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# On minimal non-supersoluble groups

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*Dedicated to the memory of Klaus Doerk (1939–2004)*

**Abstract**

The aim of this paper is to classify the finite minimal non-$p$-supersoluble groups, $p$ a prime number, in the $p$-soluble universe.

## 1. Introduction

All groups considered in this paper are finite.

Given a class $\mathfrak{X}$ of groups, we say that a group $G$ is a *minimal non-*$\mathfrak{X}$*-group* or an $\mathfrak{X}$*-critical group* if $G \notin \mathfrak{X}$, but all proper subgroups of $G$ belong to $\mathfrak{X}$. It is rather clear that detailed knowledge of the structure of $\mathfrak{X}$-critical groups could help to give information about what makes a group belong to $\mathfrak{X}$.

Minimal non-$\mathfrak{X}$-groups have been studied for various classes of groups $\mathfrak{X}$. For instance, Miller and Moreno [10] analysed minimal non-abelian groups, while Schmidt [14] studied minimal non-nilpotent groups. These groups are now known as *Schmidt groups*. Rédei classified completely the minimal non-abelian groups in [12] and the Schmidt groups in [13]. More precisely,

**Theorem 1** ([12]). *The minimal non-abelian groups are of one of the following types:*

1. $G = [V_q]C_{r^s}$, *where* $q$ *and* $r$ *are different prime numbers,* $s$ *is a positive integer, and* $V_q$ *is an irreducible* $C_{r^s}$*-module over the field of* $q$ *elements with kernel the maximal subgroup of* $C_{r^s}$*,*
2. *the quaternion group of order* $8$*,*
3. $G_{II}(q,m,n) = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}}\rangle$*, where* $q$ *is a prime number,* $m \geq 2$*,* $n \geq 1$*, of order* $q^{m+n}$*, and*

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4. *$G_{III}(q, m, n) = \langle a, b \mid a^{q^m} = b^{q^n} = [a, b]^q = [a, b, a] = [a, b, b] = 1\rangle$, where $q$ is a prime number, $m \geq n \geq 1$, of order $q^{m+n+1}$.*

We must note that there is a misprint in the presentation of the last type of groups in Huppert's book [7; Aufgabe III.22].

**Theorem 2** ([13], see also [2])**.** *Schmidt groups fall into the following classes:*

1. *$G = [P]Q$, where $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, with $q$ a prime not dividing $p - 1$ and $P$ an irreducible $Q$-module over the field of $p$ elements with kernel $\langle z^q\rangle$ in $Q$.*
2. *$G = [P]Q$, where $P$ is a non-abelian special $p$-group of rank $2m$, the order of $p$ modulo $q$ being $2m$, $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, $z$ induces an automorphism in $P$ such that $P/\Phi(P)$ is a faithful irreducible $Q$-module, and $z$ centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*
3. *$G = [P]Q$, where $P = \langle a\rangle$ is a normal subgroup of order $p$, $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, with $q$ dividing $p - 1$, and $a^z = a^i$, where $i$ is the least primitive $q$-th root of unity modulo $p$.*

Here $[K]H$ denotes the semidirect product of $K$ with $H$, where $H$ acts on $K$.

Itô [8] considered the minimal non-$p$-nilpotent groups for a prime $p$, which turn out to be Schmidt groups.

Doerk [5] was the first author in studying the minimal non-supersoluble groups. Later, Nagrebeckiĭ [11] classified them.

Let $p$ be a prime number. A group $G$ is said to be *$p$-supersoluble* whenever $G$ is $p$-soluble and all $p$-chief factors of $G$ are cyclic groups of order $p$.

Kontorovič and Nagrebeckiĭ [9] studied the minimal non-$p$-supersoluble groups for a prime $p$ with trivial Frattini subgroup. Tuccillo [15] tried to classify all minimal non-$p$-supersoluble groups in the soluble case, and gave results about non-soluble minimal non-$p$-supersoluble groups. Unfortunately, there is a gap in his paper and some groups are missing from his classification.

**Example 3.** The extraspecial group $N = \langle a, b\rangle$ of order $41^3$ and exponent 41 has automorphisms $y$ of order 5 and $z$ of order 8, given by $a^y = a^{10}$, $b^y = b^{37}$, and $a^z = b^{19}$, $b^z = a^{35}$, satisfying $y^z = y^{-1}$. The semidirect product $G$ of $N$ by $\langle x, y\rangle$ is a minimal non-supersoluble group such that the Frattini subgroup $\Phi(N)$ of $N$ is not a central subgroup of $G$. This is a minimal non-41-supersoluble group not appearing in any type of Tuccillo's result.

**Example 4.** The extraspecial group $N = \langle a, b\rangle$ of order $17^3$ and exponent 17 has an automorphism $z$ of order 32 given by $a^z = b$, $b^z = a^3$. The semidirect product $G = [N]\langle z\rangle$ is a minimal non-17-supersoluble group. It is clear that $[a, b]^z = [a, b]^{14}$ and so $[a, b]$ does not belong to the centre of $G$. This is another group missing in Tuccillo's work.

**Example 5.** The automorphism group of the extraspecial group of order $7^3$ and exponent 7 has a subgroup isomorphic to the symmetric group $\Sigma_3$ of degree 3. The corresponding semidirect product is a minimal non-7-supersoluble group not corresponding to any case of Tuccillo's work.

**Example 6.** Let $E = \langle x_1, x_2\rangle$ be an extraspecial group of order 125 and exponent 5. This group has two automorphisms $\alpha$ and $\beta$ given by $x_1^\alpha = x_2^4$, $x_2^\alpha = x_1$, $x_1^\beta = x_1^2$, and $x_2^\beta = x_2^3$ generating a quaternion group $H$ of order 8 such that the corresponding semidirect product $[E]H$ is a minimal non-5-supersoluble group. This group is also missing in [15].

**Example 7.** With the same notation as in Example 6, the automorphisms $\beta$ and $\gamma$ defined by $x_1^\gamma = x_2$, $x_2^\gamma = x_1$ generate a dihedral group $D$ of order 8. The corresponding semidirect product $[E]D$ is a minimal non-5-supersoluble group not appearing in [15].

