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Sums of Toeplitz products with harmonic symbols

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Abstract

On the Bergman space of the unit disk, we consider a class of operators which contain sums of finitely many Toeplitz products with harmonic symbols. We give characterizations of when an operator in that class has finite rank or is compact. Our results provide a unified way of treating several known results.

1. Introduction

Let D denote the unit disk of the complex plane $\mathbb{C}$. The Bergman space L^2_a is the closed subspace of the usual Lebesgue space $L^2 = L^2(D, A)$ consisting of all holomorphic functions on D where the measure dA is the normalized area measure on D. We let P be the Hilbert space orthogonal projection from L^2 onto L^2_a. For a bounded measurable function u on D, the *Toeplitz operator* T_u with *symbol* u is defined by

$$T_u f = P(uf)$$

for functions $f \in L^2_a$. Clearly, T_u is a bounded linear operator on L^2_a. In this paper we are mainly concerned with harmonic symbols. So, we introduce the notation h^∞ for the space of all bounded harmonic functions on D. Also, we let H^∞ denote the space of all bounded holomorphic functions on D.

In a recent paper [11], Guo, Sun and Zheng characterized finite rank (semi-) commutators of two Toeplitz operators with harmonic symbols. Motivated by such results, we consider in this paper a more general class of operators which contain sums of finitely many Toeplitz products with harmonic

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symbols. More explicitly, we consider operators T of the form

$$T = T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j} \tag{1.1}$$

where $u_j, v_j \in h^\infty$ for each j and λ is a finite sum of finite products of h^∞-functions. We first investigate the problem of when an operator of this type has finite rank. In addition, we also consider the problem of when such an operator is compact on L^2_a. Our results generalize several known results concerning (semi-) commutators of Toeplitz operators with harmonic symbols.

To state our results, we introduce some notation. Given $f, g \in L^2_a$, we let $f \otimes g$ be the rank one operator on L^2_a defined by

$$\big(f \otimes g\big)h = \big\langle h, g\big\rangle f, \qquad h \in L^2_a$$

where the notation $\langle\ ,\ \rangle$ denotes the inner product in L^2. We will often use the letter z not only to denote points in D, but also to denote the identity function on D.

Our first result is a characterization for operators of the form (1.1) to have finite rank in terms of symbols and functions that generate their ranges. In case $\lambda = 0$, our result is as follows.

Theorem 1.1. *Let* $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$ *and* $x_1, \dots, x_n, y_1, \dots, y_n \in L^2_a$. *Then*

$$\sum_{j=1}^{N} T_{u_j} T_{v_j} = \sum_{j=1}^{n} x_j \otimes y_j$$

if and only if the following two conditions hold:

(a) $\displaystyle\sum_{j=1}^{N} u_j v_j = \big(1 - |z|^2\big)^2 \sum_{j=1}^{n} x_j \overline{y_j}.$

(b) $\displaystyle\sum_{j=1}^{N} \big[\,\overline{P\overline{u_j}} - u_j(0)\big]\big[\,Pv_j - v_j(0)\big] = 0.$

This will be deduced as a special case of a more general result Theorem 3.5. As an immediate consequence, we give a characterization of when an operator T in (1.1) is the zero operator (Theorems 3.7 and 3.8). These special cases are also new. We also apply Theorem 3.5 to recover theorems concerning finite rank sums of finitely many (semi-)commutators (Corollaries 3.9 and 3.10) and finite rank Toeplitz products (Corollary 3.11), which have been (essentially) noticed in [11].

Note that, although we have Theorem 1.1, whether or not there are examples of functions satisfying conditions (a) and (b) above is another separate problem in general. For example, the case $N = 1$ admits only trivial examples by Corollary 3.11. When $N > 1$, however, it turns out that there are nontrivial examples; see the examples at the end of Section 3.

Our next result is a characterization of compactness of operators under consideration. To state it, we introduce more notation. Given $a \in D$, we let φ_a denote the standard Möbius map on D. Namely,

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}, \qquad z \in D.$$

Also, we let $\widetilde{\Delta}$ denote the *invariant* Laplacian on D defined by

$$\widetilde{\Delta}\psi = (1 - |z|^2)^2 \Delta\psi$$

for C^2-functions ψ on D where Δ is the ordinary Laplacian. This invariant Laplacian is easily seen to be Möbius invariant by a direct calculation. The notation C_0 stands for the class of all continuous functions ψ on D such that $\psi(a) \to 0$ as $|a| \to 1$.

In case $\lambda = 0$, our result concerning compactness is as follows.

Theorem 1.2. *Let $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\displaystyle\sum_{j=1}^{N} T_{u_j} T_{v_j}$ *is compact.*

(b) $\displaystyle\sum_{j=1}^{N} \widetilde{\Delta}[\overline{P\overline{u}_j} P v_j] \in C_0$ *and* $\displaystyle\sum_{j=1}^{N} u_j v_j \in C_0$.

(c) $\displaystyle\sum_{j=1}^{N} u_j v_j \in C_0$ *and*

$$\lim_{|a|\to 1} \int_D \left| \sum_{j=1}^{N} \left[\overline{P(\overline{u_j \circ \varphi_a})} - u_j(a) \right] \left[P(v_j \circ \varphi_a) - v_j(a) \right] \right| dA = 0.$$

This will also be deduced from a more general result Theorem 4.3. As applications of Theorem 4.3, we also obtain compact versions of all the results mentioned earlier; see Theorem 4.4, Corollaries 4.5 and 4.6 . These corollaries generalize the main results in [15] and [16].

In Section 2 we collect some basic facts and known results that we use later. In Section 3 we prove a more general version of Theorem 1.1 and derive its applications. As a preliminary step, we give a characterization for harmonicity of functions of certain type. At the end of the section we construct examples related to Theorem 1.1. In Section 4 we prove compact versions of all the results obtained in Section 3.

2. Preliminaries

Throughout the section we let $a \in D$ denote an arbitrary point, unless otherwise specified.

Since every point evaluation is a bounded linear functional on L^2_a, there corresponds to every $a \in D$ a unique function $K_a \in L^2_a$ which has following reproducing property:

$$f(a) = \langle f, K_a \rangle \tag{2.1}$$

for $f \in L^2_a$. The function K_a is the well-known Bergman kernel and its explicit formula is given by

$$K_a(z) = \frac{1}{(1 - \overline{a}z)^2}, \qquad z \in D.$$

We let k_a denote the normalized kernel, namely,

$$k_a(z) = \frac{1 - |a|^2}{(1 - \overline{a}z)^2}, \qquad z \in D.$$

By the reproducing property (2.1) we see that the projection P can be realized as an integral operator

$$Pu(a) = \langle u, K_a \rangle$$

for $u \in L^2$. Moreover, this integral representation allows us to extend P to L^1. It is well known that

$$Pf = f, \qquad P(\overline{f}K_a) = \overline{f(a)}K_a \tag{2.2}$$

for holomorphic functions $f \in L^1$. Here, $L^p = L^p(D, A)$ denotes the usual Lebesgue space. See [12, Chapter 1] for details of what have been mentioned above and related facts.

We now recall the well-known Berezin transform, which is one of the main tools in the theory of Toeplitz operators. Let $\mathfrak{L}(L^2_a)$ be the algebra of bounded linear operators on L^2_a. The *Berezin transform* of $S \in \mathfrak{L}(L^2_a)$ is the function $B[S]$ on D defined by

$$B[S](a) = \langle Sk_a, k_a \rangle.$$

For $u \in L^\infty$, we simply let $Bu = B[T_u]$. Since $|\varphi_a'(z)|^2 = |k_a(z)|^2$ by a straightforward calculation, we have

$$Bu(a) = \langle uk_a, k_a \rangle = \int_D (u \circ \varphi_a)\, dA. \tag{2.3}$$

The integral representation (2.3) allows us to extend the notion of the Berezin transform to functions $u \in L^1$. Note that the mean value property yields $Bu = u$ for harmonic functions $u \in L^1$. Also, it is known that the Berezin transform commutes with the invariant Laplacian:

$$B[\widetilde{\Delta} u] = \widetilde{\Delta}(Bu) \tag{2.4}$$

when $u \in L^1 \cap C^2(D)$ and $\widetilde{\Delta} u \in L^1$; see [2, Lemma 1].

The Berezin transform turns out to provide a compactness criterion for certain classes of operators. Here, we consider operators S which are finite sums of finite products of Toeplitz operators with bounded symbols. Thus, such an operator S is of the form

$$S = \sum_{i=1}^{M} T_{u_{i1}} \cdots T_{u_{iN_i}} \tag{2.5}$$

where each $u_{ij} \in L^\infty$. The compactness of operators of this form is characterized by the boundary vanishing property of the Berezin transform as in the next theorem.

