Rev. Mat. Iberoamericana **24** (2008), no. 1, 71–90

Transformations between surfaces in $\mathbb{R}^4$ with flat normal and/or tangent bundles

Angel Montesinos-Amilibia

Abstract

We exhibit several transformations of surfaces M in $\mathbb{R}^4$: a transformation of flat surfaces that gives surfaces with flat normal bundle (semiumbilical surfaces); and its inverse that from a semiumbilical surface obtains a flat surface; then a one-parameter family of transformations f on flat semiumbilical immersed surfaces (FSIS), such that $df(T_pM)$ is totally orthogonal to T_pM, and that give FSIS. This family satisfies a Bianchi type of permutability property.

1. Introduction

Among surfaces in $\mathbb{R}^4$, those of flat tangent bundle and those of flat normal bundle have received considerable attention, especially those that have both properties.

As a precedent, in [2] we know of a transformation that takes a hyperspherical surface (hence, of flat normal bundle), and gets a flat surface. We present here a transformation that takes any surface with flat normal bundle without inflection points (*semiumbilical* surface) and converts it to its *evolute,* which results in a flat surface; the condition (no inflection points) is meant to shun the possibility that the map go to infinity, as happens at a point with zero curvature when defining the evolute of a plane curve.

Then there is a kind of inverse, that is a transformation that takes any immersed flat surface in $\mathbb{R}^4$ and gives (in the region where that transformation is an immersion) its *envelope,* which is a semiumbilical surface.

Thus, it seems that the differential equations that define semiumbilical surfaces in $\mathbb{R}^4$ are essentially the same as those that define flat surfaces.

2000 Mathematics Subject Classification: 53A05.
Keywords: Flat, semiumbilical surfaces in $\mathbb{R}^4$, Bianchi permutability, Bäcklund transformation, evolute.

By combining both types of transformations we get a transformation, $f_t : M \to f_t(M)$, which depends on a real parameter t (and on the choice of a vector field that must satisfy a differential equation) and yields a flat semiumbilical immersed surface (from now on, an FSIS) from another FSIS. These transformations satisfy an analogous to the Bianchi permutability theorem for Bäcklund transformations (see [3] for a detailed introduction and [1] for a description in a modern context).

All the transformations $f : M \to f(M)$ so far described for FSIS in $\mathbb{R}^4$ are "orthogonal" in the sense that the tangent plane of $f(M)$ at $f(p)$ is the orthogonal complement of the tangent plane of M at p. The composition of two such transformations gives a "parallel" transformation, that is one such that the tangent plane of $f(M)$ at $f(p)$ is parallel to the tangent plane of M at p. These transformations depend on two real parameters (and on the choice of some vector field that must satisfy a differential equation).

2. Basic concepts and notation

In the following, M will be a surface immersed in $\mathbb{R}^n$, $n \geq 4$. However, since all of our study will be local, one can without loss of rigor assume that M is an embedded surface. On M we have the tangent bundle $\pi : TM \to M$, and the *normal bundle* given by

$$NM = \cup_{p\in M}(T_pM)^{\perp}, \quad \pi_N : NM \to M,$$

where $(T_pM)^{\perp}$ denotes the subspace of $T_p\mathbb{R}^n$ orthogonal to T_pM. Its fibre upon $p \in M$ will be denoted by $N_pM = (T_pM)^{\perp}$. Usually we will consider T_pM and N_pM as vector subspaces of $\mathbb{R}^n$. We will use a dot to mean the standard inner product. If $X \in T_p\mathbb{R}^n$, we will have $X = X^{\top} + X^{\perp}$, with $X^{\top} \in T_pM$, $X^{\perp} \in N_pM$.

The Lie algebra of vector fields on a manifold M will be denoted $\mathfrak{X}(M)$, and if E is the total space of a vector bundle over M, ΓE will stand for the $C^{\infty}(M)$-module of its differentiable sections. Usually, if s is a section of a fiber bundle, s_p will be its value at p.

The ordinary directional derivative of functions on $\mathbb{R}^n$ will be written as D_X. But note that it may have a broader meaning of which we will have a frequent use. In fact, if S is a submanifold of $\mathbb{R}^n$, $p \in S$, $X_p \in T_pS$ and $f : S \to \mathbb{R}^m$ is a differentiable map, then $D_{X_p}f \in \mathbb{R}^m$ will be defined as $df(X_p) \in T_{f(p)}\mathbb{R}^m \approx \mathbb{R}^m$. This defines the map Df that sends $X \in \mathfrak{X}(M)$ to the map $D_Xf : S \to \mathbb{R}^m$. For vector fields on $\mathbb{R}^n$, D is a metric (that is $Dg = 0$, where g is the metric tensor field) linear connection with zero torsion and curvature.

There is another useful viewpoint of D. Let S be a submanifold of $\mathbb{R}^n$, $X \in \mathfrak{X}(S)$ and $u : S \to \mathbb{R}^n$ be a smooth map. Then u may be regarded as a section of the $\mathbb{R}^n$-fibred vector bundle induced over S by the inclusion $i : S \to \mathbb{R}^n$. The directional derivative $D_X u$ is thus the covariant derivative defined by the linear connection induced on this bundle by the standard Levi-Civita connection on $\mathbb{R}^n$. Since the curvature of the standard connection vanishes, the same happens with the curvature of the induced connection. In other words, if $Y \in \mathfrak{X}(S)$, then

$$D_X D_Y u - D_Y D_X u - D_{[X,Y]} u = 0.$$

The second fundamental form of M, α, may be defined at p as the symmetric bilinear form $\alpha_p : T_pM \times T_pM \to N_pM$ given by $\alpha_p(X_p, Y_p) = (D_{X_p} Y)^{\perp}$, where the value of $Y \in \mathfrak{X}(M)$ at p is Y_p. In fact, $(D_{X_p} Y)^{\perp}$ depends only on the value of Y at p.

The map $\mathfrak{X}(M) \times \Gamma(NM) \to \Gamma(NM)$ given by $\nabla^{\perp}_X u = (D_X u)^{\perp}$ is a metric linear connection. If its curvature tensor field is zero everywhere, we say that the normal bundle of M is flat. The map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $\nabla^{\top}_X Y = (D_X Y)^{\top}$ is a torsionless metric linear connection. If its curvature (that is Gauss curvature of M) is zero everywhere, we say that M is flat.

Let PT_pM be the projective space of vector lines of T_pM. The second fundamental form defines a map $\eta_p : PT_pM \to N_pM$, by

$$\eta_p([t]) = \eta_p(t) = \frac{\alpha_p(t,t)}{t \cdot t}, \quad t \in T_pM \backslash \{0\}.$$

The image of η_p is an ellipse in N_pM, that may be degenerate, called the *curvature ellipse* at p. If the ellipse degenerates to a point at p, then we say that p is *umbilic*. If the ellipse lies in an affine line (i.e. it degenerates to a segment or to a point), we say that p is a *semiumbilic* point. If, in addition, a line containing the ellipse passes by the origin of N_pM, we say that p is a *point of inflection*. If $n = 4$ and the origin of N_pM lies out of the curvature ellipse at p, the directions t of T_pM such that $\eta_p(t)$ determines a line tangent to the ellipse, are called *asymptotic directions*. If $n \geq 4$ and the ellipse degenerates to a segment (not a point), the directions t of T_pM such that $\eta_p(t)$ is and end of that segment are also called asymptotic directions; they are mutually orthogonal.

The following facts are well known or easily proved (see for instance [5], [6] and [4]). The point p is semiumbilic iff the curvature of $\nabla^{\perp}$ vanishes at p. So, M is totally semiumbilic iff its normal bundle is flat.

