabla u|^{p-1}\log|\nabla u|\,dH^1 \leq C
$$

and if $p > 2$ then

$$
(4.2)\qquad \int_{\partial\Omega} |\nabla u|^{p-1}\log|\nabla u|\,dH^1 \geq C
$$

where the constant C depends only on F.

Proof. We proceed as in [2]. Note that if $\eta \in \mathbb{R}^2$ and $|\eta| = 1$ then $\zeta = \langle \nabla u, \eta \rangle$ is a strong solution to

$$L\zeta = \nabla \cdot \big((p-2)|\nabla u|^{p-4} \langle \nabla u, \nabla \zeta \rangle \nabla u + |\nabla u|^{p-2} \nabla \zeta \big) = 0$$

in $\Omega \cap N$ since $\nabla u \neq 0$. In other words

$$L\zeta = \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i} \big(a_{ik}(x) \zeta_{x_k}(x) \big) = 0 \tag{4.3}$$

where

$$a_{ik}(x) = |\nabla u|^{p-4} \big((p-2) u_{x_i} u_{x_k} + \delta_{ik} |\nabla u|^2 \big)(x) \tag{4.4}$$

and δ_{ij} is the Kronecker δ. Note that

$$Lu = (p-1) \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \tag{4.5}$$

Since the equation is rotationally invariant we can assume that $\nabla u(x) = (|\nabla u(x)|, 0)$. Let $v = \log |\nabla u(x)|$. Then

$$v_{x_k} = |\nabla u|^{-2} \sum_{l=1}^{2} u_{x_l} u_{x_l x_k}$$

and so

$$Lv = \sum_{i,k=1}^{2} \frac{\partial (a_{ik} v_{x_k})}{\partial x_i} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(|\nabla u|^{-2} \sum_{k,l=1}^{2} a_{ik} u_{x_l} u_{x_l x_k} \right).$$

Using (4.3) on the righthand side we get

$$\begin{aligned} Lv = & -2|\nabla u|^{-4} \sum_{i,k,l,m=1}^{2} a_{ik} (u_{x_l} u_{x_l x_k} u_{x_m} u_{x_m x_i}) \\ & + |\nabla u|^{-2} \sum_{i,k,l=1}^{2} a_{ik} u_{x_l x_i} u_{x_l x_k} = T_1 + T_2 \,. \end{aligned} \tag{4.6}$$

From the definiton of the a_{ik}'s and our assumption that $\nabla u(x) = (|\nabla u(x)|, 0)$ we see at x

$$a_{11} = (p-1)|\nabla u|^{p-2}, a_{22} = |\nabla u|^{p-2} \text{ and } a_{12} = a_{21} = 0 \tag{4.7}$$

and also from (4.5)

$$(p-1) u_{x_1 x_1} + u_{x_2 x_2} = 0 \,. \tag{4.8}$$

Using this in the definitions of T_1, T_2 we obtain at x

$$T_1 = -2|\nabla u|^{p-4}((p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2)$$

and

$$T_2 = p|\nabla u|^{p-4}((p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2)$$

and we conclude

$$Lv = (p-2)|\nabla u|^{p-4}((p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2) \tag{4.9}$$

so $Lv \le 0$ when $1 < p < 2$ and $Lv \ge 0$ when $p > 2$. Since u is smooth and $\nabla u \neq 0$ and $\partial\Omega$ is smooth we can apply the divergence theorem to the vector field whose ith component is

$$u \sum_{k=1}^{2} a_{ik} v_{x_k} - v \sum_{k=1}^{2} a_{ik} u_{x_k}$$

in the region $\Omega \backslash G$ where G is a region with smooth boundary which contains the set F in its interior. If $1 < p < 2$ we obtain