Theorem 2.1 ([5]). *Let S be as in* (2.5). *Then S is compact if and only if $B[S] \in C_0$.*

In conjunction with Theorem 2.1 we record here the following identity for easier reference later:

$$B[T_u T_v] - uv = B[\overline{g}h] - \overline{g}h \tag{2.6}$$

for $u, v \in h^\infty$ such that $u = f + \overline{g}$, $v = h + \overline{k}$ where f, g, h, k are holomorphic functions on D. This easily follows from (2.2).

Recall that the pseudohyperbolic distance between two points $z, w \in D$ is defined by $|\varphi_z(w)|$. Let $\mathscr{A} \subset L^\infty$ denote the algebra of all functions that are uniformly continuous with respect to the pseudohyperbolic distance. It is not hard to see that h^∞-functions are Lipschitz continuous with respect to the pseudohyperbolic distance. So, in particular, we have $h^\infty \subset \mathscr{A}$. We remark in passing that Coburn [10] has recently proved a more general result asserting that the Berezin transform $B[S]$ of an arbitrary $S \in \mathfrak{L}(L^2_a)$ is Lipschitz continuous with respect to the pseudohyperbolic distance. For Toeplitz operators with symbols in $\mathscr{A}$, the compactness has been recently characterized by the boundary vanishing property of symbol functions as in the next theorem.

Theorem 2.2 ([9]). *Let $\sigma \in \mathscr{A}$. Then T_σ is compact if and only if $\sigma \in C_0$.*

Let U_a denote the isometry on L^2_a defined by

$$U_a f = \big(f \circ \varphi_a\big) k_a$$

for $f \in L^2_a$. It is easily seen that $U_a U_a = I$ and thus $U_a^{-1} = U_a$. Now, being an invertible linear isometry, U_a is unitary. Thus, since $k_a = -\varphi_a'$, a direct calculation yields

$$B[U_a S U_a] = B[S] \circ \varphi_a \tag{2.7}$$

for $S \in \mathfrak{L}(L^2_a)$ and

$$U_a (S_1 \cdots S_N) U_a = (U_a S_1 U_a) \cdots (U_a S_N U_a) \tag{2.8}$$

for $S_1, \dots, S_N \in \mathfrak{L}(L^2_a)$. Also, for $u \in L^\infty$, it is well known that

$$U_a T_u U_a = T_{u \circ \varphi_a}\,; \tag{2.9}$$

see, for example, [4] or [5] (where U_a is defined with an extra factor -1).

The following theorem is taken from [11, Theorem 2].

Theorem 2.3 ([11])**.** *Suppose that $u \in L^\infty$ and $u = \sum f_j \overline{g_j}$ for finitely many holomorphic functions f_j and g_j on D. If T_u has finite rank, then $u = 0$.*

3. Finite rank operators

In this section, we prove a more general version of Theorem 1.1, derive some applications and construct some examples. In order to do so, we give a characterization for harmonicity of functions which are finite sums of products of an holomorphic function and a co-holomorphic function. For that purpose we first make an observation that characterizes two holomorphic mappings having mutually orthogonal ranges.

We start with the well-known "complexification" lemma; see, for example, the proof of [7, Lemma 10] or the proof of [3, Theorem II].

Lemma 3.1. *Let Ω be a domain in $\mathbb{C}^n$ and assume that Φ is holomorphic on $\Omega \times \Omega^*$ where $\Omega^* = \{\overline{z} : z \in \Omega\}$. If $\Phi(z, \overline{z}) = 0$ for all $z \in \Omega$, then $\Phi = 0$ on $\Omega \times \Omega^*$.*

Given a positive integer N, we let I_N denote the $N \times N$ identity matrix and S_N denote the set of all permutations on $\{1, \dots, N\}$. Given vectors $a = (a_1, \dots, a_N), b = (b_1, \dots, b_N) \in \mathbb{C}^N$, we let $a \cdot \overline{b} = \sum_{j=1}^N a_j \overline{b_j}$ denote the Hermitian inner product of a and b on $\mathbb{C}^N$. Also, we let a^t denote the transpose of a and put $a_\sigma = (a_{\sigma_1}, \dots, a_{\sigma_N})$ for $\sigma \in S_N$. In the exposition below the dimension N in these notations might vary and dimensions involved should be clear from the context.

Let Ω be a domain in $\mathbb{C}^n$ and consider holomorphic mappings

$$F, G : \Omega \to \mathbb{C}^N$$

such that the ranges are mutually orthogonal, i.e., $F \cdot \overline{G} = 0$. One may easily modify the proof of [3, Theorem II] to see that there exists an orthonormal basis $\{\mu_1, \dots, \mu_N\}$ of $\mathbb{C}^N$ such that F and G are of the form

$$F = \big(F \cdot \overline{\mu_1}, \dots, F \cdot \overline{\mu_k}, 0, \dots, 0\big)$$

and

$$G = \big(0, \dots, 0, G \cdot \overline{\mu_{k+1}}, \dots, G \cdot \overline{\mu_N}\big)$$

for some k relative to the orthonormal basis $\{\mu_1, \dots, \mu_N\}$. Here, we provide some more characterizations, which seem (to us) more concrete, as in the next theorem.

Theorem 3.2. *Given finitely many holomorphic functions $f_1, ..., f_N$ and $g_1, ..., g_N$ on a domain $\Omega \subset \mathbb{C}^n$, put $F = (f_j)$ and $G = (g_j)$. Then the following statements are equivalent:*

(a) $F \cdot \overline{G} = 0$ *on* Ω.

(b) There exist some positive integer $k < N$, some permutation $\sigma \in S_N$ and some $(N-k) \times k$ matrix A such that

$$F_\sigma^t = \begin{pmatrix} I_k \\ A \end{pmatrix} \big(f_{\sigma_1}, \dots, f_{\sigma_k}\big)^t$$

and

$$G_\sigma^t = \begin{pmatrix} -A^* \\ I_{N-k} \end{pmatrix} \big(g_{\sigma_{k+1}}, \dots, g_{\sigma_N}\big)^t$$

where $A^* = \overline{A^t}$.

(c) There exist vectors $\mu_1, \dots, \mu_N, \tau_1, \dots, \tau_N \in \mathbb{C}^N$ such that

$$F = \sum_{j=1}^N \mu_j f_j, \quad G = \sum_{i=1}^N \tau_i g_i$$

and

$$\mu_i \cdot \overline{\tau_j} = 0, \qquad i, j = 1, \dots, N.$$

Proof. We first assume (a) and prove (b). Let $\Omega^* = \{\overline{z} : z \in \Omega\}$ and consider a holomorphic function Φ on $\Omega \times \Omega^*$ defined by

$$\Phi(z, w) = F(z) \cdot \overline{G(\overline{w})}$$

for $(z, w) \in \Omega \times \Omega^*$. Then, since $\Phi(z, \overline{z}) = 0$ for $z \in \Omega$ by assumption, we see that Φ identically vanishes on $\Omega \times \Omega^*$ by Lemma 3.1.

We may assume that functions f_j, g_j are all nontrivial. Choose a maximal collection of functions $\{f_{j_1}, \dots, f_{j_k}\}$ subject to the condition that $\{f_{j_1}, \dots, f_{j_k}\}$ is linearly independent. Note that $\{f_1, \dots, f_N\}$ is linearly dependent, because $\Phi = 0$. So, we have $k < N$. Put $m = N - k$ for convenience. Now, after permutation if necessary, we may assume that $\{f_1, \dots, f_k\}$ is linearly independent. Now, since $\{f_1, \dots, f_k, f_j\}$ is linearly dependent for each $j > k$ by maximality, there exists some $m \times k$ matrix A such that

$$(f_{k+1}, \dots, f_N)^t = A(f_1, \dots, f_k)^t$$

which yields the desired representation of F. Put $\widetilde{F} = (f_1, \dots, f_k)$. Then, inserting the above into the identity $\Phi = 0$, we have

$$0 = F(z) \cdot \overline{G(w)} = \widetilde{F}(z)(I_k, A^t)\overline{G(w)^t}$$

for $z, w \in \Omega$. Since $\{f_1, \dots, f_k\}$ is linearly independent, it follows that

$$(I_k, A^t)\overline{G^t} = 0,$$

or equivalently,

$$(g_1, \dots, g_k)^t = -A^*(g_{k+1}, \dots, g_N)^t$$

which yields the desired representation of G.