By looking at these examples, we see that the classification of minimal non-$p$-supersoluble groups given in [15] is far from being complete. In our examples, the Frattini subgroup of the Sylow $p$-subgroup is not a central subgroup, contrary to the claim in [15; 1.7].

The aim of this paper is to give the complete classification of minimal non-$p$-supersoluble groups in the $p$-soluble universe. This restriction is motivated by the following result.

**Proposition 8.** *Let $G$ be a minimal non-$p$-supersoluble group. Then either $G/\Phi(G)$ is a simple group of order divisible by $p$, or $G$ is $p$-soluble.*

Our main theorem is the following:

**Theorem 9.** *The minimal $p$-soluble non-$p$-supersoluble groups for a prime $p$ are exactly the groups of the following types:*

**Type 1:** *Let $q$ be a prime number such that $q$ divides $p - 1$. Let $C$ be a cyclic group of order $p^s$, with $s \geq 1$, and let $M$ be an irreducible $C$-module over the field of $q$ elements with kernel the maximal subgroup of $C$. Consider a group $E$ with a normal $q$-subgroup $F$ contained in the Frattini subgroup of $E$ and $E/F$ isomorphic to the semidirect product $[M]C$. Let $N$ be an irreducible $E$-module over the field of $p$*

*elements with kernel the Frattini subgroup of $E$. Let $G = [N]E$ be the corresponding semidirect product. In this case, $\Phi(G)_p$, the Sylow $p$-subgroup of $\Phi(G)$, which coincides with the Frattini subgroup of a Sylow $p$-subgroup of $E$, is a central subgroup of $G$ and $\Phi(G)_q$, the Sylow $q$-subgroup of $\Phi(G)$, is equal to $\Phi(E)$, which coincides with the Frattini subgroup of a Sylow $q$-subgroup of $E$ and centralises $N$.*

**Type 2:** *$G = [P]Q$, where $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, with $q$ a prime not dividing $p-1$, and $P$ is an irreducible $Q$-module over the field of $p$ elements with kernel $\langle z^q\rangle$ in $Q$.*

**Type 3:** *$G = [P]Q$, where $P$ is a non-abelian special $p$-group of rank $2m$, the order of $p$ modulo $q$ being $2m$, $q$ is a prime, $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, $z$ induces an automorphism in $P$ such that $P/\Phi(P)$ is a faithful and irreducible $Q$-module, and $z$ centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

**Type 4:** *$G = [P]Q$, where $P = \langle a_0, a_1, \dots, a_{q-1}\rangle$ is an elementary abelian $p$-group of order $p^q$, $Q = \langle z\rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^f$ is the highest power of $q$ dividing $p-1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where $i$ is a primitive $q^f$-th root of unity modulo $p$.*

**Type 5:** *$G = [P]Q$, where $P = \langle a_0, a_1\rangle$ is an extraspecial group of order $p^3$ and exponent $p$, $Q = \langle z\rangle$ is cyclic of order $2^r$, with $2^f$ the largest power of $2$ dividing $p-1$ and $r > f \geq 1$. Define $a_1 = a_0^z$ and $a_1^z = a_0^i x$, where $x \in \langle [a_0, a_1]\rangle$ and $i$ is a primitive $2^f$-th root of unity modulo $p$.*

**Type 6:** *$G = [P]E$, where $E$ is a $2$-group with a normal subgroup $F$ such that $F \leq \Phi(E)$ and $E/F$ is isomorphic to a quaternion group of order $8$ and $P$ is an irreducible module for $E$ with kernel $F$ over the field of $p$ elements of dimension $2$, where $4 \mid p-1$.*

**Type 7:** *$G = [P]E$, where $E$ is a $2$-group with a normal subgroup $F$ such that $F \leq \Phi(E)$ and $E/F$ is isomorphic to a quaternion group of order $8$, $P$ is an extraspecial group of order $p^3$ and exponent $p$, where $4 \mid p-1$, and $P/\Phi(P)$ is an irreducible module for $E$ with kernel $F$ over the field of $p$ elements.*

**Type 8:** *$G = [P]E$, where $E$ is a $q$-group for a prime $q$ with a normal subgroup $F$ such that $F \leq \Phi(E)$ and $E/F$ is isomorphic to a group $G_{II}(q, m, 1)$ of Theorem 1, $P$ is an irreducible $E$-module of dimension $q$ over the field of $p$ elements with kernel $F$, and $q^m$ divides $p-1$.*

**Type 9:** *$G = [P]E$, where $E$ is a $2$-group with a normal subgroup $F$ such that $F \leq \Phi(E)$ and $E/F$ is isomorphic to a group $G_{II}(2, m, 1)$ of Theorem 1, $P$ is an extraspecial group of order $p^3$ and exponent $p$ such* *that $P/\Phi(P)$ is an irreducible $E$-module of dimension 2 over the field of $p$ elements with kernel $F$, and $2^m$ divides $p-1$.*

**Type 10:** *$G = [P]E$, where $E$ is a $q$-group for a prime $q$ with a normal subgroup $F$ such that $F \leq \Phi(E)$ and $E/F$ is isomorphic to an extraspecial group of order $q^3$ and exponent $q$, with $q$ odd, $P$ is an irreducible $E$-module over the field of $p$ elements with kernel $F$ and dimension $q$, and $q$ divides $p-1$.*

**Type 11:** *$G = [P]MC$, where $C$ is a cyclic subgroup of order $r^{s+t}$, with $r$ a prime number and $s$ and $t$ integers such that $s \geq 1$ and $t \geq 0$, normalising a Sylow $q$-subgroup $M$ of $G$, $M/\Phi(M)$ is an irreducible $C$-module over the field of $q$ elements, $q$ a prime, with kernel the subgroup $D$ of order $r^t$ of $C$, and $P$ is an irreducible $MC$-module over the field of $p$ elements, where $q$ and $r^s$ divide $p-1$. In this case, $\Phi(G)_{p'}$, the Hall $p'$-subgroup of $\Phi(G)$, coincides with $\Phi(M) \times D$ and centralises $P$.*

