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C^1 extensions of functions and stabilization of Glaeser refinements

Bo'az Klartag and Nahum Zobin

Abstract

Given an arbitrary set $E \subset \mathbb{R}^n$, $n \geq 2$, and a function $f : E \to \mathbb{R}$, consider the problem of extending f to a C^1 function defined on the entire $\mathbb{R}^n$. A procedure for determining whether such an extension exists was suggested in 1958 by G. Glaeser. In 2004 C. Fefferman proposed a related procedure for dealing with the much more difficult cases of higher smoothness. The procedures in question require iterated computations of some bundles until the bundles stabilize. How many iterations are needed? We give a sharp estimate for the number of iterations that could be required in the C^1 case. Some related questions are discussed.

1. Introduction

In 1934 Hassler Whitney published three ground-breaking papers [10, 11, 12], all dealing with various aspects of extending a function defined on a subset of $\mathbb{R}^n$ to a smooth function on the whole $\mathbb{R}^n$.

In [11] Whitney gave a complete description of traces of C^m functions on an arbitrary compact subset E of the real axis $\mathbb{R}$. It is well known that C^m functions on $\mathbb{R}$ are characterized by continuity properties of their m-th divided differences. These properties are obvious necessary conditions for extendability of f to a C^m function on $\mathbb{R}$. The fundamental result of [11] asserts that these conditions are also sufficient for such an extension to exist.

A generalization of this result to higher dimensions turned out to be very difficult. Only in 1958 was a significant progress made – G. Glaeser [8] proved an analog of Whitney's Theorem for C^1 functions on $\mathbb{R}^n$, $n > 1$. Glaeser introduced the notion of a paratangent bundle which gave him the tools to tackle the problem in the case of smoothness one.

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There was virtually no progress along these lines until 2002 when Bierstone, Milman and Pawłucki [1] proved an analog of Whitney's Theorem for higher dimensions and higher smoothness, but only for an important special class of sets – subanalytic subsets E of $\mathbb{R}^n$. They introduced the notion of iterated paratangent bundles which allowed them to formulate and prove the result. See also [2] for further developments.

A different way of attacking the problem – based on Lipschitz selections of set-valued mappings – was suggested and pursued by Yu. Brudnyi and P. Shvartsman (see, e.g., [3, 4]). Their methods allowed them to settle various cases of smoothness $2-\varepsilon$ (continuously differentiable functions with Lipschitz-type conditions on the first derivatives).

A related Whitney problem, discussed in [12] – description of open sets $E \subset \mathbb{R}^n$, allowing extension of C^{m-1} functions with bounded derivatives of order m to functions of the same class on $\mathbb{R}^n$ – was solved by the second author [13, 14] using quite different methods.

An impressive breakthrough was achieved in 2003-2005 by Charles Fefferman. In a series of papers – [5, 6, 7] and others – Fefferman developed a powerful approach which allowed him to prove a series of fundamental results on Whitney problems and their far reaching generalizations.

In particular, he gave a remarkable extension of Whitney's Theorem to higher dimensions and higher degrees of smoothness – for an arbitrary compact subset of $\mathbb{R}^n$. This constitutes a solution to an old, fundamental problem of Whitney.

A key ingredient in Fefferman's description of traces of $C^m(\mathbb{R}^n)$ functions on compact subsets of $\mathbb{R}^n$ was the notion of a Glaeser refinement. This notion, introduced by Fefferman, is related to the notions of paratangent and iterated paratangent bundles, introduced by Glaeser [8] and Bierstone–Milman–Pawłucki [1, 2].

In this article we study Glaeser refinements, in the case of smoothness one. We are especially interested in their stabilization properties, and we substantially improve some earlier results in this direction. Our main result is a construction of a set $E \subset \mathbb{R}^n$ such that a special bundle, closely related to C^1 extensions, needs "almost maximal possible" number of refinements until it stabilizes. We also present accompanying results, as well as another proof of Glaeser's Extension Theorem.

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2. Preliminaries

2.1. Bundles and sections

Let $\mathcal{P}_n^m$ denote the space of m-**jets** on $\mathbb{R}^n$ (= the space of polynomials on $\mathbb{R}^n$ of total degree $\leq m$). For any $x \in \mathbb{R}^n$ we have a ring structure on $\mathcal{P}_n^m$, we define $\mathcal{P}_n^m(x)$ to be the quotient ring of the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ over the ideal of polynomials vanishing at x, together with all derivatives of total order $\leq m$.

Let $E \subset \mathbb{R}^n$ be a compact set, and for any $x \in E$ let $H(x) \subset \mathcal{P}_n^m(x)$ be a (non-empty) affine subspace. With some abuse of standard terminology, we call the collection $\{H(x)\}_{x\in E}$ a **bundle** $H(E)$ of m-jets over E. We shall also use the term "m-bundle" for $H(E)$. For $x \in E$, the set $H(x)$ is called the **fiber** of $H(E)$ at x. In [7] the affine space $H(x)$ is always a coset over an ideal in $\mathcal{P}_n^m(x)$.

A **section** of an m-bundle $H(E)$ is a C^m-function $F : \mathbb{R}^n \to \mathbb{R}$ such that at each point $x \in E$ we have

$$J_x^m F \in H(x).$$

Here $J_x^m F$ denotes the m**-jet of the function** F at the point x, i.e.

$$(J_x^m F)(z) = \sum_{\beta\in\mathbb{Z}_+^n, |\beta|\leq m} (\partial^\beta F)(x)\frac{(z-x)^\beta}{\beta!}.$$

Here $|\beta|$ stands for the sum of coordinates of $\beta \in \mathbb{Z}_+^n$.

An m-bundle whose fibers are linear (and not merely affine) subspaces of $\mathcal{P}_n^m$, is called a **homogeneous** m**-bundle**. A homogeneous bundle always admits a section – the zero function.

2.2. Extension problems and standard bundles

All extension problems considered by Whitney (and many related problems, see [7]) can be reformulated in terms of the existence of a section of a suitable bundle. For example, the problem of extending a function $f : E \to \mathbb{R}$ to a function from $C^m(\mathbb{R}^n)$ translates into the problem of existence of a section of the m-bundle $H_f(E)$, whose fibers are defined as follows:

$$H_f(x) = \{P \in \mathcal{P}_n^m(x) : P(x) = f(x)\}. \tag{2.1}$$

We call $H_f(E)$ a **standard** m**-bundle**. In [7] the space $H_f(x)$ is called a trivial holding space for f. In this paper we mostly deal with standard 1-bundles. These bundles, or holding spaces, are closely related to problems of C^1 extensions.

We also consider particular standard m-bundles $h(E)$ associated with the function $f = 0$, i.e., bundles with fibers

$$h(x) = \{P \in \mathcal{P}_n^m(x) : P(x) = 0\}.$$

These are obviously homogeneous bundles – fibers of these bundles are linear subspaces (even ideals) in $\mathcal{P}_n^m(x)$. We call such bundles **homogeneous standard m-bundles**.

2.3. Glaeser refinements

Let us recall Fefferman's definition of Glaeser refinements [7].

Let $|\cdot|$ be the standard Euclidean norm in $\mathbb{R}^n$. Let $B(x,\delta)$ denote the open ball in $\mathbb{R}^n$, of radius δ, centered at x.

Let us note that we use the same symbol $|\cdot|$ to denote several similar but different things – the standard Euclidean norm in $\mathbb{R}^n$, the absolute value in $\mathbb{R}$, the sum of coordinates of a vector in $\mathbb{Z}^n$. It is always clear from the context what is meant so we hope this does not cause any confusion.

Definition 2.1. *Given an m-bundle $H(E)$, an integer $k \geq 1$, and $x_0 \in E$, define the set $H'_k(x_0)$ as follows:*

An m-jet P_0 belongs to $H'_k(x_0)$ if and only if $P_0 \in H(x_0)$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, \ldots, x_k \in E \cap B(x_0,\delta)$ there exist $P_1 \in H(x_1), \ldots, P_k \in H(x_k)$ such that $\forall\, i,j,\ 0 \leq i,j \leq k,\ \forall\, \alpha \in \mathbb{Z}_+^n$,

$$|\alpha| \leq m, \quad |\partial^\alpha (P_i - P_j)(x_j)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|}. \tag{2.2}$$

The set $H'_k(x_0)$, if non-empty, is an affine subspace of $\mathcal{P}_n^m(x_0)$.

If the sets $H'_k(x)$ are non-empty for all $x \in E$, then $H'_k(E) = \{H'_k(x)\}_{x\in E}$ is an m-bundle, called the **Glaeser k-refinement** *of $H(E)$. In this case the m-bundle $H(E)$ is called* **k-refinable**.

For the purpose of this definition, we agree that $0^0 = 0$.

One can iterate this procedure, thus arriving at **higher k-refinements** $H_k^i(E)$, $i = 2, 3, \ldots$. More precisely, $H_k^{i+1}(E)$ is defined as the Glaeser k-refinement of the bundle $H_k^i(E)$:

$$H_k^{i+1}(E) = (H_k^i)'_k(E).$$

Let us note that the refinement of a homogeneous m-bundle is again a homogeneous m-bundle.

2.4. Existence of sections

By Taylor's Theorem, if there exists a section F of an m-bundle $H(E)$, then the jet $J_x^m F$ belongs to $H'_k(x)$ for any $x \in E, k \geq 1$. So, if a bundle admits a section then this bundle is k-refinable for any $k \geq 1$. Moreover, in this case F is also a section of the bundle $H'_k(E)$, as well as of all higher refinements $H^i_k(E)$. So, if for some $i \geq 1$ the bundle $H^i_k(E)$ is k-nonrefinable then the initial bundle $H(E)$ does not allow sections.

For example, since a homogeneous bundle always admits a section, then a homogeneous bundle has refinements of all orders.

As is described in [7] (see also [1, 8]), the iterated application of the procedure of Glaeser k-refinement has an important stabilization property: for any bundle $H(E)$ and any $k \geq 1$, after finitely many consecutive Glaeser k-refinements, we either arrive at a k-nonrefinable bundle, or at a k-**steady** bundle, i.e., a bundle which is its own Glaeser k-refinement. This observation is crucial because of the following fundamental result [7]:

Theorem 2.2. (C. Fefferman [7]) *There exists a constant k depending only on m and n such that the following is true:*

Let $H(E)$ be an m-bundle over a compact subset $E \subset \mathbb{R}^n$, such that

- *$H(E)$ is k-steady,*
- *The fibers $H(x)$ are cosets over ideals in $\mathcal{P}^m_n(x)$.*

Then $H(E)$ admits a section.

One can show (see [7]) that fibers of all k-refinements of a standard m-bundle are cosets over ideals in the corresponding $\mathcal{P}^m_n(x)$.

In particular, Theorem 2.2 suggests that the extendability of a given function on $E \subset \mathbb{R}^n$ to a C^m function on $\mathbb{R}^n$ can be checked as follows:

Corollary 2.3. (Fefferman's Extendability Test) *Let $f : E \to \mathbb{R}$ be a function on a compact set $E \subset \mathbb{R}^n$. Construct the initial m-bundle $H^0(E)$ – the standard m-bundle $H_f(E)$. Compute the consecutive Glaeser k-refinements $H^i_k(E)$ (with constant k from Theorem 2.2) of the initial bundle until you arrive either at a k-nonrefinable bundle, or at a k-steady bundle $H(E)$. The function f admits a C^m extension to $\mathbb{R}^n$ if and only if you arrive at a steady bundle.*

The case $m = 1$ in the above described procedure is essentially Glaeser's C^1 Extension Theorem [8].

2.5. Stabilization numbers and their estimates

The computation of Glaeser k-refinements is the central ingredient in Fefferman's Extendability Test. Its complexity depends upon the number k, so one should try to take k (satisfying Fefferman's Theorem 2.2) as small as possible.