Next, we assume (b) and prove (c). Put $m = N - k$. Let $a_i = (a_{i1}, \dots, a_{ik})$ be the i-th row of A and $b_j = (a_{1j}, \dots, a_{mj})^t$ be the j-th column of A. Also, let e^j_ℓ be the j-th row of I_ℓ. Then we have by assumption

$$F_\sigma = \sum_{j=1}^{k} (e^j_k, b^t_j) f_{\sigma_j} \quad \text{and} \quad G_\sigma = \sum_{i=1}^{m} (-\overline{a_i}, e^i_m) g_{\sigma_{k+i}}.$$

So, taking vectors

$$\widetilde{\mu}_j = \begin{cases} (e^j_k, b^t_j) & \text{if } j \le k \\ 0 & \text{if } j > k \end{cases}$$

and

$$\widetilde{\tau}_i = \begin{cases} 0 & \text{if } i \le k \\ (-\overline{a_i}, e^i_m) & \text{if } i > k, \end{cases}$$

we obtain

$$F_\sigma = \sum_{j=1}^{N} \widetilde{\mu}_j f_{\sigma_j} \quad \text{and} \quad G_\sigma = \sum_{i=1}^{N} \widetilde{\tau}_i g_{\sigma_i}.$$

Note that

$$\widetilde{\mu}_j \cdot \overline{\widetilde{\tau}_i} = (e^j_k) \cdot \overline{(-\overline{a_i})} + (b^t_j) \cdot \overline{(e^i_m)} = -a_{ij} + a_{ij} = 0$$

for all $i > k$ and $j \leq k$. So, (c) holds.

Finally, the implication (c) $\Longrightarrow$ (a) is straightforward. The proof is complete. ■

Now, we give the following characterization, which will be a key tool in proving Theorem 1.1.

Theorem 3.3. *Let $f_1, \dots, f_N$ and $g_1, \dots, g_N$ be finitely many holomorphic functions on D. If $\sum_{j=1}^{N} f_j \overline{g_j}$ is harmonic on D, then*

$$\sum_{j=1}^{N} \big(f_j - f_j(0)\big)\big(\overline{g_j} - \overline{g_j(0)}\big) = 0$$

holds on D.

Proof. Let $F = (f_j)$ and $G = (g_j)$. Assuming $F(0) = G(0) = 0$ without loss of generality, we need to prove $F \cdot \overline{G} = 0$ on D. Since $\sum_{j=1}^{N} f_j \overline{g_j}$ is harmonic on D, we have $F' \cdot \overline{G'} = 0$ on D. It follows from Theorem 3.2 that there exist vectors $\mu_1, \dots, \mu_N, \tau_1, \dots, \tau_N \in \mathbb{C}^N$ such that

$$F' = \sum_{j=1}^{N} \mu_j f_j', \quad G' = \sum_{i=1}^{N} \tau_i g_i'$$

and

$$\mu_i \cdot \overline{\tau_j} = 0, \qquad i, j = 1, \dots, N.$$

Now, since $F(0) = G(0) = 0$, we have

$$F = \sum_{j=1}^{N} \mu_j f_j, \quad G = \sum_{i=1}^{N} \tau_i g_i$$

and thus

$$F \cdot \overline{G} = \sum_{i,j=1}^{N} (\mu_i \cdot \overline{\tau_j}) f_i \overline{g_j} = 0$$

as desired. The proof is complete. ■

The Bloch space $\mathcal{B}$ is the space of all holomorphic functions f on D for which

$$\sup_{z\in D}\left(1-|z|^2\right)\left|f'(z)\right| < \infty.$$

It is easily seen that Bloch functions are of logarithmic growth near the boundary and thus $\mathcal{B} \subset L^p$ for all $0 < p < \infty$. In particular, we have

$$f\overline{g} \in L^2 \qquad \text{and} \qquad \widetilde{\Delta}(f\overline{g}) \in L^\infty \tag{3.1}$$

for functions $f, g \in \mathcal{B}$. Also, it is well known that, given a function $u = f + \overline{g} \in h^\infty$ where f, g are holomorphic functions on D, we have $f, g \in \mathcal{B}$; see, for example, [15].

In what follows, given $u, v \in h^\infty$, we let

$$Q_{u,v}(\,\cdot\,, a) = \left[\,\overline{P(\overline{u\circ\varphi_a})} - u(a)\right]\left[P(v\circ\varphi_a) - v(a)\right]$$

for $a \in D$ and put

$$R_{u,v} = Q_{v,u}.$$

More explicitly, if $u = f + \overline{g}$ and $v = h + \overline{k}$ where $f, g, h, k \in \mathcal{B}$, then

$$\begin{aligned} Q_{u,v}(\,\cdot\,, a) &= \left[\overline{g\circ\varphi_a} - \overline{g(a)}\,\right]\left[h\circ\varphi_a - h(a)\right] \\ R_{u,v}(\,\cdot\,, a) &= \left[f\circ\varphi_a - f(a)\right]\left[\overline{k\circ\varphi_a} - \overline{k(a)}\,\right] \end{aligned} \tag{3.2}$$

for $a \in D$. Also, we let $\mathscr{F}$ denote the class of all functions λ of the form

$$\lambda = \sum_{i=1}^{M} u_{i_1}u_{i_2}\cdots u_{i_{N_i}} \tag{3.3}$$

where each $u_{i_j} \in h^\infty$. We need the following simple fact.

Lemma 3.4. *$\widetilde{\Delta}\lambda \in L^\infty$ for each $\lambda \in \mathscr{F}$.*

Proof. Let $\lambda \in \mathscr{F}$. We may assume $\lambda = u_1\cdots u_N$ where each $u_j \in h^\infty$. Let $\partial = \frac{\partial}{\partial z}$ and put

$$\begin{aligned} \|\lambda\|_* &= \sup_{z\in D}\left(1-|z|^2\right)\left[|\partial\lambda(z)| + |\overline{\partial}\lambda(z)|\right], \\ \|\lambda\|_{**} &= \sup_{z\in D}\left(1-|z|^2\right)^2\left|\partial\overline{\partial}\lambda(z)\right| \end{aligned}$$

for simplicity. It is clear that $\|\lambda\|_* < \infty$ and $\|\lambda\|_{**} < \infty$ in case $N = 1$. Now, given $u \in h^\infty$, an elementary calculation yields inequalities

$$\begin{aligned} \|\lambda u\|_* &\le \|u\|_\infty\|\lambda\|_* + \|\lambda\|_\infty\|u\|_*, \\ \|\lambda u\|_{**} &\le \|u\|_\infty\|\lambda\|_{**} + \|\lambda\|_*\|u\|_*. \end{aligned}$$

So, an induction on N shows that $\|\lambda\|_* < \infty$ and thus $\|\lambda\|_{**} < \infty$ for arbitrary N. The proof is complete. ■

Now, we are ready to prove the following more general version of Theorem 1.1.

Theorem 3.5. *Let* $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$, $x_1, \dots, x_n, y_1, \dots, y_n \in L^2_a$ *and* $\lambda \in \mathscr{F}$. *Then*

$$T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j} = \sum_{j=1}^{n} x_j \otimes y_j \tag{3.4}$$

if and only if the following two conditions hold:

(a) $\lambda + \sum_{j=1}^{N} u_j v_j = \left(1 - |z|^2\right)^2 \sum_{j=1}^{n} x_j \overline{y_j}.$

(b) $\lambda + \sum_{j=1}^{N} \overline{P\overline{u_j}}\, P v_j$ *is harmonic.*

The idea of the proof of the necessity below comes from the argument in [11].

Proof. For each $j = 1, \dots, N$, we write $u_j = f_j + \overline{g_j}$ and $v_j = h_j + \overline{k_j}$ where f_j, g_j, h_j and k_j are all in $\mathcal{B}$. Also, we write

$$\left(1 - |z|^2\right)^2 \sum_{j=1}^{n} x_j \overline{y}_j = \left(1 - 2z\overline{z} + z^2 \overline{z}^2\right) \sum_{j=1}^{n} x_j \overline{y}_j = \sum_{j=1}^{3n} \alpha_j \overline{\beta_j}$$

where α_j, β_j are all in L^2_a.

