Rev. Mat. Iberoamericana **23** (2007), no. 3, 1125–1140

Smooth rough paths and applications to Fourier analysis

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Abstract

We show rough path estimates for smooth L^p functions whose derivatives are in L^q. We also give applications to Fourier analysis.

1. Introduction

Let $p, q \geq 1$ be real numbers and let n be a natural number. Suppose that $F : \mathbb{R} \to \mathbb{R}^n$ is a smooth path and that it satisfies the following integrability conditions:

$$\begin{aligned} \|F\|_p &= \|F\|_{p,\mathbb{R}} = \left\{ \int_{\mathbb{R}} |F(t)|^p dt \right\}^{1/p} < \infty, \\ \|F'\|_q &= \|F'\|_{q,\mathbb{R}} = \left\{ \int_{\mathbb{R}} \left| \frac{d}{dt} F(t) \right|^q dt \right\}^{1/q} < \infty. \end{aligned}$$

Does F have finite r-variation on $\mathbb{R}$ for some r? This is not a trivial question even when F is a smooth path. In fact, the length of the path can be infinite and the uniform estimate refers to functions defined on the whole real line $\mathbb{R}$. Our aim is to answer a general version of this question, that is, to estimate the rough path norm of F. In other words, we will show that there exist *smooth rough paths*. We will also show new conditions for the pointwise convergence of classical and nonlinear Fourier transforms. Our idea is to consider Fourier transforms as the limits of the solutions to the corresponding differential equations and to use the general framework of rough path theory to see how the limits behave.

In the next section, we will present the basic concepts of rough path theory; we refer for details to Lyons [2] and Lyons-Qian [3]. We will state our main theorems in Section 3 and 4; Sections 5 and 6 are devoted to applications.

2000 Mathematics Subject Classification: 26A45, 42A20.
Keywords: rough paths, Fourier transforms.

2. Basic definitions

We present the basic concepts of rough path theory. Though rough path theory works in a rather general framework, we will concentrate only on the essential definitions needed in our context.

Let $X(t) : I \to \mathbb{R}^n$ be a smooth function defined on a closed interval $I = [a, b]$ (where a or b may be infinite). We are interested in the oscillations of the iterated integrals of X:

$$X^i_{u,v} = \int_{u<t_1<\cdots<t_i<v} dX_{t_1} \otimes \cdots \otimes dX_{t_i} \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n, \quad i = 1, 2, \ldots$$

In general rough path theory, how to define such iterated integrals for non-smooth functions is a subtle issue. But, in our case, there is no problem because the function X is smooth.

The integrals are well-defined and satisfy an important algebraic relation, *i.e.*, Chen's identity (for details, see [3, p. 30]). Notice that $X^1_{u,v}$ is simply $X(v) - X(u)$.

First we need the definition of a control function.

Definition 1 *Let Δ_I denote the simplex $\{(u, v) : u \leq v,\ u, v \in I\}$. We call a bounded continuous function $\omega : \Delta_I \to [0, \infty)$ a* control function *if $\omega(u, u) = 0$ for all $u \in I$ and it is super-additive, i.e.,*

$$\omega(u, v) + \omega(v, w) \leq \omega(u, w)$$

for all $u \leq v \leq w$ $(u, v, w \in I)$.

We introduce a fixed norm $|\cdot|$ on $\mathbb{R}^n$ and we fix some compatible family of tensor norms on $\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$, also denoted by $|\cdot|$. Next we define our main concept.

Definition 2 *Let $r \geq 1$ be a constant and denote the integer part by $[r]$. We call $X(t)$ a (smooth) r-*rough path *(or simply, a* rough path*) if there exists a control function $\omega : \Delta_I \to [0, \infty)$ such that*

$$\left|X^i_{u,v}\right| \leq \omega(u, v)^{i/r} \tag{2.1}$$

for any $i = 1, \ldots, [r]$ and any $u \leq v \in I$.

Note that the first level estimate, *i.e.*, $|X(v) - X(u)| \leq \omega(u, v)^{1/r}$ means that X has finite r-variation on I, because we can sum up the oscillations for any partition $(u \leq u_0 < \cdots < u_k \leq v)$ of I by the super additivity:

$$\sum_{n=0}^{k-1} |X(u_{n+1}) - X(u_n)|^r \leq \sum_{n=0}^{k-1} \omega(u_n, u_{n+1}) \leq \omega(u, v) < \infty.$$

Rough path theory provides tools for estimating the r-variation of higher iterated integrals in terms of the r-variation of the lower iterated integral. With these estimates, one can define an integral and consider the differential equations driven by the rough paths (see [2], [3]).