(4.10)

$$0 \ge \int_{\Omega\backslash G} u \left(\sum_{k=1}^{2} \frac{\partial}{\partial x_i}(a_{ik} v_{x_k}) \right) - v \left(\sum_{k=1}^{2} \frac{\partial}{\partial x_i}(a_{ik} u_{x_k}) \right) dx$$
$$= \int_{\partial\Omega} |\nabla u|^{p-1} \log |\nabla u| \, dH^1 + \int_{\partial G} \sum_{i=1}^{2} \left(u \sum_{k=1}^{2} a_{ik} v_{x_k} - v \sum_{k=1}^{2} a_{ik} u_{x_k} \right) \eta_i \, dH^1$$

where η is the outward unit normal for $\Omega\backslash G$ on ∂G and we used the fact that $u = 0$ on $\partial\Omega$ and $\eta = -\frac{\nabla u}{|\nabla u|}$ on $\partial\Omega$. This gives (1.9) and (1.10) where the constant is determined by the integral over ∂G which is independent of Ω. ■

Remember that ψ is a C^∞ function on $\mathbb{R}$ with $\max \psi = 1$ and support in the unit interval. Also, in section 3 σ_0, $0 < \sigma_0 \le 10^{-3}$ was chosen so that (3.14) was true. Finally, for a given ϵ, $0 < \epsilon \le \sigma_0 \, r_0$ was chosen so small that the estimates in section 3 are true for $0 < r \le r_0$. We describe the construction of D in more detail. We only describe the case of "pushing out" since the other case is similar. Let D_0 be a domain such that $F \subset D_0$ and the function u_0 which satisfies (1.5a)-(1.5b) for D_0 also satisfies $|\nabla u_0| > 1$ on ∂D_0. Let $\rho = d(\partial\Omega, F)$. Let $\epsilon_0 = \sigma_0$ and $\epsilon_k = 2^{-k}\epsilon_0$ for $k = 1, 2, \ldots$. Choose a covering $L_1 = \{B(z_0^i, t_0^i)\}$, $1 \le i \le k_0$ of ∂D_0 such that $t_0^i \le 1/2$ for all i and

$$2 \sum_{i=1}^{k_0} t_0^i \le H^1(\partial D_0) + \frac{1}{2}$$

Since D_0 is compact we can assume $k_0 < \infty$. Let $2r_1' > 0$ be the distance from ∂D_0 to $\mathbb{R}^2 \setminus \bigcup_1^{k_0} B(z_0^i, t_0^i)$. Set $\Omega = D_0$, $\epsilon = \epsilon_1$ and $r_1 = \min\{r_1', r_0(\epsilon_1, K_1, K_2), 10^{-9}\rho\}$ where K_1 and K_2 are defined relative to D_0, u_0 as in section 3. Then we do as in section 3 to obtain $D_1 = \Omega'$. Now suppose for some $m \geq 1$ we have defined $\{D_k\}_0^m$, $\{L_k\}_0^m$, $\{r_k'\}_0^m$ and $\{r_k\}_0^m$. Let $L_{m+1} = \{B(z_m^i, t_m^i)\}_1^{k_m}$ be a covering of ∂D_m such that $t_m^i \leq 2^{-(m+1)}$, $1 \leq i \leq k_m$ and

$$2\sum_{i=1}^{k_m} t_m^i \leq H^1(\partial D_m) + 2^{-(m+1)} \tag{4.11}$$

Let $2r_m' > 0$ be the distance between ∂D_m and $\mathbb{R}^2 \setminus \bigcup_1^{k_m} B(z_m^i, t_m^i)$. Let $\Omega = D_m$, $\epsilon = \epsilon_m$ and $r = r_{m+1} = \min\{r_m', r_0(\epsilon_{m+1}, K_1, K_2), 10^{-4m} r_m\}$ where K_1 and K_2 are defined relative to D_m, u_m as in Section 3. Then we do as in Section 3 to obtain $D_{m+1} = \Omega' \supset D_m$. By induction we get $\{D_k\}_0^\infty$, $\{L_k\}_1^\infty$, $\{r_k'\}_1^\infty$ and $\{r_k\}_1^\infty$. Finally define D to be the union of the sets D_k

Lemma 13. *Let D, D_k, $k = 1, 2, \ldots$ be as above. Then D is a quasicircle which is not convex. For D_k we have $\mu_k(\partial D_k) \leq C$ where C is inedependent of k and μ_k is the measure corresponding to u_k as in* (1.2).