**Type 12:** *$G = [P]MC$, where $C$ is a cyclic subgroup of order $2^{s+t}$, with $s$ and $t$ integers such that $s \geq 1$ and $t \geq 0$, normalising a Sylow $q$-subgroup $M$ of $G$, $q$ a prime, $M/\Phi(M)$ is an irreducible $C$-module over the field of $q$ elements with kernel the subgroup $D$ of order $2^t$ of $C$, and $P$ is an extraspecial group of order $p^3$ and exponent $p$ such that $P/\Phi(P)$ is an irreducible $MC$-module over the field of $p$ elements, where $q$ and $2^s$ divide $p-1$. In this case, $\Phi(G)_{p'}$, the Hall $p'$-subgroup of $\Phi(G)$, is equal to $\Phi(M) \times D$ and centralises $P$.*

From Proposition 8 and Theorem 9 we deduce immediately that a minimal non-$p$-supersoluble group is either a Frattini extension of a non-abelian simple group of order divisible by $p$, or a soluble group.

As a consequence of Theorem 9, bearing in mind that minimal non-supersoluble groups are soluble by [5] and minimal non-$p$-supersoluble groups for a prime $p$, we obtain the classification of minimal non-supersoluble groups:

**Theorem 10.** *The minimal non-supersoluble groups are exactly the groups of Types 2 to 12 of Theorem 9, with $r$ dividing $q-1$ in the case of groups of Type 11.*

The classification of minimal non-$p$-supersoluble groups can be applied to get some new criteria for supersolubility. A well-known theorem of Buckley [4] states that if a group $G$ has odd order and all its subgroups of prime order are normal, then $G$ is supersoluble. The next generalisation follows easily from our classification:

**Theorem 11.** *Let $G$ be a group whose subgroups of prime order permute with all Sylow subgroups of $G$ and no section of $G$ is isomorphic to the quaternion group of order* 8*. Then $G$ is supersoluble.*

As a final remark, we mention that Tuccillo [15] also gave some partial results for Frattini extensions of non-abelian simple groups of order divisible by $p$. Looking at the results of Section 4 of that paper, it seems that the classification of minimal non-$p$-supersoluble groups in the general finite universe is a hard task.

## 2. Preliminary results

First we gather the main properties of a minimal non-supersoluble group. They appear in Doerk's paper [5].

**Theorem 12.** *Let $G$ be a minimal non-supersoluble group. We have:*

1. *$G$ is soluble.*
2. *$G$ has a unique normal Sylow subgroup $P$.*
3. *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
4. *The Frattini subgroup $\Phi(P)$ of $P$ is supersolubly embedded in $G$, i. e., there exists a series $1 = N_0 \leq N_1 \leq \cdots \leq N_m = \Phi(P)$ such that $N_i$ is a normal subgroup of $G$ and $|N_i/N_{i-1}|$ is prime for $1 \leq i \leq m$.*
5. *$\Phi(P) \leq \mathrm{Z}(P)$; in particular, $P$ has class at most* 2*.*
6. *The derived subgroup $P'$ of $P$ has at most exponent $p$, where $p$ is the prime dividing $|P|$.*
7. *For $p > 2$, $P$ has exponent $p$; for $p = 2$, $P$ has exponent at most* 4*.*
8. *Let $Q$ be a complement to $P$ in $G$. Then $Q \cap \mathrm{C}_G\big(P/\Phi(P)\big) = \Phi(G) \cap \Phi(Q) = \Phi(G) \cap Q$.*
9. *If $\overline{Q} = Q/\big(Q \cap \Phi(G)\big)$, then $\overline{Q}$ is a minimal non-abelian group or a cyclic group of prime power order.*

In [6; VII, 6.18], some properties of critical groups for a saturated formation in the soluble universe are given. This result has been extended to the general finite universe by the first author and Pedraza-Aguilera. Recall that if $\mathfrak{F}$ is a formation, the $\mathfrak{F}$-residual of a group $G$, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of $G$ such that $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$.

**Lemma 13** ([3; Theorem 1 and Proposition 1]). *Let $\mathfrak{F}$ be a saturated formation.*

1. *Assume that $G$ is a group such that $G$ does not belong to $\mathfrak{F}$, but all its proper subgroups belong to $\mathfrak{F}$. Then $\mathrm{F}'(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$, where $\mathrm{F}'(G) = \mathrm{Soc}\big(G \bmod \Phi(G)\big)$, and $\mathrm{F}'(G) = G^{\mathfrak{F}}\Phi(G)$. In addition, if the derived subgroup of $G^{\mathfrak{F}}$ is a proper subgroup of $G^{\mathfrak{F}}$, then $G^{\mathfrak{F}}$ is a soluble group. Furthermore, if $G^{\mathfrak{F}}$ is soluble, then $\mathrm{F}'(G) = \mathrm{F}(G)$, the Fitting subgroup of $G$. Moreover $(G^{\mathfrak{F}})' = T \cap G^{\mathfrak{F}}$ for every maximal subgroup $T$ of $G$ such that $G/\mathrm{Core}_G(T) \notin \mathfrak{F}$ and $\mathrm{F}'(G)T = G$.*
2. *Assume that $G$ is a group such that $G$ does not belong to $\mathfrak{F}$ and there exists a maximal subgroup $M$ of $G$ such that $M \in \mathfrak{F}$ and $G = M\,\mathrm{F}(G)$. Then $G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of $G$, $G^{\mathfrak{F}}$ is a $p$-group for some prime $p$, $G^{\mathfrak{F}}$ has exponent $p$ if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, either $G^{\mathfrak{F}}$ is elementary abelian or $(G^{\mathfrak{F}})' = \mathrm{Z}(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.*

It is clear that the class $\mathfrak{F}$ of all $p$-supersoluble groups for a given prime $p$ is a saturated formation [7; VI, 8.3]. Thus Lemma 13 applies to this class.

The following series of lemmas is also needed in the proof of Theorem 9.