Remark. We emphasize that in this paper we study smooth paths through rough path estimates. Rough path theory is usually applied to non-smooth, actually very rough functions, and we usually consider the r-variation of functions (or of their iterated integrals) on a finite interval. In this paper, we are working on the whole real line $I = \mathbb{R}$. The point is to control the behavior of the global oscillations, especially near infinity, using uniform estimates on ω.

3. The estimate for the variational norm

First we consider the first level estimate. Our goal is to show that, under the norm assumptions on F and F', there exists $C < \infty$ such that for any $(u, v) \in \Delta_{\mathbb{R}}$ and some $r \geq 1$,

$$|F(v) - F(u)|^r \leq \omega(u, v) < C.$$

The r-variation can be estimated weakly. Suppose that a smooth function $G : \mathbb{R} \to \mathbb{R}$ satisfies the estimates $\|G\|_p < \infty$ and $\|G'\|_q < \infty$. Let us write the function G as the sum of the positive part $G^+ := G \vee 0$ and the negative part $G^- := (-G) \vee 0$. Then, almost everywhere, we have that

$$(G^+)' = G' \quad \text{if } G > 0, \quad (G^+)' = 0 \quad \text{if } G \leq 0.$$

Since $G^+ \geq 0$ and $r \geq 1$, we have

$$\left|G^+(t) - G^+(s)\right|^r \leq \left|G^+(t)^r - G^+(s)^r\right|.$$

Then we can estimate the right hand side by means of an integral as follows:

$$\begin{aligned}
|G^+(t)^r - G^+(s)^r| &= \left|\int_{s<u<t} d\left\{G^+(u)^r\right\}\right| \\
&= \left|\int_{s<u<t} r\, G^+(u)^{r-1}\, (G^+)'(u)\, du\right| \\
&\leq r \int_{s<u<t} G^+(u)^{r-1} \left|(G^+)'(u)\right| du.
\end{aligned}$$

The negative part G^- also has the same estimate.

By Jensen's inequality,

$$\begin{aligned} |G(t)-G(s)|^r &\le \left(|G^+(t)-G^+(s)|+|G^-(t)-G^-(s)|\right)^r \\ &\le 2^{r-1}\left(|G^+(t)-G^+(s)|^r+|G^-(t)-G^-(s)|^r\right), \end{aligned}$$

so that

$$|G(t)-G(s)|^r \le r\,2^{r-1}\int_{s<u<t}|G(u)|^{r-1}\,|G'(u)|du.$$

It is an easy exercise in the Hahn-Banach Theorem to see that if the above estimate holds for escalar functions, it holds for Banach valued functions F.

Now, a simple application of Hölder's inequality gives that

$$\begin{aligned} \|F\|_{r\text{-var},[s,t]} &= \sup_{s<u_1<\cdots<u_m<t}\sum_{j=1}^{m-1}|F(u_{j+1})-F(u_j)|^r \\ &\le r2^{r-1}\int_{s<u<t}|F(u)|^{r-1}\,|F'(u)|\,du \\ &\le r2^{r-1}\left(\int_{s<u<t}|F(u)|^{(r-1)\alpha}\,du\right)^{1/\alpha}\left(\int_{s<u<t}|F'(u)|^{\beta}\,du\right)^{1/\beta}, \end{aligned}$$

where $1/\alpha+1/\beta=1$.

Therefore the path F has finite r-variation on $\mathbb{R}$ if $p=(r-1)\alpha$ and $q=\beta$, because

$$\begin{aligned} \|F\|_{r\text{-var}} = \|F\|_{r\text{-var},\mathbb{R}} &\le r2^{r-1}\left\{\int_{-\infty}^{\infty}|F(u)|^p du\right\}^{1/\alpha}\left\{\int_{-\infty}^{\infty}|F'(u)|^q du\right\}^{1/\beta} \\ &= r2^{r-1}\|F\|_p^{r-1}\|F'\|_q<\infty. \end{aligned}$$

Note that $(r-1)/p+1/q=1$.

Then, setting

$$\omega(s,t)=r2^{r-1}\left\{\frac{r-1}{p}\|F\|_{p,[s,t]}^p+\frac{1}{q}\|F'\|_{q,[s,t]}^q\right\},$$

ω is a control function for the r-variation of F by the elementary inequality:

$$x^{1/a}y^{1/b}\le\frac{x}{a}+\frac{y}{b}$$

for any $x,y\ge 0$ and $1/a+1/b=1$, $a>1$.