Proof. To prove that D is a quasicircle it suffices to show that ∂D_m satisfies the Ahlfors three point condition for $m = 1, 2, \ldots$ with constant independent of m. Once we have proved this we get a sequence $\{f_m\}$ of quaisconformal mappings of $\mathbb{R}^2$ with

$$f_m(\partial B(0,1)) = \partial D_m \text{ and } |(f_m)_{\bar{z}}| \leq k|(f_m)_z| \tag{4.12}$$

where $0 < k < 1$ is independent of m. Since a subsequence of $\{f_m\}$ converges uniformly on compact subsets of $\mathbb{R}^2$ to a quasiconformal $f : \mathbb{R}^2 \to \mathbb{R}^2$ we see that (4.12) holds with f_m, D_m replaced by f, D. To show that ∂D_m satisfies the Ahlfors three point condition independent of m we first find a constant C such that $|z_1 - z_3| < C|z_1 - z_2|$ for z_1, z_2, z_3 on the graph of ψ and z_3 between z_1 and z_2. Now suppose z_1, z_2, z_3 lie on ∂D_m and $|z_1 - z_2| < 10r_m$. Let ξ be a function whose graph is after a rotation and translation $\partial D_m \cap B(z_1, 10r_m)$. By (3.1) the distance $|\xi(x) - \psi(x)|$ is less than $10^{-9}\epsilon_m^4|x|$ which implies that the graph of ξ and therefore $\partial D_m \cap B(z_1, 10r_m)$ satisfies the Ahlfors three point condition with a slightly larger constant C but still independent of m. If $|z_1 - z_2| > 10r_m$ we find $k < m$ such that $|z_1 - z_2| < 10r_k$ but $|z_1 - z_2| > 100r_{k+1}$. Let z^* be the projection of $z \in \partial D_m$ on ∂D_k. Then $|z_1 - z_2| > |z_1^* - z_2^*| - \eta r_k$ where η is small and likewise $|z_1 - z_3| < |z_1^* - z_3^*| + \eta r_k$. From this it follows that $|z_1 - z_3| < 2C|z_1 - z_3|$ for all m.

To prove that $\mu_k(\partial D_k) \leq C$ where C is independent of k we recall that $\mu(B(x,r)) \leq cr^{2-p}(\max_{B(x,2r)} u)^{p-1}$ for any measure defined by (1.2). This estimate is proved in [5] and our claim follows immediately by covering the boundaries of the domains D_k with balls and then applying the estimate in each ball since $u_k(x) \leq 1$ for all $x \in D_k$. To see that the domain is not convex note that the function ψ can be chosen so that D_m has the property that there exist points $x, y \in D_m$ such that

$$\max_{t\in[0,1]} d(tx + (t-1)y, D_m) > \frac{\epsilon_m^4 r_m}{8} > 5^{4m} r_{m+1} \frac{\epsilon_0}{8} > r_{m+1}$$

if m is large enough. It is clear from the construction described above that if $z \in D$ then $d(z, D_m) < r_{m+1}$ so the line segment between x and y does not lie in D. However $x, y \in D$ so D is not convex. ■

The proof of Theorem 1 follows from the above lemmas and the argument at the end of section 1 once we prove (1.14). The proof that $H^1|_{\partial D_m} \to H^1|_{\partial D}$ in [16] applies to our case without change. For completeness we give a brief outline. First show that there exists a mapping h_m from ∂D_m to ∂D_{m+1} which satisfies

$$|h_m(x) - h_m(z)| \geq (1 - cr_m^{1/2})|x - z|.$$

Then let

$$p_j(x) = \lim_{k\to\infty} h_k \circ \cdots \circ h_{j+1}(x) \text{ for } x \in \partial D_j.$$