First suppose (3.4) holds. Note that we have by (2.6)

$$B[T_{u_j} T_{v_j}] = B[h_j \overline{g_j}] + \left(f_j h_j + \overline{g_j} \overline{k_j}\right) + f_j \overline{k_j}$$

for each j. Also, note that

$$B[x_j \otimes y_j] = \left(1 - |z|^2\right)^2 x_j \overline{y_j}$$

for each j. Thus, taking the Berezin transforms of both sides of (3.4), we obtain

$$B\lambda + \sum_{j=1}^{N} B[h_j \overline{g_j}] + \sum_{j=1}^{N} \left(f_j h_j + \overline{g_j} \overline{k_j}\right) + \sum_{j=1}^{N} f_j \overline{k_j} = \sum_{j=1}^{3n} \alpha_j \overline{\beta_j}. \tag{3.5}$$

Let

$$\sigma = \widetilde{\Delta} \lambda + \sum_{j=1}^{N} \widetilde{\Delta}(h_j \overline{g_j}).$$

Note that σ is bounded by (3.1) and Lemma 3.4. Now, applying the invariant Laplacian to both sides of (3.5), we have by (2.4)

$$B\sigma = \sum_{j=1}^{3n} \widetilde{\Delta}(\alpha_j\overline{\beta_j}) - \sum_{j=1}^{N} \widetilde{\Delta}(f_j\overline{k_j}) = \big(1-|z|^2\big)^2\bigg(\sum_{j=1}^{3n} \alpha_j'\overline{\beta_j'} - \sum_{j=1}^{N} f_j'\overline{k_j'}\bigg). \tag{3.6}$$

Dividing by $(1-|z|^2)^2$, we obtain

$$\int_D \frac{\sigma(\zeta)}{|1-z\overline{\zeta}|^4}\,dA(\zeta) = \sum_{j=1}^{3n} \alpha_j'(z)\overline{\beta_j'(z)} - \sum_{j=1}^{N} f_j'(z)\overline{k_j'(z)}$$

for $z \in D$. Now, by Lemma 3.1, we have

$$\int_D \frac{\sigma(\zeta)}{(1-z\overline{\zeta})^2(1-\overline{w}\zeta)^2}\,dA(\zeta) = \sum_{j=1}^{3n} \alpha_j'(z)\overline{\beta_j'(w)} - \sum_{j=1}^{N} f_j'(z)\overline{k_j'(w)}$$

for every $z, w \in D$. Differentiate both sides of the above as many times as needed with respect to $\overline{w}$ variable and then insert $w = 0$. The result is

$$\begin{aligned} T_\sigma\zeta^\ell(z) &= \int_D \frac{\sigma(\zeta)\zeta^\ell}{(1-z\overline{\zeta})^2}\,dA(\zeta) \\ &= \sum_{j=1}^{3n} a_{j\ell}\alpha_j'(z) + \sum_{j=1}^{N} b_{j\ell}f_j'(z), \qquad \ell = 0,1,2,\dots \end{aligned}$$

for some coefficients $a_{j\ell}$ and $b_{j\ell}$. Now, by the same argument as in the proof of [11, Proposition 4], we see that T_σ has finite rank. Note that σ can be represented as a sum of finitely many products of a holomorphic function and a co-holomorphic function, because $\lambda \in \mathscr{F}$. So, Theorem 2.3 gives $\sigma = 0$. Namely, the function $\lambda + \sum_{j=1}^{N} h_j\overline{g_j}$ is harmonic. Accordingly, noting that

$$Pv_j = h_j + \overline{k_j(0)} \quad \text{and} \quad P\overline{u_j} = g_j + \overline{f_j(0)}, \tag{3.7}$$

we conclude (b). Also, it follows from (3.6) that the function

$$\sum_{j=1}^{N} f_j\overline{k_j} - \sum_{j=1}^{3n} \alpha_j\overline{\beta_j}$$

is harmonic. Since harmonic L^1-functions are invariant under the Berezin transform, it follows that

$$\sum_{j=1}^{N} f_j\overline{k_j} - \sum_{j=1}^{3n} \alpha_j\overline{\beta_j} = \sum_{j=1}^{N} B[f_j\overline{k_j}] - \sum_{j=1}^{3n} B[\alpha_j\overline{\beta_j}].$$

Combining this with (3.5), we obtain

$$\begin{aligned}B\lambda + \sum_{j=1}^{N} B[u_jv_j] &= B\lambda + \sum_{j=1}^{N} \big(f_jh_j + \overline{g_jk_j}\big) + \sum_{j=1}^{N} B\big[f_j\overline{k_j} + h_j\overline{g_j}\big]\\ &= \sum_{j=1}^{N} B[f_j\overline{k_j}] + \sum_{j=1}^{3n} \alpha_j\overline{\beta_j} - \sum_{j=1}^{N} f_j\overline{k_j}\\ &= \sum_{j=1}^{3n} B[\alpha_j\overline{\beta_j}].\end{aligned}$$

So, we conclude (a), because the Berezin transform is one-to-one (see, for example, [12, Chapter 2]).

Now, suppose (a) and (b). Note that the set $\{K_a : a \in D\}$ spans a dense subset of L^2_a. So, to prove (3.4), it is sufficient to show

$$\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j}T_{v_j}\Big]K_a = \sum_{j=1}^{n}(x_j \otimes y_j)K_a \tag{3.8}$$

for all $a \in D$. Let $a \in D$ be an arbitrary point. First, note that we have by (2.2)

$$\begin{aligned}T_{u_j}T_{v_j}K_a &= P\big[\big(f_j + \overline{g_j}\big)\big(h_j + \overline{k_j(a)}\big)K_a\big]\\ &= P\big[\big(f_jh_j + h_j\overline{g_j} + \overline{g_j(a)k_j(a)}\big)K_a\big] + f_j\overline{k_j(a)}K_a\\ &= P\big[\big(f_jh_j + h_j\overline{g_j} + \overline{g_jk_j}\big)K_a\big] + f_j\overline{k_j(a)}K_a\\ &= P\big[\big(u_jv_j - f_j\overline{k_j}\big)K_a\big] + f_j\overline{k_j(a)}K_a\end{aligned}$$

for each j. Also, note that $(x_j \otimes y_j)K_a = x_j\overline{y_j(a)}$ for each j. So, by (a), in order to prove (3.8), it is necessary and sufficient to show

$$\sum_{j=1}^{3n} P[\alpha_j\overline{\beta_j}K_a] - \sum_{j=1}^{N} P[f_j\overline{k_j}K_a] = \sum_{j=1}^{n} x_j\overline{y_j(a)} - \sum_{j=1}^{N} f_j\overline{k_j(a)}K_a. \tag{3.9}$$

Since the function $\lambda + \sum_{j=1}^{N} h_j\overline{g_j}$ is harmonic by (b) and (3.7), the function $\lambda + \sum_{j=1}^{N}(u_jv_j - f_j\overline{k_j})$ is also harmonic. Note that $\sum_{j=1}^{N}(u_jv_j - f_j\overline{k_j}) \in L^2$, because $f_jk_j \in L^2_a$ by (3.1) for each j. Thus, we have by (a)

$$\sum_{j=1}^{3n} \alpha_j\overline{\beta_j} - \sum_{j=1}^{N} f_j\overline{k_j} = F + \overline{G}$$

for some holomorphic functions $F, G \in L^2_a$.

Thus, multiplying by K_a and then applying the projection P to both sides of the above, we obtain by (2.2)

$$\sum_{j=1}^{3n} P[\alpha_j \overline{\beta_j} K_a] - \sum_{j=1}^{N} P[f_j \overline{k_j} K_a] = [F + \overline{G(a)}]K_a.$$

Meanwhile, we have by Lemma 3.1

$$F + \overline{G(a)} = \sum_{j=1}^{3n} \alpha_j \overline{\beta_j(a)} - \sum_{j=1}^{N} f_j \overline{k_j(a)} = K_a^{-1} \sum_{j=1}^{n} x_j \overline{y_j(a)} - \sum_{j=1}^{N} f_j \overline{k_j(a)}.$$

Combining these equalities, we obtain (3.9). The proof is complete. ■

Taking $\lambda = 0$ in Theorem 3.5, we obtain Theorem 1.1 which we restate here for convenience.

Theorem 3.6. *Let* $u_1, ..., u_N, v_1, ..., v_N \in h^\infty$ *and* $x_1, ..., x_n, y_1, ..., y_n \in L^2_a$. *Then*

$$\sum_{j=1}^{N} T_{u_j} T_{v_j} = \sum_{j=1}^{n} x_j \otimes y_j \tag{3.10}$$

if and only if the following two conditions hold:

(a) $\sum_{j=1}^{N} u_j v_j = \left(1 - |z|^2\right)^2 \sum_{j=1}^{n} x_j \overline{y_j}.$

(b) $\sum_{j=1}^{N} Q_{u_j, v_j}(\,\cdot\,, 0) = 0.$

Proof. The theorem follows from Theorem 3.5 and the fact that

$$\sum_{j=1}^{N} \overline{P\overline{u_j}}\, P v_j$$

is harmonic if and only if (b) holds by Theorem 3.3. ■

As another special case of Theorem 3.5, we have the following characterizations for operators under consideration to be the zero operator.

Theorem 3.7. *Let* $u_1, \cdots, u_N, v_1, \cdots, v_N \in h^\infty$ *and* $\lambda \in \mathscr{F}$. *Then*

$$T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j} = 0$$

if and only if the following two conditions hold:

(a) $\lambda + \sum_{j=1}^{N} u_j v_j = 0.$

(b) $\sum_{j=1}^{N} R_{u_j,v_j}(\cdot, 0) = 0.$

Proof. If $\lambda + \sum_{j=1}^{N} u_j v_j = 0$, then it is easily seen that $\lambda + \sum_{j=1}^{N} \overline{P\overline{u_j}}\, P v_j$ is harmonic if and only if $\sum_{j=1}^{N} P u_j \overline{P\overline{v_j}}$ is harmonic. Thus the theorem holds by Theorems 3.5 and 3.3. The proof is complete. ■

In case $N = 1$ Theorem 3.7 is known to hold for general $\lambda \in L^\infty$ (see [1, Corollary 1]) and we do not know whether such a general result holds for arbitrary N.