We can also check that the estimate holds for $p=\infty$ or $q=\infty$, if we use the supremum norm $\|\cdot\|_\infty$. But the estimate is nonsense for $p=q=\infty$.

We summarize the above argument in the following theorem:

Theorem 3 *Let $1 < p, q < \infty$. If $F \in L^p(\mathbb{R}^n)$ and $F' \in L^q(\mathbb{R}^n)$, then there exists a control function ω on $\Delta_{\mathbb{R}}$ such that*

$$\left| \int_{s<u<t} dF(u) \right|^r \leq \omega(s,t), \quad \text{for} \quad r = p\Big(1 - \frac{1}{q}\Big) + 1,$$

and

$$\omega(-\infty, \infty) \leq r\, 2^{r-1} \bigg(\Big(1 - \frac{1}{q}\Big) \|F\|_p^p + \frac{1}{q} \|F'\|_q^q \bigg)$$

In particular, $\|F\|_{r\text{-var}} < \infty$. The result also holds for $r = 2$ if $p = \infty$ and $q = 1$, and for $r = p + 1$ if $q = \infty$.

Moreover, if $r < 2$, i.e., if $\frac{1}{p} + \frac{1}{q} > 1$, then F is a $[r]$-rough path, i.e., a 1-rough path.

Observe that if $r < 2$, then it is already well known from [2], [3] that one can establish estimates on the variation of high order integrals.

4. The estimate for the area process

Next we want to discuss a second level estimate.

Observe that for $1/p + 1/q < 1$, no estimate is in general possible, essentially because the integral

$$\int_{-\infty}^{\infty} F(u) F'(u) du$$

may be infinite. For example, consider the special case of $p = q > 2$ and the following function $H : \mathbb{R} \to \mathbb{R}^2$:

$$H(t) = (H_1(t),\, H_2(t)) = (R(t) \cos t,\, R(t) \sin t),$$

where both the radius $R(t) : \mathbb{R} \to \mathbb{R}$ and the derivative $R'(t)$ are in L^p for some $p > 2$, but $R(t)$ is not in L^2. Then, we can easily see that H satisfies our condition, i.e., $\|H\|_p + \|H'\|_p < \infty$. However, the area can explode to infinity:

$$\left| \int_{-\infty}^{\infty} (H_2 dH_1 - H_1 dH_2) \right| = \int_{-\infty}^{\infty} R(t)^2 dt = \infty$$

and we can not have finite r-variational norm for any r.

Our aim in this paper is to provide an estimate for the area when $1/p + 1/q = 1$. We have the following theorem:

Theorem 4 *Let $1/p + 1/q = 1$, $p > 1$. If $F \in L^p(\mathbb{R}^n)$ and $F' \in L^q(\mathbb{R}^n)$, then the estimate for the area*

$$\left| \iint_{s<u<v<t} dF(u) \otimes dF(v) \right| \le C\omega(s,t), \quad (-\infty \le s < t \le \infty)$$

holds for a constant C and the same control function ω as in Theorem 3. In particular, the function F is a 2-rough path.

It is enough to estimate the antisymmetric tensor component of

$$\iint_{s<u<v<t} dF(u) \otimes dF(v),$$

since the symmetric component is

$$\frac{1}{2}(F(s) - F(t)) \otimes (F(s) - F(t))$$

and this is already controlled using the argument in section 3. To control the injective tensor norm on $\iint_{s<u<v<t} dF(u) \otimes dF(v)$, it is enough to consider all projections onto planar paths. So, without loss of generality, we will assume that F takes its values in $\mathbb{R}^2$. Clearly the result generalizes (by changing C) to any tensor norm if F takes its values in a finite dimensional Banach space. In the infinite dimensional case, our estimates work for the injective tensor norm.

Suppose that a smooth path Z in $\mathbb{R}^2$ satisfies the condition $\|Z\|_p + \|Z'\|_q < \infty$. We want to estimate the off-diagonal part

$$A_{st} = \frac{1}{2} \int_{s<v<t} (Z(v) - Z(s)) \times dZ(v),$$

where "$\times$" means that $(x, y) \times (z, w) = xw - yz$ for (x, y) and $(z, w) \in \mathbb{R}^2$.