**Lemma 14.** *Let $N$ be a non-abelian special normal $p$-subgroup of a group $G$, $p$ a prime, such that $N/\Phi(N)$ is a minimal normal subgroup of $G/\Phi(N)$. Assume that there exists a series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = \Phi(N)$ with $N_i$ normal in $G$ for all $i$ and cyclic factors $N_i/N_{i-1}$ of order $p$ for $1 \leq i \leq t$. Then $N/\Phi(N)$ has order $p^{2m}$ for an integer $m$.*

**Proof.** The result holds if $N$ is extraspecial by [6; A, 20.4]. Assume that $N$ is not extraspecial. Let $T = N_1$ be a minimal normal subgroup of $G$ contained in $\Phi(P)$, then $T$ has order $p$. It is clear that $(N/T)' = N'/T$ and $\Phi(N/T) = \Phi(N)/T$. Consequently $(N/T)' = \Phi(N/T)$. On the other hand, $\Phi(N/T) = \Phi(N)/T = \mathrm{Z}(N)/T \leq \mathrm{Z}(N/T)$. If $\Phi(N/T) \neq \mathrm{Z}(N/T)$, then $\mathrm{Z}(N/T) = N/T$ because $N/\Phi(N)$ is a chief factor of $G$, but this implies that $N/T$ is abelian, in particular, $T = N'$ and $N$ is extraspecial, a contradiction. Therefore $G/T$ satisfies the hypothesis of the lemma and $N/T$ is non-abelian. By induction, $(N/T)/\Phi(N/T) \cong N/\Phi(N)$ has order $p^{2m}$. ■

**Lemma 15.** *Let $G$ be a group, and let $N$ be a normal subgroup of $G$ contained in $\Phi(G)$. If $p$ is a prime and $G$ is a minimal non-$p$-supersoluble group, then $G/N$ is a minimal non-$p$-supersoluble group.*

*Conversely, if $G/N$ is a minimal non-$p$-supersoluble group, $N \leq \Phi(G)$, and there exists a series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = N$ with $N_i$ normal in $G$*

*for all $i$ and whose factors $N_i/N_{i-1}$ are either cyclic of order $p$ or $p'$-groups for $1 \le i \le t$, then $G$ is a minimal non-$p$-supersoluble group.*

**Proof.** Assume that $G$ is a minimal non-$p$-supersoluble group and $N \le \Phi(G)$. If $M/N$ is a proper subgroup of $G/N$, then $M$ is a proper subgroup of $G$. Hence $M$ is $p$-supersoluble, and so is $M/N$. If $G/N$ were $p$-supersoluble, since $N \le \Phi(G)$, $G$ would be $p$-supersoluble, a contradiction. Therefore $G/N$ is minimal non-$p$-supersoluble.

Conversely, assume that $G/N$ is a minimal non-$p$-supersoluble group, $N \le \Phi(G)$, and that there exists a series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = N$ with $N_i$ normal in $G$ for all $i$ and factors $N_i/N_{i-1}$ cyclic of order $p$ or $p'$-groups for $1 \le i \le t$. It is clear that $G$ cannot be $p$-supersoluble. Let $M$ be a maximal subgroup of $G$. Since $N \le \Phi(G)$, $N \le M$. Thus $M/N$ is $p$-supersoluble. On the other hand, it is clear that every chief factor of $M$ below $N$ is either a $p'$-group or a cyclic group of order $p$. Consequently, $M$ is $p$-supersoluble. ■

**Lemma 16** ([1]). *Let $A$ be a group, and let $B$ be a normal subgroup of $A$ of prime index $r$ dividing $p-1$, $p$ a prime. If $M$ is an irreducible and faithful $A$-module over $\mathrm{GF}(p)$ of dimension greater than $1$ and the restriction of $M$ to $B$ is a sum of irreducible $B$-modules of dimension $1$, then $M$ has dimension $r$. In this case, $M$ is isomorphic to the induced module of one of the direct summands of $M_B$ from $B$ up to $A$.*

In the rest of the paper, $\mathfrak{F}$ will denote the formation of all $p$-supersoluble groups, $p$ a prime.

**Lemma 17.** *Let $G$ be a minimal non-$p$-supersoluble group whose $p$-supersoluble residual $N = G^{\mathfrak{F}}$ is normal Sylow $p$-subgroup. Then a Hall $p'$-subgroup $R/\Phi(G)$ of $G/\Phi(G)$ is either cyclic of prime power order or a minimal non-abelian group.*

**Proof.** By Lemma 15, we can assume without loss of generality that $\Phi(G)=1$. Then, by Lemma 13, $G$ is a primitive group and $\mathrm{C}_G(N) = N$. In particular, for each subgroup $X$ of $G$, we have that $\mathrm{O}_{p',p}(XN) = N$. Let $M$ be a maximal subgroup of $R$. Then $MN$ is a $p$-supersoluble group and so $MN/\mathrm{O}_{p',p}(MN) = MN/N$ is abelian of exponent dividing $p-1$. Therefore if $R$ is non-abelian, then it is a minimal non-abelian group. Suppose that $R$ is abelian. If $R$ has a unique maximal subgroup, then $R$ is cyclic of prime power order. Assume now that $R$ has at least two different maximal subgroups. Then $R$ is a product of two subgroups of exponent dividing $p-1$. Consequently $R$ has exponent $p-1$ and so $N$ is a cyclic group of order $p$ by [6; B, 9.8], a contradiction. Therefore if $R$ is not cyclic of prime power order, $R$ must be a minimal non-abelian group and the lemma is proved. ■

**Lemma 18.** *Let $G$ be a minimal non-$p$-supersoluble group with a normal Sylow $p$-subgroup $N$ such that $G/\Phi(N)$ is a Schmidt group. Then $G$ is a Schmidt group.*

**Proof.** Let $G$ be a minimal non-$p$-supersoluble group with a normal Sylow $p$-subgroup $N$ such that $G/\Phi(N)$ is a Schmidt group. Then $G = NQ$, for a Hall $p'$-subgroup $Q$ of $G$. Moreover, since $G$ is not $p$-supersoluble and $G/\Phi(N)$ is a Schmidt group, we have that $Q$ is a cyclic $q$-group for a prime $q$ and $q$ does not divide $p-1$ by Theorem 2. Let $M$ be a maximal subgroup of $G$. If $N$ is not contained in $M$, then a conjugate of $Q$ is contained in $M$ and so we can assume without loss of generality that $M = \Phi(N)Q$. Since $q$ does not divide $p-1$ and $M$ is $p$-supersoluble, we have that $Q$ centralises all chief factors of a chief series of $M$ passing through $\Phi(N)$. But by [6; A, 12.4], it follows that $Q$ centralises $\Phi(N)$ by and so $M$ is nilpotent. If $N$ is contained in $M$, then $M$ is a normal subgroup of $G$ such that $M/\Phi(N)$ is nilpotent. By [7; III, 3.5], it follows that $M$ is nilpotent. This completes the proof. ■