Combining Theorems 3.6 and 3.7, we have the following characterization.

Theorem 3.8. *Let $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\sum_{j=1}^{N} T_{u_j} T_{v_j} = 0.$

(b) $\sum_{j=1}^{N} T_{v_j} T_{u_j} = 0.$

(c) $\sum_{j=1}^{N} Q_{u_j,v_j}(\ \cdot\ , 0) = \sum_{j=1}^{N} u_j v_j = 0.$

(d) $\sum_{j=1}^{N} R_{u_j,v_j}(\ \cdot\ , 0) = \sum_{j=1}^{N} u_j v_j = 0.$

We now apply our theorems to recover results in [11] concerning sums of finitely many (semi-)commutators. Given Toeplitz operators T_u and T_v, we let

$$\begin{aligned} \big[T_u, T_v\big] &= T_u T_v - T_v T_u, \\ \big(T_u, T_v\big] &= T_u T_v - T_{uv} \end{aligned}$$

denote the commutator and the semi-commutator, respectively.

Theorem 3.5 also has some consequences for sums of finitely many (semi-)commutators of Toeplitz operators with harmonic symbols as in the next two corollaries. For semi-commutators, we have the following consequence, which is a slightly different form of [11, Theorem 8].

Corollary 3.9. *Let $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\displaystyle\sum_{j=1}^{N} \big(T_{u_j}, T_{v_j}\big] = 0$.

(b) $\displaystyle\sum_{j=1}^{N} \big(T_{u_j}, T_{v_j}\big]$ *has finite rank.*

(c) $\displaystyle\sum_{j=1}^{N} R_{u_j, v_j}(\,\cdot\,, 0) = 0$.

Proof. The equivalence (a) $\iff$ (c) holds by Theorem 3.7 (with $\lambda = -\sum_{j=1}^{N} u_j v_j$). The implication (a) $\Longrightarrow$ (b) is trivial.

We now assume (b) and prove (a). Since $\sum_{j=1}^{N} (T_{u_j}, T_{v_j}]$ has finite rank, we have $\sum_{j=1}^{N} (T_{u_j}, T_{v_j}] = \sum_{j=1}^{n} x_j \otimes y_j$ for some functions $x_1, ..., x_n, y_1, ..., y_n \in L^2_a$. We may assume that $x_1, ..., x_n$ are linearly independent. We have $\sum_{j=1}^{n} x_j \overline{y_j} = 0$ by Theorem 3.5 and thus

$$\sum_{j=1}^{n} x_j(z) \overline{y_j(w)} = 0$$

for all $z, w \in D$ by Lemma 3.1. Since $x_1, \dots, x_n$ are linearly independent, it follows that $y_j = 0$ for all j and thus $\sum_{j=1}^{N} (T_{u_j}, T_{v_j}] = 0$. The proof is complete. ■

Since a commutator is the difference of associated semi-commutators, Corollary 3.9 yields yet another corollary for commutators as follows. Another way of deriving this corollary is to take $\lambda = 0$ in Theorem 3.5 (or Theorem 3.7).

Corollary 3.10. *Let $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\displaystyle\sum_{j=1}^{N} \big[T_{u_j}, T_{v_j}\big] = 0$.

(b) $\displaystyle\sum_{j=1}^{N} \big[T_{u_j}, T_{v_j}\big]$ *has finite rank.*

(c) $\displaystyle\sum_{j=1}^{N} Q_{u_j, v_j}(\,\cdot\,, 0) = \sum_{j=1}^{N} R_{u_j, v_j}(\,\cdot\,, 0)$.

Also, we can recover the result on finite rank Toeplitz products with harmonic symbols, which is proved in [11, Theorem 7].

Corollary 3.11. *Let $u, v \in h^\infty$. Then $T_u T_v$ has finite rank if and only if either $u = 0$ or $v = 0$.*

Proof. The sufficiency is trivial. We prove the necessity. Suppose that T_uT_v has finite rank. Let $u = f + \overline{g}$ and $v = h + \overline{k}$ where f, g, h, k are holomorphic functions. By Theorem 3.6 we have

(i) $uv = (1 - |z|^2)^2 \sum_{j=1}^{n} x_j \overline{y_j}$
(ii) $h\overline{g}$ is harmonic on D

for some finitely many functions $x_1, \dots, x_n, y_1, \dots, y_n \in L^2_a$. It follows from (i) that uv has a continuous extension on $\overline{D}$ and $uv = 0$ on ∂D (see [14, Theorem 7.2.5]). Being bounded harmonic functions, u and v have radial limits almost everywhere on ∂D. So, there are two possibilities: one is that u or v vanishes almost everywhere on ∂D and the other is that u and v vanish on some sets of positive measures on ∂D. Note that u or $\overline{v}$ is holomorphic by (ii). Therefore, in either case, we conclude either $u = 0$ or $v = 0$ on D. The proof is complete. ■

In view of Theorem 3.6, one may ask whether (3.10) can actually happen. In other words, one may ask whether there are examples of functions satisfying conditions (a) and (b) of Theorem 3.6. The answer is *yes*. For example, given $x_1, \dots, x_n, y_1, \dots, y_n \in H^\infty$, put

$$u_{j1} = x_j, \quad u_{j2} = -2zx_j, \quad u_{j3} = z^2 x_j$$
$$v_{j1} = \overline{y_j}, \quad v_{j2} = \overline{z}\,\overline{y_j}, \quad v_{j3} = \overline{z}^2\,\overline{y_j}$$

for $j = 1, \dots, n$. Then one can easily check that conditions (a) and (b) of Theorem 3.6 are satisfied and thus

$$\sum_{j=1}^{n} \sum_{i=1}^{3} T_{u_{ji}} T_{v_{ji}} = \sum_{j=1}^{n} x_j \otimes y_j.$$

In particular, the operator

$$I - 2T_z T_{\overline{z}} + T_{z^2} T_{\overline{z}^2} = 1 \otimes 1$$

is simply the point evaluation at the origin, which one may also verify by a direct calculation.

The above examples shows that, given an n-dimensional subspace X_n of L^2_a generated by bounded holomorphic functions, we can find $3n$ pairs of symbols $u_j, v_j \in h^\infty$ such that the range of $\sum_{j=1}^{3n} T_{u_j} T_{v_j}$ is precisely X_n. In prescribing ranges like that, we do not know whether we can control the number of pairs of symbols in general. However, as far as the rank is concerned, the next example shows that just two pairs of symbols are enough to produce arbitrary ranks. Note that at least two pairs of symbols are required in prescribing ranks by Corollary 3.11.

Example 3.12. *Given a positive integer n, there exist some $u_1, u_2, v_1, v_2 \in h^\infty$ such that $\sum_{j=1}^{2} T_{u_j} T_{v_j}$ has rank n.*

Proof. Let a positive integer n be given and let p_n be the polynomial of degree $(n-1)$ such that

$$z^{n+1} - (n+1)z + n = (z-1)^2 p_n.$$

An elementary calculation yields

$$p_n = \sum_{j=0}^{n-1} \bigl(z^j + z^{j-1} + \cdots + 1\bigr) = \sum_{j=0}^{n-1} \bigl(n-j\bigr) z^j.$$

Choose real numbers a, b such that $|a| + |b| < n$ and put

$$y = \frac{1}{az^{n+1} + bz + n}.$$

Note that $y \in H^\infty$, because $|a| + |b| < n$. We may choose a, b with the additional property that the polynomials $z^{n+1} - a$ and $(n+1)z + b$ have no common zeros and therefore we have

$$\bigl|z^{n+1} - a\bigr| + \bigl|(n+1)z + b\bigr| \ge \delta, \qquad z \in D \tag{3.11}$$

for some positive number δ. So, there exist some functions $h_1, h_2 \in H^\infty$ such that

$$(z^{n+1} - a)h_1 - \bigl((n+1)z + b\bigr)h_2 = 1 \tag{3.12}$$

on D by the corona theorem.