Let πZ be the projection of Z onto the line from $Z(s)$ to $Z(t)$ along the line from z to 0. Then we have

$$\begin{aligned} &\int_{s<v<t} \pi Z(v) \times d(\pi Z)(v) \\ &\quad = \int_{s<v<t} (\pi Z(v) - Z(s)) \times d(\pi Z)(v) + \int_{s<v<t} Z(s) \times d(\pi Z)(v) \\ &\quad = \int_{s<v<t} Z(s) \times d(\pi Z)(v) = \int_{s<v<t} Z(s) \times dZ(v), \end{aligned}$$

since $(\pi Z(v) - Z(s))$ and $d\pi Z(v)$ are collinear. So we have

$$2A_{s,t} = \int_{s<v<t} Z(v) \times dZ(v) - \int_{s<v<t} \pi Z(v) \times d(\pi Z)(v).$$

The first term of the right hand side is easy to handle as follows:

$$\begin{aligned}\int_{s<v<t} |Z(v)\times dZ(v)| &\le \int_{s<v<t} |Z(v)|\,|Z'(v)|\,dv\\ &\le \left\{\int_{s<v<t} |Z(v)|^p\,dv\right\}^{1/p}\left\{\int_{s<v<t} |Z'(v)|^q\,dv\right\}^{1/q}\\ &\le \omega(s,t)\,.\end{aligned}$$

The difficulty lies in the second part, i.e.,

$$\int_{s<v<t} \pi Z(v)\times d(\pi Z)(v) = \int_{s<v<t} Z(s)\times dZ(v) = Z(s)\times\{Z(t)-Z(s)\}\,.$$

There are two cases. Either

$$|Z(s)\times(Z(t)-Z(s))| \le \theta\,\omega(s,t)$$

or

$$|Z(s)\times(Z(t)-Z(s))| > \theta\,\omega(s,t)$$

for some constant θ. The first case poses no difficulty because it is already controlled by $\omega(s,t)$. For the second case, remember the first level control, i.e., for any $u\in[s,t]$,

$$|Z(u)-Z(s)| \le C\omega(s,u)^{1/2} \le C\omega(s,t)^{1/2}.$$

Then the point $Z(u)$ is relatively near to $Z(s)$; the path $\{Z(u)\,(s\le u\le t)\}$ is well separated from the origin, and this fact should ensure that the triangle area can be estimated by the control function. More precisely, we will show that if $|Z(u)-Z(s)|\le C\omega(s,t)^{1/2}$ and $|Z(s)\times(Z(t)-Z(s))\,| > \theta\omega(s,t)$, then $|\pi Z(v)\times d\pi Z(v)| < C|Z(v)\times dZ(v)|$.

Set $\theta=2$, δ to be $\omega(s,t)$, $\alpha = Z(s)$, $\beta = Z(t)-Z(s)$ and $\gamma = Z(u)$. Then the following lemma, because it does not impose an upper bound on $|\alpha|$, gives the estimate we require:

Lemma 5 *Suppose that α and β are vectors emanating from the origin O and that l is the line through α parallel to β. Then let $\pi\gamma$ be projection of γ onto l through zero. That is to say*

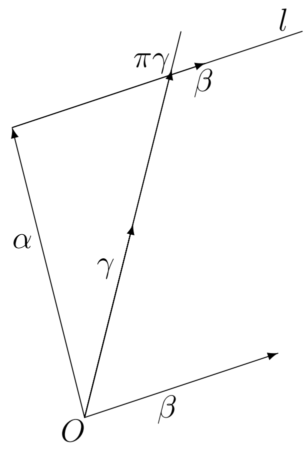
$$\begin{aligned}\pi\gamma &= r\gamma,\\ \beta\times(\alpha-r\gamma) &= 0.\end{aligned}$$

Now suppose that $|\beta\times\alpha|>\delta^2$, $|\beta|\le\delta/2$, and $|\gamma-\alpha|<\delta/2$.

Then, for an infinitesimal increment $d\gamma$ at γ, we have

$$|\gamma\times d\gamma| \ge C\,|\pi\gamma\times d\pi\gamma|\,,$$

where C is a universal constant.



Proof. Choose a unit vector $\hat{\beta}$ so that $\hat{\beta}\cdot\beta = 0$. (The symbol "$\cdot$" means the inner product.) Then