## 3. Proof of the main theorems

**Proof of Proposition 8.** By Lemma 13, $G/\Phi(G)$ has a unique minimal normal subgroup $T/\Phi(G)$ and $T = G^{\mathfrak{F}}\Phi(G)$. It follows that $T/\Phi(G)$ must have order divisible by $p$. Assume that $T/\Phi(G)$ is a direct product of non-abelian simple groups. We note that, since $G/\Phi(G)$ is a minimal non-$p$-supersoluble group by Lemma 15, $T/\Phi(G) = G/\Phi(G)$ and so $G/\Phi(G)$ is a simple non-abelian group.

Assume now that $T/\Phi(G)$ is a $p$-group. By Lemma 13, we have that $G^{\mathfrak{F}}$ is a $p$-group. In this case, $T/\Phi(G)$ is complemented by a maximal subgroup $M/\Phi(G)$ of $G/\Phi(G)$. Since $M$ is $p$-supersoluble, so is $M/\Phi(G)$. Therefore $G/\Phi(G)$ is $p$-soluble. It follows that $G$ is $p$-soluble. ■

**Proof of Theorem 9.** Assume that $G$ is a $p$-soluble minimal non-$p$-supersoluble group. By Lemma 13 and Proposition 8, $N = G^{\mathfrak{F}}$ is a $p$-group.

Assume first that $N$ is not a Sylow subgroup of $G$. By Lemma 13, $N/\Phi(N)$ is non-cyclic.

Assume that $\Phi(G) = 1$. Then $N$ is the unique minimal normal subgroup of $G$, which is an elementary abelian $p$-group, and it is complemented by a subgroup, $R$ say. Moreover, $N$ is self-centralising in $G$. This implies that $\mathrm{O}_{p',p}(G) = N = \mathrm{O}_p(G)$. Since $N$ is not a Sylow $p$-subgroup of $G$, we have that $p$ divides the order of $R$. Consider a maximal normal subgroup $M$ of $R$. Observe that $NM$ is a $p$-supersoluble group and $\mathrm{O}_{p',p}(NM) = \mathrm{O}_p(NM) = N$ because $\mathrm{O}_p(M)$ is contained in $\mathrm{O}_p(R) = 1$. Therefore $M \cong MN/\mathrm{O}_{p',p}(MN)$

is abelian of exponent dividing $p-1$. It follows that $M$ is a normal Hall $p'$-subgroup of $R$ and $|R:M| = p$ because $p$ divides $|R|$. In particular, $M$ is the only maximal normal subgroup of $R$. Moreover, if $C$ is a Sylow $p$-subgroup of $R$, then $C$ is a cyclic group of order $p$.

Let $M_0$ be a normal subgroup of $R$ such that $M/M_0$ is a chief factor of $R$. Let $X = NM_0C$. Since $X$ is a proper subgroup of $G$, we have that $X$ is $p$-supersoluble. Hence $X/\mathrm{O}_{p',p}(X)$ is an abelian group of exponent dividing $p-1$. It follows that $C \leq \mathrm{O}_{p',p}(X)$. In particular, $C = M_0C \cap \mathrm{O}_{p',p}(X)$ is a normal subgroup of $M_0C$ which intersects trivially $M_0$. We conclude that $C$ centralises $M_0$. If $M_1$ is another normal subgroup of $R$ such that $M/M_1$ is a chief factor of $R$, then $M = M_0M_1$. The same argument shows that $C$ centralises $M_1$ and so $C$ centralises $M$ as well, a contradiction because in this case $C \leq \mathrm{Z}(R)$ and then $C \leq \mathrm{O}_p(R) = 1$. Consequently $M_0$ is the unique such normal subgroup. Since $M$ is abelian, we have that $M_0 \leq \mathrm{Z}(R)$.

Now $R$ has an irreducible and faithful module $N$ over $\mathrm{GF}(p)$. By [6; B, 9.4], $\mathrm{Z}(R)$ is cyclic. In particular, $M_0$ is cyclic. We will prove next that $M_0 = 1$. In order to do so, assume, by way of contradiction, that $M$ is not a minimal normal subgroup of $R$. First of all, if $M$ is not a $q$-group for a prime $q$, then $M$ is a direct product of its Sylow subgroups, but all of them should be contained in $M_0$, a contradiction. Therefore, $M$ is a $q$-group for a prime $q$. Since $M$ has exponent dividing $p-1$, we have that $q$ divides $p-1$. If $\mathrm{Soc}(M)$ is a proper normal subgroup of $M$, then $\mathrm{Soc}(M) \leq M_0$. Since $M_0$ is cyclic, we have that $M$ is an abelian group with a cyclic socle. Therefore $M$ is cyclic. But since $q$ divides $p-1$, we have that $C$ centralises $M$ and so $C \leq \mathrm{O}_p(R) = 1$, a contradiction. Consequently $M = \mathrm{Soc}(M)$, and $M$ is a $C$-module over $\mathrm{GF}(q)$. If $M$ is not irreducible as $C$-module, then $M$ can be expressed as a direct sum of proper $C$-modules over $\mathrm{GF}(q)$. Hence $M$ has at least two maximal $C$-submodules, which yield two different chief factors $M/M_1$ and $M/M_2$ of $R$, a contradiction. Therefore $M$ is a minimal normal subgroup of $R$, $R = MC$, and $\mathrm{C}_R(M) = M$. On the other hand, $N$ is a faithful and irreducible $R$-module over $\mathrm{GF}(p)$. By Clifford's theorem [6; B, 7.3], the restriction of $N$ to $M$ is a direct sum of $|R:T|$ homogeneous components, where $T$ is the inertia subgroup of one of the irreducible components of $N$ when regarded as an $M$-module. Moreover, by [6; B, 8.3], we have that each of these homogeneous components $N_i$ is irreducible. Therefore they have dimension 1 because $N_iM$ is supersoluble for every $i$. Since $N$ is not cyclic, we have that $|R:T| > 1$. Since $M \leq T \leq R$, we have that $M = T$ and so $N$ has order $p^p$.

