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L^p decay estimates for weighted oscillatory integral operators on $\mathbb{R}$

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Abstract

In this paper, we formulate necessary conditions for decay rates of L^p operator norms of weighted oscillatory integral operators on $\mathbb{R}$ and give sharp L^2 estimates and nearly sharp L^p estimates.

1. Introduction

Suppose f and g are real-analytic, real-valued functions in a neighborhood V of the origin in $\mathbb{R}^2$ with $f(0,0) = g(0,0) = 0$ and let χ be a smooth function of compact support in V. We consider the oscillatory integral operator

$$T_\lambda \varphi(x) = \int_{\mathbb{R}} e^{i\lambda f(x,y)} |g(x,y)|^{\epsilon/2} \chi(x,y)\varphi(y)dy,$$

where ϵ is any positive number. In this paper we will study the decay rate in λ of $||T_\lambda||_{L^p \to L^p}$ as $\lambda \to \infty$.

The case where $g(x,y) = 1$ has been studied in [3], [6], [10], [11], [12], and [15]. In [10] and [11], Phong and Stein considered a case where the phase function $f(x,y)$ is a real homogeneous polynomial and they obtained sharp decay estimates for $||T_\lambda||_{L^2 \to L^2}$. In [12], they took into account of more general cases where the phase function $f(x,y)$ is a real analytic function and they proved $||T_\lambda||_{L^2 \to L^2} \sim \lambda^{-\delta}$ where δ is the reduced Newton distance of $f(x,y)$. In [15] Rychkov developed the ideas of Phong and Stein in [12] and Seeger in [16] to obtain sharp L^2 decay estimates for the case where the phase function $f(x,y)$ is a real smooth function with the condition that the formal power series expansion of f''_{xy} at the origin does not vanish. He proved $||T_\lambda||_{L^2 \to L^2} \sim \lambda^{-\delta}$, where δ is the reduced Newton distance of

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the formal power series expansion of $f(x, y)$ at the origin, with a loss of a certain power of $\log \lambda$ in the case where all solutions $r(x)$ of $f''_{xy}(x, r(x)) = 0$ have the same asymptotic fractional power series expansion with leading power 1. In [3], Greenblatt gave a new proof for the theorem of Phong and Stein in [12]. For L^p estimates, Greenleaf and Seeger obtained sharp decay estimates [6]. They considered oscillatory integral operators in $\mathbb{R}^n$ with a real smooth phase function with the assumption of two-sided fold singularities. They established sharp $L^p - L^q$ decay estimates of the oscillatory integral operators. In [17], Seeger formulated optimal L^p regularity of generalized Radon transforms on $\mathbb{R}^2$ and he obtained sharp L^p regularity estimates except endpoints. In [19], sharp L^p decay estimates for T_λ have been established excluding estimates on vertices of Newton polygon of f''_{xy}.

The case where $g = f''_{xy}$ has been studied in [13]. In [13] Phong and Stein proved best possible decay estimate, that is, $||T_\lambda||_{L^2 \to L^2} \sim \lambda^{-1/2}$ when $g(x, y) = f''_{xy}(x, y)$ and $\epsilon = 1/2$. We wish to investigate the improvement in the decay rate of $||T_\lambda||_{L^p \to L^p}$ when f is unrelated to g.

Higher dimensional case even without any damping factor has not been understood well. There have been a few L^2 estimates of special cases [1], [2], [6], [7], [9]. Sharp L^2 estimates under the assumption of two-sided fold singularities were obtained in [9]. Optimal estimates with one-sided fold singularity have been established in [2] and [4]. Related operators with various types of higher order singularities have been treated in [1], [5] and [7]. We recommend [8] as a more detailed and organized survey on this subject.

The case where the weight $g(x, y)$ is not related to $f(x, y)$ has been considered by the first author in a different context [14]. In [14] she introduced weighted Newton distance to treat the weighted integral. We shall use some notions in [14] and we briefly describe them. We start with factorizing f''_{xy} and g

$$f''_{xy}(x, y) = U_1(x, y)x^{\alpha_1}y^{\beta_1} \prod_{\nu \in I(f''_{xy})} (y - r_\nu(x)) \tag{1.1}$$

$$g(x, y) = U_2(x, y)x^{\alpha_2}y^{\beta_2} \prod_{\mu \in I(g)} (y - s_\mu(x)) \tag{1.2}$$