Now, given a nontrivial function $x \in H^\infty$, take functions $f_j, k_j \in H^\infty$ as follows:

$$\begin{aligned} &f_1 = (z^{n+1} - a)x, \quad f_2 = -\bigl((n+1)z + b\bigr)x, \\ &k_1 = z^{n+1}y, \quad k_2 = zy. \end{aligned}$$

Using these functions, we put

$$u_j = f_j, \quad v_j = h_j + \overline{k_j}, \qquad j = 1, 2.$$

Then we have by (3.7)

$$\overline{P\overline{u_j}} - u_j(0) = \overline{f_j(0)} - \overline{f_j(0)} = 0 \tag{3.13}$$

for $j = 1, 2$ and

$$\begin{aligned} u_1v_1 + u_2v_2 &= \bigl(f_1h_1 + f_2h_2\bigr) + f_1\overline{k_1} + f_2\overline{k_2} \\ &= x + \bigl(z^{n+1} - a\bigr)\overline{z}^{n+1}x\overline{y} - \bigl((n+1)z + b\bigr)\overline{z}x\overline{y} \\ &= x\overline{y}\bigl[\,\overline{y}^{-1} + (z^{n+1} - a)\overline{z}^{n+1} - \bigl((n+1)z + b\bigr)\overline{z}\bigr] \\ &= \bigl(1 - |z|^2\bigr)^2 p_n\bigl(|z|^2\bigr)x\overline{y}. \end{aligned}$$

Thus, setting

$$x_j = \big(n-j\big)z^j x, \quad y_j = z^j$$

for $j = 0, \dots, n-1$, we obtain

$$u_1v_1 + u_2v_2 = \big(1-|z|^2\big)^2 \sum_{j=0}^{n-1} x_j\overline{y}_j \tag{3.14}$$

on D. Now, having (3.13) and (3.14), we conclude

$$T_{u_1}T_{v_1} + T_{u_2}T_{v_2} = \sum_{j=0}^{n-1} x_j \otimes y_j$$

by Theorem 3.6. Since $\{x_j\}$ and $\{y_j\}$ are both linearly independent, this shows that the operator $\sum_{j=1}^{2} T_{u_j}T_{v_j}$ has rank n. The proof is complete. ■

4. Compact operators

In this section, we prove compact versions of results obtained in the previous section. For that purpose, we first recall the notion of maximal ideal space. The maximal ideal space $\mathfrak{M}$ of H^∞ is the space (endowed with the weak* topology of the dual of H^∞) of all nonzero multiplicative linear functionals on H^∞. As is well known, we have $H^\infty \subset C(\mathfrak{M})$ via the Gelfand transform. Moreover, it is known ([13, Lemma 4.4]) that $h^\infty \subset C(\mathfrak{M})$. We will use the same notation for a function $u \in h^\infty$ and its continuous extension u on the whole $\mathfrak{M}$. Identifying $z \in D$ with the multiplicative evaluation functional $f \mapsto f(z)$, we can freely regard D as a subset of $\mathfrak{M}$. The corona theorem says that D is dense in $\mathfrak{M}$.

For each $m \in \mathfrak{M}$, K. Hoffman ([13]) constructed a canonical map L_m from D into $\mathfrak{M}$. This map L_m is defined by taking a net $\{z_\alpha\}$ in D such that $z_\alpha \to m$ and defining

$$L_m(z)(h) = \lim_\alpha h \circ \varphi_{z_\alpha}(z)$$

for $z \in D$ and $h \in H^\infty$. The above limit exists and is independent of the net $\{z_\alpha\}$ provided that $z_\alpha \to m$. For each $f \in H^\infty$, the map $f \circ L_m$ is in H^∞. Moreover, if u is continuous on $\mathfrak{M}$ and $\{z_\alpha\}$ is a net converging to m in $\mathfrak{M}$, then it is known ([15, Lemma 5]) that $u \circ \varphi_{z_\alpha} \to u \circ L_m$ uniformly on compact subsets of D and thus

$$(\widetilde{\Delta} u) \circ \varphi_{z_\alpha} = \widetilde{\Delta}(u \circ \varphi_{z_\alpha}) \to \widetilde{\Delta}(u \circ L_m). \tag{4.1}$$

In particular, we have $u \circ L_m \in h^\infty$ for $u \in h^\infty$. Also, $\lambda \circ L_m \in \mathscr{F}$ for $\lambda \in \mathscr{F}$; recall that $\mathscr{F}$ is the class introduced in (3.3).

The following lemma is implicit in the proof of [6, Lemma 5.1].

Lemma 4.1. *Suppose that* $\{z_\alpha\}$ *is a net in* D *such that* $z_\alpha \to m \in \mathfrak{M}$*. Then*

$$\widetilde{\Delta}[\overline{P\overline{u}}\,Pv\,]\circ\varphi_{z_\alpha} \to \widetilde{\Delta}[\overline{P(\overline{u\circ L_m})}\,P(v\circ L_m)\,]$$

(pointwise) on D *for* $u, v \in h^\infty$*.*

Also, we need the following fact.

Lemma 4.2. *Suppose that* $\{z_\alpha\}$ *is a net in* D *such that* $z_\alpha \to m \in \mathfrak{M}$*. Let* $\lambda_1, \dots, \lambda_M \in \mathscr{F}$*. Then*

$$T_{\lambda_M\circ\varphi_{w_\alpha}}\cdots T_{\lambda_1\circ\varphi_{w_\alpha}} \to T_{\lambda_M\circ L_m}\cdots T_{\lambda_1\circ L_m}$$

in the weak operator topology.

Proof. Fix $f \in L^2_a$. Recall $\lambda_1\circ\varphi_{w_\alpha} \to \lambda_1\circ L_m$ uniformly on compact subsets of D. So, since λ_1 is bounded, the dominated convergence theorem yields

$$(\lambda_1\circ\varphi_{w_\alpha})f \to (\lambda_1\circ L_m)f \quad \text{in} \quad L^2$$

and thus

$$P[(\lambda_1\circ\varphi_{w_\alpha})f] \to P[(\lambda_1\circ L_m)f] \quad \text{in} \quad L^2_a$$

by continuity of P. This proves the lemma for $M = 1$. We now proceed by induction on M. Assume $M \ge 2$ and suppose that the lemma holds for $M - 1$. Put

$$h_\alpha = T_{\lambda_{M-1}\circ\varphi_{w_\alpha}}\cdots T_{\lambda_1\circ\varphi_{w_\alpha}} f \quad \text{and} \quad g = T_{\lambda_{M-1}\circ L_m}\cdots T_{\lambda_1\circ L_m} f$$

for simplicity. Then we have by induction hypothesis $h_\alpha \to g$ in L^2_a and thus uniformly on compact subsets of D. Now, since $\lambda_M\circ\varphi_{w_\alpha}$ is bounded and converges pointwise to $\lambda_M\circ L_m$, we have

$$\begin{aligned}&\|(\lambda_M\circ\varphi_{w_\alpha})h_\alpha - (\lambda_M\circ L_m)g\|_{L^2}\\ &\qquad \le \|\lambda_M\|_{L^\infty}\|h_\alpha - g\|_{L^2} + \|(\lambda_M\circ\varphi_{w_\alpha})g - (\lambda_M\circ L_m)g\|_{L^2} \to 0\end{aligned}$$

by the dominated convergence theorem and thus

$$P[(\lambda_M\circ\varphi_{w_\alpha})h_\alpha] \to P[(\lambda_M\circ L_m)g] \quad \text{in} \quad L^2_a$$

by continuity of P. In other words, $T_{\lambda_M\circ\varphi_{w_\alpha}}h_\alpha \to T_{\lambda_M\circ L_m}g$ in L^2_a. This completes the induction and the proof of the lemma. ■

We are now ready to prove the compact version of Theorem 3.7.

Theorem 4.3. *Let* $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$ *and* $\lambda \in \mathscr{F}$*. Then the following statements are equivalent:*

(a) $T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}$ *is compact.*

(b) $T_{\lambda\circ L_m} + \sum_{j=1}^{N} T_{u_j\circ L_m} T_{v_j\circ L_m} = 0$ *for each* $m \in \mathfrak{M} \setminus D$*.*

(c) $\widetilde{\Delta}\lambda + \sum_{j=1}^{N} \widetilde{\Delta}\big[\overline{P\overline{u_j}} Pv_j\big] \in C_0$ *and* $\lambda + \sum_{j=1}^{N} u_j v_j \in C_0$*.*

(d) $\sum_{j=1}^{N} \widetilde{\Delta}\big[Pu_j \overline{P\overline{v_j}}\big] \in C_0$ *and* $\lambda + \sum_{j=1}^{N} u_j v_j \in C_0$*.*

(e) $\sum_{j=1}^{N} \big(T_{u_j}, T_{v_j}\big]$ *is compact and* $\lambda + \sum_{j=1}^{N} u_j v_j \in C_0$*.*

(f) $\lim_{|a|\to 1} \int_D \Big|\sum_{j=1}^{N} R_{u_j,v_j}(z,a)\Big| \, dA(z) = 0$ *and* $\lambda + \sum_{j=1}^{N} u_j v_j \in C_0$*.*

We will complete the proof by proving the following sequences of implications:

$$\begin{aligned}
&(\mathrm{b}) \iff (\mathrm{c}),\\
&(\mathrm{b}) \iff (\mathrm{d}),\\
&(\mathrm{a}) \implies (\mathrm{b}) \implies (\mathrm{e}) \implies (\mathrm{a}),\\
&(\mathrm{b}) \implies (\mathrm{f}) \implies (\mathrm{e}).
\end{aligned}$$

Since proofs are somewhat long, we will prove each case separately.