However, the complexity of Fefferman's Extendability Test depends even heavier upon the number of refinements needed to arrive at the steady bundle or at a non-refinable bundle.

Definition 2.4. *Consider a set $E \subset \mathbb{R}^n$, integers $m, k \geq 1$, and an m-bundle $H(E)$. Let* $\mathrm{st} = \mathrm{st}\,(n, m, k; H(E))$ *be the natural number such that either $H_k^{\mathrm{st}}(E)$ is a k-nonrefinable bundle, or*

$$H_k^{\mathrm{st}\,+1}(E) = H_k^{\mathrm{st}}(E) \subsetneq H_k^{\mathrm{st}\,-1}(E).$$

We call this number the **stabilization number** *of the bundle $H(E)$.*

It follows from the considerations in [7] (see also [1, 8]) that for $k \geq 1$,

$$\mathrm{st}\,(n, m, k; H(E)) \leq 2 \dim \mathcal{P}_n^m + 1 \tag{2.3}$$

for any m-bundle $H(E)$ over a compact $E \subset \mathbb{R}^n$.

Definition 2.5.

$$\mathrm{ST}\,(n, m, k) = \max_{H_f(E)} \mathrm{st}\,(n, m, k; H_f(E))$$

the maximum is over all compact subsets $E \in \mathbb{R}^n$ and all **standard** *m-bundles over E. We call this quantity the* **standard stabilization number**.

Since $\dim \mathcal{P}_n^1 = n + 1$, then for a 1-bundle $H(E)$ over $E \subset \mathbb{R}^n$ we get

$$\mathrm{st}\,(n, 1, k; H(E)) \leq 2n + 3.$$

Therefore

$$\mathrm{ST}\,(n, 1, k) \leq 2n + 3.$$

There is an example (due to Glaeser) showing that

$$\mathrm{st}\,(2, 1, k; h(E)) \geq 2$$

for a homogeneous standard 1-bundle $h(E)$ over a special subset $E \subset \mathbb{R}^2$ and all $k \geq 2$, which, in particular, means that for $n \geq 2$

$$\mathrm{ST}\,(n, 1, k) \geq 2.$$

However, as C. Fefferman pointed out in his 2005 Princeton lectures, there were no examples of bundles –in any dimension and any degree of smoothness– that require at least three Glaeser refinements to arrive at stabilization.

So we see a considerable gap between the known upper and lower bounds for the standard stabilization numbers $\mathrm{ST}\,(n,1,k)$ for $n \geq 2$:

$$2 \leq \mathrm{ST}\,(n,1,k) \leq 2n+3.$$

3. Formulation of problems and results

There are two questions that one should answer in order to estimate the degree of complexity of Fefferman's Extendability Test:

Question 1. What is the minimal constant $k = k(n,m)$ that we can have in Fefferman's extendability test?

Question 2. How many Glaeser k-refinements of a *standard* m-bundle one could need before arriving at a k-steady bundle or at a k-nonrefinable bundle?

In this paper we give quite complete answers to both of these questions in the case $m = 1$, $n \geq 2$ – see Theorems 3.1, 3.2 below. Let us note that we may disregard the case $n = 1$, since in this case Whitney Theorem [11] provides a much easier computable criterion of extendability.

Theorem 3.1. *Let $H(E)$ be a standard 1-bundle over $E \subset \mathbb{R}^n$.*
(a) $H(E)$ is k-refinable ($k \geq 2$) if and only if it is 2-refinable, and

$$\forall\, k \in \mathbb{Z}_+,\ k \geq 2, \quad H_2'(E) = H_k'(E).$$

(b) The bundles $(H_2')_k^i(E)$, $i = 0, 1, \ldots,$ are k-refinable ($k \geq 1$) if and only if they are 1-refinable, and

$$\forall\, k \in \mathbb{Z}_+,\ k \geq 1, \quad (H_2')_k^i(E) = (H_2')_1^i(E).$$

Thus, in calculating Glaeser k-refinements of standard 1-bundles, it is enough to consider $k = 2$ for the first refinement and $k = 1$ for the successive refinements. These two numbers are optimal, as follows from Lemma 5.3. Our main result is the following.

Theorem 3.2. $n \leq \mathrm{ST}\,(n,1,k) \leq n+1$ *for any* $k \geq 2$, $n \geq 2$.

This result is proven in two steps. First, we prove the following:

Theorem 3.3. *There exists a compact set* $E \subset \mathbb{R}^n$, $n \geq 2$, *such that for the homogeneous standard* 1*-bundle* $h(E)$

$$\mathrm{st}\,(n, 1, k; h(E)) \geq n$$

for all $k \geq 2$. *In particular,* $ST(n, 1, k) \geq n$ *for* $n \geq 2, k \geq 1$.

Our construction of the set E is inspired by a two-dimensional example due to Glaeser [8].

Next, we improve the upper estimate (2.3), again concentrating on standard 1-bundles:

Theorem 3.4. *Let* $H(E)$ *be a standard* 1*-bundle over a compact set* $E \subset \mathbb{R}^n$. *Then*

$$\mathrm{st}\,(n, 1, k; H(E)) \leq n + 1.$$

Theorem 3.3 and Theorem 3.4 give us the assertion of our main result, Theorem 3.2.

Other results of this paper include a new simple proof of Glaeser's C^1 Extension Theorem.

The rest of the paper is organized as follows:

Section 4 deals with a reduction of the problem to homogeneous bundles. In Section 5 we give a more convenient description of Glaeser refinements of standard 1-bundles. In particular, we apply our description to prove Theorem 3.1. In Section 6 we construct a special set $E \subset \mathbb{R}^n$ and, using the results of Section 5, we compute the refinements of the homogeneous standard 1-bundle over this set, thus proving Theorem 3.3. In Section 7 we prove Theorem 3.4. Using the results of Section 5 and, to a very small extent, Section 7, we sketch another proof of Glaeser's Theorem in Section 8.

4. Homogenization and Glaeser refinements

For any $x \in \mathbb{R}^n$, the fiber of an m-bundle $H(E)$ at x is an affine subspace in $\mathcal{P}_n^m(x)$, i.e., a coset over a well defined linear subspace $I(x) \subset \mathcal{P}_n^m(x)$. As it has been already mentioned, in [7] the subspace $I(x)$ is always assumed to be an ideal.

For a non-homogeneous m-bundle $H(E)$ we consider the m-bundle

$$h(H)(E) = \{I(x)\}_{x \in E}.$$

We call this bundle $h(H)(E)$ the **homogenization** of the bundle $H(E)$.

In particular, the homogeneous standard m-bundle $h(E)$ is the homogenization of any standard m-bundle $H_f(E)$: $h(E) = h(H_f)(E)$.

What are the relations between the operations of homogenization and of k-refinement? Let us show that these operations commute (see also Theorem 3.2 in [2] for a closely related result).

Lemma 4.1. *Let $H(E)$ be a k-refinable m-bundle. Let $H'_k(E)$ be its Glaeser k-refinement. Then*

$$[h(H)]'_k(E) = h(H'_k)(E).$$

Proof. We need to show that for any $x \in E$ we have $[h(H)]'_k(x) = h(H'_k)(x)$. Choose $x_0 \in E$. Since $H'_k(x_0) \neq \emptyset$, we can choose $Q_0 \in H'_k(x_0) \subset H(x_0)$.

To prove the inclusion $[h(H)]'_k(x_0) \subset h(H'_k)(x_0)$, take $p_0 \in [h(H)]'_k(x_0)$, and let us show that $p_0 \in h(H'_k)(x_0)$, i.e., that $Q_0 + p_0 \in H'_k(x_0)$.

Choose any $\varepsilon > 0$. Take $\delta > 0$ small enough so that for any $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$ there exist $p_i \in I(x_i)$, $Q_i \in H(x_i)$, $i = 1, 2, \ldots, k$, such that for any $|\beta| \leq m,\ i, j = 0, 1, \ldots, k$, we have

$$|\partial^\beta (p_i - p_j)(x_j)| < \frac{1}{2}\varepsilon |x_i - x_j|^{m-|\beta|},$$

$$|\partial^\beta (Q_i - Q_j)(x_j)| < \frac{1}{2}\varepsilon |x_i - x_j|^{m-|\beta|}.$$

This immediately implies that

$$|\partial^\beta ((Q_i + p_i) - (Q_j + p_j))(x_j)| < \varepsilon |x_i - x_j|^{m-|\beta|}.$$

So $Q_0 + p_0 \in H'_k(x_0)$.

To prove the inclusion $[h(H)]'_k(x_0) \supset h(H'_k)(x_0)$, take $p_0 \in h(H'_k)(x_0)$, and show that $p_0 \in [h(H)]'_k(x_0)$.

Consider $\tilde{Q}_0 = p_0 + Q_0 \in H'_k(x_0) \subset H(x_0)$. Take any $\varepsilon > 0$, and choose $\delta > 0$ small enough so that for any $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$ there exist $Q_i, \tilde{Q}_i \in H(x_i)$, $i = 1, 2, \ldots, k$, such that for any $|\beta| \leq m,\ i, j = 0, 1, \ldots, k$, we have

$$|\partial^\beta (Q_i - Q_j)(x_j)| < \frac{1}{2}\varepsilon |x_i - x_j|^{m-|\beta|},$$

$$|\partial^\beta (\tilde{Q}_i - \tilde{Q}_j)(x_j)| < \frac{1}{2}\varepsilon |x_i - x_j|^{m-|\beta|}.$$

Consider $p_i = \tilde{Q}_i - Q_i \in I(x_i),\ i = 1, \ldots, k$. We see that for any $|\beta| \leq m,\ i, j = 0, 1, \ldots, k$,

$$|\partial^\beta (p_i - p_j)(x_j)| < \varepsilon |x_i - x_j|^{m-|\beta|},$$

which proves that $p_0 \in [h(H)]'_k(x_0)$. ∎

Lemma 4.2. *Let $H(E)$ be an m-bundle over $E \subset \mathbb{R}^n$. Let $h(H)(E)$ be the homogenization of this bundle. Then either*

$$\mathrm{st}\,(n,m,k;H(E)) = \mathrm{st}\,(n,m,k;h(H)(E)),$$

or $H^i_k(E)$ is not k-refinable for some $i \le \mathrm{st}\,(n,m,k;h(H)(E))$.

Proof. Assume that $H^i(E)$ is k-refinable for all $i \le N = \mathrm{st}\,(n,m,k;h(H)(E))$. Then for any $x \in E$ there exists $P(x) \in H^{N+1}_k(x) \subset H^N_k(x) \subset \cdots \subset H(x)$. Therefore for any $i \le N+1$ and for any $x \in E$, due to the definition of homogenization and Lemma 4.1,

$$H^i_k(x) = P(x) + h(H^i_k)(x) = P(x) + [h(H)]^i_k(x).$$

Since for any $x \in E$ we have $[h(H)]^N_k(x) = [h(H)]^{N+1}_k(x)$, we conclude that $H^N_k(x) = H^{N+1}_k(x)$. Also, since $[h(H)]^{N-1}_k(x) \supsetneqq [h(H)]^N_k(x)$ for some $x \in E$, we conclude that $H^{N-1}_k(E) \supsetneqq H^N_k(E)$. ■

Remark. Lemma 4.2 shows that in order to compute $\mathrm{ST}\,(n,m,k)$ we may restrict ourselves to consideration of homogeneous standard bundles.