where $I(h)$ denotes a set whose elements are used to index roots of h and U_i $i = 1, 2$ are real analytic functions with $U_i(0, 0) \neq 0$. We assume that index sets $I(f''_{xy})$ and $I(g)$ are disjoint. α_i's and β_i's are non-negative integers and $r_\nu(x)$'s and $s_\mu(x)$'s are Puiseux series of the form

$$r_\nu(x) = c_\nu x^{a_\nu} + O(x^{b_\nu}) \quad \text{and} \quad s_\mu(x) = c_\mu x^{a_\mu} + O(x^{b_\mu})$$

where for any $\eta \in I(f''_{xy}) \cup I(g)$, $b_\eta > a_\eta$ are rational numbers and $c_\eta \neq 0$. We re-index the combined set of distinct exponents a_ν and a_μ with $\nu \in I(f''_{xy})$ and $\mu \in I(g)$ into increasing order so that

$$0 < a_1 < a_2 < \cdots < a_N.$$

For $l \in \{1, ..., N\}$ we define

$$\begin{aligned} m_l &= \#\{\nu \in I(f''_{xy}) \ : \ r_\nu(x) = c_\nu x^{a_l} + \cdots , \quad c_\nu \neq 0\} \\ n_l &= \#\{\mu \in I(g) \ : \ s_\mu(x) = c_\mu x^{a_l} + \cdots , \quad c_\mu \neq 0\}, \end{aligned}$$

where $\#A$ denotes the cardinality of a set A. We call m_l and n_l generalized multiplicities of f''_{xy} and g, respectively, corresponding to the exponent a_l. Now we define

$$\begin{aligned} A_l &= \alpha_1 + \textstyle\sum_{i=1}^{l} a_i m_i, \qquad & B_l &= \beta_1 + \textstyle\sum_{i=l+1}^{N} m_i \\ C_l &= \alpha_2 + \textstyle\sum_{i=1}^{l} a_i n_i, \qquad & D_l &= \beta_2 + \textstyle\sum_{i=l+1}^{N} n_i. \end{aligned}$$

Then $\{(A_l, B_l)\}$ and $\{(C_l, D_l)\}$ are sets of vertices of the Newton diagrams of f''_{xy} and g, respectively. The number of common roots of f''_{xy} and g is an important information to obtain optimal estimates. To extract the information we use a coordinate transformation η given by

$$\eta : (x, y) \mapsto (x, y - q(x))$$

where q is a convergent real-valued Puiseux series in a neighborhood of the origin. For $f''_{xy} \circ \eta$ and $g \circ \eta$ we can define previous notions such as A_l, B_l, C_l, D_l, and a_l in the same way. To avoid confusion we use notations $A_l(\eta)$, $B_l(\eta)$, $C_l(\eta)$, $D_l(\eta)$, and $a_l(\eta)$ to specify the coordinate transformation η. For the sake of simplicity we define $E_l(\eta)$ and $F_l(\eta)$ as

$$\begin{aligned} E_l(\eta) &= A_l(\eta) + a_l(\eta) B_l(\eta) \\ F_l(\eta) &= C_l(\eta) + a_l(\eta) D_l(\eta) \end{aligned}$$

For a coordinate transform $\eta : (x, y) \mapsto (x, y - q(x))$ we define $\mathcal{E}_{l,\eta}$ as

$$\begin{aligned} \mathcal{E}_{l,\eta} = \big\{ \deg\left(r(x) - q(x)\right) | \, y = r(x) \text{ is a root of } f''_{xy}(x, y) = 0 \text{ or } g(x, y) = 0, \\ r(x) = c x^{a_l} + \cdots (c \neq 0) \text{ and } \deg(r(x) - q(x)) \geq a_l \} \end{aligned}$$

where $\deg(p(x))$ is the degree of the leading term of a Puiseux series p. For $a_{l'}(\eta) \in \mathcal{E}_{l,\eta}$ we define

$$H_{l,l'}(\eta) = E_{l'}(\eta) + 1 + 2a_{l'}(\eta) - a_l.$$

We define E_l, F_l, and H_l as

$$E_l = E_l(\mathbf{id}), \quad F_l = F_l(\mathbf{id}), \quad \text{and} \quad H_l = H_{l,l'}(\mathbf{id})$$