Proof of (b) $\iff$ (c). First we assume (b) and prove (c). It is sufficient to show that, for a given net $\{w_\alpha\}$ in D converging to some $m \in \mathfrak{M}\setminus D$, we have

$$\widetilde{\Delta}\Big[\lambda + \sum_{j=1}^{N} \overline{P\overline{u_j}} Pv_j\Big](w_\alpha) \to 0 \tag{4.2}$$

and

$$\Big[\lambda + \sum_{j=1}^{N} u_j v_j\Big](w_\alpha) \to 0. \tag{4.3}$$

So, fix a net $\{w_\alpha\}$ in D such that $w_\alpha \to m$ for some $m \in \mathfrak{M} \setminus D$. To prove (4.2), note that we have

$$\widetilde{\Delta}\Big[\lambda \circ L_m + \sum_{j=1}^{N} \overline{P(\overline{u_j \circ L_m})} P(v_j \circ L_m)\Big] = 0$$

by assumption (b) and Theorem 3.5. Thus, we have (4.2) by (4.1) and Lemma 4.1 (with evaluation at the origin). Also, note that

$$\Big[\lambda+\sum_{j=1}^{N}u_jv_j\Big]\circ L_m=0$$

holds by assumption (b) and Theorem 3.5. Thus, we have (4.3) by a similar argument.

Now, we assume (c) and prove (b). Let $m\in\mathfrak{M}\backslash D$ and choose a net $\{w_\alpha\}$ in D such that $w_\alpha\to m$. Fix an arbitrary point $a\in D$. Put $z_\alpha=\varphi_{w_\alpha}(a)$ and $m_a=L_m(a)$. Since $h(z_\alpha)=h\big(\varphi_{w_\alpha}(a)\big)\to m_a(h)$ for $h\in H^\infty$, we have $z_\alpha\to m_a$ in $\mathfrak{M}$.

By the Schwarz lemma there are rotations, say $W_{a,\alpha}$, such that

$$\varphi_{z_\alpha}=\varphi_{w_\alpha}\circ\varphi_a\circ W_{a,\alpha}.$$

Since the set of rotations is compact, we may assume $W_{a,\alpha}$ converges to some rotation W_a. Now, for a given function $f\in H^\infty$, since $f\circ\varphi_{w_\alpha}\to f\circ L_m$ uniformly on compact subsets of D, we see that $f\circ\varphi_{z_\alpha}\to f\circ L_m\circ\varphi_a\circ W_a$ on D. Thus, $L_{m_a}=L_m\circ\varphi_a\circ W_a$. It follows that

$$\widetilde{\Delta}\big[\,\overline{P(\overline{u\circ L_{m_a}})}\,P(v\circ L_{m_a})\big]=\widetilde{\Delta}\big[\overline{P(\overline{u\circ L_m})}\,P(v\circ L_m)\big]\circ\varphi_a\circ W_a \tag{4.4}$$

for $u,v\in h^\infty$ by the Möbius invariance of $\widetilde{\Delta}$.

Note that $m\in\mathfrak{M}\setminus D$ implies $|w_\alpha|\to 1$ and thus $|z_\alpha|\to 1$. Now, since $|z_\alpha|\to 1$, we obtain by (4.1), (4.4) and Lemma 4.1 (with evaluation at the origin)

$$0=\lim_\alpha\widetilde{\Delta}\Big[\lambda+\sum_{j=1}^{N}\overline{P\overline{u_j}}\,Pv_j\Big](z_\alpha)=\widetilde{\Delta}\Big[\lambda\circ L_m+\sum_{j=1}^{N}\overline{P(\overline{u_j\circ L_m})}\,P(v_j\circ L_m)\Big](a).$$

Since $a\in D$ is arbitrary, this shows that $\lambda\circ L_m+\sum_{j=1}^{N}\overline{P(\overline{u_j\circ L_m})}\,P(v_j\circ L_m)$ is harmonic. Also, since $\lambda+\sum_{j=1}^{N}u_jv_j\in C_0$, we have $\lambda\circ L_m+\sum_{j=1}^{N}(u_jv_j)\circ L_m=0$. Hence, by Theorem 3.5, we conclude (b). The proof is complete. ■

Proof of (b) $\Longleftrightarrow$ (d). By Theorems 3.7 and 3.3 we have (b) if and only if

$$\Big[\lambda+\sum_{j=1}^{N}u_jv_j\Big]\circ L_m=0$$

and

$$\widetilde{\Delta}\Big[\sum_{j=1}^{N}P(u_j\circ L_m)\,\overline{P(\overline{v_j\circ L_m})}\Big]=0$$

for each $m\in\mathfrak{M}\setminus D$. Thus, following the proof of (b) $\Longleftrightarrow$ (c), we see that (b) and (d) are equivalent. The proof is complete. ■

Proof of (a) $\Longrightarrow$ (b) $\Longrightarrow$ (e) $\Longrightarrow$ (a). First, we assume (a) and prove (b). Let $m \in \mathfrak{M} \setminus D$. Since the set $\{k_a : a \in D\}$ spans a dense subset of L^2_a, it is sufficient to show that

$$\Big[T_{\lambda\circ L_m} + \sum_{j=1}^{N} T_{u_j\circ L_m} T_{v_j\circ L_m}\Big] k_a = 0, \qquad a \in D. \tag{4.5}$$

Fix $a \in D$ and choose a net $\{w_\alpha\}$ in D such that $w_\alpha \to m$. Then, since

$$T_{\lambda\circ\varphi_{w_\alpha}} + \sum_{j=1}^{N} T_{u_j\circ\varphi_{w_\alpha}} T_{v_j\circ\varphi_{w_\alpha}} \longrightarrow T_{\lambda\circ L_m} + \sum_{j=1}^{N} T_{u_j\circ L_m} T_{v_j\circ L_m}$$

in the weak operator topology by Lemma 4.2, we have

$$\begin{aligned}\Big\|\Big[T_{\lambda\circ L_m} + \sum_{j=1}^{N} T_{u_j\circ L_m} T_{v_j\circ L_m}\Big] k_a\Big\|_{L^2} &= \lim_\alpha \Big\|\Big[T_{\lambda\circ\varphi_{w_\alpha}} + \sum_{j=1}^{N} T_{u_j\circ\varphi_{w_\alpha}} T_{v_j\circ\varphi_{w_\alpha}}\Big] k_a\Big\|_{L^2}\\ &= \lim_\alpha \Big\|U_{w_\alpha}\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big] U_{w_\alpha} k_a\Big\|_{L^2} \qquad \text{(by (2.8) and (2.9))}\\ &= \lim_\alpha \Big\|\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big] U_{w_\alpha} k_a\Big\|_{L^2}.\end{aligned}$$

Note that $|w_\alpha| \to 1$, because $m \in \mathfrak{M} \setminus D$. Thus, it is easily seen that $U_{w_\alpha} k_a$ converges to 0 weakly in L^2_a. Hence, the compactness of $T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}$ yields (4.5).

Next, we assume (b) and prove (e). The second part of assertion (e) is contained in the implication (b) $\Longrightarrow$ (c), which is proved above. By Theorem 2.2 we see that $T_\lambda + T_{\sum_{j=1}^{N} u_j v_j}$ is compact. So, in order to prove (e), we need to prove that $T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}$ is compact. By Theorem 2.1, it is sufficient to prove

$$B\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big] \in C_0. \tag{4.6}$$

Suppose not. Then there is a net $\{w_\alpha\}$ in D converging to some $m \in \mathfrak{M} \setminus D$ such that

$$\limsup_\alpha \Big| B\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big](w_\alpha)\Big| > 0. \tag{4.7}$$

Note that we have by Lemma 4.2

$$B\Big[T_{\lambda\circ\varphi_{w_\alpha}} + \sum_{j=1}^{N} T_{u_j\circ\varphi_{w_\alpha}} T_{v_j\circ\varphi_{w_\alpha}}\Big] \longrightarrow B\Big[T_{\lambda\circ L_m} + \sum_{j=1}^{N} T_{u_j\circ L_m} T_{v_j\circ L_m}\Big]$$

pointwise on D. It follows that

$$\begin{aligned} 0 &= B\Big[T_{\lambda\circ L_m} + \sum_{j=1}^{N} T_{u_j\circ L_m} T_{v_j\circ L_m}\Big](0) \\ &= \lim_\alpha B\Big[T_{\lambda\circ\varphi_{w_\alpha}} + \sum_{j=1}^{N} T_{u_j\circ\varphi_{w_\alpha}} T_{v_j\circ\varphi_{w_\alpha}}\Big](0) \\ &= \lim_\alpha B\Big[U_{w_\alpha}\Big(T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big)U_{w_\alpha}\Big](0) \qquad (\text{by} \quad (2.9)) \\ &= \lim_\alpha B\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big]\big(\varphi_{w_\alpha}(0)\big) \qquad (\text{by} \quad (2.7)) \\ &= \lim_\alpha B\Big[T_\lambda + \sum_{j=1}^{N} T_{u_j} T_{v_j}\Big](w_\alpha), \end{aligned}$$

which contradicts (4.7). Hence we have (4.6), as desired.