$$\begin{aligned}
\pi\gamma &= \frac{\hat{\beta}\cdot\alpha}{\hat{\beta}\cdot\gamma}\gamma, \\
d\pi\gamma &= \frac{\left(\hat{\beta}\cdot\gamma\right)\left(\hat{\beta}\cdot\alpha\right)d\gamma - \left(\hat{\beta}\cdot\alpha\right)\left(\hat{\beta}\cdot d\gamma\right)\gamma}{\left(\hat{\beta}\cdot\gamma\right)^2} \\
&= \frac{\hat{\beta}\cdot\alpha\left\{\left(\hat{\beta}\cdot\gamma\right)d\gamma - \left(\hat{\beta}\cdot d\gamma\right)\gamma\right\}}{\left(\hat{\beta}\cdot\gamma\right)^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\pi\gamma\times d\pi\gamma &= \frac{\hat{\beta}\cdot\alpha}{\hat{\beta}\cdot\gamma}\gamma\times\frac{\left(\hat{\beta}\cdot\gamma\right)\left(\hat{\beta}\cdot\alpha\right)d\gamma - \left(\hat{\beta}\cdot\alpha\right)\left(\hat{\beta}\cdot d\gamma\right)\gamma}{\left(\hat{\beta}\cdot\gamma\right)^2} \\
&= \frac{\hat{\beta}\cdot\alpha}{\hat{\beta}\cdot\gamma}\gamma\times\frac{\left(\hat{\beta}\cdot\gamma\right)\left(\hat{\beta}\cdot\alpha\right)d\gamma}{\left(\hat{\beta}\cdot\gamma\right)^2}
\end{aligned}$$

and using the homogeneity and orthogonality of $\hat{\beta}$ and β one has this

$$\begin{aligned}
&= \frac{\left(\hat{\beta}\cdot\alpha\right)^2}{\left(\hat{\beta}\cdot\gamma\right)^2}\gamma\times d\gamma \\
&= \frac{|\beta\times\alpha|^2}{|\beta\times\gamma|^2}\gamma\times d\gamma.
\end{aligned}$$

We want to estimate of the ratio of the above equation, i.e., $|\beta\times\alpha|^2/|\beta\times\gamma|^2$. We do not have an upper bound on $|\alpha|$.

Let Θ be the angle between α and β. By hypothesis

$$\delta^2 < |\beta \times \alpha| = |\beta|\,|\alpha|\,|\sin\Theta|,$$

and $|\beta| \leq \delta/2$, so dividing the expression above by $|\alpha|\,\delta^2$,

$$\frac{1}{|\alpha|} \leq \frac{|\sin\Theta|\ |\beta|}{\delta^2} \leq \frac{|\sin\Theta|}{2\delta} \leq \frac{1}{2\delta}.$$

On the other hand, we have $|\alpha| \geq 2\delta$ and $|\gamma - \alpha| < \delta/2$, so

$$1 - \frac{1}{4} = \frac{3}{4} \leq \frac{|\gamma|}{|\alpha|} \leq \frac{5}{4} = 1 + \frac{1}{4}.$$

Next let Φ be the angle between α and γ. Then we can estimate it simply,

$$\begin{aligned} |\alpha|\ |\sin\Phi| &\leq |\gamma - \alpha| \\ |\sin\Phi| &\leq \frac{\delta}{2\,|\alpha|}. \end{aligned}$$

Then, we have

$$|\sin\Phi| \leq \frac{\delta}{2|\alpha|} \leq \frac{|\sin\Theta|}{4} \leq \frac{1}{4}.$$

Now we can estimate the angle Ψ between β and γ with the estimates above. Note that the inequality

$$\sin(a+b) \leq \sin a + \sin b$$

holds for positive a and b, providing $\max[a,b] \leq \pi/2$. Therefore we have

$$\frac{3}{4}\,|\sin\Theta| \leq |\sin\Psi|.$$

Putting it together, we get

$$\frac{(\beta\times\alpha)^2}{(\beta\times\gamma)^2} = \left(\frac{|\alpha|\sin\Theta}{|\gamma|\sin\Psi}\right)^2 \leq \left(\frac{4}{3}\right)^4.$$

Hence

$$\pi\gamma \times d\pi\gamma = C_{\gamma,d\gamma}\,\gamma\times d\gamma$$

where $|C_{\gamma,d\gamma}| \leq \left(\frac{4}{3}\right)^4$. The estimate is patently uniform and so proves the lemma. ∎

Let us return to the proof of the theorem. From Lemma 5 we have the following slightly surprising estimate:

$$\left|\int_{s<v<t} \pi Z(v) \times d(\pi Z)(v)\right| \leq \left(\frac{4}{3}\right)^4 \int_{s<v<t} |Z(v) \times Z'(v)|\, dv.$$

Now we can estimate the area process as follows.

$$\begin{aligned} |A_{st}| &= \frac{1}{2}\left|\int_s^t Z(u) \times Z'(u)du - \int_s^t \pi Z(u) \times d(\pi Z)(u)\right| \\ &\leq \frac{1}{2}\left|\int_s^t Z(u) \times Z'(u)du\right| + \frac{1}{2}\left|\int_s^t \pi Z(u) \times d(\pi Z)(u)\right| \\ &\leq \frac{1}{2}\left(1 + \max\left\{1, \left(\frac{4}{3}\right)^4\right\}\right) \int_s^t |Z(u) \times Z'(u)|\, du \\ &=: C \int_s^t |Z(u) \times Z'(u)|\, du \\ &\leq C\omega(s,t). \end{aligned}$$

And the theorem is proved.