Let a, b be the half-axes of the curvature ellipse at p; then the curvature of $\nabla^{\top}$, or equivalently its Gauss curvature K, vanishes at p iff the origin of N_pM belongs to the sphere in N_pM of radius $\sqrt{a^2 + b^2}$ centered at the center of the ellipse. Thus, if p is semiumbilic, $K_p = 0$ iff $\eta_p(t_1)$ and $\eta_p(t_2)$ are orthogonal, where t_1 and t_2 are the asymptotic directions at p. If, in addition, p is not of inflection, $\eta_p(t_1)$ and $\eta_p(t_2)$ are linearly independent.

3. Some facts on semiumbilical surfaces

The following characterization of semiumbilic points will be crucial for our results. It differs from the first one that I know, that of Wong [7]. The reason is that for surfaces in $\mathbb{R}^n$ with $n \geq 5$, Wong condition of being semiumbilic is satisfied "almost everywhere", that is, in addition to the points that are semiumbilic for us, whenever the curvature ellipse does not degenerate and the affine plane containing it does not pass by the origin.

Proposition 3.1. *Let M be an immersed surface in $\mathbb{R}^n$, $n \geq 4$, $p \in M$ be a non umbilic point and g denote the first fundamental form of M. Then there is a 2-dimensional vector subspace E_p of N_pM that contains the curvature ellipse at p and there is a vector $c_p \in E_p$ such that $c_p \cdot \alpha_p = g_p$, iff p is a non inflection semiumbilic point. Moreover if such a vector c_p exists, it is unique.*

Proof. If such a plane E_p and vector c_p exist, then for each unit vector $t \in T_pM$ we will have $c_p \cdot \alpha_p(t,t) = c_p \cdot \eta_p(t) = g_p(t,t) = 1$. Therefore, the height of all points of the curvature ellipse at p over the hyperplane H_p of N_pM with normal c_p is constant and equal to $\frac{1}{|c_p|}$. Therefore, the curvature ellipse lies in the line of intersection of E_p with the affine hyperplane parallel to H_p at that distance, so that p is semiumbilic and obviously it cannot be of inflection without being umbilic.

Conversely, if p is a non inflection semiumbilic point, let n_p be the point of the line containing the curvature ellipse at p (a segment) nearest to the origin of N_pM. Since p is not an inflection point, $n_p \neq 0$. Let $c_p = \frac{n_p}{n_p \cdot n_p}$. Since $n_p \cdot \eta_p(t) = n_p \cdot n_p$, for all unit vectors $t \in T_pM$, se will have $c_p \cdot \eta_p(t) = 1 = g_p(t,t)$. Since $c_p \cdot \eta_p$ is a quadratic form on T_pM, we conclude that its corresponding bilinear symmetric form $c_p \cdot \alpha_p$ is equal to g_p. The uniqueness of c_p is obvious. ■

Wong only requires the existence of a vector $c_p \in N_pM$ such that $c_p \cdot \alpha_p = g_p$. In $\mathbb{R}^4$ we can dispense with the requirement of a plane E_p to contain c_p because N_pM does the job. Thus

Corollary 3.2. *Let M be an immersed surface in $\mathbb{R}^4$, $p \in M$ be a non umbilic point and g denote the first fundamental form of M. Then there is a unique vector $c_p \in N_pM$ such that $c_p \cdot \alpha_p = g_p$ iff p is a non inflection semiumbilic point.*

Note that if $X, Y \in \mathfrak{X}(M)$ then $Y \cdot X = c \cdot \alpha(X, Y) = c \cdot D_X Y = -Y \cdot D_X c$. Hence

$$(D_X c)^\top = -X.$$

We describe now the curvature ellipse in more concrete terms. If (t_1, t_2) is a local orthonormal frame of TM, we put $b_1 = \eta(t_1) = \alpha(t_1, t_1)$, $b_2 = \eta(t_2) = \alpha(t_2, t_2)$, $b_3 = \alpha(t_1, t_2)$. If $t \in \mathfrak{X}(M)$ is a unit vector field, we will have $t = t_1 \cos\theta + t_2 \sin\theta$. Then, we have $\eta(t) = b_1 \cos^2\theta + b_2 \sin^2\theta + b_3 \sin 2\theta$. After an easy calculation we get

$$\eta(t) = H + B\cos 2\theta + C \sin 2\theta,$$

where

$$H = \frac{1}{2}(b_1 + b_2), \quad B = \frac{1}{2}(b_1 - b_2), \quad C = b_3$$

are smooth local sections of NM.

H is called *mean curvature vector (field)* and it does not depend on the choice of the orthonormal frame (t_1, t_2). The other two sections B and C do depend on it. In a region where the ellipse does not degenerate to a point or a circle, the frame (t_1, t_2) can be locally chosen so that the major half-axis of the ellipse be B and the minor, C. That is $|B| \geq |C|$, and $B \cdot C = 0$. If M is semiumbilical, let E_p be the plane passing by the origin of N_pM that contains the curvature ellipse at p. We denote by $J : E_p \to E_p$ any one of the isometries of E_p such that $Ju_p \cdot u_p = 0, \forall u \in NM$. Then J is defined up to a sign.

Proposition 3.3. *Let M be an immersed semiumbilical surface in $\mathbb{R}^n$. Let the local orthonormal frame (t_1, t_2) of TM satisfy $b_3 = \alpha(t_1, t_2) = 0$. Then the section*

$$c = \frac{JB}{H \cdot JB},$$

where $B = \frac{1}{2}(\alpha(t_1, t_1) - \alpha(t_2, t_2))$, $H = \frac{1}{2}(\alpha(t_1, t_1) + \alpha(t_2, t_2))$ is well defined and satisfies $c \cdot \alpha = g$.

Proof. Since there are no inflection points in M, the curvature ellipse at each point is a segment not collinear with the origin, that is b_1 and b_2 are linearly independent. The frame (t_1, t_2) is simply a local frame of asymptotic directions. We have first $H \cdot JB = \frac{1}{4}(b_1 + b_2) \cdot (Jb_1 - Jb_2) = -\frac{1}{2} b_1 \cdot Jb_2 \neq 0$ because b_1 and b_2 are linearly independent. Therefore,

$$c \cdot \alpha(t_1, t_1) = \frac{Jb_1 - Jb_2}{-b_1 \cdot Jb_2} \cdot b_1 = 1 = t_1 \cdot t_1.$$

In the same manner we get $c \cdot \alpha(t_2, t_2) = t_2 \cdot t_2 = 1$. Since $\alpha(t_1, t_2) = 0$, we have finally $c \cdot \alpha = g$. ∎

The following result will be used afterwards, in the context of surfaces with both bundles, normal and tangent, flat.