Assume now that $\Phi(G) \neq 1$. In this case, $\overline{G} = G/\Phi(G)$ is a minimal non-$p$-supersoluble group by Lemma 15 and $\Phi(\overline{G}) = 1$. We observe that $N\Phi(G)/\Phi(G)$ cannot be a Sylow $p$-subgroup of $G/\Phi(G)$, because otherwise

$NH$, where $H$ is a Hall $p'$-subgroup of $G$, would be a proper supplement to $\Phi(G)$ in $G$, which is impossible. In particular, if $T$ is a normal subgroup of $G$ contained in $\Phi(G)$, then the $p$-supersoluble residual $NT/T$ of $G/T$ is not a Sylow $p$-subgroup of $G/T$. Therefore $\overline{G}$ has the above structure. Since

$$N\Phi(G) = \mathrm{F}(G), \quad \mathrm{F}\big(G/\Phi(G)\big) = \mathrm{F}(G)/\Phi(G), \quad \text{and} \quad \Phi\big(\mathrm{F}(G)/\Phi(G)\big) = 1,$$

we have that $\overline{N} = (\overline{G})^{\mathfrak{F}} = N\Phi(G)/\Phi(G)$ satisfies

$$\begin{aligned}\overline{N}/\Phi(\overline{N}) &= \big(N\Phi(G)/\Phi(G)\big)/\Phi\big(N\Phi(G)/\Phi(G)\big) \\ &= \big(\mathrm{F}(G)/\Phi(G)\big)/\Phi\big(\mathrm{F}(G)/\Phi(G)\big),\end{aligned}$$

which is isomorphic to $\mathrm{F}(G)/\Phi(G) = N\Phi(G)/\Phi(G)$, and the latter is $G$-isomorphic to $N/\big(N \cap \Phi(G)\big) = N/\Phi(N)$ by Lemma 13. Assume that $\Phi(N) \neq 1$. By Lemma 14, we have that $N/\Phi(N)$ has square order. But this order is equal to $|\overline{N}/\Phi(\overline{N})| = p^p$, which implies that $p = 2$. This contradicts the fact that $q$ divides $p-1$. Therefore $\Phi(N) = 1$. Now we will prove that $\Phi(G)_p$, the Sylow $p$-subgroup of $\Phi(G)$, is a central cyclic subgroup of $G$. Assume first that $\Phi(G)_{p'}$, the Hall $p'$-subgroup of $\Phi(G)$, is trivial. We have that $G/\Phi(G) = \overline{N}\,\overline{M}\,\overline{C}$, where $\overline{C}$ is a cyclic group of order $p$, $\overline{M}$ is an irreducible and faithful module for $\overline{C}$ over $\mathrm{GF}(q)$, $q$ a prime dividing $p-1$, and $\overline{N}$ is an irreducible and faithful module for $\overline{M}\,\overline{C}$ over $\mathrm{GF}(p)$ of dimension $p$. Let $N$, $M$, and $C$ be, respectively, preimages of $\overline{N}$, $\overline{M}$, and $\overline{C}$ by the canonical epimorphism from $G$ to $G/T$. We can assume that $N = G^{\mathfrak{F}}$ and $M$ is a Sylow $q$-subgroup of $G$. Since $\overline{C}$ is cyclic of order $p$, we can find a cyclic subgroup $C$ of $G$ such that $\overline{C} = C\Phi(G)/\Phi(G)$. Consider now a chief factor $H/K$ of $G$ contained in $\Phi(G)_p$. Then $G/\mathrm{C}_G(H/K)$ is an abelian group of exponent dividing $p-1$ and $H/K$ is centralised by a Sylow $p$-subgroup of $G/K$; in particular, $G/\mathrm{C}_G(H/K)$ is isomorphic to a factor group of a group with a unique normal subgroup of index $p$. It follows that $\mathrm{C}_G(H/K) = G$, that is, $H/K$ is a central factor of $G$. Now $N$ centralises $\Phi(G)$ because $\Phi(N) = 1 = N \cap \Phi(G)$ and $M$ is a $q$-group stabilising a series of $\Phi(G)$. By [6; A, 12.4], $M$ centralises $\Phi(G)$. Moreover $C$ normalises $M$ because $M\Phi(G) = M \times \Phi(G)$ is normalised by $C$. In particular, $MC$ is a subgroup of $G$. Since $G = N(MC)$ and $N$ is a minimal normal subgroup of $G$, it follows that $MC$ is a maximal subgroup of $G$. Hence $\Phi(G)$ is contained in $MC$ and so in $C$. This implies that $\Phi(C) \leq \mathrm{Z}(G)$. In the general case, we have that

$$\Phi(G)/\Phi(G)_{p'} \leq \mathrm{Z}\big(G/\Phi(G)_{p'}\big).$$

Then $[G, \Phi(G)_p] \leq \Phi(G)_{p'}$. Therefore $\Phi(G)_p \leq \mathrm{Z}(G)$. On the other hand, it is clear that $\Phi(G)_p$ is a proper subgroup of $C$. Thus $\Phi(G)_p \leq \Phi(C)$ and so $\Phi(G)_p \leq \Phi(MC)$. Now $\Phi(G)_{p'} = \Phi(G)_q$, the Sylow $q$-subgroup of $\Phi(G)$, is contained in $M$ and $M/\Phi(G)_{p'}$ is elementary abelian. Hence $\Phi(M) \leq \Phi(G)_{p'}$. Moreover, by Maschke's theorem [6; A, 11.4], the elementary abelian group $M/\Phi(M)$ admits a decomposition

$$M/\Phi(M) = \Phi(G)_{p'}/\Phi(M) \times A/\Phi(M),$$

where $A$ is normalised by $C$. In this case, $R = MC = A\bigl(C\Phi(G)_{p'}\bigr)$. Since $C$ normalises $A$, we have that $AC$ is a subgroup of $G$. Therefore $N(AC)$ is a subgroup of $G$ and so $G = (NAC)\Phi(G)_{p'}$. We conclude that $G = NAC$. By order considerations, we have that $M = A$ and so $\Phi(M) = \Phi(G)_{p'}$.