5. Analysis of refinements of a standard 1-bundle

5.1. The first refinement

Let $H(E) = H_f(E)$ be a standard 1-bundle over E. A fiber $H_f(x_0)$ of such bundle consists of 1-jets of the form $f(x_0) + \langle u, x - x_0 \rangle,\ u \in \mathbb{R}^n$. We identify this 1-jet with the vector $u \in \mathbb{R}^n$, i.e.,

$$f(x_0) + \langle u, x - x_0 \rangle \ \leftrightarrow\ u.$$

Let us rewrite the definition of the Glaeser k-refinement for the case of a standard 1-bundle: a 1-jet $f(x_0) + \langle u_0, x - x_0 \rangle$ belongs to $H^1_k(x_0)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$ one can find $u_1, \ldots, u_k \in \mathbb{R}^n$ such that

$$(5.1)\qquad \forall\, i,j,\ 0 \le i,j \le k, \quad |f(x_i) + \langle u_i, x_j - x_i \rangle - f(x_j)| < \varepsilon |x_i - x_j|$$

and

$$(5.2)\qquad \forall\, i,j,\ 0 \le i,j \le k, \quad |u_i - u_j| < \varepsilon.$$

Lemma 5.1. *Let* $H(E) = H_f(E)$ *be a standard* 1*-bundle. Let* $x_0 \in E$ *and* $k \geq 2$. *A vector* $u \in \mathbb{R}^n$ *belongs to* $H^1_k(x_0)$ *(meaning, the* 1*-jet* $f(x_0) + \langle u, x - x_0\rangle$ *belongs to* $H^1_k(x_0)$*) if and only if*

$$\lim_{\delta\to 0^+} \sup_{x,y\in E\cap B(x_0,\delta)} \frac{|f(y) + \langle u, x-y\rangle - f(x)|}{|x-y|} = 0. \tag{5.3}$$

In particular, for a standard 1*-bundle, the Glaeser* 2*-refinement equals the Glaeser* k*-refinement for any* $k \geq 2$.

Proof. Assume that (5.3) holds. Given $\varepsilon > 0$ choose $\delta > 0$ so that

$$\sup_{x,y\in E\cap B(x_0,\delta)} \frac{|f(y) + \langle u, x-y\rangle - f(x)|}{|x-y|} < \varepsilon. \tag{5.4}$$

For any $x_1, \dots, x_k \in E \cap B(x_0, \delta)$ put $u_i = u$, for $i = 1, \dots, k$. Let us check that conditions (5.1) and (5.2) hold. The left hand side of the inequality (5.2) is simply zero, and hence it trivially holds. Regarding (5.1), condition (5.4) implies that for $i, j \in \{0, \dots, k\}$,

$$|f(x_j) + \langle u, x_i - x_j\rangle - f(x_i)| < \varepsilon|x_i - x_j|$$

which is exactly (5.1). Hence the requirements of (2.2) are satisfied, and $u \in H^1_k(x_0)$.

We now move to the "only if" part. Assume that the converse is true. Then $u \in H^1_k(x_0)$ but there are sequences $E \ni x_\nu \to x_0, E \ni y_\nu \to x_0$ such that for all ν,

$$|f(y_\nu) + \langle u, x_\nu - y_\nu\rangle - f(x_\nu)| > \varepsilon_0|x_\nu - y_\nu| \tag{5.5}$$

for some $\varepsilon_0 > 0$. Given ε, $0 < \varepsilon < \varepsilon_0/2$, we will show that (5.1), (5.2) cannot be satisfied for any choice of $\delta > 0$. For any $\delta > 0$, take ν large enough so that $x_\nu, y_\nu \in B(x, \delta)$. Assume we associate with x_ν, y_ν the vectors $u_\nu \in H_f(x_\nu)$, $v_\nu \in H_f(y_\nu)$, respectively. Then by (5.2),

$$|u_\nu - u|, |v_\nu - u| < \varepsilon < \frac{\varepsilon_0}{2}$$

and hence

$$|f(y_\nu) + \langle u_\nu, x_\nu - y_\nu\rangle - f(x_\nu)| > |f(y_\nu) + \langle u, x_\nu - y_\nu\rangle - f(x_\nu)| - \frac{\varepsilon_0}{2}|x_\nu - y_\nu|.$$

Combining this with (5.5), we conclude that

$$|f(y_\nu) + \langle u_\nu, x_\nu - y_\nu\rangle - f(x_\nu)| > \frac{1}{2}\varepsilon_0|x_\nu - y_\nu| > \varepsilon|x_\nu - y_\nu|.$$

We conclude that it is impossible to associate vectors from $H_f(x_\nu), H_f(y_\nu)$ with x_ν, y_ν to satisfy (5.1) and (5.2), in contradiction to the assumption that $u \in H^1_k(x_0)$. This finishes the proof. ■

For standard 1-bundles $H(E)$ we define $H^1(E) := H^1_k(E)$, where k is any integer ≥ 2. This definition makes sense, as $H^1_k(E)$ does not depend on k, as long as $k \geq 2$, by Lemma 5.1.

Note that $H^1_1(E)$ might be different from $H^1_2(E)$, as follows from the following lemma.

Lemma 5.2. *Let $h(E)$ be the homogeneous standard 1-bundle over the set*

$$E = \{(x, y) \in \mathbb{R}^2; |y| \leq x^2,\ 0 \leq x \leq 1\}.$$

Then

$$h^1_2(0,0) = \{0\},$$
$$h^1_1(0,0) = \{P \in \mathcal{P}^1_2 : P(0,0) = 0, \partial_x P(0,0) = 0\},$$

so

$$h^1_2(0,0) \neq h^1_1(0,0).$$

The proof of this lemma is just a straightforward checking.

5.2. Tangent vectors and *E*-gradients

In this section we reformulate Lemma 5.1 in the terminology of Glaeser paratangent bundles (see [8, 1]).

Definition 5.3. *A vector $v \in \mathbb{R}^n$ is called* **tangent** *to the set $E \subset \mathbb{R}^n$ at the point $x \in E$, if there exist $y_i, z_i \in E$, $y_i \to x, z_i \to x$, such that*

$$|v| \frac{y_i - z_i}{|y_i - z_i|} \xrightarrow{i \to \infty} v.$$

Let $T_x(E)$ denote the set of all vectors tangent to E at $x \in E$.

Note that if a vector v is tangent to E at x then for any $\lambda \in \mathbb{R}$ the vector λv is also tangent to E at x. On the other hand, a sum of two vectors tangent to E at x is not necessarily tangent to E at x. So the set $T_x(E)$ is closed under dilation but not under addition.

In the case of a homogenous bundle, Lemma 5.1 can be rewritten as follows:

Lemma 5.4. *Let $h(E)$ be a homogeneous standard 1-bundle. Then*

$$\forall\, x \in E \quad h^1(x) = T_x(E)^{\perp}.$$

To obtain Lemma 5.4, simply plug in $f = 0$ in (5.3).

Definition 5.5. *A function* $f : E \to \mathbb{R}$ *is called* E-**differentiable** *at* $x \in E$ *if there exists a vector* $u \in \mathbb{R}^n$ *such that for every vector* $v \in T_x(E)$, $|v| = 1$, *we have*

$$\langle u, v\rangle = \lim \left\{ \frac{f(y_i) - f(z_i)}{|y_i - z_i|} : y_i, z_i \in E,\ y_i, z_i \longrightarrow x,\ \frac{y_i - z_i}{|y_i - z_i|} \to v \right\}.$$

This vector, if it exists, is uniquely defined modulo $T_x(E)^\perp$.

We call such vector u *an* E-**gradient** *of* f *at* $x \in E$. *The set of* E-*gradients of* f *at* $x \in E$ *is denoted by* $(\nabla_E f)(x)$.

Lemma 5.1 may be reformulated as follows:

Lemma 5.6. *Let* $H(E) = H_f(E)$ *be a standard* 1-*bundle over* $E \subset \mathbb{R}^n$. *Then* $H^1(x_0) \neq \emptyset$ *if and only if* f *is* E-*differentiable at* $x_0 \in E$. *If* $H^1(x_0) \neq \emptyset$, *then*

$$\{u \in \mathbb{R}^n : f(x_0) + \langle u, x - x_0\rangle \in H^1(x_0)\} = (\nabla_E f)(x_0).$$

5.3. Two examples of computation of the first refinement

An additional, more geometric, reformulation of Lemma 5.1 reads as follows:

Lemma 5.7. *Let* $h(E)$ *be a homogeneous standard* 1-*bundle over a compact set* $E \subset \mathbb{R}^n$. *Then a vector* u *belongs to* $h^1(x)$ *if and only if for any distinct* $y_i, z_i \in E$, $y_i \to x, z_i \to x$, *the angle between* u *and the segment* $[y_i, z_i]$ *goes to* $\pi/2$.

Definition 5.8. *We say that an infinite set* $E \subset \mathbb{R}^n$ *is* **sticking to the line** l *near* $x \in l$ *if*

- x *is a non-isolated point in* E,
- *the acute angle between the segment* $[y_i, z_i]$ *and the line* l *goes to zero, as* $E \ni y_i, z_i \to x$.

In other words, E is sticking to a line l near x, if the set $T_x(E)$ is the line parallel to l. By Lemma 5.4 we arrive at the following:

Lemma 5.9. *Let a compact set* $E \subset \mathbb{R}^n$ *stick to the line* l *near* $x \in E$. *Consider the homogeneous standard* 1-*bundle* $h(E)$. *Then*

$$h^1(x) = l^\perp.$$

Definition 5.10. *We say that an infinite set* $E \subset \mathbb{R}^n$ *is* **sparsely sticking to the lines** l_1, l_2 *near* $x \in l_1 \cap l_2$, *if*

- x *is a non-isolated point in* E,
- *The minimum, over* $\nu = 1, 2$ *of the acute angles, formed by the segment* $[y_i, z_i]$ *with the line* l_ν $(\nu = 1, 2)$, *goes to zero as* $E \ni y_i, z_i \to x$,
- E *is not sticking to the line* ℓ_1, *nor to the line* ℓ_2, *at* x.

In other words, E is sparsely sticking to lines l_1, l_2 near x, if the set $T_x(E)$ consists of two lines, one parallel to l_1, the other parallel to l_2. The next lemma now follows from Lemma 5.4.

Lemma 5.11. *Let a compact set $E \subset \mathbb{R}^n$ be sparsely sticking to the lines l_1, l_2 near $x \in E$. Consider the homogeneous standard 1-bundle $h(E)$. Then*

$$h^1(x) = l_1^\perp \cap l_2^\perp.$$

5.4. Higher refinements

Our next lemma analyzes further refinements of our standard 1-bundle $H(E) = H_f(E)$.

Lemma 5.12. *Let $H(E)$ be a standard 1-bundle. Let $i \geq 2, k \geq 1$. Let $x_0 \in E$. Then a vector u belongs to $H_k^i(x_0)$ (meaning, as always, that the 1-jet $f(x_0) + \langle u, x - x_0 \rangle$ belongs to $H_k^i(x_0)$) if and only if*

$$\lim_{\delta \to 0^+} \sup_{x \in E \cap B(x_0,\delta)} \inf_{v \in H_k^{i-1}(x)} |u - v| = 0. \tag{5.6}$$

In particular, the Glaeser 1-refinement of $H^1(E)$ of order $i \geq 1$ equals the Glaeser k-refinement of $H^1(E)$ of the same order i, for any $k \geq 1$.