Lemma 3.4. *Let M be a semiumbilical immersed surface in $\mathbb{R}^n$. Let the local orthonormal frame (t_1, t_2) of TM satisfy $b_3 = \alpha(t_1, t_2) = 0$ and put $b_1 = \alpha(t_1, t_1)$, $b_2 = \alpha(t_2, t_2)$. Then*

$$b_1 \cdot \nabla^\perp_{t_2} c = b_2 \cdot \nabla^\perp_{t_1} c = 0.$$

Proof. We have $0 = \alpha(t_1, t_2) = (D_{t_1} t_2)^\perp = (D_{t_2} t_1)^\perp$. Therefore, $D_{t_1} t_2$, $D_{t_2} t_1 \in \mathfrak{X}(M)$. Now,

$$\begin{aligned} b_1 \cdot \nabla^\perp_{t_2} c = b_1 \cdot D_{t_2} c &= \alpha(t_1, t_1) \cdot D_{t_2} c = (D_{t_1} t_1)^\perp \cdot D_{t_2} c \\ &= D_{t_2}((D_{t_1} t_1)^\perp \cdot c) - c \cdot D_{t_2} (D_{t_1} t_1)^\perp \\ &= D_{t_2}(c \cdot \alpha(t_1, t_1)) - c \cdot D_{t_2}(D_{t_1} t_1 - (D_{t_1} t_1)^\top) \\ &= D_{t_2}(1) - c \cdot D_{t_2} D_{t_1} t_1 + c \cdot \alpha(t_2, (D_{t_1} t_1)^\top) \\ &= -c \cdot (D_{t_1} D_{t_2} t_1 + D_{[t_2, t_1]} t_1) + t_2 \cdot D_{t_1} t_1. \end{aligned}$$

Now we observe that $[t_2, t_1]$, $D_{t_2} t_1 \in \mathfrak{X}(M)$, whence

$$b_1 \cdot \nabla^\perp_{t_2} c = -t_1 \cdot D_{t_2} t_1 - [t_2, t_1] \cdot t_1 - t_1 \cdot D_{t_1} t_2 = 0,$$

because $t_1 \cdot D_{t_2} t_1 = 0$ and $[t_2, t_1] = D_{t_2} t_1 - D_{t_1} t_2$. In the same manner we get $b_2 \cdot \nabla^\perp_{t_1} c = 0$. ■

Let M be an immersed surface in $\mathbb{R}^n$ and $p \in M$. Then it is clear that $\dim \alpha(T_p M \times T_p M) < 2$ iff p is an inflection point.

Proposition 3.5. *1. Let M be an immersed surface in $\mathbb{R}^n$, $p \in M$ and $\mu : T_p M \to \mathrm{Hom}(T_p M, N_p M)$ be the map given by*

$$\mu(A)(X) = \alpha_p(A, X)\,.$$

Then, there is some non-vanishing vector $A \in T_p M$ such that $\mu(A) = 0$, iff the ellipse at p is a segment, one of whose ends is the origin of $N_p M$.

2. *Let M be an immersed surface in $\mathbb{R}^n$ and $p \in M$. There is some vector $A \in T_p M$ such that $\mu(A)$ is one-to-one iff p is not an inflection point.*
3. *Let $M \subset \mathbb{R}^4$ be an FSIS and $p \in M$. Then, for the vector $A \in T_p M$ there exist a non-vanishing vector $X \in T_p M$ such that $\alpha_p(A, X) = 0$ iff A is parallel to an asymptotic direction at p.*

Proof. (1) Let $t_2 \in T_pM$ be a unit vector such that $\mu(t_2) = 0$ and let (t_1, t_2) be an orthonormal basis of T_pM. Then $b_2 = b_3 = 0$. Therefore the curvature ellipse at p is given by $\frac{1}{2}b_1(1 + \cos 2\theta)$ and this proves our claim.

(2) If p an inflection point, it is clear the space generated by α_p is a subspace of the line containing the curvature ellipse. Hence $\mu(A)$ can never be one-to-one.

Reciprocally, assume that for any non-vanishing $A \in T_pM$, there is a non-vanishing $X \in T_pM$ such that $\alpha(A, X) = 0$. In particular, with the above notation, there is some linear combination rt_1+st_2 such that $\alpha(t_1, rt_1+st_2) = rb_1 + sb_3 = 0$. Therefore b_1 and b_3, and by the same reason b_2 and b_3, are linearly dependent. If $b_3 \neq 0$, then b_1 and b_2 are multiples of b_3 and this proves our claim. Let us denote by $t^\natural$ the 1-form defined by any vector field $t \in \mathfrak{X}(M)$ as $t^\natural(X) = t \cdot X,\ X \in \mathfrak{X}(M)$. If $b_3 = 0$, then for any $r, s \in \mathbb{R}$ we have that $\mu(rt_1 + st_2) = rb_1 \otimes t_1^\natural + sb_2 \otimes t_2^\natural$ cannot be one-to-one. But this obviously implies that b_1 and b_2 are linearly dependent, so that p is an inflection point.

(3) Now if the orthonormal basis (t_1, t_2) is given by asymptotic directions we have $b_3 = 0,\ b_1 \cdot b_2 = 0$, and b_1, b_2 are linearly independent. If $X = X^1t_1 + X^2t_2$, we have $\alpha_p(A, X) = A^1X^1b_1 + A^2X^2b_2$. If $X^1 \neq 0$ then $\alpha_p(X, A) = 0$ implies $A^1 = 0$, whence A is parallel to the asymptotic direction t_2. And if $X^2 \neq 0$ then A is parallel to the asymptotic direction t_1. ■

4. Transformations between surfaces with flat tangent bundle and surfaces with flat normal bundle in $\mathbb{R}^4$

Let M be an immersed submanifold on $\mathbb{R}^n$. We put $d^\perp, d^\top$ to denote the covariant differentials. That is, $d^\perp$ acts upon a 0-form u with values in NM, that is a section of NM, by

$$(d^\perp u)(X) = \nabla_X^\perp u = (D_X u)^\perp.$$

It acts upon any 1-form β on M with values in NM by giving a 2-form $d^\perp\beta$ with values in NM as follows

$$\begin{aligned} d^\perp\beta(X, Y) &= \nabla_X^\perp(\beta(Y)) - \nabla_Y^\perp(\beta(X)) - \beta([X, Y]) \\ &= (D_X\beta(Y))^\perp - (D_Y\beta(X))^\perp - \beta([X, Y]), \end{aligned}$$

and so on.

The definition of $d^\top$, that acts upon forms on M with values in TM, is similar. The following Lemma is probably well known:

Lemma 4.1. *Let M be an immersed k-dimensional submanifold of $\mathbb{R}^n$ diffeomorphic to an open k-ball.*

1. *If the normal bundle NM is flat and β is an 1-form on M with values in NM then $d^\perp\beta = 0$ iff $\beta = d^\perp u$ for some $u \in \Gamma(NM)$ that is determined up to the addition of a parallel section of NM;*

2. *If M is flat and β is an 1-form on M with values in TM then $d^\top\beta = 0$ iff $\beta = d^\top X$ for some $X \in \mathfrak{X}(M)$ that is determined up to the addition of a parallel vector field on M.*

Proof. (1) Since NM is flat and M is diffeomorphic to an open ball, there is a global parallel frame $u_{k+1}, \ldots, u_n$ of NM. We can write $\beta = \sum_{i=k+1}^n u_i \otimes \beta^i$, where the β^i are ordinary 1-forms on M. Then $d^\perp\beta = \sum_{i=k+1}^n u_i \otimes d\beta^i$. Thus $d^\perp\beta = 0$ iff $d\beta^i = 0, i = k+1, \ldots, n$; that is iff there are functions $b^i \in C^\infty(M)$ such that $\beta = \sum_{i=k+1}^n u_i \otimes db^i$. If we put $u = \sum_{i=k+1}^n b^i u_i$, we have

$$\beta(X) = \sum_{i=k+1}^{n} X(b^i)u^i = \nabla_X^\perp\Big(\sum_{i=k+1}^{n} b^i u_i\Big) = \nabla_X^\perp u = (d^\perp u)(X),$$

as claimed. It is clear that the addition of a parallel normal section to u preserves the condition.