Now let $G$ be a minimal non-$p$-supersoluble group such that $N$ is a Sylow $p$-subgroup of $G$. Let $Q$ be a Hall $p'$-subgroup of $G$. Then $G = NQ$. Denote with bars the images in $\overline{G} = G/\Phi(G)$. By Lemma 13, $\overline{N} = N\Phi(G)/\Phi(G)$ is a minimal normal subgroup of $\overline{G} = G/\Phi(G)$ and either $N$ is elementary abelian, or $N' = \mathrm{Z}(N) = \Phi(N)$. Note that $\Phi(N) = \Phi(G)_p$, the Sylow $p$-subgroup of $\Phi(G)$, because $\Phi(N)$ is contained in $\Phi(G)_p$ and $\overline{N}$ is a chief factor of $G$. Assume that $\Phi(G)_{p'}$, the Hall $p'$-subgroup of $\Phi(G)$, is not contained in $\Phi(Q)$. Then there exists a maximal subgroup $A$ of $Q$ such that $Q = A\Phi(G)_{p'}$. In this case, $G = NQ = NA\Phi(G)_{p'}$ and so $G = NA$. It follows that $A = Q$ by order considerations, a contradiction. Therefore $\Phi(G)_{p'} \leq \Phi(Q)$. We also note that since

$$\overline{Q} = Q\Phi(G)/\Phi(G) \cong Q/\Phi(G)_q,$$

where $\Phi(G)_q$ is the Sylow $q$-subgroup of $\Phi(G)$, has an irreducible and faithful module $\overline{N} = N/\Phi(N)$ over $\mathrm{GF}(p)$, we have that $\mathrm{Z}(\overline{Q})$ is cyclic by [6; B, 9.4].

By Lemma 17 we have that the Hall $p'$-subgroup $\overline{Q}$ of $\overline{G}$ is either a cyclic group of prime power order or a minimal non-abelian group.

Suppose that $\overline{Q} = \langle \bar{z} \rangle$ is a cyclic group of order a power of a prime number, $q$ say. Since this group is isomorphic to $Q/\Phi(G)_q$ and $\Phi(G)_q \leq \Phi(Q)$, we have that $Q$ is a cyclic group of $q$-power order, $Q = \langle z \rangle$ say.

Suppose that the order of $\bar{z}$ is $q^f$. Then $q^{f-1}$ divides $p-1$. If $\bar{z}^q = 1$, then $\overline{G}$ is a Schmidt group. By Lemma 18, $G$ is a Schmidt group. By Theorem 2, $G$ is a group of Type 2 if $\Phi(N) = 1$, or 3 if $\Phi(N) \neq 1$.

Assume now that $f \geq 2$. In this case, $q$ divides $p-1$ and, by Lemma 16, we have that $\overline{N}$ has order $p^q$. Let $a_0 \in \overline{N} \setminus 1$. Let $a_i = a_0^{z^i}$ for $1 \leq i \leq q-1$, then $a_0^{z^q} = a_0^i$, where $i$ is a $q^{f-1}$-root of unity modulo $p$. It follows that

$$(a_0^{z^{q^{f-1}}}) = a_0^{i^{q^{f-2}}}.$$

If $i$ is not a primitive $q^{f-1}$-th root of unity modulo $p$, we have that $i^{q^{f-2}} \equiv 1 \pmod{p}$. In particular, $a_0^{z^{q^{f-1}}} = a_0$, which contradicts the fact that the order of $\bar{z}$ is $q^f$. If $\Phi(N) = 1$, then we obtain a group of Type 4. If $\Phi(N) \neq 1$, then $\overline{N}$ has square order by Lemma 14 and so $q = 2$. Hence $N$ is an extraspecial group of order $p^3$ and exponent 3, and $G$ is a group of Type 5.

Assume now that $Q$ is not cyclic. In this case, $\overline{Q}$ is a minimal non-abelian group by Lemma 17. Let $x$ be an element of $\overline{Q}$. Since $\overline{N}\langle x\rangle$ is a $p$-supersoluble group, we have that the order of $x$ divides $p-1$. It follows that the exponent of $\overline{Q}$ divides $p-1$. Since $\overline{N} = N/\Phi(N)$ is an irreducible and faithful $\overline{Q}$-module over $\mathrm{GF}(p)$ of dimension greater than 1 and the restriction of $\overline{N}$ to every maximal subgroup of $\overline{Q}$ is a sum of irreducible modules of dimension 1, we have that $\overline{N}$ has order $p^q$ by Lemma 16.

Suppose that $\overline{Q}$ is a $q$-group for a prime $q$. By Theorem 1,

$$\text{either}\quad \overline{Q} \cong Q_8, \quad \text{or}\quad \overline{Q} \cong G_{\mathrm{II}}(q,m,n), \quad \text{or}\quad \overline{Q} \cong G_{\mathrm{III}}(q,m,n).$$

Suppose that $\overline{Q}$ is isomorphic to a quaternion group $Q_8$ of order 8. In this case, $q = 2$, $|\overline{N}| = p^2$ and $\exp(\overline{Q}) = 4$ divides $p-1$. If $\Phi(N) = 1$, then we have a group of Type 6. Assume that $\Phi(N) \neq 1$. In this case, $N$ is an extraspecial group of order $p^3$ and exponent $p$ and so $G$ is a group of Type 7.