Proof. Assume that $u \in H_k^i(x_0)$. Then by (5.2), for any $\varepsilon > 0$ there is $\delta > 0$ such that for $x \in E \cap B(x_0, \delta)$ there exists $v \in H_k^{i-1}(x)$ with $|u - v| < \varepsilon$. Thus $\sup_{x \in E \cap B(x_0,\delta)} \inf_{v \in H_k^{i-1}(x)} |u - v| < \varepsilon$. This proves that (5.6) is necessary for u to belong to $H_k^i(x_0)$. Why is (5.6) sufficient? Suppose that (5.6) holds. Then in particular,

$$0 \leq \inf_{v \in H_k^{i-1}(x_0)} |u - v| \leq \lim_{\delta \to 0^+} \sup_{x \in E \cap B(x_0,\delta)} \inf_{v \in H_k^{i-1}(x)} |u - v| = 0.$$

Since $H_k^{i-1}(x_0)$ is an affine subspace, it is closed and $u \in H_k^{i-1}(x_0)$. Given $\varepsilon > 0$, fix $\delta > 0$ so that

$$\sup_{x \in E \cap B(x_0,\delta)} \inf_{v \in H_k^{i-1}(x)} |u - v| < \frac{\varepsilon}{2}. \tag{5.7}$$

Note also that since $u \in H_k^{i-1}(x_0) \subset H_k^1(x_0)$, by Lemma 5.1 we may assume that $\delta > 0$ is also chosen to satisfy

$$\sup_{\substack{x,y \in E \cap B(x_0,\delta) \\ x \neq y}} \frac{|f(y) + \langle u, x - y \rangle - f(x)|}{|x - y|} < \frac{\varepsilon}{2}. \tag{5.8}$$

Now, given $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$ select $u_1 \in H_k^{i-1}(x_1), \ldots, u_k \in H_k^{i-1}(x_k)$ such that $|u_i - u| < \varepsilon/2$. Condition (5.2) is automatically satisfied, and we need to check condition (5.1). This follows from (5.8), as

$$|f(x_i) + \langle u_i, x_j - x_i \rangle - f(x_j)| < |f(x_i) + \langle u, x_j - x_i \rangle - f(x_j)| + \\ + |\langle u - u_i, x_j - x_i \rangle| < \varepsilon |x_i - x_j|. \qquad \blacksquare$$

For standard 1-bundles we define, for $i \geq 2$,

$$H^i(E) := H_k^i(E)$$

for any $k \geq 1$, and the definition makes sense.

Proof of Theorem 3.1. Lemmas 5.1 and 5.12 contain all assertions of Theorem 3.1. Lemma 5.2 shows that the constants of Lemmas 5.1, 5.12 are optimal. ■

In all further considerations we skip the reference to the number k for standard 1-bundles.

5.5. An example of computation of higher refinements

The following result directly follows from Lemma 5.12:

Corollary 5.13. *Let $h(E)$ be a homogeneous standard 1-bundle over $E \subset \mathbb{R}^n$ and $i \geq 2$. Fix $x \in E$. Let $k \geq 1$, and assume there exist finitely many subsets $A_j \subset E, j = 1, 2, \ldots, k$, integers $S(j) \geq 1, j = 1, \ldots, k$, and non-zero vectors $E_{js} \in \mathbb{R}^n$ for $1 \leq j \leq k, 1 \leq s \leq S(j)$, such that*

(i) $\forall\, y \in A_j \quad h^{i-1}(y) = \bigcap_{s=1}^{S(j)} e_{js}(y)^{\perp}$, *for some vectors $e_{js}(y) \in \mathbb{R}^n$ that satisfy the following: For any $1 \leq j \leq k, 1 \leq s \leq S(j)$ and $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$y \in A_j, |y - x| < \delta \quad \Rightarrow \quad |e_{js}(y) - E_{js}| < \varepsilon.$$

(ii) *for some $\delta > 0$ the set $E \cap B(x, \delta) \setminus \{x\}$ is the disjoint union of non-empty sets $A_j \cap B(x, \delta), j = 1, 2, \ldots, k$.*

Then

$$h^i(x) = h^{i-1}(x) \cap \left(\bigcap_{\substack{j,s: 1 \leq j \leq k, \\ 1 \leq s \leq S(j)}} E_{js}^{\perp} \right).$$

Proof. Let $u \in h^{i-1}(x)$ satisfies that $u \perp E_{js}$ for all $1 \leq j \leq k, 1 \leq s \leq S(j)$. By (i) and (ii), it is easy to see that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$y \in E \cap B(x, \delta) \quad \Rightarrow \quad \inf_{v \in h^{i-1}(y)} |u - v| < \varepsilon$$

and hence $u \in h^i(x)$.

For the other direction, assume on the contrary that $u \in h^i(x)$, but $\langle u, E_{js} \rangle \neq 0$ for some $1 \leq j \leq k, 1 \leq s \leq S(j)$. Then there exists a sequence $y_1, y_2, \ldots \in A_j$ such that $y_m \to x$, and such that $|\langle u, e_{js}(y_m) \rangle| > \varepsilon$ for all m. Therefore $\inf_{v \in h^{i-1}(y_m)} |u - v|$ does not tend to zero, in contradiction to Lemma 5.12. ■

Remark. Lemmas 5.1 and 5.12 suggest that the actual effect of Glaeser refinements on standard 1-bundles is smaller than it seems at first glance. The first refinement is geometric (dealing with the geometry of the set E near the point x_0), while all the higher refinements are merely concerned with the selection of continuous sections of the bundle $h^1(E)$.

6. Proof of Theorem 3.3

6.1. An overview of the construction

We shall construct a compact set $E \subset \mathbb{R}^n$, $n \geq 2$, such that the following holds:

Sticking Conditions. For each non-isolated point x of E (except of one) there will be two lines $l_1(x)$ and $l_2(x)$ passing through x such that E is sparsely sticking to these lines near x. At the only exceptional non-isolated point (we choose it to be the origin) there will be only one line $l_1(0)$ such that E will stick to it near the origin.

This will enable us to easily compute the first refinement of the homogeneous standard 1-bundle $h(E)$ at all non-isolated points, using Lemma 5.9 and Lemma 5.11. Let Iso E denote the set of isolated points of E. Then

$$h^1(0) = l_1(0)^\perp,$$
$$\forall\, 0 \neq x \in E \setminus \text{Iso } E,\ h^1(x) = l_1(x)^\perp \cap l_2(x)^\perp.$$

Obviously,
$$\forall\, x \in \text{Iso } E,\ h^1(x) = h(x) = \mathbb{R}^n.$$

Higher refinements are easy to compute, thanks to Lemma 5.12 and Corollary 5.13.

Let us fix the coordinates $X_1, \ldots, X_n$ in $\mathbb{R}^n$. We let $e_1, \ldots, e_n$ denote the unit vectors of the respective coordinate axis.

We choose the line $l_1(x)$ to be the same for all non-isolated points of E, and we let it coincide with the X_1-axis. In particular, all non-isolated points of E will be on the X_1-axis, and more precisely, we place all such points on the positive X_1-semi-axis.

Let F be the part of E belonging to the X_1-axis. F contains all non-isolated points of E, and it will also contain some of the isolated points of E. So for each non-zero non-isolated point $x \in F$ we have $h^1(x) = e_1^\perp \cap l_2(x)^\perp$.

6.2. Preparation for construction of *F*

Next, we will describe the construction of F. A quick definition of the set F could have been

$$\{a_{i_1} + a_{i_1+i_2} + a_{i_1+i_2+i_3} + \cdots + a_{i_1+\cdots+i_n}; i_1, \ldots, i_n \in \{1, 2, \ldots\} \cup \{\infty\}\} \tag{6.1}$$

where $a_k = 2^{-2^k}$ for integer k and $a_\infty = 0$. However, this definition is not intuitive and its properties might not be clear for some readers, so in the following few sections we present a detailed, instructive construction of a set F in the spirit of (6.1). We will not use the set (6.1) in any place.

The set F is going to be the union of decreasing sequences, each imbedded in its own open interval, which will be called cluster. Each such sequence will be rapidly converging to the left end of the cluster. The sequences are going to be the shifts and truncations of one basic sequence, and the clusters are going to form a tree with respect to inclusion, with clusters "of the same level" well separated from each other.

To define these we fix a pair of rapidly decreasing sequences of positive numbers $a_k, b_k, k = 1, 2, \ldots$, in the interval $(0, 1)$ such that

$$\forall\, k \geq 3 \quad 0 < b_k < a_{k-1} - a_k,\ b_k \leq a_k^2,\ a_k \leq a_{k-1}^2,\ a_k < \frac{1}{10} a_{k-1}, \tag{6.2}$$

$$\lim_{k\to\infty} \frac{a_k + b_k}{a_{k-1}} = \lim_{k\to\infty} \frac{b_k}{a_k - (a_{k+1} + b_{k+1})} = 0. \tag{6.3}$$

With such choices of sequences, intervals of the form $(a_k, a_k + b_k)$ are well separated from each other, on scales proportional to their lengths.

For instance, we may set $a_k = 2^{-2^k},\ b_k = 2^{-2^{k+1}}$. The basic sequence – a building block of our construction – will be the set

$$\mathcal{B} = \{a_k; k = 1, 2, \ldots\} \subset (0, 1).$$

For $x = a_k \in \mathcal{B}$ we define $C^\circ(x)$ to be the interval

$$C^\circ(x) = (a_k, a_k + b_k) \subset (0, 1).$$

Note that $\{C^\circ(x)\}_{x\in\mathcal{B}}$ is a disjoint family of intervals, that do not intersect $\mathcal{B}$.

Furthermore, given an interval $(s,t) \subset \mathbb{R}$, $s < t$, we will denote by $\mathcal{B}_{(s,t)}$ a suitable adaptation (shift and truncation) of the set $\mathcal{B}$ to the interval (s,t), namely,

$$\begin{aligned}\mathcal{B}_{(s,t)} &= \{s + x : x \in \mathcal{B}, x < t - s\} \\ &= \{s + a_k : k = k_0, k_0 + 1, \ldots, k_0 = \min\{k : s + a_k < t\}\} \subset (s,t).\end{aligned}$$

So, the sequence $\mathcal{B}_{(s,t)}$ is a subset of the interval (s,t), and it is rapidly decreasing to s. Also, for $x = s + a_k \in \mathcal{B}_{(s,t)}$ we set

$$\begin{aligned}C_{(s,t)}(x) &= \{s + y; y \in C^{\circ}(x - s), y < t - s\} \\ &= (s + a_k, \min\{s + a_k + b_k, t\}) \subset (s,t).\end{aligned}$$

Note that $x \in C_{(s,t)}(x)$. Still, $\{C_{(s,t)}(x)\}_{x \in \mathcal{B}_{(s,t)}}$ is a disjoint family of well separated intervals that do not intersect $\mathcal{B}_{(s,t)}$.

6.3. Construction of F

The set F is the intersection of E with the X_1-axis. In order to ease our notation, we identify this axis with $\mathbb{R}$, i.e., for $t \in \mathbb{R}$, $t \in F$ should be interpreted as $te_1 \in F$.

The construction of the set F is inductive. We define disjoint sets $F_0, \ldots, F_n \subset \mathbb{R}$, their union will constitute the set F. Additionally, for each point $x \in F_i$, $0 \leq i \leq n$, we define a **cluster**, an open interval $C(x) \subset \mathbb{R}$. The following properties will hold:

1. The clusters $\{C(x)\}_{x \in F_i}$ are pairwise disjoint and do not intersect F_i.
2. All limit points of F_i lie in $\bigcup_{j=i+1}^{n} F_j$, and every point of $\bigcup_{j=i+1}^{n} F_j$ will be a limit point for F_i. The set $\cup_{j=i}^{n} F_j$ is closed.

Having established these properties, we conclude that F_i is the set of isolated points of $\bigcup_{j=i}^{n} F_j$, i.e. $F_i = \mathrm{Iso} \bigcup_{j=i}^{n} F_j$. Start by setting

$$F_n = \{0\} \subset \mathbb{R}.$$

The cluster $C(0)$ that is associated with $0 \in F_n$ is the interval $(0,1)$. The two properties above trivially hold.