where $\mathbf{id}$ is the identity map on $\mathbb{R}^2$. Here we remark that since $\mathcal{E}_{l,\mathbf{id}} = \{a_l\}$, $a_{l'}(\mathbf{id}) = a_l$ so $H_l = E_l + 1 + a_l$. To describe optimal decay rate of $||T_{\lambda,\epsilon}||_{L^p \to L^p}$ we shall need the following notations. Let $K = [0,1] \times \mathbb{R}$. For $a_{l'}(\eta) \in \mathcal{E}_{l,\eta}$ we define subsets $\mathcal{A}_0$, $\mathcal{A}_l$, and $\mathcal{A}_{l,l'}(\eta)$ of K as

$$\begin{aligned}
\mathcal{A}_0 &= \left\{ \left(\frac{1}{p}, \alpha\right) \in K : \alpha \leq \frac{1}{p}, \text{ and } \alpha \leq 1 - \frac{1}{p} \right\}, \\
\mathcal{A}_l &= \left\{ \left(\frac{1}{p}, \alpha\right) \in K : \alpha \leq \frac{\epsilon F_l + 2a_l}{2H_l} + \frac{1-a_l}{H_l} \cdot \frac{1}{p} \right\}, \\
\mathcal{A}_{l,l'}(\eta) &= \left\{ \left(\frac{1}{p}, \alpha\right) \in K : \alpha \leq \frac{\epsilon F_{l'}(\eta) + 2a_{l'}(\eta)}{2H_{l,l'}(\eta)} + \frac{1-a_l}{H_{l,l'}(\eta)} \frac{1}{p} \right\}.
\end{aligned}$$

Here we note that if $\eta = \mathbf{id}$ and $a_l = a_{l'}(\eta)$, then $\mathcal{A}_{l,l'}(\eta) = \mathcal{A}_l$. We set

$$\mathcal{A}_1 = \bigcap_l \mathcal{A}_l \quad \text{and} \quad \mathcal{A}_2 = \bigcap_\eta \bigcap_{\substack{l,l'\,; \\ a_{l'}(\eta) \in \mathcal{E}_{l,\eta}}} \mathcal{A}_{l,l'}(\eta).$$

Now we finally define $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_0 \cap \mathcal{A}_1 \cap \mathcal{A}_2.$$

From the definitions it is clear that $\mathcal{A}_1$ is a special case of $\mathcal{A}_2$ where $\eta = \mathbf{id}$ so $\mathcal{A}_2 \subset \mathcal{A}_1$. Actually it is not necessary to define those two sets in a separate way. Here we separately define $\mathcal{A}_1$ and $\mathcal{A}_2$ because we want to simplify notations in the proof of the first step of each theorem and give clear ideas of proofs.

Remark 1.1 When we define $A_{l,l'}(\eta)$ we include the case where $a_{l'}(\eta) = \infty$. In this case we assume that

$$\mathcal{A}_{l,l'}(\eta) = \left\{ \left(\frac{1}{p}, \alpha\right) \in K : \alpha \leq \frac{\epsilon D_{l'}(\eta) + 2}{2(B_{l'}(\eta) + 2)} \right\}. \tag{1.3}$$

Theorem 1.2 (Necessity) *If T_λ is bounded on $L^p(\mathbb{R})$ with $||T_\lambda||_{L^p \to L^p} \leq O(\lambda^{-\alpha})$, then $(1/p, \alpha) \in \mathcal{A}$.*

Remark 1.3 The definition of domain $\mathcal{A}$ has been motivated from earlier works in [12], [17] and [19]. We write

$$||T_\lambda||_{L^p \to L^p} = \sup_{\varphi \in L^p, \psi \in L^{p'}} \frac{< T_\lambda \varphi, \psi >}{||\varphi||_{L^p} ||\psi||_{L^{p'}}}.$$

To find necessary conditions for L^p decay estimates we have to consider the case where the oscillation of the phase function λf does not play any role even if λ is very large. This situation happens when φ and ψ are supported in small intervals whose lengths depend on λ, f, and g so that $|\lambda f(x,y)| \sim c\lambda^{-1}$ and $|g(x,y)|$ is bounded below when x and y are in the support of ψ and φ, respectively. To be more precise we fix $\lambda \geq \lambda_0$ for some λ_0, sufficiently large. A set of the form

$$B = \{(x,y) \in \text{supp}\chi \mid a \leq x \leq b,\ c \leq y \leq d\}$$

is defined to be a "testing box" if there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ depending on B satisfying

$$\sup_{(x,y)\in B} |\lambda(f(x,y) - F_1(x) - F_2(y))| < \frac{\pi}{4}.$$

Set $I_1 = [a,b]$ and $I_2 = [c,d]$. If $\mathfrak{F}$ denote the class of all testing boxes, then

$$||T_\lambda||_{L^p \to L^p} \geq \max\left\{ \sup_{B\in\mathfrak{F}} \left\{ |I_1|^{1-\frac{1}{p}} |I_2|^{\frac{1}{p}} \inf_{(x,y)\in B} |g(x,y)|^{\epsilon/2} \right\}, \lambda^{-1/2} \right\}.$$