5. Applications to Fourier analysis

5.1. Pointwise convergence of Fourier transform

First we show the essential idea by means of the following simple example. Let f be a function on $\mathbb{R}$ and $\hat{f}$ be the Fourier transform. Recall that if

$$g(k) = \widehat{\left(\frac{f(x)}{x}\right)}(k),$$

then we have

$$g'(k) = c\hat{f}(k),$$

where c is a constant. By our first theorem, the path g has finite global p-variation if the function g and its derivative g' are in L^p $(p \geq 1)$. (Note that $r = p(1 - 1/p) + 1 = p$.) Therefore $g(R)$ has a limit as $R \to \pm\infty$ (and the limit is zero).

Since

$$g(R) - g(R') = \int_{R'}^{R} g'(u)du$$

for any $R' < R$, we have that

$$\int_{R'}^{R} e^{ik0}\hat{f}(k)dk = \int_{R'}^{R} \hat{f}(k)dk$$

has a limit as $R \to \infty$ and $R' \to -\infty$, so we have the inverse Fourier transform at $x = 0$.

In the special case $p = 2$, since g and g' are in L^2 if and only if f and $f(x)/x$ are in L^2, we have a verifiable condition for the pointwise convergence of the inverse Fourier transform.

5.2. A non-commutative version of the pointwise convergence problem

We can use rough path theory to easily prove the existence of limits in some nonlinear oscillating problems which we describe below. For this nonlinear case, we will need to control not only the variation of the path, but also the area (see [2] and [3]). In the case of $p = q = 2$, we have the second level rough path estimate by Theorem 4.

Let γ be a complex valued function, *i.e.*, $\gamma : \mathbb{R} \to \mathbb{C}$, that satisfies our conditions $\gamma, \gamma' \in L^2$. Suppose that $\mathbb{C} \hookrightarrow \mathcal{G}$, where $\mathcal{G}$ is a Lie algebra of a Lie group G. We can compute

$$\Gamma_{R,R'} = \overrightarrow{\exp} \int_{R'}^{R} d\gamma$$

(*i.e.*, the solution of the differential equation on the Lie group: $(d/dR)\Gamma_{R,R'} = \gamma'_R$). The limit exists as $R \to \infty, R' \to -\infty$. Therefore, if a complex valued function f on $\mathbb{R}$ satisfies f and $f(x)/x \in L^2$, the limit of $\overrightarrow{\exp} \int_{R'}^{R} \hat{f}$ exists.

Though this example may seem artificial, we will see its relevance in Section 6.

5.3. Radial behavior of a harmonic function in the unit disc

Let $f(z) = f(e^{i\theta})$ be a function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \simeq \{0 \leq \theta \leq 2\pi\}$ and consider the harmonic function $\tilde{f}$ on the disk $D = \{|z| < 1\}$ whose boundary data is f. Let P_r be the Poisson kernel operator, so that $P_r f(e^{i\theta}) = \tilde{f}((1-r)e^{i\theta})$. If we write f as the Fourier series $\sum_n a_n e^{in\theta}$, the classical Fourier theory says that the Abel sum $P_{1-r} f(e^{i\theta}) = \sum a_n r^{|n|} e^{in\theta}$ converges to $f(\theta)$ as $r \nearrow 1$ under suitable conditions for f on $\mathbb{T}$.

Suppose that $f \in L^2(T)$, *i.e.*, $\sum_n |a_n|^2 < \infty$. The following estimate for the square of the derivative $\nabla \tilde{f}$ holds by Green's theorem:

$$\int_D |\nabla \tilde{f}(z)|^2 \log \frac{1}{|z|} dz \leq \int_{\mathbb{T}} |f(e^{i\theta}) - f(0)|^2 d\theta < \infty, \tag{5.1}$$

where $f(0)$ is the mean value

$$1/(2\pi) \int_{\mathbb{T}} f(e^{i\theta}) d\theta$$

(see Garnett [1]). This implies that the gradient of $\tilde{f}$ is in L^2 in the hyperbolic geometry along almost all radial geodesics. Therefore, we can apply our theorem to obtain the rough path property if we have the corresponding L^2 estimate for the function itself in the hyperbolic geometry, namely

$$\int_{1/2}^{1} \left|\tilde{f}(re^{i\theta}) - f(e^{i\theta})\right|^2 \frac{dr}{1-r}, \quad \text{a.s. } \theta,$$

or more directly,

$$\int_{0}^{1/2} |Q_s f - f|^2 \frac{ds}{s} < \infty, \quad \text{a.s. } \theta,$$

where $Q_s f = \sum_n a_n e^{-s|n|} e^{in\theta}$.