Let $1 \leq i \leq n$. Having constructed a set F_i and a family of disjoint clusters $\{C(x)\}_{x \in F_i}$, let us describe the construction of F_{i-1} and $\{C(x)\}_{x \in F_{i-1}}$. We set

$$F_{i-1} = \bigcup_{x \in F_i} \mathcal{B}_{C(x)}.$$

This is a disjoint union, since the clusters $\{C(x)\}_{x \in F_i}$ are disjoint. For any $y \in F_{i-1}$ there is a unique $x \in F_i$ such that $y \in \mathcal{B}_{C(x)}$. We now define the cluster of y to be

$$C(y) = C_{C(x)}(y) \subset C(x).$$

In other words, each cluster $C(x),\ x \in F_i$ gives rise to a sequence of pairwise disjoint clusters, contained in $C(x)$. Therefore the clusters $\{C(y)\}_{y\in F_{i-1}}$ are disjoint. By the construction, the clusters $\{C(y)\}_{y\in F_{i-1}}$ are also disjoint from F_{i-1}.

It is also straightforward to verify that $\cup_{j=i-1}^n F_j$ is closed, that all limit points of F_{i-1} are in $\bigcup_{j=i}^n F_j$, and that every point of $\bigcup_{j=i}^n F_j$ is a limit point of F_{i-1}.

This finishes the construction of the sets $F_0, \dots, F_n$.

Next, set

$$F = \bigcup_{i=0}^{n} F_i.$$

Clearly, F_0 is the set of isolated points of F. For any $x \in F$ we have a cluster $C(x)$, that was defined in the construction of $F_0, \dots, F_n$. See Figure 1 for a schematic drawing of the clusters of the set F.

The following property of our construction will be substantially used later:

$$\text{(6.4)} \qquad \forall\, x \in F_i\ \exists\, \delta > 0 \quad (F \cap B(x,\delta)) \setminus \{x\} \subset \bigcup \{C(y) :\ y \in \mathcal{B}_{C(x)}\}.$$

In particular,

$$\text{(6.5)} \qquad \forall\, \{y_k\}_{k=1}^{\infty} \subset F,\ y_k \to x \in F_i\ \ y_k \neq x,$$
$$\exists\, k_0 \in \mathbb{N} \quad \forall\, k > k_0, \quad y_k > x, \quad y_k \in \bigcup_{j=1}^{i-1} F_j.$$

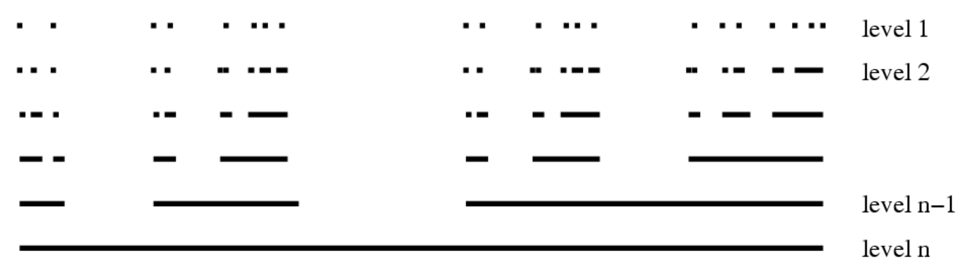

Figure 1

6.4. A tree structure on the set of clusters

Property (6.12) means that there is a natural tree structure, induced by the inclusion relation, on the family of clusters $\{C(x)\}_{x\in F}$. Indeed, we say that $C(y)$ is the child of $C(x)$ if

$$y \in \mathcal{B}_{C(x)}.$$

We may of course also speak about parents, descendants and ancestors of a cluster, as is customary when dealing with trees. Note that the root of this tree is the cluster $C(0) = (0,1)$ that the leaves are $\{C(x)\}_{x \in F_0}$, and that if $C(y)$ is a child of $C(x)$ then $x < y \in C(x)$. There is a simple criterion to check the tree relation between $C(x)$ and $C(y)$, for $x, y \in F$, to be described as follows. The cluster $C(y)$ is a descendant of $C(x)$ if and only if

$$C(y) \subset C(x). \tag{6.6}$$

As follows from (6.12), the cluster $C(y)$ is not a descendant of $C(x)$ and $C(x)$ is not a descendant of $C(y)$ if and only if

$$C(x) \cap C(y) = \emptyset \tag{6.7}$$

Since there is a one-to-one correspondence $x \mapsto C(x)$ between the points of F and the clusters, we may transfer the tree structure from the set of clusters to the set F.

6.5. The proximity function and properties of F

Definition 6.1. *For $x \in F$ let*

$$p(x) = |C(x)|,$$

where $|C(x)|$ denotes the length of the interval $C(x)$. We call the function $p: F \to \mathbb{R}_+$ the **proximity function**.

An important property of the proximity function is the following:

$$F \ni y_k \to x, \ y_k \neq x, \ \Rightarrow \ p(y_k) \to 0. \tag{6.8}$$

This property immediately follows from the observation that if $F \ni y_k \to x$, then for every i, $0 \leq i \leq n$, the subsequence $\{y_k\}_{k=1}^{\infty} \cap F_i$, if infinite, also converges to x from the right. Since the clusters $C(y_k)$, $y_k \in F_i$, whose left ends are y_k, are disjoint, then their lengths $p(y_k)$ must tend to zero.

Since $\mathcal{B}_{C(x)} \subset C(x)$, and $|C(x)| = p(x)$, then

$$\forall\, y = x + a_k \in \mathcal{B}_{C(x)} \quad a_k \leq p(x). \tag{6.9}$$

For a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$ we write $\operatorname{dist}(x, A) = \inf_{y \in A} |x - y|$ for the distance between x and A. The distance between two sets $A, B \subset \mathbb{R}^n$ is of course $\operatorname{dist}(A, B) = \inf_{x \in A} \operatorname{dist}(x, B)$.

The following statements can be verified in a straightforward manner, from (6.2) and (6.3):

(6.10) $\forall\, x, y \in F \;\; C(x) \cap C(y) = \emptyset \;\Rightarrow\; \text{dist}\,[C(x), C(y)] \geq \max\{p(x), p(y)\}.$

$$\lim_{\substack{C(x)\ni y_1, y_2 \to x \\ C(x)\cap C(y)\neq\emptyset}} \frac{\text{dist}\,(C(y_1), C(y_2))}{\max\{p(y_1), p(y_2)\}} = \infty. \tag{6.11}$$

A useful property that is evident from the construction of F is that

(6.12) $\forall\, x, y \in F \quad C(x) \cap C(y) \neq \emptyset \;\Rightarrow\; C(x) \subset C(y)$ or $C(y) \subset C(x)$.

6.6. Cylinders associated with clusters

Let $C(x)$, $x \in F$, be a cluster. Then $C(x) \subset \mathbb{R}e_1$, $|C(x)| = p(x)$. Consider the **cylinder** $P(x) \subset \mathbb{R}^n$, defined as follows:

$$P(x) = \{z \in \mathbb{R}^n : \langle z, e_1\rangle e_1 \in \overline{C(x)},\, \text{dist}\,(z, C(x)) \leq p(x)^2\} \tag{6.13}$$

where $\overline{C(x)}$ is the closure of $C(x)$. Note that $x \in \overline{C(x)}$. Note that the cluster $C(x)$ is the axis of the cylinder $P(x)$. $P(x)$ has height $p(x)$ in the e_1-direction, its base is a ball in $\mathbb{R}^{n-1}$ of radius $p(x)^2$.

If $C(y) \subset C(x)$ then $p(y) \leq p(x)$ and consequently $P(y) \subset P(x)$. Therefore, there is a one-to-one inclusion-preserving correspondence between the set of clusters and the set of cylinders. So the set of all cylinders also gets a tree structure. In particular, two cylinders are either disjoint or one is contained in another. The cylinders $P(x)$, $x \in F_i$, are disjoint, and for each cylinder $P(x)$, $x \in F_i, i < n$, there exists a unique parent cylinder $P(y)$, $y \in F_{i+1}$, such that $P(x) \subsetneq P(y)$. The disjoint cylinders are well separated from each other:

$$P(x) \cap P(y) = \emptyset \quad\Rightarrow\quad \text{dist}\,(P(x), P(y)) \geq \max\{p(x), p(y)\}.$$

The following lemma will be needed for verification of the sticking properties of our construction.

Lemma 6.2. *Let $A \in P(x)$, $B \in P(y)$, $P(x) \cap P(y) = \emptyset$. Then the acute angle α between the segment $[A, B]$ and the vector e_1 satisfies the estimate*

$$|\sin\alpha| \leq 2\max\{p(x), p(y)\}.$$

Proof.

$$|\sin\alpha| = \max_{e\perp e_1, |e|=1} \frac{|\langle B - A, e\rangle|}{|B - A|} \leq \frac{\text{dist}\,(B, C(y)) + \text{dist}\,(A, C(x))}{|B - A|}$$
$$\leq \frac{p(y)^2 + p(x)^2}{\max\{p(y), p(x)\}} \leq 2\max\{p(x), p(y)\}.$$

■

6.7. Construction of $E \setminus F$

To each $x \in \bigcup_{j=1}^{n-1} F_j$ we assign a line $l_2(x)$, passing through x, in the direction $e(x) \in \mathbb{R}^n$. The choice of the direction vectors $e(x)$ of the lines goes as follows:

$$\forall\, x \in F_1 \quad e(x) = \frac{1}{\sqrt{1+p(x)^2}}(e_1 + p(x)e_2), \tag{6.14}$$

$$\forall\, x \in F_i, \quad 2 \le i \le n-1, \qquad e(x) = \frac{1}{\sqrt{1+p(x)^2+p(x)^4}}(e_1 + p(x)e_i + p(x)^2 e_{i+1}). \tag{6.15}$$

Let us note that the vectors $e(x)$ are of unit lengths, and for the angle $\gamma(x)$ between the vectors $e(x)$ and e_1 we have

$$|\tan\gamma(x)| \le 2p(x).$$

From this we deduce that

$$|\sin\gamma(x)| \le |\tan\gamma(x)| \le 2p(x). \tag{6.16}$$

An important consequence of this and of (6.8) is the following:

$$y_k \in F,\ y_k \ne x,\ y_k \to x \in F \quad \Rightarrow \quad \gamma(y_k) \to 0. \tag{6.17}$$

Let us define another sequence of numbers:

$$\lambda_k = \sqrt{a_{k-1}(a_k + b_k)}.$$

This sequence is sparsely intertwined between the intervals $\{C^0(a_k)\}_{k\ge 1}$, i.e.,

$$a_k + 2b_k < \lambda_k < a_{k-1} - b_{k-1}, \tag{6.18}$$

$$\lim_{k\to\infty} \frac{\lambda_k}{a_k + b_k} = \infty, \qquad \lim_{k\to\infty} \frac{\lambda_k}{a_{k-1}} = 0. \tag{6.19}$$

For each $0 \ne x \in F \setminus F_0$ we will place a sequence of points $Y(x)$, converging to x, on the line $l_2(x)$:

$$Y(x) = \{x + \lambda_k e(x) :\ k \ge 1,\ \ x + \lambda_k e(x) \in P(x)\}. \tag{6.20}$$

Note that the condition $x + \lambda_k e(x) \in P(x)$ holds for all integers $k > k_0(x)$, for an appropriate number $k_0(x) > 0$. In addition, if $C(y)$ is a descendant of $C(x)$, then the intertwining condition (6.18) on λ_k guarantees that

$$Y(x) \cap P(y) = \emptyset \quad \text{and} \quad \inf_{z\in Y(x), w\in P(y)} |z - w| > 0 \tag{6.21}$$

Finally, set

$$E = F \cup \left(\bigcup_{0\ne x\in F\setminus F_0} Y(x) \right).$$

6.8. Verification of the Sticking Conditions

For each $z \in E$ there exists a unique smallest cylinder $P(x(z))$, $x(z) \in F$, containing z. We call such cylinder the **holder** of z. If $z \in F$ then $x(z) = z$, and the cylinder $P(z)$ is the holder of z. If $z \in E \setminus F$, then there exists a unique $y \in F$ such that $z \in Y(y)$. In that case, by (6.21), we have $x(z) = y$, and $P(y)$ is the holder of z. Therefore, the holders of $z_1, z_2 \in E$ coincide if and only if z_1, z_2 belong to the same set $Y(x) \cup \{x\}$ for some $x \in F$. In this case $x(z_1) = x(z_2) = x$. Note that for any $z \in E$ we have that $x(z) \leq \langle z, e_1 \rangle$.