Since we have a weight $|g(x,y)|^{\epsilon/2}$ in our operator, we have to choose the testing box carefully so that $|g(x,y)|^{\epsilon/2}$ has a lower bound in terms of λ. If not, we just have a trivial bound. If $g \equiv 1$, then it is known that $\mathcal{A}$ is an image of the reduced Newton polygon by a map $(m,n) \mapsto (\frac{m}{m+n}, \frac{1}{m+n})$ in [19]. T_λ is called a damped oscillatory integral operator if $g = f''_{xy}$. This case has been studied by Phong and Stein in [13]. Their results show that $\mathcal{A}$ is a triangular region with vertices $(0,0)$, $(1,0)$, and $(1/2,1/2)$ if $g = f''_{xy}$ and $\epsilon \geq 1$. When $g = f''_{xy}$ and $\epsilon < 1$, the region $\mathcal{A}$ can be obtained by interpolation of results in [13] and [19].

Theorem 1.4 (L^2 estimates) *If $(1/2, \alpha) \in \mathcal{A}$, then*

$$||T_\lambda||_{L^2 \to L^2} \leq O(\lambda^{-\alpha}).$$

Theorem 1.5 (L^p estimates) *If $(1/p, \alpha) \in \text{int}(\mathcal{A})$, then we have*

$$||T_\lambda||_{L^p \to L^p} \leq O(\lambda^{-\alpha}).$$

Remark 1.6 In Theorem 1.5 we only have estimates in the interior of $\mathcal{A}$. During the proof of the theorem one can easily observe that we have estimates on some part of the boundary of $\mathcal{A}$. We shall discuss this in detail in part 1 of the final remark.

2. Proof of Theorem 1.2

In this section we shall prove Theorem 1.2. The idea of the proof is described in Remark 1.3.

Proof of Theorem 1.2. Suppose that T_λ is bounded on L^p with

$$||T_\lambda||_{L^p \to L^q} \leq O(\lambda^{-\alpha}).$$

First we shall show that $(1/p, \alpha) \in \mathcal{A}_1$. Suppose $f''_{xy}(x, y) = \sum_{p,q\geq 0} c_{pq} x^p y^q$. Then we have

$$\begin{aligned} f(x,y) &= \sum_{p,q\geq 0} c_{pq} \frac{x^{p+1} y^{q+1}}{(p+1)(q+1)} + F_1(x) + F_2(y) \\ &= \sum_{p,q\geq 1} \tilde{c}_{pq} x^p y^q + F_1(x) + F_2(y) \end{aligned}$$

where $F_1(x)$ and $F_2(y)$ are real analytic. Note that the Newton diagram of $\sum_{p,q\geq 1} \tilde{c}_{pq} x^p y^q$ is same as the reduced Newton diagram of f. We fix l and recall $H_l = A_l + a_l B_l + a_l + 1$. Let $R > 0$ and c_1 be constants to be specified. Now, for large positive λ, we define the function φ_λ, ψ_λ by

$$\varphi_\lambda(y) = \begin{cases} e^{-i\lambda F_2(y)} & \text{if } R \leq y\lambda^{a_l/H_l} \leq R + c_1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_\lambda(x) = \begin{cases} e^{-i\lambda F_1(x)} & \text{if } R \leq x\lambda^{1/H_l} \leq R + c_1 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for any $\epsilon > 0$, in the support of $\varphi_\lambda(y)\psi_\lambda(x)$ we have:

$$\left|\lambda f(x,y) - \lambda F_1(x) - \lambda F_2(y) - \sum{}' \tilde{c}_{pq} R^q\right| < \epsilon,$$