The proof for the second claim is analogous. ■

Corollary 4.2. *Let M be an immersed k-dimensional submanifold of $\mathbb{R}^n$, with NM flat, and diffeomorphic to an open k-dimensional ball, and let $X \in \mathfrak{X}(M)$ denote any vector field. Then the following claims are true, and also those obtained from them by interchanging TM by NM, $^\top$ by $^\perp$, etc.*

1. *Let $A \in \mathfrak{X}(M)$ and β be the 1-form on M with values in NM given by $\beta(X) = (D_X A)^\perp$. Then, there is a section $u \in \Gamma(NM)$ such that $(D(A-u))^\perp = 0$ iff $d^\perp\beta = 0$. The section u is determined up to a parallel section of NM.*

2. *Let $A \in \mathfrak{X}(M)$ be parallel. Then there is $u \in \Gamma(NM)$ such that $(D(A-u))^\perp = 0$. If in addition M is flat, then there is $B \in \mathfrak{X}(M)$ such that $(D(u-B))^\top = 0$.*

3. *Let $S : M \to \mathbb{R}^n$ be C^∞ and such that $(DS)^\top = 0$, and $A \in \mathfrak{X}(M)$ be such that $(D(S-A))^\perp = 0$. Then, there is some $z \in \Gamma(NM)$ such that $(D(A-z))^\perp = 0$.*

Proof. (1) is an immediate consequence of 4.1. (2) Let $\beta(X) = (D_X A)^\perp$. Then

$$\begin{aligned} d^\perp\beta(X,Y) &= \big(D_X(D_Y A)^\perp - D_Y(D_X A)^\perp - D_{[X,Y]}A\big)^\perp \\ &= \big(D_X(D_Y A) - D_Y(D_X A) - D_{[X,Y]}A\big)^\perp = 0, \end{aligned}$$

because $(D_X A)^\perp = D_X A - (D_X A)^\top = (D_X A)$, and because $\mathbb{R}^n$ is flat. Our claim follows from (1).

Now, suppose that M is flat and let β be the 1-form on M with values in TM given by $\beta(X) = (D_X u)^\top$. We need only prove that $d^\top\beta = 0$. We have:

$$\begin{aligned} d^\top\beta(X,Y) &= (D_X\beta(Y))^\top - (D_Y\beta(X))^\top - \beta([X,Y]) \\ &= \big(D_X(D_Y u)^\top - D_Y(D_X u)^\top - D_{[X,Y]}u\big)^\top \\ &= \big(- D_X(D_Y u)^\perp + D_Y(D_X u)^\perp\big)^\top \\ &= \big(- D_X(D_Y A)^\perp + D_Y(D_X A)^\perp\big)^\top \\ &= \big(- D_X(D_Y A) + D_Y(D_X A)\big)^\top \\ &= (-D_{[X,Y]}A)^\top = 0, \end{aligned}$$

because A is parallel.

(3) Let us define $\beta : \mathfrak{X}(M) \to \Gamma(NM)$ by $\beta(X) = (D_X A)^\perp$. Then we have

$$\begin{aligned} d^\perp\beta(X,Y) &= \big(D_X\beta(Y) - D_Y\beta(X) - D_{[X,Y]}A\big)^\perp \\ &= \big(D_X(D_Y A)^\perp - D_Y(D_X A)^\perp - D_{[X,Y]}A\big)^\perp \\ &= \big(D_X(D_Y S)^\perp - D_Y(D_X S)^\perp - D_{[X,Y]}S\big)^\perp \\ &= \big(D_X(D_Y S) - D_Y(D_X S) - D_{[X,Y]}S\big)^\perp = 0, \end{aligned}$$

because $(D_X S)^\perp = D_X S - (D_X S)^\top = D_X S$. ■

A good part of our results are based in the following Lemma:

Lemma 4.3. *Let U, V be two surfaces immersed in $\mathbb{R}^4$ and let $f : U \to V$ be a diffeomorphism such that for any $p \in U$ we have $T_pU = N_qV$, where $q = f(p)$. Then,*

1. *A section Y of TU satisfies $\nabla^\top Y = 0$ (is parallel) iff $\tilde{Y} = Y \circ f^{-1}$, which is a section of NV, satisfies $\nabla^\perp\tilde{Y} = 0$ (is parallel). And a section u of NU is parallel iff $u \circ f^{-1}$, which is a section of TV, is parallel.*
2. *Let us denote by α and $\tilde{\alpha}$ the second fundamental forms of U and V, respectively. Then, for any $p \in U$, $X_p, Y_p \in T_pU$, $u_p \in N_pU$, we have*

$$u_p \cdot \alpha_p(X_p, Y_p) = -X_p \cdot \tilde{\alpha}_q(u_p, df(Y_p)).$$

Proof. In the following let us denote with the same letter, crowned by a tilde, functions on $f(U)$ that correspond to functions on U.

(1) Let us put $\phi = f^{-1}$. We have $\tilde{Y}_q = (Y \circ \phi)_{f(p)} = Y_p \in T_pU = N_qV$; hence $\tilde{Y}$ is a section of NV. Let $X \in T_pU = N_qV$, $u \in N_pU = T_qV$. We will have

$$\begin{aligned} X \cdot \tilde{\nabla}^{\perp}_u \tilde{Y} &= X \cdot D_u \tilde{Y} = X \cdot d\tilde{Y}(u) = X \cdot dY(d\phi(u)) \\ &= X \cdot D_{d\phi(u)} Y = X \cdot \nabla^{\top}_{d\phi(u)} Y, \end{aligned}$$

and now both claims are evident.

(2) Let $u, v \in \Gamma(NU)$ and $X \in \mathfrak{X}(U)$. Then $u \circ \phi, v \circ \phi \in \mathfrak{X}(f(U))$ and $X \circ \phi \in \Gamma(Nf(U))$. Thus

$$(X \circ \phi) \cdot \tilde{\alpha}(u \circ \phi, v \circ \phi) = -(u \circ \phi) \cdot D_{v \circ \phi}(X \circ \phi).$$

Now we evaluate this at q. We obtain

$$\begin{aligned} X_p \cdot \tilde{\alpha}_q(u_p, v_p) &= -u_p \cdot D_{v_p}(X \circ \phi) = -u_p \cdot d(X \circ \phi)(v_p) \\ &= -u_p \cdot dX(d\phi(v_p)) = -u_p \cdot D_{d\phi(v_p)} X = -u_p \cdot \alpha_p(X_p, d\phi(v_p)). \end{aligned}$$

Putting $v_p = df(Y_p)$ we obtain our claim. ■

For the previous Lemma we did need that the ambient space were $\mathbb{R}^4$ because then

$$2 = \dim(T_pU) = \dim(T_qV) = \dim(N_pU) = \dim(N_qV),$$

so that the equality $T_pU = N_qV$ could make sense at all. In the following Lemma this condition is not necessary. The proof is similar.

Lemma 4.4. *Let U, V be two surfaces immersed in $\mathbb{R}^n$ and let $f : U \to V$ be a diffeomorphism such that for any $p \in U$ we have $T_pU = T_qV$, where $q = f(p)$. Then,*

1. *A section Y of TU is parallel iff $\tilde{Y} = Y \circ f^{-1}$, which is a section of TV, is parallel. And a section u of NU is parallel iff $u \circ f^{-1}$, which is a section of NV, is parallel.*

2. *Let us denote by α and $\tilde{\alpha}$ the second fundamental forms of U and V, respectively. Then, for any $p \in U$, $X_p, Y_p \in T_pU$, we have*

$$\alpha_p(X_p, Y_p) = \tilde{\alpha}_q(df(X_p), Y_p).$$

Now we begin to study the conditions to have diffeomorphisms as those used in the preceding Lemmas.