Since $z \in P(x(z))$ then $\langle z, e_1 \rangle e_1 \in \overline{C(x(z))}$. Let $E \ni z_r \overset{r\to\infty}{\longrightarrow} x \in F$. By (6.5) and (6.21), necessarily $z_r \in P(x)$ for sufficiently large r. We conclude that $x(z_r) \geq x$ for r large enough. Note also that $x(z_r) \leq \langle z_r, e_1 \rangle \to x$ and hence $x(z_r) \to x$. By (6.8) we see that

$$E \ni z_r \to x, \ x(z_r) \neq x \ \Rightarrow \ p(x(z_r)) \to 0. \tag{6.22}$$

To verify the sticking conditions we need to estimate the angles between segments $[A_r, B_r]$, where $E \ni A_r, B_r \to x \in F$, and at least one of the vectors $e_1, e(x)$.

Lemma 6.2 tells us what happens if the holders $P(x(A_r)), P(x(B_r))$ of A_r, B_r are disjoint.

What if the holders are not disjoint? Then the holders either coincide or one is a proper part of another.

The first case is very easy.

Lemma 6.3. *Let $A, B \in E$, $A \neq B$, have the same holder: $x(A) = x(B) = x$. Then the segment $[A, B]$ is parallel to the vector $e(x)$. In particular, the acute angle between $[A, B]$ and e_1 satisfies the estimate*

$$|\sin \alpha| \leq 2p(x)$$

Now assume that $A, B \in E$ have distinct intersecting holders. Then one of the holders is a subset of the other, assume $P(x(B)) \subsetneq P(x(A))$. We conclude that $A \in \{x(A)\} \cup Y(x(A))$, and B belongs to $P(x_k)$ for some $x_k \in \mathcal{B}_{C(x(A))}$.

Two case exist: either $A = x(A)$ or otherwise $A = x(A) = \lambda_\ell e(x(A))$ for some ℓ. The first case is a limiting case of the second one, so we may confine our attention to the case where $A = x(A) + \lambda_\ell e(x(A))$ and $B \in P(x_k), x_k = x(A) + a_k e_1 \in \mathcal{B}_{C(x(A))}$.

Note that since $x(A) + a_k e_1 \in C(x(A))$ then $a_k < |C(x(A))| = p(x(A))$. For $\ell \leq k$, our definitions imply

$$\begin{aligned} |A - B| &\geq \operatorname{dist}(A, P(x_k)) \geq \operatorname{dist}(\langle A, e_1 \rangle e_1, C(x_k)) \\ &\geq \frac{\lambda_\ell}{\sqrt{1 + p(x)^2 + p(x)^4}} - (a_k + b_k). \end{aligned}$$

Note that $p(x_k) \le b_k$ and that $p(x) \le 1$. Combining with (6.2), we get that when $\ell \le k$,

$$|A-B| \ge \frac{\sqrt{a_{\ell-1}(a_\ell+b_\ell)}}{2} - (a_k+b_k) \ge \frac{\sqrt{a_{k-1}(a_k+b_k)}}{4} \tag{6.23}$$

provided that $k > k_0$ by (6.19), for some universal constant k_0. Similarly, if $\ell > k$, then,

$$|A-B| \ge \operatorname{dist}\left(\langle A, e_1\rangle e_1, C(x_k)\right) \ge a_k - \frac{\lambda_\ell}{\sqrt{1+p(x)^2}}.$$

Combining with (6.2), we get that when $\ell > k$,

$$|A-B| \ge a_k - \lambda_\ell \ge a_k - \lambda_{k+1} > \frac{1}{2}a_k \tag{6.24}$$

provided that $k > k_0$, by (6.19).

Lemma 6.4. *Let $A = x + \lambda_\ell e(x) \in P(x)$, and $B \in P(x_k)$, $x_k = x + a_k e_1 \in \mathcal{B}_{C(x)}$. Assume that $\ell, k \ge k_0$, for some number k_0. If $\ell \le k$ then we have for the angle α between $[A,B]$ and $e(x)$:*

$$|\sin\alpha| \le \tau(k_0)$$

where $\tau(k_0)$ is some function of k_0 such that $\tau(k_0) \to 0$ as $k_0 \to \infty$.

If $\ell > k$ then we have for the angle α between $[A,B]$ and e_1,

$$|\sin\alpha| \le \tau(k_0)$$

for the same function $\tau(k_0)$. Same holds also if $A = \lim_{\ell\to\infty} x + \lambda_\ell e(x) = x$.

Proof. Let $\ell \le k$. Recall that $l_2(x)$ is the line through x in direction $e(x)$. Then,

$$|\sin\alpha| = \frac{\operatorname{dist}(B, l_2(x))}{|A-B|} \le \frac{\operatorname{dist}(x,B)}{|A-B|} \le \frac{\max_{D\in P(x_k)} \operatorname{dist}(x,D)}{|A-B|}$$
$$= \frac{\sqrt{(a_k+b_k)^2 + p(x_k)^4}}{|A-B|}.$$

Combining with (6.23) and the fact that $p(x_k) \le b_k$,

$$|\sin\alpha| \le 8\sqrt{\frac{a_k+b_k}{a_{k-1}}}.$$

By (6.3), the right hand side tends to zero when k_0 (and hence also k) tends to infinity.

Regarding the case $\ell > k$, recall that $l_1(x)$ is the line through x in direction e_1. Let $l_1(x) + B$ be the line through B in direction e_1. Then,

$$\begin{aligned} |\sin\alpha| &= \frac{\operatorname{dist}(A, B + l_1(x))}{|A - B|} \leq \frac{d(A, l_1(x)) + d(B, l_1(x))}{|A - B|} \\ &\leq \frac{\lambda_l 2p(x) + p(x_k)^2}{|A - B|} < 4\frac{\lambda_{k+1} + b_k}{a_k}. \end{aligned}$$

The right hand side tends to zero as k_0, and hence k, tend to infinity. This finishes the proof. The case

$$A = x = \lim_{\ell\to\infty} x(A) + \lambda_\ell e(x)$$

follows by continuity. ■

Now we can prove the following result:

Lemma 6.5. *The set E sparsely sticks to the lines $l_1(x)$ (the X_1-axis) and $l_2(x)$ near every $x,\ x \neq 0,\ x \in F \setminus F_0$. The set E also sticks to the line $l_1(0)$ near* 0.

Proof. Take any $x \in F \setminus \operatorname{Iso} F$. Let $E_1 = Y(x)$, $E_2 = E \setminus E_1$. Obviously, E_1 sticks to $l_2(x)$ near x, since $E_1 \subset l_2(x)$.

Let us show that E_2 sticks to $l_1(x)$ near x. Let $A_r, B_r \in E_2,\ A_r, B_r \to x$. Let $P(y_r)$ be the holder of A_r, let $P(z_r)$ be the holder of B_r. Since the projection of A_r onto the X_1-axis belongs to $C(y_r)$ and these projections converge to x from the right, then $y_r \to x$. Similarly, $z_r \to x$. Note that $y_r, z_r \neq x$, since $A_r, B_r \notin E_1$. Therefore $p(y_r), p(z_r) \to 0$. There are two cases: If $P(y_r)$ and $P(z_r)$ are disjoint, then the angle between $[A_r, B_r]$ and e_1 goes to zero because of Lemma 6.2. Otherwise, by Lemma 6.4, the minimal angle between $[A_r, B_r]$ and the three lines $e_1, l_2(y_r), l_2(z_r)$ goes to zero as $r \to 0$. Since $p(y_r), p(z_r) \to 0$, we conclude that the angle between $[A_r, B_r]$ and e_1 goes to zero as $r \to \infty$.

We need to consider only the case where $A_r \in E_1, B_r \in E_2$ are such that $A_r, B_r \to x$. Since $A_r \in E_1$ then $A_r = x + \lambda_\ell e(x)$. Since $B_r \in E_2$ then $B_r \in P(x_k)$, $x_k \in \mathcal{B}_{C(x)}$. So we find ourselves in the situation of Lemma 6.4. Applying this Lemma, we note that if $\ell \leq k$ and r is very large, then the angle between $[A_r, B_r]$ and $e(x)$ is very small. Similarly, if $\ell > k$, and r is very large, then by Lemma 6.4 the angle between $[A_r, B_r]$ and e_1 is very small. We conclude that as $r \to \infty$, the minimal angle of $[A_r, B_r]$ with the directions $e_1, e_2(x)$ goes to zero as $r \to \infty$. The Lemma is proven. ■

6.9. Computation of refinements of $h(E)$

Using Lemmas 5.9 and 5.11, we are able to compute the first refinements of $h(E)$ at $E \setminus \text{Iso } E = \bigcup_{j=1}^{n-1} F_j$:

Corollary 6.6.

$$h^1(0) = e_1^\perp,$$

$$\forall\, x \in \bigcup_{j=1}^{n-1} F_j \quad h^1(x) = e_1^\perp \cap l_2(x)^\perp.$$

Remark. Few features of the lines $l_2(x)$ were used in the construction. The only important property is that

$$F \ni x_n \to x \in F \quad \Rightarrow \quad l_2(x_n) \to sp\{e_1\}.$$

Corollary 6.6 holds for any such choice of lines.

Using the locality of the definition of refinements, and the fact that $h^0(x) = \mathbb{R}^n$ for all $x \in E$ for a homogeneous standard 1-bundle, we see that

$$\forall\, x \in \text{Iso } E = (E \setminus F) \cup F_0 \quad h^1(x) = h^2(x) = \cdots = \mathbb{R}^n = 0^\perp. \tag{6.25}$$

From Corollary 6.6 and the definition of $e(x)$ we get

$$\forall\, x \in F_1 \quad h^1(x) = e_1^\perp \cap e(x)^\perp = e_1^\perp \cap e_2^\perp, \tag{6.26}$$

$$\forall\, x \in F_j,\ 2 \le j \le n-1, \quad h^1(x) = e_1^\perp \cap e(x)^\perp = e_1^\perp \cap (e_j + p(x) e_{j+1})^\perp. \tag{6.27}$$

From this we are able to compute higher refinements using Corollary 5.13, since all first refinement fibers are represented in the form used in this Corollary.

It is important to note that for every $x \in F_j$ there exists a neighborhood of this point which does not contain any other points from $\bigcup_{k=j}^n F_j$, so if $E \ni y_s \to x,\ y_s \ne x$, then we may assume that all y_s belong to $E \setminus \bigcup_{k=j}^n F_j$.

In the case $n = 2$, Corollary 6.6 implies that $h^1(0) = e_1^\perp$, while for $x \in F_1$ we have $h^1(x) = \{0\}$. Since any neighborhood of zero contains points from F_1, Lemma 5.12 implies that $h^2(0) = \{0\}$ and thus $h^1(0) \ne h^2(0)$ and

$$\text{st}\,(2, 1; h(E)) \ge 2,$$

as promised in Theorem 3.3.

We may confine our attention to the case $n \ge 3$.