If

$$e_j = \prod_{m=j+1}^{\infty} (1 - cr_m^{1/2})$$

it follows that

$$e_j|x - y| \leq |p_j(x) - p_j(y)|, \qquad x, y \in \partial D_j,$$

and if q_j is the inverse of p_j we have

$$|q_j(x) - q_j(y)| \leq e_j^{-1}|x - y| \tag{4.13}$$

when $x, y \in \partial D$. Next we use Kirsbraun's Theorem (see [7]) to obtain an extension of q_j to $\mathbb{R}^2$ such that (4.13) holds whenever $x, y \in \mathbb{R}^2$. Let $\nu(E) = H^1(q_j^{-1}(E) \cap \partial D)$. Then we have

$$H^1(E \cap \partial D_j) \leq e_j \nu(E)$$

Also note that it follows from the definition of the r_m's that $e_j \to 1$ when $j \to \infty$. Let $g \geq 0$ be a continuous function. Then it follows from the change of variables formula that

$$e_j \int_{\partial D_j} g\, dH^1 \le \int_{\mathbb{R}^n} g\, d\nu = \int_{\partial D} g \circ q_j\, dH^1 \tag{4.14}$$

If we let $j \to \infty$ then $q_j(x) \to x$ uniformly on compact subsets of $\mathbb{R}^n$ so

$$\int_{\partial D} g \circ q_j\, dH^1 \to \int_{\partial D} g\, dH^1$$

Hence from (4.14) we have

$$\limsup_{k\to\infty} \int_{\partial D_n} g\, dH^1 \le \int_{\partial D} g\, dH^1$$

From our construction of D it follows that

$$H^1(\partial D) \le \liminf_{m\to\infty} H^1(\partial D_m)$$

If $0 \le g \le 1$ then it follows that

$$\begin{aligned} H^1(\partial D) &\le \liminf_{k\to\infty} H^1(\partial D_{n_k}) \\ &\le \liminf_{k\to\infty} \int_{\partial D_k} g\, dH^1 + \limsup_{k\to\infty} \int_{\partial D_k} (1-g)\, dH^1 \\ &\le \limsup_{k\to\infty} \int_{\partial D_n} g\, dH^1 + \limsup_{k\to\infty} \int_{\partial D_n} (1-g)\, dH^1 \\ &\le \int_{\partial D} g\, dH^1 + \int_{\partial D} (1-g)\, dH^1 = H^1(\partial D) \end{aligned}$$

Thus equality holds everywhere so

$$\lim_{k\to\infty} \int_{\partial D_n} g\, dH^1 = \int_{\partial D} g\, dH^1$$

which is what we wanted to prove.

To show that $\mu_k \to \mu$ we note that if we are pushing out then $u(x) < \epsilon$ on ∂D_n for n large enough. Therefore $u(x) < u_n(x) + \epsilon$ in D_n in other words $u(x) - u_n(x) < \epsilon$ in D_n. Elsewhere $u_n(x) = 0$ and $u(x) < \epsilon$ so $u_n \to u$ uniformly. Since the measures μ_n are bounded we have a subsequence which is weakly convergent to some measure ν. Now

$$\begin{aligned} \int \phi\, d\nu &= \lim_{n\to\infty} \int \phi\, d\mu_n = \lim_{n\to\infty} \int_N |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \phi \rangle\, dx \\ &= \int_N \lim_{n\to\infty} |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \phi \rangle\, dx \\ &= \int_N |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle\, dx = \int \phi\, d\mu \end{aligned} \tag{4.15}$$

where N is some neighborhood containing ∂D and ∂D_n if n is large enough and $\phi \in C_0^\infty(N)$. It follows that $\nu = \mu$ which is what we wanted to show.

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