Finally, the implication (e) $\Longrightarrow$ (a) holds by Theorem 2.2. The proof is complete. ■

Before proceeding, we recall the well-known notion of Hankel operators. For $u \in L^\infty$, the *Hankel operator* H_u with symbol u is the operator on L^2_a defined by $H_u f = (I-P)(uf)$. The relation between Toeplitz operators and Hankel operators is given by the well-known identity: $(T_u, T_v] = H^*_{\overline{u}} H_v$. Using this identity, one can easily verify that the semi-commutator of Toeplitz operators T_u and T_v with harmonic symbols are represented as an integral operator:

$$\big(T_u, T_v\big] f(a) = \int_D \frac{\Lambda_{u,v}(z,a)}{(1-a\overline{z})^2} f(z)\, dA(z), \qquad a \in D$$

where

$$\Lambda_{u,v}(z,a) = \big[Pu(z) - Pu(a)\big]\big[\,\overline{P\overline{v}(z)} - \overline{P\overline{v}(a)}\,\big].$$

The kernel $\Lambda_{u,v}$ is closely related with $R_{u,v}$ in the sense that

$$\Lambda_{u,v}\big(\varphi_a(z), a\big) = R_{u,v}(z,a),$$

which can be seen by a little manipulation.

Proof of (b) $\Longrightarrow$ (f) $\Longrightarrow$ (e). First, we assume (b) and prove (f). We only need to prove the first part of (f). Suppose that the first part of (f) fails to hold. Then there is a net $\{w_\alpha\}$ in D converging to some $m \in \mathfrak{M} \setminus D$ such that

$$\limsup_{\alpha} \int_D \Big| \sum_{j=1}^{N} R_{u_j,v_j}(z, w_\alpha) \Big| \, dA(z) > 0. \tag{4.8}$$

Note that we have by Lemma 4.2

$$P(u \circ \varphi_{w_\alpha}) - u(w_\alpha) \longrightarrow P(u \circ L_m) - u \circ L_m(0) \qquad \text{in} \quad L^2_a$$

for each $u \in h^\infty$. Thus, applying this to functions u_j and $\overline{v_j}$, we obtain

$$\begin{aligned} &\sum_{j=1}^{N} R_{u_j,v_j}(\,\cdot\,, w_\alpha) \longrightarrow \\ &\qquad \sum_{j=1}^{N} \big[P(u_j \circ L_m) - u_j \circ L_m(0)\big] \big[\overline{P(\overline{v_j} \circ L_m)} - v_j \circ L_m(0)\big] \end{aligned} \tag{4.9}$$

in L^1. Meanwhile, we have by (b) and Theorem 3.7

$$\sum_{j=1}^{N} \big[P(u_j \circ L_m) - u_j \circ L_m(0)\big] \big[\overline{P(\overline{v_j} \circ L_m)} - v_j \circ L_m(0)\big] = 0,$$

which, together with (4.9), is a contradiction to (4.8).

Next, we assume (f) and prove (e). For each $r \in (0,1)$, define $S_r : L^2_a \to L^2$ by

$$S_r f(a) = \chi_{rD}(a) \int_D \frac{\Lambda(z,a)}{(1 - a\overline{z})^2} f(z) \, dA(z), \qquad a \in D$$

for $f \in L^2_a$ where $\Lambda = \sum_{j=1}^{N} \Lambda_{u_j,v_j}$ and χ_{rD} denotes the usual characteristic function of the set rD. Now, following the proof of of [8, Theorem 1] (or, easily modifying the proof of [6, Theorem 1.3]), one can verify that each S_r is compact and that

$$\Big\| \sum_{j=1}^{N} (T_{u_j}, T_{v_j}] - S_r \Big\|^2 \le C \sup_{a \in D \setminus rD} \left\{ \int_D \big|\Lambda\big(\varphi_a(z), a\big)\big| \, dA(z) \right\}^{1/14}$$

for some constant C independent of r.

Note that $\Lambda(\varphi_a(z), a) = \sum_{j=1}^{N} R_{u_j,v_j}(z,a)$. So, the operator $\sum_{j=1}^{N} (T_{u_j}, T_{v_j}]$ is approximated in the strong operator topology by compact operators, so it is compact. The proof is complete. ■

In case $N=1$ some characterizations in Theorem 4.3 are already known for more general λ. Namely, the conditions (a), (d) and (e) are known to be equivalent for general $\lambda \in \mathscr{A}$, and a version for general $\lambda \in L^\infty$ is also known; see [9, Theorem 4.4]. We do not know whether such general results hold for arbitrary N.

In case $\lambda = 0$ in Theorem 4.3, note that we have

$$\sum_{j=1}^N T_{u_j\circ L_m} T_{v_j\circ L_m} = 0 \iff \sum_{j=1}^N T_{v_j\circ L_m} T_{u_j\circ L_m} = 0$$

for $m \in \mathfrak{M} \setminus D$ by Theorem 3.8. Thus we have the following consequence of Theorem 4.3, which contains Theorem 1.2 and is the compact version of Theorem 3.8.

Theorem 4.4. *Let $u_1, \dots, u_N, v_1, \dots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\sum_{j=1}^N T_{u_j} T_{v_j}$ *is compact.*

(b) $\sum_{j=1}^N \widetilde{\Delta}[\overline{P\overline{u}_j} P v_j] \in C_0$ *and* $\sum_{j=1}^N u_j v_j \in C_0$.

(c) $\lim_{|a|\to 1} \int_D \left| \sum_{j=1}^N Q_{u_j,v_j}(z,a) \right| dA(z) = 0$ *and* $\sum_{j=1}^N u_j v_j \in C_0$.

Moreover, each of the above conditions is equivalent to the similar condition with u_j and v_j replaced by each other.

Remark. One may also directly prove the implication (c) $\Longrightarrow$ (a) in the above theorem as follows. Note that we have by (3.2), (2.3) and (2.6)

$$\sum_{j=1}^N \int_D Q_{u_j,v_j}(z,a)\, dA(z) = \sum_{j=1}^N \big(B[T_{u_j} T_{v_j}] - u_j v_j \big)(a)$$

for $a \in D$. So, assuming (c), we have $\sum_{j=1}^N \big(B[T_{u_j} T_{v_j}] - u_j v_j \big) \in C_0$. Combining this with the assumption $\sum_{j=1}^N u_j v_j \in C_0$, we have $\sum_{j=1}^N B\big[T_{u_j} T_{v_j}\big] \in C_0$. So, (a) holds by Theorem 2.1.

Another special case $\lambda = -\sum_{j=1}^N u_j v_j$ in Theorem 4.3 yields the following corollary for sums of finitely many semi-commutators, which is the compact version of Corollary 3.9.

Corollary 4.5. *Let $u_1, \ldots, u_N, v_1, \ldots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\displaystyle\sum_{j=1}^{N} \big(T_{u_j}, T_{v_j}\big]$ *is compact.*

(b) $\displaystyle\sum_{j=1}^{N} \widetilde{\Delta}\big[Pu_j\overline{P\overline{v_j}}\big] \in C_0$.

(c) $\displaystyle\lim_{|a|\to 1} \int_D \Big|\sum_{j=1}^{N} R_{u_j,v_j}(z,a)\Big|\, dA(z) = 0.$

For sums of finitely many commutators, one can use Theorem 4.4 or Corollary 4.5 to derive the following compact version of Corollary 3.10.

Corollary 4.6. *Let $u_1, \ldots, u_N, v_1, \ldots, v_N \in h^\infty$. Then the following statements are equivalent:*

(a) $\displaystyle\sum_{j=1}^{N} \big[T_{u_j}, T_{v_j}\big]$ *is compact.*

(b) $\displaystyle\sum_{j=1}^{N} \widetilde{\Delta}\big[\,\overline{P\overline{u_j}}\,Pv_j - Pu_j\overline{P\overline{v_j}}\big] \in C_0$.

(c) $\displaystyle\lim_{|a|\to 1} \int_D \Big|\sum_{j=1}^{N} Q_{u_j,v_j}(z,a) - R_{u_j,v_j}(z,a)\Big|\, dA(z) = 0.$

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