where the sum $\sum'$ is taken over (p, q) that belong to the face of the reduced Newton diagram with equation $p + a_l q = H_l$, as long as c_1 is taken to be small in terms of $\sum' |\tilde{c}_{pq}| R^q$ and then λ is taken to be large. To prove the claim, first we note that if $0 < c_1 < R$ is sufficiently small then we have

$$\begin{aligned} \left|\sum{}' \tilde{c}_{pq}(\lambda x^p y^q - R^q)\right| &\leq \sum{}' |\tilde{c}_{pq}||\lambda x^p y^q - 1| \\ &\leq \sum{}' |\tilde{c}_{pq}|[(1 + \frac{c_1}{R})^q (1 + c_1)^p - 1] \cdot R^q \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Also, because of the convex nature of the Newton diagram, $p + a_l q > C$ for all other (p, q) such that $\tilde{c}_{pq} \neq 0$, so,

$$\lambda \Big| \sum_{(p,q);p+a_l q \neq H_l} \tilde{c}_{pq} x^p y^q \Big| < \frac{\epsilon}{2}.$$

If we take, say $\epsilon < \pi/2$ then this shows that

$$\begin{aligned} | < T_\lambda \varphi_\lambda, \psi_\lambda > | &= \left| \int_{\mathbb{R}^2} e^{i\lambda f(x,y)} |g(x,y)|^{\frac{\epsilon}{2}} \chi(x,y) \varphi_\lambda(y) \overline{\psi_\lambda(x)} dy dx \right| \\ &= \left| \int_{(x,y)\in S_\lambda} e^{i[\lambda f(x,y) - \lambda F_1(x) - \lambda F_2(y)]} \chi(x,y) |g(x,y)|^{\frac{\epsilon}{2}} dy dx \right| \\ &= \left| \int_{(x,y)\in S_\lambda} e^{i[\lambda f(x,y) - \lambda F_1(x) - \lambda F_2(y) - \sum' \tilde{c}_{pq} R^q]} |g(x,y)|^{\frac{\epsilon}{2}} dy dx \right| \end{aligned}$$

where $S_\lambda = \{(x,y) | 1 \leq \lambda^{1/H_l} x \leq 1 + c_1, R \leq y \lambda^{a_l/H_l} \leq R + c_1\}$. Hence we have

$$| < T_\lambda \varphi_\lambda, \psi_\lambda > | \geq C \int_{(x,y)\in S_\lambda} \chi(x,y) |g(x,y)|^{\frac{\epsilon}{2}} dy dx.$$

Let $R > 2 \cdot \max\{|c|; y = cx^{a_l} + \cdots$ is a root of $g\}$ and $R > 1$. Then $g(x,y) \sim |x|^{C_l} |y|^{D_l}$ on the support of $\varphi_\lambda(y)\psi_\lambda(x)$. We therefore have

$$| < T_\lambda \varphi_\lambda, \psi_\lambda > | \geq C \lambda^{-\frac{F_l}{H_l} \cdot \frac{\epsilon}{2}} \lambda^{-\frac{a_l + 1}{H_l}}$$

as $\lambda \to \infty$. Hence, we have

$$\begin{aligned} \frac{| < T_\lambda \varphi_\lambda, \psi_\lambda > |}{||\varphi_\lambda||_p \cdot ||\psi_\lambda||_{p'}} &\geq C \frac{\lambda^{-\frac{F_l}{H_l} \cdot \frac{\epsilon}{2}} \lambda^{-\frac{a_l+1}{H_l}}}{\lambda^{-\frac{a_l}{H_l p} - \frac{1}{H_l}(1 - \frac{1}{p})}} \\ &\geq C \lambda^{-\frac{\epsilon F_l + 2a_l}{2H_l} - \frac{1 - a_l}{H_l} \frac{1}{p}}, \end{aligned}$$

which implies

$$\alpha \leq \frac{\epsilon F_l + 2a_l}{2H_l} + \frac{1 - a_l}{H_l} \frac{1}{p}.$$

Therefore $(1/p, \alpha) \in \mathcal{A}_1$.