Proposition 4.5. *Let M be an immersed surface in $\mathbb{R}^n$. Then the following statements are equivalent:*

1. *M is flat;*
2. *Given any point $p \in M$, there is an open neighborhood U of p in M and a vector field $e \in \mathfrak{X}(U)$ such that $\nabla^\top_X e = X$ for all $X \in \mathfrak{X}(U)$;*
3. *Given any point $p \in M$, there is an open neighborhood U of p in M and a vector field $e \in \mathfrak{X}(U)$ such that for any $q \in U$ we have $df(T_qU) \subset N_qU$, where $f : U \to \mathbb{R}^n$ is the map defined by $f(q) = q - e_q$, that is $f = \mathrm{id} - e$.*

Proof. (1) $\Leftrightarrow$ (2). If M is flat, there are two orthonormal parallel vector fields $X, Y \in \mathfrak{X}(M)$ in an open neighborhood U of p. One sees easily that the 1-forms $X^\natural, Y^\natural$ given by $X^\natural(A) = X \cdot A$, $Y^\natural(A) = Y \cdot A$ are closed. Therefore in any neighborhood U of p diffeomorphic to a ball, there are functions $u, v \in C^\infty(U)$ such that $X = \operatorname{grad} u$, $Y = \operatorname{grad} v$. The vector field $e = uX + vY$ satisfies the required property. In fact $(D_Z e)^\top = Z(u)X + Z(v)Y = (\operatorname{grad} u \cdot Z)X + (\operatorname{grad} v \cdot Z)Y = (Z \cdot X)X + (Z \cdot Y)Y = Z$, because X, Y are parallel and orthonormal. This field e is determined up to the addition of a parallel tangent vector field on U. Conversely, if $e \in \mathfrak{X}(U)$ satisfies the condition, then for any $X, Y \in \mathfrak{X}(U)$ we have

$$R^\top(X,Y)e = \nabla^\top_X \nabla^\top_Y e - \nabla^\top_Y \nabla^\top_X e - \nabla^\top_{[X,Y]} e = \nabla^\top_X Y - \nabla^\top_Y X - [X,Y] = 0,$$

because $\nabla^\top$ is torsionless. Since e can vanish only at isolated points and the dimension of M is 2, we conclude that $R^\top = 0$.

(2) $\Leftrightarrow$ (3). Let $e : U \to TU$ be a local section of TM and let $X, Y \in T_qM$, $q \in U$. Then $Y \cdot df(X) = Y \cdot (X - de(X)) = Y \cdot (X - D_X e) = Y \cdot (X - \nabla^\top_X e)$ and now our claim is evident. ■

Note that e is the image, by an isometric chart, of the radius vector field. It shares with the radius vector in $\mathbb{R}^2$ the property $e = \frac{1}{2}\operatorname{grad}(e \cdot e)$. In fact, if $X \in \mathfrak{X}(M)$, we have $\frac{1}{2}\operatorname{grad}(e \cdot e) \cdot X = \frac{1}{2}X(e \cdot e) = e \cdot D_X e = e \cdot X$, whence that property follows. In the following we will call such a vector field a *radius vector*.

For M flat, and assuming that the radius vector e is defined in all of M, let $f = \mathrm{id} - e : M \to \mathbb{R}^n$ and assume that for some non vanishing $X \in T_pM$ we have $df(X) = 0$. This would be equivalent to say that for any $u \in N_pM$ we had $0 = u \cdot df(X) = u \cdot (X - D_X e) = -u \cdot D_X e = -u \cdot \alpha_p(e_p, X) = 0$, that is $\alpha_p(e_p, X) = 0$, and this is only possible if $\{0\} \neq \ker \alpha_p(e_p, \,) : T_pM \to N_pM$. If $n = 4$, df would generically not be one-to-one only along some curves; if $n > 4$, f would be an immersion generically outside isolated points, and so on. However, I shall not dwell on this point.

Assume that in fact f be an immersion, and let $\phi = (f|_U)^{-1}$ for some open $U \subset M$ such that $f|_U : U \to V = f(U)$ is a diffeomorphism. Then $\phi = \mathrm{id}_V + e \circ \phi$, as it can be proved easily. Since $T_qV = df(T_pU) \subset N_pU$, we will have $T_pU \subset N_qV$. Hence, $c = e \circ \phi$ is a section of NV and the map $\phi = \mathrm{id} + c$ satisfies $d\phi(T_qV) = T_pU \subset N_qV$. This motivates the following proposition.

Proposition 4.6. *Let M be an immersed surface in $\mathbb{R}^n$ with normal bundle NM and first and second fundamental forms g and α, respectively. Let $c \in \Gamma(NM)$ and put $f = \mathrm{id} + c : M \to \mathbb{R}^n$. Then $c \cdot \alpha = g$ iff for each $p \in M$ we have $df(T_pM) \subset N_pM$.*

Proof. Let $p \in M$, $X, Y \in T_pM$. Then

$$\begin{aligned} Y \cdot (df)_p(X) &= Y \cdot (X + (dc)_p(X)) = Y \cdot (X + D_X c) \\ &= Y \cdot X - c \cdot D_X Y = g(X, Y) - c \cdot \alpha(X, Y), \end{aligned}$$

and our claim is now evident. ■

Assume now that $c \in \Gamma(NM)$ satisfies $c \cdot \alpha = g$, and let $f = \mathrm{id} + c : M \to \mathbb{R}^n$. If $p \in M$, let us study the condition for df_p not being one-to-one. This happens iff there is some non vanishing vector $X \in T_pM$ such that, for all $u \in N_pM$ the following holds: $u \cdot df_p(X) = u \cdot (X + D_X c) = u \cdot \nabla^{\perp}_X c = 0$. That is iff $\{0\} \neq \ker\, (\nabla^{\perp} c)_p : X \in T_pM \mapsto \nabla^{\perp}_X c \in N_pM$. As before, we see that for $n = 4$ it fails generically to be an immersion only on some curves, etc.

The condition $c_p \cdot \alpha_p = g_p$ says that the height of the curvature ellipse with respect to the vector hyperplane of N_pM orthogonal to c_p is constant and equal to $\frac{1}{|c_p|}$. If $n = 4$ this happens only if the ellipse degenerates to a point, not the origin, or to an affine segment not collinear with the origin. If $n > 4$ this may occur almost always, because it is equivalent to require only that the least affine subspace of N_pM that contains the curvature ellipse does not pass by the origin.

The next two Theorems, that are part of our main results, explain why from now on in this section we consider only surfaces in $\mathbb{R}^4$. Roughly, they establish a transformation of a surface with flat normal bundle to a flat surface, and a transformation that takes a flat surface and converts it to a surface with flat normal bundle.

Theorem 4.7. *Let M be a semiumbilical surface immersed in $\mathbb{R}^4$. Let $S : M \to \mathbb{R}^4$ be a smooth map such that $(DS)^{\top} = 0$. Let c be the section of NM described in* 3.3 *that satisfies $c \cdot \alpha = g$, and assume that $f = \mathrm{id} + c + S : M \to \mathbb{R}^4$ is an immersion. Then $f(M)$ is an immersed flat surface, and we say it is an* evolute *of M. If, in addition, M is flat, then $f(M)$ is semiumbilical.*

Proof. Let U be an open subset of M for which $f : U \to V = f(U)$ is a diffeomorphism. Taking account of the dimensions, if $p \in M$ and $q = f(p)$, we conclude that $df(T_pU) = N_pU = T_qV$ and $df^{-1}(T_qV) = T_pU = N_qV$. Our claims are now a consequence of 4.5 and 4.3. In fact, M is flat iff there is a non-vanishing parallel vector field on M, and all its points are semiumbilic iff its normal bundle is flat, that is iff it admits a non-vanishing parallel section. In both cases, due to the dimension 2 of those bundles. The question whether $f(M)$ has inflection points when M is semiumbilical and flat may be settled with the same technique that will be used in Theorem 5.3 under a more general context. ∎

In the same manner we have

Theorem 4.8. *Let M be a surface immersed in $\mathbb{R}^4$. Let $S : M \to \mathbb{R}^4$ be a smooth map such that $(DS)^\top = 0$. Let $e \in \mathfrak{X}(M)$ be a radius vector (hence, M is flat) and put $f = \mathrm{id} - e + S : M \to \mathbb{R}^4$. Then, if f is an immersion, the immersed surface $f(M)$ is semiumbilical and we will say that it is an* envelope *of M. If, in addition, M is semiumbilical, then $f(M)$ is flat.*

Note that any parallel vector field $S \in \mathfrak{X}(M)$ does the job required in the above statements. Also, by means of 4.2, one may find many maps S with the required property and that are neither vertical nor horizontal.