Lemma 6.7. *Assume $n \geq 3$.*

(1) *For each $i = 2, \ldots, n-1$ and for each $x \in F_j$, $j = i+1, i+2, \ldots, n-1$, we have*

$$h^i(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp,$$
$$h^i(0) = e_1^\perp \cap \cdots \cap e_{n-1}^\perp.$$

(2) *For each $i = 2, \ldots,$ and for each $x \in F_j$, $j = 1, 2, \ldots, \min\{i, n-1\}$, we have*

$$h^i(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap e_j^\perp \cap e_{j+1}^\perp.$$

Proof. Induction by i.

Basis: $i = 2$.

• *Proof of (2) for $i = 2$.* We have to compute the second refinements at the points of F_1 and F_2. Let first consider $x \in F_1$.

Let $A_0 = \text{Iso } E$. By (6.25),

$$\forall\, y \in A_0 \quad h^1(y) = \mathbb{R}^n = 0^\perp.$$

There is a punctured neighborhood of x in which the only points from E belong to A_0. By Corollary 5.13,

$$\forall\, x \in F_1 \quad h^2(x) = h^1(x) \cap 0^\perp = e_1^\perp \cap e_2^\perp,$$

according to (6.26).

Let us compute $h^2(x)$ for $x \in F_2$. Let $A_0 = \text{Iso } E, A_1 = F_1$. It immediately follows from the construction that A_0, A_1 satisfy the condition (ii) of Corollary 5.13. By (6.26),

$$\forall\, y \in A_0 \quad h^1(y) = 0^\perp,$$
$$\forall\, y \in A_1 \quad h^1(y) = e_1^\perp \cap e_2^\perp.$$

So we are in the situation considered in Corollary 5.13. Also, $h^1(x) = e_1^\perp \cap e(x)^\perp$. By Corollary 5.13 we get

$$\begin{aligned} h^2(x) &= h^1(x) \cap (0^\perp \cap e_1^\perp \cap e_2^\perp) = (e_1^\perp \cap e(x)^\perp) \cap (e_1^\perp \cap e_2^\perp) \\ &= (e_1^\perp \cap e_2^\perp) \cap (e_1 + p(x)e_2 + p(x)^2 e_3)^\perp = e_1^\perp \cap e_2^\perp \cap e_3^\perp. \end{aligned}$$

The assertion (2) is proven for $i = 2$.

• *Proof of (1) for* $i = 2$. Let $x \in F_j$ for $i + 1 \leq j \leq n$ (possibly $x = 0$). Let $A_0 = \text{Iso } E, A_1 = F_1, \dots, A_{j-1} = F_{j-1}$. It immediately follows from the construction that $A_0, A_1, \dots, A_{j-1}$ satisfy the condition (ii) of Corollary 5.13. By (6.26), (6.27),

$$\begin{aligned} \forall\, y \in A_0 \quad & h^1(y) = 0^\perp, \\ \forall\, y \in A_1 \quad & h^1(y) = e_1^\perp \cap e_2^\perp, \\ \forall\, y \in A_2 \quad & h^1(y) = e_1^\perp \cap (e_2 + p(y)e_3)^\perp, \\ & \vdots \\ \forall\, y \in A_{j-1} \quad & h^1(y) = e_1^\perp \cap (e_{j-1} + p(y)e_j)^\perp, \end{aligned}$$

and $p(y) \to 0$, as $y \to x$. Assume now that $j \leq n-1$. Then

$$h^1(x) = e_1^\perp \cap e(x)^\perp.$$

By Corollary 5.13

$$\begin{aligned} h^2(x) &= h^1(x) \cap \big(0^\perp \cap (e_1^\perp \cap e_2^\perp) \cap (e_1^\perp \cap e_3^\perp) \cap \dots \cap (e_1^\perp \cap e_{j-1}^\perp)\big) \\ &= e_1^\perp \cap \dots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp, \end{aligned}$$

and the assertion (1) is proven for $j \leq n-1$.

The case $j = n$ includes only the point $x = 0$. In this case $h^1(0) = e_1^\perp$ and Corollary 5.13 imply

$$h^2(0) = e_1^\perp \cap \dots \cap e_{j-1}^\perp = e_1^\perp \cap \dots \cap e_{n-1}^\perp,$$

meaning that assertion (1) is proven for $i = 2$.

Inductive step $i - 1 \mapsto i$.

Assume we have already proven the Lemma for $i - 1$. Let us prove it for i. Take any $j \geq 1$, consider any $x \in F_j$.

Let $A_0 = \text{Iso } E, A_1 = F_1, \dots, A_{j-1} = F_{j-1}$.

We have to consider three cases: $j > i$, $j = i$ and $j < i$.

• *Case* $j < i$. By our assumptions,

$$\forall\, y \in A_k,\ 1 \leq k \leq j-1 < i-1, \quad h^{i-1}(y) = e_1^\perp \cap \dots \cap e_{k+1}^\perp.$$

Also,

$$h^{i-1}(x) = e_1^\perp \cap \dots \cap e_{j+1}^\perp.$$

Applying Corollary 5.13 as before, we immediately conclude that

$$h^i(x) = e_1^\perp \cap \dots \cap e_{j+1}^\perp.$$

- *Case* $j = i$. Again, by our assumptions

$$\forall\, y \in A_k,\ 1 \le k \le j-1 = i-1, \quad h^{i-1}(y) = e_1^\perp \cap \cdots \cap e_{k+1}^\perp.$$

Also, by our assumptions,

$$h^{i-1}(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp.$$

So, using Corollary 5.13 we see that

$$h^i(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp \cap \bigcap_{k \le j-1} e_{k+1} = e_1^\perp \cap \cdots \cap e_j^\perp \cap e_{j+1}^\perp.$$

- *Case* $j > i$. By our assumptions,

$$\forall\, y \in A_k,\ 1 \le k \le i-1 < j-1, \quad h^{i-1}(y) = e_1^\perp \cap \cdots \cap e_{k+1}^\perp.$$

$$\forall\, y \in A_k,\ j > k > i-1, \quad h^{i-1}(y) = e_1^\perp \cap \cdots \cap e_{k-1}^\perp \cap (e_k + p(y)e_{k+1})^\perp,$$

In addition, if $j \le n-1$ then

$$h^{i-1}(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp.$$

and if $j = n$ then $x = 0$ and

$$h^{i-1}(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp.$$

Applying Corollary 5.13 as before, we immediately conclude that

$$h^i(x) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp \cap \bigcap_{k \le i-1} e_{k+1}^\perp \cap \bigcap_{i-1<k<j} e_k^\perp$$

$$= e_1^\perp \cap \cdots \cap e_{j-1}^\perp \cap (e_j + p(x)e_{j+1})^\perp$$

in the case $j \le n-1$, and that

$$h^i(0) = e_1^\perp \cap \cdots \cap e_{j-1}^\perp.$$

■

6.10. Completion of the proof of Theorem 3.3

By Lemma 6.7, for any $x \in F_{n-1}$ we have $h^{n-1}(x) = \{0\}$. Since there is a sequence $x_1, x_2, \cdots \in F_{n-1}$ that tends to zero, Corollary 5.13 implies that $h^n(0) = \{0\}$. However, by Lemma 6.7 we have $h^{n-1}(0) \neq \{0\}$ and hence $h^{n-1}(0) \supsetneqq h^n(0)$, so

$$\mathrm{st}\,(n, 1; h(E)) \ge n,$$

and Theorem 3.3 is proven. ■

7. Proof of Theorem 3.4

Let $E \subset \mathbb{R}^n$ be a compact set. Let $h(E)$ be the homogeneous standard 1-bundle over E. We shall prove that

$$\mathrm{st}\,(n, 1; h(E)) \leq n + 1.$$

Due to Lemma 4.2, this will mean that

$$\mathrm{ST}\,(n, 1) \leq n + 1.$$

For $x \in E$ and $i \in \mathbb{N}$ we denote

$$J^i(x) = h^i(x)^{\perp}. \tag{7.1}$$

For a subspace $G \subset \mathbb{R}^n$, we denote $S(G) = \{x \in G; |x| = 1\}$, the unit sphere in G.

Lemma 7.1. *Fix $i \geq 1$, $x \in E$ and let $u \in \mathbb{R}^n$. Then $u \in h^{i+1}(x)$ if and only if, for any sequences $x_k \in E, v_k \in S(\mathbb{R}^n)$ such that $x_k \to x$ and $v_k \in J^i(x_k)$, we have*

$$\langle u, v_k \rangle \stackrel{k\to\infty}{\longrightarrow} 0.$$

Proof. By Lemma 5.12, $u \in h^{i+1}(x)$ if and only if

$$\lim_{\delta\to 0^+} \sup_{y \in E \cap B(x,\delta)} \inf_{w \in h^i(y)} |u - w| = 0.$$

The condition (7.1) implies that

$$\inf_{w \in H^i(y)} |u - w| = \sup_{v \in S(J^i(y))} |\langle u, v \rangle|\,.$$

Therefore we may reformulate Lemma 5.12 as

$$u \in h^{i+1}(x) \quad \Leftrightarrow \quad \lim_{\delta\to 0^+} \sup_{y \in E \cap B(x,\delta)} \sup_{v \in S(J^i(y))} |\langle u, v \rangle| = 0,$$

and the lemma follows by a standard argument. ■

Since $E \subset \mathbb{R}^n$, we may view $J^i(E)$ as a subset of $\mathbb{R}^{2n}$:

$$J^i(E) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n; x \in E; y \in J^i(x)\}.$$

We define $\overline{J^i(E)}$ to be the closure of $J^i(E) \subset \mathbb{R}^{2n}$. For $x \in E$ we set

$$\bar{J}^i(x) = \{y \in \mathbb{R}^n : (x, y) \in \overline{J^i(E)}\}.$$

Lemma 7.2. *For $i \geq 2, x \in E$,*

$$J^i(x) = \text{span } \bar{J}^{i-1}(x).$$

Proof. Let $u \in h^i(x)$ and let $0 \neq v \in \bar{J}^{i-1}(x)$. We will show that $\langle u, v \rangle = 0$. By the definition of $\bar{J}^{i-1}(x)$, there exist a sequence $E \ni x_k \to x$ and vectors $v_k \in J^{i-1}(x_k)$ such that $v_k \to v$. We may assume that $|v_k| = |v|$ for all k. Lemma 7.1 implies that

$$\left\langle u, \frac{v}{|v|} \right\rangle = \lim_{k \to \infty} \left\langle u, \frac{v_k}{|v_k|} \right\rangle = 0.$$

Hence $u \perp \bar{J}^{i-1}(x)$ and consequently,

$$h^i(x) \subset \left[\bar{J}^{i-1}(x)\right]^{\perp}.$$

Next, let $u \in \mathbb{R}^n$ be such that $u \perp \bar{J}^{i-1}(x)$. Assume on the contrary that $u \notin h^i(x)$. By Lemma 7.1 there exist a sequence $x_k \to x$ and $v_k \in S(J^{i-1}(x_k))$ such that for all k,

$$|\langle u, v_k \rangle| > \varepsilon_0$$

for some $\varepsilon_0 > 0$. Passing to a subsequence, if necessary, we may assume that there exists $v \in S(\mathbb{R}^n)$ such that $v_k \to v$. Note that by definition of $\overline{J^{i-1}}$ as the closure, $v \in \bar{J}^{i-1}(x)$. Yet,

$$|\langle u, v \rangle| = \lim_{k \to \infty} |\langle u, v_k \rangle| \geq \varepsilon_0 > 0$$

in contradiction to $u \perp \bar{J}^{i-1}(x)$. This shows that

$$h^i(x) = \left[\bar{J}^{i-1}(x)\right]^{\perp}.$$

Since $[J^i(x)]^{\perp} = h^i(x)$, we conclude that

$$J^i(x) = \text{span } \bar{J}^{i-1}(x)$$

and the lemma is proven. ■

Lemma 7.3. *Let $E \ni x_r \to x \in E$, and let $k, l > 0$ be integers. Assume that $\bar{J}^k(x_r)$ contains an l-dimensional subspace. Then $\bar{J}^k(x)$ also contains an l-dimensional subspace.*

Proof. Choose an l-dimensional subspace $U_r \subset \bar{J}^k(x_r)$. Recall that the Grassmannian $G_{n,l}$ of l-dimensional linear subspaces in the n-dimensional space is compact, with the Hausdorff metric

$$\text{dist}\,(E_1, E_2) = \max\left\{ \sup_{x \in S(E_1)} \inf_{y \in S(E_2)} |x - y|, \sup_{x \in S(E_2)} \inf_{y \in S(E_1)} |x - y| \right\}. \tag{7.2}$$

Hence we may select a subsequence U_{r_j} that converges to some l-dimensional subspace U. The set $\bar{J}^k(E)$ is closed, and hence necessarily $U \in \bar{J}^k(x)$. ■

Let

$$E_k = \{x \in E; \bar{J}^k(x) \neq \bar{J}^{k-1}(x)\},$$

and let $\overline{E_k}$ be the closure of E_k. We claim that $E_{k+1} \subset \overline{E_k}$ (and hence also $\overline{E_{k+1}} \subset \overline{E_k}$). Indeed, if $x \notin \overline{E_k}$, then there is a neighborhood of x in which $\overline{J^k} = \overline{J^{k-1}}$, and because our operations are local, we obtain $\bar{J}^l(x) = \bar{J}^{k-1}(x)$ for any $l \geq k-1$.