We show that $(1/p, \alpha) \in \mathcal{A}_1$. Let r be a root of $f''_{xy}(x,y) = 0$ or $g(x,y) = 0$ in (1.1) and (1.2) and set $r(x) = cx^{a_l} + \cdots$. We choose a coordinate transform $\eta : (x,y) \mapsto (x, y - q(x))$ with convergent Puiseux series q of real coefficients. We choose $a_{l'}(\eta)$ so that $a_l \leq a_{l'}(\eta)$. Here we assume that the lowest degree term of q is x^{a_l} because to define $\mathcal{A}_{l,l'}(\eta)$ we assume that $a_{l'}(\eta) \geq a_l$. Suppose $r(x) = \tilde{r}(x) + O(|x|^{a_{l'}(\eta)})$. We define φ_λ and ψ_λ as

$$\varphi_\lambda(y) = \begin{cases} e^{-i\lambda F_2(y)} & \text{if} \quad \tilde{r}(\lambda^{-1/H_{l,l'}(\eta)}) + R\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)} \leq y \\ & \qquad \leq \tilde{r}(\lambda^{-1/H_{l,l'}(\eta)}) + 2R\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_\lambda(x) = \begin{cases} e^{-i\lambda F_1(x)} & \text{if } \lambda^{-1/H_{l,l'}(\eta)} \leq x \leq \lambda^{-1/H_{l,l'}(\eta)} + c_1\lambda^{-(a_{l'}(\eta)-a_l+1)/H_{l,l'}(\eta)} \\ 0 & \text{otherwise} \end{cases}$$

where c_1 and R are constants, and F_1, F_2 are real-valued functions to be specified later. On the support of $\varphi_\lambda(y)\psi_\lambda(x)$ we have

$$|y - r(x)| \leq |\tilde{r}(\lambda^{-1/H_{l,l'}(\eta)}) + 2R\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)} - \tilde{r}(x) + O(\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)})|.$$

Suppose $\tilde{r}(x) = \alpha x^{a_l} + \beta x^{b_l} + \cdots$ where without loss of generality $\alpha > 0, \beta > 0$. Then

$$\begin{aligned} |y - r(x)| &\leq |\alpha\lambda^{-a_l/H_{l,l'}(\eta)} + \beta\lambda^{-b_l/H_{l,l'}(\eta)} + 2R\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)} - \alpha\lambda^{-a_l/H_{l,l'}(\eta)} \\ &\quad - \beta[\lambda^{-1/H_{l,l'}(\eta)}(1 + c_1\lambda^{-(a_{l'}(\eta)-a_l)/H_{l,l'}(\eta)} + O(\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)})]| \\ &\leq 3R\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)} \end{aligned}$$

and

$$\begin{aligned} |y - r(x)| &\geq |R\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)} + \tilde{r}(\lambda^{-1/H_{l,l'}(\eta)}) \\ &\quad - \alpha[\lambda^{-1/H_{l,l'}(\eta)}(1 + c_1\lambda^{-(a_{l'}(\eta)-a_l)/H_{l,l'}(\eta)})]^{a_l} \\ &\quad - \beta\lambda^{-b_l/H_{l,l'}(\eta)} + o(\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)})| \\ &\geq \frac{R}{2}\lambda^{-a_{l'}(\eta)/H_{l,l'}(\eta)}. \end{aligned}$$

Let $(x_0(\lambda), y_0(\lambda))$ be a fixed point on the support of $\varphi_\lambda(y)\psi_\lambda(x)$. Then for any (x, y) in the support

$$\begin{aligned} \int_{x_0}^{x}\int_{y_0}^{y} f''_{xy}(s,t)\,dt\,ds &= \int_{x_0}^{x}[f'_x(s,y) - f'_x(s,y_0)]\,ds \\ &= f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0). \end{aligned} \tag{2.1}$$

Let $F_2(y) = f(x_0(\lambda), y)$, $F_1(x) = f(x, y_0(\lambda)) - f(x_0(\lambda), y_0(\lambda))$. We notice that for (s,t) in the support of $\varphi_\lambda(y)\psi_\lambda(x)$,

$$\begin{aligned} \left|f''_{xy}(s,t)\right| &\sim |t - \tilde{r}(s)|^{B_{l'}(\eta)}|s|^{A_{l'}(\eta)} \\ &\sim R^{B_{l'}(\eta)}\lambda^{-\frac{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)}{H_{l,l'}(\eta)}} \\ &= R^{B_{l'}(\eta)}\lambda^{-\frac{E_{l'}(\eta)}{H_{l,l'}(\eta)}}. \end{aligned}$$