The first of the following two results will be used later. For proving them, we use transformations as described above, but the results in themselves do not claim for those transformations; on the other hand, a direct proof would seem to demand heavy calculations.

Proposition 4.9. *Let M be a flat surface in $\mathbb{R}^4$ without inflection points and $S : M \to \mathbb{R}^4$ be a smooth map such that $(DS)^\top = 0$. Then, there is a unique vector field $A \in \mathfrak{X}(M)$ such that $(D(S - A))^\perp = 0$.*

Proof. First we show the uniqueness of A. The condition may be read also as $(D_XA)^\perp = \alpha(A, X) = (D_XS)^\perp$ for any $X \in \mathfrak{X}(M)$. Now, if $p \in M$, the right hand of this equation defines an element β of $\mathrm{Hom}(T_pM, N_pM)$ by $\beta(X_p) = (D_{X_p}S)^\perp$, for $X_p \in T_pM$. Since p is not an inflection point, 3.5(1) assures us that A_p, if it exists, is unique.

Let $e_p \in T_pM$ be such that $\mu(e_p)$ is one to one (see 3.5(2)), and for some open neighborhood U of p, let $e \in \mathfrak{X}(U)$ be a radius vector such that its value at p is e_p.

Let $0 \le \epsilon \in \mathbb{R}$. We define the map $f_\epsilon = \mathrm{id} - e + \epsilon S : U \to \mathbb{R}^4$. Assume that $0 \neq X \in T_pM$. Then $df_0(X) = (df_0(X))^\perp = -\alpha_p(e_p, X) = -\mu(e_p)(X) \neq 0$. Hence, f_0 is an immersion in a neighborhood of p. Thus by taking $\epsilon > 0$ sufficiently small, we see that f_ϵ is an immersion on a neighborhood of p that we shall keep denoting by U.

The immersed surface $f_\epsilon(U)$ is semiumbilical. Let us call $\tilde{c} \in \Gamma(Nf_\epsilon(U))$ the section such that $\tilde{c}\cdot\tilde{\alpha} = \tilde{g}$, where the tildes mean that we are referring to $f_\epsilon(U)$. Let us define $A \in \mathfrak{X}(U)$ by $\epsilon A = -\tilde{c}\circ f_\epsilon + e$. Thus $\tilde{c} = e\circ f_\epsilon^{-1} - \epsilon A\circ f_\epsilon^{-1}$. In the formula of Lemma 4.3(2) let $p \in U$ be an arbitrary point and replace $\tilde{c}_q$ by X. Then

$$u \cdot df_\epsilon(Y) = -u \cdot \alpha_p(\tilde{c}_q, Y),$$

that is:

$$\begin{aligned}
0 &= \alpha(\tilde{c}_q, Y) + df_\epsilon(Y) = \alpha(\tilde{c}_q, Y) + df_\epsilon(Y)^\perp \\
&= \alpha(\tilde{c}_q, Y) - (D_Y e)^\perp + \epsilon(D_Y S)^\perp \\
&= \alpha(\tilde{c}_q - e_p, Y) + \epsilon(D_Y S)^\perp \\
&= \alpha(-\epsilon A_p, Y) + \epsilon(D_Y S)^\perp \\
&= \epsilon(D_Y(S - A))^\perp .
\end{aligned}$$

Since Y is arbitrary and $p \in U$ is arbitrary, we see that $(D(A-S))^\perp = 0$ in U. The uniqueness of A allows us to extend its existence to all of M. ■

Proposition 4.10. *Let M be a semiumbilical surface in $\mathbb{R}^4$ diffeomorphic to an open ball, such that for any $p \in M$ the map $(Dc)^\perp : X \in T_pM \mapsto (D_Xc)^\perp \in N_pM$ is one-to-one, and let $S : M \to \mathbb{R}^4$ be a smooth map such that $(DS)^\top = 0$. Then, there is a smooth section $z \in \Gamma(NM)$ such that $(D(S-z))^\perp = 0$; it is determined up to the addition of a parallel section.*

Proof. Let us define $f_\epsilon : M \to \mathbb{R}^4$ by means of $f_\epsilon = \mathrm{id} + c + \epsilon S$, for $\epsilon \in \mathbb{R}$. Then, $df_\epsilon(T_pM) \subset N_pM$ for any $p \in M$ and as a consequence, if $X \in T_pM$, we have

$$df_\epsilon(X) = df_\epsilon(X)^\perp = (D_Xc)^\perp + \epsilon(D_XS)^\perp .$$

Since $(Dc)^\perp$ is one to one, there is some $\epsilon > 0$ and some open neighborhood U of p such that f_ϵ is an immersion on U. Then the surface $f_\epsilon(U)$ is flat and we can take U so that there is some radius vector $\tilde{e}$ on $f_\epsilon(U)$. Let $\phi = f_\epsilon^{-1}$. We put $\phi = \mathrm{id} - \tilde{e} + B\circ\phi$, where $B : U \to \mathbb{R}^4$ is some smooth map. Then we have easily that $(\tilde{D}(B\circ\phi))^\top = 0$, that is $(DB)^\perp = 0$. Since $\phi = \mathrm{id} - c\circ\phi - \epsilon S\circ\phi = \mathrm{id} - \tilde{e} + B\circ\phi$, we have $\tilde{e} = (c + \epsilon S + B)\circ\phi \in \mathfrak{X}(f_\epsilon(U))$. Hence, $c + \epsilon S + B \in \Gamma(NU)$. Since $c \in \Gamma(NU)$, there is some $z \in \Gamma(NU)$ such that $\epsilon S + B = \epsilon z$. and we have $(D(S-z))^\perp = -\frac{1}{\epsilon}(DB)^\perp = 0$. It is clear that the difference between two such sections z is a parallel section of NU. One can now extend the solution to the whole M because M is diffeomorphic to a ball. ■

5. Transformations of surfaces with tangent and normal bundles both flat

In this section, M will be an FSIS and c will be defined as in 3.3.

As we have recalled, there is, in a neighborhood of any point of M, an orthonormal frame (t_1, t_2) of asymptotic directions, such that, if $b_1 = \alpha(t_1, t_1) = (D_{t_1} t_1)^{\perp}$, $b_2 = \alpha(t_2, t_2) = (D_{t_2} t_2)^{\perp}$, $b_3 = \alpha(t_1, t_2) = (D_{t_1} t_2)^{\perp}$, we have:

1. b_1 and b_2 are linearly independent at each point;
2. $b_1 \cdot b_2 = 0$, $b_3 = 0$.