If f is not only in L^2 but it is in the log Sobolev space on the boundary, which is a slightly stronger requirement, we can verify the needed estimate as follows:

Theorem 6 *If the boundary data f is in the log Sobolev space on $\mathbb{T}$, i.e., $\sum_n |a_n|^2 \log |n| < \infty$, then the restriction of f to almost every radius is a 2-rough path.*

Proof. Now we only need to estimate the following:

$$\begin{aligned} \int_0^{2\pi} d\theta \int_0^1 ds \frac{|Q_s f - f|^2}{s} &= \int_0^{2\pi} d\theta \int_0^1 \frac{ds}{s} \left| \sum_n a_n (e^{-s|n|} - 1) e^{in\theta} \right|^2 \\ &= \sum_n |a_n|^2 \int_0^1 \frac{(1 - e^{-s|n|})^2}{s} ds. \end{aligned}$$

We can dominate $1 - e^{-s|n|}$ by $s|n|$ in $[0, 1/|n|]$ and by 1 in $[1/|n|, 1]$, so that

$$\int_0^1 \frac{(1 - e^{-s|n|})^2}{s} \leq \int_0^{1/|n|} \frac{s^2 n^2}{s} ds + \int_{1/|n|}^1 \frac{ds}{s} = n^2 \frac{1}{2n^2} + \log |n| = \frac{1}{2} + \log |n|.$$

Therefore, if $\sum_n |a_n|^2 \log |n| < \infty$, we have

$$\int_0^1 \frac{|Q_s f - f|^2}{s} ds < \infty, \quad \text{a.e. } \theta.$$

By Theorem 4, this estimate and (5.1) mean that almost all radial behaviors $r \mapsto \tilde{f}(re^{i\theta})$ are rough paths. ■

Note that the rough path property is independent of the geometry of the space because it is defined by the supremum over all choices of the partition. This theorem ensures that $\tilde{f}$ has a limit as $r \to 1$ because it is a rough path. Therefore we should be able to study the differential equation driven by $\tilde{f}$ along the radius by rough path theory.

6. A further problem –rough path approach to nonlinear Fourier analysis

In this section, we will roughly explain the concept of the nonlinear Fourier transform according to T. Tao and C. Thiele ([6]), and show our idea to study it by rough path theory.

We define the nonlinear Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ under reasonable conditions.

Let us define the matrix-valued Dirac operator $L = L(f)$ by

$$L = \begin{pmatrix} D & -\bar{f} \\ f & -D \end{pmatrix},$$

where $D = d/dt$ is the differentiation operator. We consider the following eigenfunction problem:

$$L \begin{pmatrix} \phi(k;t) \\ \psi(k;t) \end{pmatrix} = ik \begin{pmatrix} \phi(k;t) \\ \psi(k;t) \end{pmatrix}.$$

Note that k is a real number because the operator L is anti-selfadjoint. We suppose that the eigenfunctions have some regularity and the boundary conditions $\psi(k;t) \sim e^{ikt}, \psi(k;t) \sim 0$ as $t \to +\infty$.

It is natural to set

$$\phi(k;t) = a(k;t)e^{ikt} \quad \text{and} \quad \psi(k;t) = b(k;t)e^{-ikt},$$

considering the free case $f \equiv 0$. Then the equation becomes

$$\frac{d}{dt}a(k;t) = b(k;t)\overline{f(t)}e^{-2ikt}, \tag{6.1}$$

$$\frac{d}{dt}b(k;t) = a(k;t)f(t)e^{2ikt} \tag{6.2}$$

with the boundary conditions $a(k;+\infty) = 1$ and $b(k;+\infty) = 0$. We define the nonlinear Fourier transform of f as

$$\widehat{f}\,[k] = \begin{pmatrix} a(k;-\infty) \\ b(k;-\infty) \end{pmatrix},$$

if the solutions of (6.1) and (6.2) exist.