By the same reason if (x,y) is in the support of $\varphi_\lambda(y)\psi_\lambda(x)$, then

$$|g(x,y)| \sim \lambda^{-\frac{F_{l'}(\eta)}{H_{l,l'}(\eta)}}.$$

Therefore we have

$$\begin{aligned}\left|\int_{x_0}^{x}\int_{y_0}^{y} f''_{xy}(s,t)dtds\right| &\sim R^{B_{l'}(\eta)+1}\lambda^{-\frac{E_{l'}(\eta)}{H_{l,l'}(\eta)}}\lambda^{-\frac{a_{l'}(\eta)}{H_{l,l'}(\eta)}}\cdot c_1\lambda^{-\frac{a_{l'}(\eta)-a_l+1}{H_{l,l'}(\eta)}}\\ &\sim R^{B_{l'}(\eta)+1}\cdot c_1\cdot\lambda^{-1}.\end{aligned}$$

By choosing c_1 sufficiently small, we can ensure that for some $0<\epsilon<\pi/4$

$$|\lambda f(x,y)-\lambda f(x_0,y)-\lambda f(x,y_0)+\lambda f(x_0,y_0)|<\epsilon.$$

Hence we have

$$\begin{aligned}|<T_\lambda\varphi_\lambda,\psi_\lambda>| &\geq \int_{(x,y)\in S_\lambda}|g(x,y)|^{\frac{\epsilon}{2}}dydx\\ &\geq \lambda^{-\frac{\epsilon F_{l'}(\eta)}{2H_{l,l'}(\eta)}}\lambda^{-\frac{a_{l'}(\eta)}{H_{l,l'}(\eta)}}\lambda^{-\frac{a_{l'}(\eta)-a_l+1}{H_{l,l'}(\eta)}}.\end{aligned}$$

This yields

$$\frac{|<T_\lambda\varphi_\lambda,\psi_\lambda>|}{||\varphi_\lambda||||\psi_\lambda||}\geq C\lambda^{-\frac{\epsilon F_{l'}(\eta)+2a_{l'}(\eta)}{2H_{l,l'}(\eta)}-\frac{1-a_l}{H_{l,l'}(\eta)}\frac{1}{p}},$$

which implies

$$\alpha\leq\frac{\epsilon F_{l'}(\eta)+2a_{l'}(\eta)}{2H_{l,l'}(\eta)}+\frac{1-a_l}{H_{l,l'}(\eta)}\frac{1}{p}.$$

Therefore $(1/p,\alpha)\in\mathcal{A}_2$.

Finally we shall show that $(1/p,\alpha)\in\mathcal{A}_0$. There exists (x_0,y_0) such that $|g(x_0,y_0)|\geq k>0$. Let

$$F_1(x)=\sum_{i=1}^{\infty}\frac{(\partial_x^i f)(x_0,y_0)}{i!}(x-x_0)^i$$

and

$$F_2(y)=\sum_{j=1}^{\infty}\frac{(\partial_y^j f)(x_0,y_0)}{j!}(y-y_0)^j.$$

We define $\psi_\lambda(x)$ and $\varphi_\lambda(y)$ by

$$\varphi_\lambda(y)=\begin{cases} e^{-i\lambda F_2(y)} & \text{if}\quad y_0\leq y\leq y_0+\lambda^{-1}\\ 0 & \text{otherwise,}\end{cases}$$

$$\psi_\lambda(x)=\begin{cases} e^{-i\lambda F_1(x)} & \text{if}\quad x_0\leq x\leq x_0+c_1\\ 0 & \text{otherwise.}\end{cases}$$

By choosing a small number $c_1 > 0$ we have

$$|\lambda(f(x,y) - f(x_0,y_0) - F_1(x) - F_2(y))| \leq \pi/4.$$

Hence we have

$$\left| e^{-i\lambda f(x_0,y_0)} \int T_\lambda \varphi_\lambda(x)\psi_\lambda(x)dx \right| \geq C\lambda^{-1}$$

and

$$||f_\lambda||_{L^p} \sim \lambda^{-1/p} \quad \text{and} \quad ||g_\lambda||_{L^{p'}} \sim 1.$$

Therefore we have $\alpha \leq 1 - 1/p$. By exchanging the role of f_λ and g_λ we have $\alpha \leq 1/p$. This shows that $(1/p, \alpha) \in \mathcal{A}_0$. ■