With this notation, we have:

Lemma 5.1. *Let M be an FSIS in $\mathbb{R}^4$. Then,*

1. *There is a unique vector field $j \in \mathfrak{X}(M)$ such that for any $X \in \mathfrak{X}(M)$ we have $\alpha(j, X) = (D_X j)^{\perp} = (D_X c)^{\perp} = \nabla^{\perp}_X c$, and it is given by*

$$j = \frac{1}{2}\operatorname{grad}(c \cdot c) = \frac{b_1 \cdot D_{t_1} c}{b_1 \cdot b_1} t_1 + \frac{b_2 \cdot D_{t_2} c}{b_2 \cdot b_2} t_2.$$

2. *Let $U \subset M$ be an open subset diffeomorphic to a ball, and assume that $e \in \mathfrak{X}(U)$ is a radius vector. Then there is a section $k \in \Gamma NU$ such that $(D(e-k))^{\perp} = 0$. It is determined up to the addition of a parallel section of NU. Also, we have $(D(c+e))^{\top} = 0$.*

Proof. (1) If such a vector field j exists, then

$$j \cdot X = c \cdot \alpha(j, X) = c \cdot \nabla^{\perp}_X c = c \cdot D_X c = \frac{1}{2} D_X(c \cdot c) = \frac{1}{2} d(c \cdot c)(X)$$
$$= \frac{1}{2}\operatorname{grad}(c \cdot c) \cdot X.$$

Therefore, if it exists, j is unique and is given by $\frac{1}{2}\operatorname{grad}(c \cdot c)$. Its existence is not evident. We can write $j = j^1 t_1 + j^2 t_2$ and must have $\alpha(j, t_1) = j^1 b_1 = \nabla^{\perp}_{t_1} c$. This reduces to the following two conditions

i) $j^1 b_1 \cdot b_1 = b_1 \cdot D_{t_1} c$, that is $j^1 = \frac{b_1 \cdot D_{t_1} c}{b_1 \cdot b_1}$.

ii) $j^1 b_1 \cdot b_2 = b_2 \cdot D_{t_1} c$.

Condition i) determines j^1. Since $b_1 \cdot b_2 = 0$, condition ii) can be met iff $b_2 \cdot D_{t_1} c = 0$, but this is true by 3.4. The same happens to j^2, and this proves our claims.

(2) We define an 1-form β on U with values in NU by $\beta(X) = (D_X e)^\perp$. Let us prove that $d^\perp\beta = 0$. We have

$$\begin{aligned} d^\perp\beta(X,Y) &= (D_X\beta(Y))^\perp - (D_Y\beta(X))^\perp - (D_{[X,Y]}e)^\perp \\ &= \big(D_X(D_Y e)^\perp - D_Y(D_X e)^\perp - D_{[X,Y]}e\big)^\perp \\ &= \big(D_X D_Y e - D_X Y - D_Y D_X e + D_Y X - D_{[X,Y]}e\big)^\perp \\ &= (-[X,Y])^\perp = 0, \end{aligned}$$

because $D_X D_Y - D_Y D_X - D_{[X,Y]} = 0$ and $D_X Y - D_Y X - [X,Y] = 0$. By 4.2 our first claim is true. As for the second, if $X, Y \in \mathfrak{X}(M)$ we have

$$Y \cdot (D_X(c+e))^\top = Y \cdot D_X c + Y \cdot X = -c \cdot \alpha(Y,X) + Y \cdot X = 0.$$

■

Theorem 5.2. *Let M be an FSIS in $\mathbb{R}^4$, assume that the vector fields j and k of* 5.1 *are defined in all of M and let the C^∞ map $S : M \to \mathbb{R}^4$ satisfy $(DS)^\perp = 0$. If the map $f : M \to \mathbb{R}^4$ is defined by*

$$f = \mathrm{id} + t_1(e-k) + t_2(c-j) + S, \qquad \textit{with } t_1, t_2 \in \mathbb{R},$$

then $df(T_pM) \subset T_pM$, $\forall p \in M$. Moreover, if U is the open subset of M where f is an immersion, then $f(U)$ is an FSIS.

Proof. Let $u \in N_pM$, $X \in T_pM$. We will have

$$\begin{aligned} u \cdot df(X) &= u \cdot (X + t_1 D_X(e-k) + t_2 D_X(c-j) + D_X S) \\ &= u \cdot \big(t_1(D_X(e-k))^\perp + t_2(D_X(c-j))^\perp + (D_X S)^\perp\big) = 0. \end{aligned}$$

Therefore, $df(T_pM) \subset T_pM$.

Now $f(U)$ is flat and with flat normal bundle as a consequence of 4.4. In fact, from a parallel local reference t_1, t_2 of TU and a parallel local reference u_3, u_4 of NU we can obtain the parallel references $t_i \circ f^{-1}, i = 1, 2$, and $u_i \circ f^{-1}, i = 3, 4$, of $Tf(U)$ and $Nf(U)$, respectively. We need still to prove that there are no inflection points in $f(U)$. Let $p \in U$, $q = f(p)$, $X, Y \in T_pU$. Then, by Lemma 4.4,(2) we have

$$\alpha_p(X,Y) = \tilde{\alpha}_q(df(X), Y).$$

Therefore, the dimension of the subspaces generated by α_p and $\tilde{\alpha}_q$ must be the same. Hence, $f(M)$ is semiumbilical. ■

We say that this transformation is of *parallel* type.

Now, we exhibit a transformation that sends each tangent space to its orthogonal: this is another of our main results.

Theorem 5.3. *Let M be an FSIS in $\mathbb{R}^4$. Let c be the section of NM such that $c\cdot\alpha = g$, $e \in \mathfrak{X}(M)$ be a radius vector, and $t \in \mathbb{R}$. Let $S : M \to \mathbb{R}^4$ be a smooth map such that $(D_X S)^\top = 0$ and $A \in \mathfrak{X}(M)$ be the unique vector field such that $(D(S-A))^\perp = 0$ (4.9). Put $f = \mathrm{id} + tc - (1-t)e + S : M \to \mathbb{R}^4$. Then $df(T_pM) \subset N_pM$ for any $p \in M$, the open subset $U \subset M$ where f is an immersion is the set of points where $tj - (1-t)e + A$ is not parallel to an asymptotic direction, and $f(U)$ is an FSIS. Also, if ϕ denotes a local inverse of f, we have*

1. *$\tilde{c} = ((1-t)e - tj - A) \circ \phi$.*
2. *Let k be a (local) section of NU such that $(D_X(e-k))^\perp = 0$. Then we have that $\tilde{e} = (tc - (1-t)k + z) \circ \phi \in \mathfrak{X}(f(U))$, where $z \in \Gamma(NM)$ is such that $(D(A-z))^\perp = 0$ (see 4.2(3)), is a radius vector.*

Proof. Let $X, Y \in T_pM$. Then:

$$\begin{aligned} X \cdot df(Y) &= X \cdot (Y + tD_Y c - (1-t)D_Y e + D_Y S) \\ &= X \cdot Y - tX \cdot Y - (1-t)X \cdot Y + X \cdot (D_Y S)^\top = 0, \end{aligned}$$

so that $df(T_pM) \subset N_pM$, as claimed.

Then,

$$\begin{aligned} df(Y) = df(Y)^\perp &= t(D_Y c)^\perp - (1-t)(D_Y e)^\perp + (D_Y S)^\perp \\ &= \alpha(Y, tj - (1-t)e + A). \end{aligned}$$

By 3.5(3), we see that f fails to be an immersion only at the points where $tj - (1-t)e + A$ is parallel to an asymptotic direction.

$f(U)$ is flat and with flat normal bundle. Now by 4.3(2) we have

$$u \cdot \alpha_p(X, Y) = -X \cdot \tilde{\alpha}_q(u, df(Y)),$$

for any $u \in N_pM$. If $p \in U$ then the curvature ellipse at $q = f(p)$ lies in an affine line of $N_q f(U)$. If q is an inflection point of $f(U)$ (this includes the case that it be umbilic) then that line passes by the origin (the ellipse itself passes by it because $f(U)$ is flat).

Hence, for any $Y \in T_pM$ and $u \in N_pM$, $\tilde{\alpha}(df(Y), u)$ lies in the vector line containing the ellipse. If X is non zero and orthogonal to that line, then $u \cdot \alpha(X, Y) = 0$ for all Y, u. Hence $\mu(X) = 0$ (see 3.5(1)) and p must be an inflection point, against our hypotheses.