It is not difficult, at least heuristically, to see that if f is small, then

$$a(k;-\infty) \sim 1 + \frac{1}{2}|\hat{f}[k]|^2; \quad b(k;-\infty) \sim \hat{f}[k],$$

where $\hat{f}$ is the usual Fourier transform. In this sense, the classical Fourier transform is a linearization of the nonlinear Fourier transform at the origin. One can also show that the nonlinear Fourier transform shares quite many properties with the linear one: linearity, homogeneity, symmetries, Riemann-Lebesgue lemma, etc. The most important property is the following nonlinear Plancherel formula:

$$\int_{-\infty}^{\infty} \log(1 + |b(k; -\infty)|^2) dk = \pi \int_{\mathbb{R}} |f(t)|^2 dt.$$

In principle, this formula should allow the extension of the definition of the nonlinear Fourier transform to the whole of L^2 (this relation holds at least for any compactly supported f). However, this approach to define the transform for all L^2 functions has not been completed. For partial results and further arguments, see Sylvester and Winebrenner [4] and also [5].

Next, we explain our approach to the study of the differential equations (6.1) and (6.2) via rough path estimates. We rewrite those equations as follows:

$$d\begin{pmatrix} \overline{a(k;t)} \\ b(k;t) \end{pmatrix} = \overline{\begin{pmatrix} \dfrac{b(k;t)}{a(k;t)} \end{pmatrix}} e^{ikt} dF(t), \quad (\, F'(t) = f(t)\,), \tag{6.3}$$

or, equivalently, in real Euclidean space terms, as

$$\begin{pmatrix} a_1 \\ -a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ -b_2 & b_1 \\ a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} d \begin{pmatrix} \widetilde{F_1}(t) \\ \widetilde{F_2}(t) \end{pmatrix}, \tag{6.4}$$

where

$$d\begin{pmatrix} \widetilde{F_1}(t) \\ \widetilde{F_2}(t) \end{pmatrix} = \begin{pmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{pmatrix} d \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix}$$

and $F(t) = F_1(t) + iF_2(t)$ with real valued functions F_1 and F_2.

This expression suggests the possibility of studying the nonlinear Fourier transform through the input "rough signal" $d\widetilde{F} = e^{ikt}dF$ with rough path theory. In fact, if the input signal is a rough path, the output (the solution of the differential equation) is also a rough path under reasonable conditions, because of the fundamental theorem of rough paths (see [2], [3]); and if so, the limit at infinity would exist. This provides precise conditions for the pointwise convergence of the nonlinear Fourier transform.

Theorem 7 *If $F \in L^p$ and $f \in L^q$ for $1/p + 1/q \geq 1$, then the nonlinear Fourier transform of f at $k = 0$ is well defined*

(cf. Section 5.2. We need a rough path property here; simple r-variation is insufficient).

We have also the condition for $k \neq 0$; we can use the modulation symmetry of the nonlinear Fourier transform:

$$\widehat{F}[k - k_0] = \widehat{F_{k_0}}[k],$$

where $F_{k_0}(x) = F(x)e^{-2ik_0 x}$.

But this argument states only a sufficient condition for the existence of the transform at each point k. Therefore we want to know a general condition to ensure that the input signal is a rough path (for almost every k). In other words, we need to know the rough path property of the *classical* Fourier transform to study the nonlinear Fourier transform. For example, we want to know a reasonable condition for F to ensure that

$$\left| \int_s^t e^{iku} dF(u) \right|^2 \leq \omega(s,t), \tag{6.5}$$

$$\left| \iint_{s<u<v<t} e^{iku} dF(u) \otimes e^{ikv} dF(v) \right| \leq \omega(s,t), \tag{6.6}$$

for any $-\infty \leq s < t \leq \infty$. However, as the almost everywhere pointwise convergence of the classical Fourier series for L^2 functions is hard to establish, the above problem should not be easy.

Another approach to get such a general condition should be to extend the definition with a complex number. Let us consider the same differential equation (6.3) with the complex parameter $z = k + im$ instead of the real k. Then, the integrand becomes $e^{iz} = e^{ikt}e^{-mt}$, which is a rough path of $t \in [0, \infty]$ on the upper half plane $\{z = k + im : m > 0\}$.

Therefore, by general rough path theory, if the driving signal F is a rough path, the solution of the differential equation is also a rough path. More precisely, because e^{-mt} has bounded variation on $[0, \infty]$, the pair (F, e^{-mt}), the truncated signatures and the projection like the twice iterated integral

$$\iint d(e^{-mt})dX$$

are rough paths. Therefore, we can now consider the limit as $t \to +\infty$, and the limit exists. In a similar way, we can go back to $-\infty$ through the lower half plane. It means that we can define a general concept of the non-linear Fourier transform of F as a pair of two rough paths (or the limits at $\pm\infty$) on the upper and lower half planes. However, we do not know when this extended definition coincides with the reasonable $\widehat{F}[k]$ for real k.

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Recibido: 17 de abril de 2006

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