Lemma 7.4. *If $x \in \overline{E_k}$ then $\bar{J}^k(x)$ contains a k-dimensional subspace.*

Proof. By induction. Begin with the case $k = 1$. Note that for any $x \in E, u \in \mathbb{R}^n$ and $0 \neq t \in \mathbb{R}$,

$$u \in J^1(x) \quad \Leftrightarrow \quad tu \in J^1(x) \tag{7.3}$$

because $J^1(x)$ is a subspace. The condition (7.3) is closed, and hence also

$$u \in \bar{J}^1(x) \quad \Leftrightarrow \quad tu \in \bar{J}^1(x).$$

Therefore, whenever $x \in E_1$, we have $\bar{J}^1(x) \neq \bar{J}^0(x) = \{0\}$ and $\bar{J}^1(x)$ contains a one-dimensional subspace. If $y \in \overline{E_1}$ then there is a sequence $E_1 \ni y_r \to y$, and by Lemma 7.3, also $\bar{J}^1(y)$ contains a one-dimensional subspace.

Assume validity for $k-1$, and prove for k. Let $x \in E_k$. If there was a neighborhood $B(x, \delta)$ such that $J^k(y) = \bar{J}^{k-1}(y)$ for any $y \in E \cap B(x, \delta)$, then $\bar{J}^k(x) = \bar{J}^{k-1}(x)$ and $x \notin E_k$, since our operations are local. We conclude that for any $\delta > 0$ there is $y_\delta \in E \cap B(x, \delta)$ such that

$$J^k(y_\delta) \neq \bar{J}^{k-1}(y_\delta). \tag{7.4}$$

Hence $y_\delta \in E_k \subseteq \overline{E_{k-1}}$. Note that $J^k(y_\delta) = \operatorname{span} \bar{J}^{k-1}(y_\delta)$, and that by induction, $\bar{J}^{k-1}(y_\delta)$ contains a $k-1$ dimensional subspace.

Together with (7.4) we conclude that $\dim(J^k(y_\delta)) \geq k$. Clearly also $\dim(\bar{J}^k(y_\delta)) \geq k$, and by Lemma 7.3, $\dim(\bar{J}^k(x)) \geq k$. Hence, whenever $x \in E_k$, necessarily $\dim(\bar{J}^k(x)) \geq k$. For $x \in \overline{E_k}$ take a sequence $E_k \ni x_r \to x$ and use Lemma 7.3 to obtain

$$\dim(\bar{J}^k(x)) \geq k. \qquad \blacksquare$$

Proof of Theorem 3.4. Lemma 7.4 implies that $E_{n+1} = \emptyset$. Hence, for any $x \in E$ we have $\bar{J}^{n+1}(x) = \bar{J}^n(x)$. This implies by Lemma 7.2 that $J^{n+2}(x) = J^{n+1}(x)$ for any $x \in E$, and hence $h^{n+2}(E) = h^{n+1}(E)$ and $n+1$ Glaeser refinements are always sufficient. $\blacksquare$

8. A new proof of Glaeser's Theorem

Let us sketch a quick proof of Glaeser's C^1 Extension Theorem [8], based on Michael's Continuous Selection Theorem (see, e.g., [9, pp. 181-184]) and Whitney's Extension Theorem ([10], case $m = 1$ of Theorem 1).

Theorem 8.1. (E. Michael) *Let $E \subset \mathbb{R}^n$, and for any $x \in E$ let $h(x) \subset \mathbb{R}^n$ be a non-empty convex set. Assume that for every $x \in E$ and $u \in h(x)$,*

$$\lim_{\delta \to 0^+} \sup_{y \in E \cap B(x,\delta)} \inf_{v \in h(y)} |v - u| = 0.$$

Then there exists a continuous map $p : E \to \mathbb{R}^n$ such that $p(x) \in h(x)$ for every $x \in E$.

Theorem 8.2. (H. Whitney) *Let $E \subset \mathbb{R}^n$. For any $x \in E$ let $p_x \in \mathcal{P}_n^1$ be a 1-jet. Assume that for any $x \in E$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in E \cap B(x, \delta)$ then,*

$$|(p_{x_1} - p_{x_2})(x_2)| < \varepsilon |x_1 - x_2|, \quad |\nabla(p_{x_1} - p_{x_2})(x_2)| < \varepsilon.$$

Then there exists a C^1 function f such that $J_x^1 f = p_x$ for any $x \in E$.

Glaeser's result [8] is equivalent to the following

Theorem 8.3. *Consider a function $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}^n$, $n \geq 2$. Let $H^0(E) = H_f(E)$ be the standard 1-bundle. Let $H^1(E)$ be its Glaeser 2-refinement. For $i > 1$ let $H^{i+1}(E)$ be the Glaeser 1-refinement of $H^i(E)$. The function f extends to a C^1 function on $\mathbb{R}^n$ if and only if $H^n(E)$ is 1-refinable.*

Proof. If f extends to a C^1 function on $\mathbb{R}^n$, then the bundle $H^0(E)$ allows a section, so it has refinements of all orders. Therefore $H^n(E)$ is refinable.

Now assume that $H^n(E)$ is 1-refinable. Therefore $\forall\, x \in E$, $H^{n+1}(x) \neq \emptyset$. By Theorem 3.4, $H^{n+2}(E) = H^{n+1}(E)$.

As before, for each $i \geq 0$ consider

$$\mathfrak{h}^i(x) = \{u \in \mathbb{R}^n : f(x) + \langle u, y - x \rangle \in H^i(x)\}.$$

Obviously,

$$\forall\, x \in E \quad \mathfrak{h}^{n+2}(x) = \mathfrak{h}^{n+1}(x).$$

Corollary 5.12 implies that the non-empty convex sets $\mathfrak{h}^{n+1}(x)$, $x \in E$, satisfy the conditions of Theorem 8.1. Hence it is possible to choose a continuous section, i.e. a continuous map $E \ni x \mapsto u_x \in \mathfrak{h}^{n+1}(x)$. Therefore, for any $x_0 \in E, \varepsilon > 0$ there is $\delta_1(\varepsilon, x) > 0$ such that for $x, y \in E \cap B(x_0, \delta_1(\varepsilon, x))$,

$$|u_x - u_y| < \varepsilon.$$

For any $x \in E$, we know, in particular, that $f(x) + \langle u_x, y - x\rangle \in H^1(x)$. By Lemma 5.1, for any $\varepsilon > 0$ there is $\delta_2(\varepsilon, x) > 0$ such that if $x_1, x_2 \in E \cap B(x, \delta_2(\varepsilon))$, then

$$|f(x_1) + \langle u_x, x_2 - x_1\rangle - f(x_2)| < \varepsilon|x_1 - x_2|.$$

Let

$$\delta(\varepsilon, x) = \min\left\{\delta_1\left(\frac{\varepsilon}{2}, x\right), \delta_2\left(\frac{\varepsilon}{2}, x\right)\right\}.$$

If $x_1, x_2 \in E \cap B(x, \delta)$ then,

$$\begin{aligned} &|f(x_1) + \langle u_{x_1}, x_2 - x_1\rangle - f(x_2)| \\ &\leq \ |f(x_1) + \langle u_x, x_2 - x_1\rangle - f(x_2)| + |x_1 - x_2||u_x - u_{x_1}| < \varepsilon|x_1 - x_2| \end{aligned}$$

and also $|u_{x_1} - u_{x_2}| < \varepsilon$. We conclude that $p_x(y) = f(x) + \langle u_x, y - x\rangle$ satisfy the conditions of Theorem 8.2, and hence a C^1 extension exists. ■

Remark. We could avoid using Theorem 3.4, whose proof is quite complicated, and use instead a much easier estimate (2.3). This will require replacing the condition of refinability of $H^n(E)$ by a seemingly stronger (but, actually, equivalent – due to Theorem 3.4) condition of refinability of $H^{2n+2}(E)$.

References

[1] Bierstone, E., Milman, P. and Pawłucki, W.: Differentiable functions defined in closed sets. A problem of Whitney. *Invent. Math.* **151** (2003), 329–352.

[2] Bierstone, E., Milman, P. and Pawłucki, W.: Higher-order tangents and Fefferman's paper on Whitney's extension problem. *Ann. of Math. (2)* **164** (2006), 361–370.

[3] Brudnyi, Yu. and Shvartsman, P.: Generalizations of Whitney's extension theorem. *Internat. Math. Res. Notices* **3** (1994), 129–139.

[4] Brudnyi, Yu. and Shvartsman, P.: Whitney's extension problem for multivariate $C^{1,\omega}$-functions. *Trans. Amer. Math. Soc.* **353** (2001), 2487–2512.

[5] Fefferman, C. L.: A sharp form of Whitney's extension theorem. *Ann. of Math. (2)* **161** (2005), 509–577.

[6] Fefferman, C. L.: A generalized sharp Whitney theorem for jets. *Revista Mat. Iberoamericana* **21** (2005), 577–688.

[7] Fefferman, C. L.: Whitney's extension problem for C^m. *Ann. of Math. (2)* **164** (2006), 313–359.

[8] GLAESER, G.: Étude de quelques algèbres tayloriennes. *J. Analyse Math.* **6** (1958), 1–124.

[9] HOLMES, R. B.: *Geometric functional analysis and its applications*. Graduate Texts in Mathematics **24**. Springer-Verlag, New York-Heidelberg, 1975.

[10] WHITNEY, H.: Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.* **36** (1934), 63–89.

[11] WHITNEY, H.: Differentiable functions defined in closed sets. I. *Trans. Amer. Math. Soc.* **36** (1934), 369–387.

[12] WHITNEY, H.: Functions differentiable on the boundaries of regions. *Ann. of Math. (2)* **35** (1934), 482–485.

[13] ZOBIN, N.: Whitney's problem on extendability of functions and an intrinsic metric. *Adv. Math.* **133** (1998), 96–132.

[14] ZOBIN, N.: Extension of smooth functions from finitely connected planar domains. *J. Geom. Anal.* **9** (1999), 491–511.

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Bo'az Klartag
School of Mathematics
Institute for Advanced Study
1 Einstein Dr.
08540, Princeton, NJ, USA
klartag@ias.edu

Nahum Zobin
Department of Mathematics
College of William and Mary, PO Box 8795
23187-8795, Williamsburg, VA, USA
zobin@math.wm.edu

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