Since $f(U)$ is an FSIS, there is $\tilde{c} \in \Gamma(Nf(U))$ such that $\tilde{c} \cdot \tilde{\alpha} = \tilde{g}$. The proof of the formula $\tilde{c} = \big((1-t)e - tj - A\big) \circ \phi$ is almost the same as that used in the last part of the proof of 4.9.

(2) By 4.2(3) we know that there is locally some section $z \in \Gamma(NU)$ such that $D(A-z)^{\perp} = 0$. Let $u, v \in T_q f(U) = N_p U$. Then

$$\begin{aligned} v \cdot \tilde{\nabla}_u^{\top} \tilde{e} &= v \cdot D_u(tc - (1-t)k + z) \circ \phi = v \cdot D_{d\phi(u)}(tc - (1-t)k + z) \\ &= v \cdot D_{d\phi(u)}(tj - (1-t)e + A) = -v \cdot D_{d\phi(u)}(\tilde{c} \circ f) = -v \cdot D_u \tilde{c} \\ &= v \cdot u, \end{aligned}$$

whence $\tilde{\nabla}_u^{\top} \tilde{e} = u$, as desired. ■

Now we will see that the composition of two transformations of orthogonal type is one of parallel type, as we could expect, and also we want to prove a permutability theorem in the same vein of the Bianchi permutability theorem for Bäcklund transformations (see [1]). For this we need some more notation. First we note that in the definition of the map f of the preceding Theorem the ambiguity in the choice of e, that is the addition of a parallel vector field on M, may be absorbed in the ambiguity of the choice of S. Thus, we may consider that e is uniquely defined, so that we may describe the map f as $F(t, S) : M \to \mathbb{R}^4$ and it is given by

$$F(t, S) = \mathrm{id} + tc - (1-t)e + S,$$

where $t \in \mathbb{R}$, $e \in \mathfrak{X}(M)$ is a radius vector and $S : M \to \mathbb{R}^4$ satisfies $(D_X S)^{\top} = 0$, $\forall X \in \mathfrak{X}(M)$.

Theorem 5.4. *With the above notation, assume that $F(t_1, S_1) : M \to \mathbb{R}^4$ and $F(t_2, S_2) : M \to \mathbb{R}^4$ are immersions and, for the sake of brevity, that they are diffeomorphisms. Let us put $M_i = F(t_i, S_i)(M)$, $i = 1, 2$. Then there exists locally a transformation $F(t_2, \tilde{S}_1) : M_1 \to \mathbb{R}^4$ and a transformation $F(t_1, \tilde{S}_2) : M_2 \to \mathbb{R}^4$ such that*

$$F(t_2, \tilde{S}_1) \circ F(t_1, S_1) = F(t_1, \tilde{S}_2) \circ F(t_2, S_2)$$

and this composition is one of the transformations of parallel type described in 5.2.

Proof. Let us put $\phi_i = F(t_i, S_i)^{-1}$, $i = 1, 2$. Then we have for M_i the fields $\tilde{c}_i = ((1-t_i)e - t_i j - A_i) \circ \phi_i$, where $A_i \in \mathfrak{X}(M_i)$ satisfies $(D(S_i - A_i))^{\perp} = 0$, and $\tilde{e}_i = (tc - (1-t_i)k_i + v_i) \circ \phi_i$, where $(D(A_i - v_i))^{\perp} = 0$. Now, let $s_i : M \to \mathbb{R}^4$ be such that $(Ds_i)^{\perp} = 0$. Then, if we put $\tilde{S}_i = s_i \circ \phi_i$ we have for any $u_i \in T_{q_i} M_i$ that $(D_{u_i} \tilde{S}_i)^{\top} = 0$, as it is easily proved (here, $^{\perp}$ refers to M_i). We shall compute the compositions

$$F(t_2, \tilde{S}_1) \circ F(t_1, S_1), \quad F(t_1, \tilde{S}_2) \circ F(t_2, S_2),$$

and prove that by a convenient choice of s_i we can make those compositions to be equal.

Let $p \in M$ and $q = F(t_1, S_1)(p)$. We have that the first composition maps $p \in M$ to

$$
\begin{aligned}
F(t_2,&\tilde{S}_1)(p + t_1c_p - (1-t_1)e_p + S_{1p}) \\
&= p + t_1c_p - (1-t_1)e_p + S_{1p} + t_2\tilde{c_1}_q - (1-t_2)\tilde{e_1}_q + (s_1 \circ \phi_1)_q \\
&= p + t_1c_p - (1-t_1)e_p + S_{1p} + t_2((1-t_1)e_p - t_1j_p - A_{1p}) \\
&\quad - (1-t_2)(t_1c_p - (1-t_1)k_{1p} + v_{1p}) + s_{1p} \\
&= p + t_1t_2(c_p - j_p) - (1-t_1)(1-t_2)(e_p - k_{1p}) \\
&\quad - t_2A_{1p} - (1-t_2)v_{1p} + S_{1p} + s_{1p}.
\end{aligned}
$$

Now

$$
-t_2A_1 - (1-t_2)v_1 + S_1 + s_1 = (1-t_2)(A_1 - v_1) + (S_1 - A_1) + s_1,
$$

and $\big(D((1-t_2)(A_1 - v_1) + S_1 - A_1)\big)^{\perp} = 0$ by the hypotheses (on passing, this proves also our last claim, that is that those compositions are of the type described in 5.2). Hence, we can take $s_1 = -(1-t_2)(A_1 - v_1) + A_1 - S_1$ and $s_2 = -(1-t_1)(A_2 - v_2) + A_2 - S_2$, because then clearly $(Ds_i)^{\perp} = 0$, as needed. Then both compositions are made equal to

$$
\mathrm{id} + t_1t_2(c - j) - (1-t_1)(1-t_2)(e - k)\,.
$$

■

References

[1] BURSTALL, F. E.: Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems. In *Integrable systems, geometry, and topology*, 1–82. AMS/IP Stud. Adv. Math. **36**, Amer. Math. Soc., Providence, RI, 2006.

[2] DO CARMO, M. AND DAJCZER, M.: Local isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$. *J. Reine Angew. Math.* **442** (1993), 205–219.

[3] DARBOUX, G.: *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*,vol. III, chap. XII. Chelsea Publishing Co, Bronx, New York, 1972.

[4] MELLO, L. F.: Line fields on surfaces immersed in $\mathbb{R}^4$. In *EQUADIFF 2003*, 951–956. World Sci. Publ., Hackensack, NJ, 2005.

[5] NUÑO-BALLESTEROS, J. J.: Submanifolds with a non-degenerate parallel normal vector field in Euclidean spaces. In *Singularity theory and its applications*, 311–332. Adv. Stud. Pure Math. **43**. Math. Soc. Japan, Tokyo, 2006.

[6] ROMERO-FUSTER, M. C. AND SÁNCHEZ-BRINGAS, F.: Umbilicity of surfaces with orthogonal asymptotic lines in $\mathbb{R}^4$. *Differential Geom. Appl.* **16** (2002), 213–224.

[7] Wong, Y.-C.: A note on the first normal space of a V_m in an R_n. *Bull. Amer. Math. Soc.* **51** (1945), 997–1000.

Recibido: 20 de abril de 2006
Revisado: 27 de octubre de 2006

Angel Montesinos-Amilibia
Departamento de Geometría y Topología
Universidad de Valencia
Vicente A. Estellés 1
46100 Burjasot (Valencia), Spain
montesin@uv.es

Work partially supported by DGCYT grant no. BFM2003-02037.