Suppose that $\overline{Q}$ is isomorphic to

$$G_{\mathrm{II}}(q,m,n) = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}}\rangle,$$

where $m \geq 2$, $n \geq 1$, of order $q^{m+n}$. Since $\overline{Q}$ has an irreducible and faithful module $\overline{N}$, we have that $\mathrm{Z}(\overline{Q})$ is cyclic by [6; B, 9.4]. Since $\langle a^p, b^p\rangle \leq \mathrm{Z}(\overline{Q})$ and $m \geq 2$, we have that $b^p = 1$ and so $n = 1$. Hence $q^m$ divides $p-1$. If $\Phi(N) = 1$, then we obtain a group of Type 8. If $\Phi(N) \neq 1$, then $N$ is non-abelian and so $|\overline{N}|$ is a square by Lemma 14. It follows that $q = 2$ and $G$ is a group of Type 9.

Suppose now that $\overline{Q}$ is isomorphic to

$$G_{\mathrm{III}}(q,m,n) = \langle a, b \mid a^{q^m} = b^{q^n} = [a,b]^q = [a,b,a] = [a,b,b] = 1\rangle,$$

where $m \geq n \geq 1$, of order $q^{m+n+1}$. Since $G_{\mathrm{III}}(2,1,1) \cong G_{\mathrm{II}}(2,2,1)$, we can assume that $(q,m,n) \neq (2,1,1)$.

As before, $\mathrm{Z}(\overline{Q})$ is cyclic. Consider $\langle a^q, b^q, [a,b]\rangle$, which is contained in $\mathrm{Z}(\overline{Q})$. If $m \geq 2$, then $\langle a^q, [a,b]\rangle$ is cyclic. Since $[a,b]$ has order $p$, we have that $[a,b] = a^{qt}$ for a natural number $t$. But hence $a^b = a^{1+qt}$ and so $\langle a\rangle$ is a normal subgroup of $G$. Therefore $|\overline{Q}| = |\langle a,b\rangle| = |\langle a\rangle\langle b\rangle| \leq q^{m+n}$, a contradiction. Consequently $m = 1$. It follows that $\overline{Q}$ is an extraspecial group of order $q^3$ and exponent $q$. If $\Phi(N) \neq 1$, then $\overline{N}$ has square order, but this implies that $q = 2$, a contradiction. Consequently, $\Phi(N) = 1$ and we have a group of Type 10.

Assume now that $\overline{Q}$ is a minimal non-abelian group which is not a $q$-group for any prime $q$. Then $\overline{Q}$ is isomorphic to $[V_q]C_{r^s}$, where $q$ and $r$ are different primes numbers, $s$ is a positive integer, and $V_q$ is an irreducible $C_{r^s}$-module over the field of $q$ elements with kernel the maximal subgroup of $C_{r^s}$. Since $\overline{N}V_q$ is a $p$-supersoluble subgroup, it follows that the restriction of $\overline{N}$ to $V_q$ can be expressed as a direct sum of irreducible modules of dimension 1. By Lemma 16, we have that $\overline{N}$ has dimension $r$. We know that $\Phi(G)_{p'} \leq \Phi(Q)$ and $\Phi(G)_p = \Phi(N)$. Since $\overline{Q}$ is isomorphic to $Q/\Phi(G)_{p'}$, and this group is $r$-nilpotent, $Q$ is $r$-nilpotent. Consequently $Q$ has a normal Sylow $q$-subgroup $M$. On the other hand, $\Phi(G)_q$, the Sylow $q$-subgroup of $\Phi(G)$, is contained in $M$ and $M/\Phi(G)_q$ is elementary abelian. This implies that $\Phi(M)$ is contained in $\Phi(G)_q$. Let $C$ be a Sylow $r$-subgroup of $G$. Then, by Maschke's theorem [6; A, 11.4],

$$M/\Phi(M) = \Phi(G)_q/\Phi(M) \times A/\Phi(M)$$

for a subgroup $A$ of $M$ normalised by $C$. Then $Q = (AC)\Phi(G)_q = AC$ and so $A = M$. Consequently $\Phi(M) = \Phi(G)_q$. Now the Sylow $r$-subgroup $\Phi(G)_r$ of $\Phi(G)$ is contained in $C$. If $\Phi(G)_r$ were not contained in $\Phi(C)$, there would exist a maximal subgroup $T$ of $C$ such that $C = T\Phi(G)_r$. This would imply $Q = MT$ and $T = C$, a contradiction. Hence $\Phi(G)_r$ is contained in $\Phi(C)$ and $C$ is cyclic. Moreover $\Phi(G)_r$ centralises $M$.

If $\Phi(N) = 1$, then we have a group of Type 11. If $\Phi(N) \neq 1$, then $r = 2$ and $N$ is an extraspecial group of order $p^3$ and exponent $p$. This is a group of Type 12.

Conversely, it is clear that the groups of Types 1 to 12 are minimal non-$p$-supersoluble. ■

**Proof of Theorem 10.** It is clear that all groups of the statement of the theorem are minimal non-supersoluble. Conversely, assume that a group is minimal non-supersoluble. Hence it is soluble, and so its $p$-supersoluble residual is a $p$-group by Proposition 8. Note that groups of Type 1 in Theorem 9 are not minimal non-supersoluble. On the other hand, groups of Type 11 are not minimal non-supersoluble when $r$ does not divide $q-1$, because in this case the subgroup $MC$ is not supersoluble. ■

**Proof of Theorem 11.** Assume that the result is false. Choose for $G$ a counterexample of least order. Since the property of the statement is inherited by subgroups, it is clear that $G$ must be a minimal non-supersoluble group, and so a minimal non-$p$-supersoluble group for a prime $p$. In particular, the $p$-supersoluble residual $N = G^{\mathfrak{F}}$ of $G$ is a $p$-group. Suppose that $N$ has exponent $p$. The hypothesis implies that every subgroup of $N$ is normalised by $\mathrm{O}^p(G)$. In particular, $N/\Phi(N)$ is cyclic, a contradiction.

Consequently $p = 2$ and the exponent of $N$ is 4. By Theorem 9, the only group with $\mathfrak{F}$-residual of exponent 4 is a group of Type 3. But in this case either $N/\Phi(N)$ has order 4 and $N$ must be isomorphic to the quaternion group of order 8, because the dihedral group of order 8 does not have any automorphism of odd order, or $N/\Phi(N)$ has order greater than 4. In the last case, $N$ has an extraspecial quotient, which has a section isomorphic to a quaternion group of order 8, final contradiction. ■

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