3. Proof of Theorem 1.4

The proof of Theorem 1.3 follows the main ideas in [12] and [13]. Namely, one writes T_λ as a sum of almost orthogonal operators

$$T_\lambda = \sum T_{jk}^\lambda$$

where T_{jk}^λ will be defined later. The dyadic rectangles $[2^{-j},2^{-j+1}] \times [2^{-k},2^{-k+1}]$ in the definition of T_{jk}^λ can be divided into two categories, depending on their proximity to the zero varieties of f''_{xy} and g. If a rectangle is located away from these zero varieties, then the L^2-norm of T_{jk}^λ may be estimated using a combination of the operator Van der Corput lemma in [12, Section 3] and [13, Lemma 1] and Schur's lemma. Near a branch of the zero varieties, one needs a finer resolution of T_{jk}^λ to operators supported on "curved rectangles" adapted to that branch. It is then possible to determine the sizes of f''_{xy} and g on these finer domains, so that the operator Van der Corput and Schur's lemmas can again be used. The resolution process terminates in a finite number of steps, since a real-analytic function can only vanish to finite order in a small neighborhood of the origin. Moreover the steps followed at the finer levels of decomposition match closely those in the first step. We therefore present in detail only the computations for the initial stage of recursion. Calculations for the successive steps are left to the interested reader.

Proof of Theorem 1.4. Recall that the quantities $a_l, a_{l'}(\eta), A_l, B_l, C_l, D_l$, etc. can be read off the generalized Newton diagrams of f''_{xy} and g. Without loss of generality, let $a_1 \geq 1$. We write

$$T_{jk}^\lambda \varphi(x) = \int_{\mathbb{R}} e^{i\lambda f(x,y)} |g(x,y)|^{\epsilon/2} \chi(x,y)\chi_j(x)\chi_k(y)\varphi(y)dy$$

where

$$\chi_i(z) = \begin{cases} 1 & \text{if } 2^{-i} \le z \le 2^{-i+1} \\ 0 & \text{otherwise.} \end{cases}$$

We consider four ranges of j, k:

- $a_l j \ll k \ll a_{l+1} j$;
- $k \ll a_1 j$;
- $k \gg a_N$;
- $k \approx a_l j$,

where $A \ll B$, $A \gg B$, and $A \approx B$ mean that $A + C < B$, $A > B + C$, and $A - C < B < A + C$ respectively for some $C > 0$ which makes the following arguments hold true. Since the treatments of the first three cases are similar, we only consider two cases: $a_l j \ll k \ll a_{l+1} j$; $k \approx a_l j$.

Case 1: $a_l j \ll k \ll a_{l+1} j$

In this case

$$|f''_{xy}(x, y)| \sim 2^{-A_l j} 2^{-B_l k}, \quad |g(x, y)| \sim 2^{-C_l j} 2^{-D_l k}$$

on the support of $\chi_j(x)\chi_k(y)$. The operator Van der Corput lemma in [12, Section 3] and [13, Lemma 1] yields

$$||T_{jk}|| \le C(\lambda 2^{-A_l j - B_l k})^{-1/2} 2^{-\epsilon(C_l j + D_l k)/2}, \tag{3.1}$$

and by using Schur's lemma we obtain

$$||T_{jk}|| \le C 2^{-(j+k)/2} 2^{-\epsilon(C_l j + D_l k)/2}. \tag{3.2}$$

If we put $k = a_l j + r$ with $0 \ll r \ll (a_{l+1} - a_l) j$, we can rewrite (3.1) and (3.2) as

$$\begin{aligned} ||T_{jk}|| &\le \min\left\{\lambda^{-1/2} 2^{j(A_l - \epsilon C_l)/2} 2^{k(B_l - \epsilon D_l)/2},\ 2^{-j(1+\epsilon C_l)/2} 2^{-k(1+\epsilon D_l)/2}\right\} \\ &\le \min\left\{\lambda^{-1/2} 2^{j(E_l - \epsilon F_l)/2} 2^{r(B_l - \epsilon D_l)/2},\ 2^{-j(1+a_l+\epsilon F_l)/2} 2^{-r(1+\epsilon D_l)/2}\right\}. \end{aligned}$$

First we assume

$$\lambda^{-1/2} 2^{j(E_l - \epsilon F_l)/2} 2^{r(B_l - \epsilon D_l)/2} \le 2^{-j(1+a_l+\epsilon F_l)/2} 2^{-r(1+\epsilon D_l)/2},$$

which is equivalent to

$$2^{jH_l/2} \le \lambda^{1/2} 2^{-r(1+B_l)/2}$$

i.e.,

$$2^{j/2} \le \lambda^{\frac{1}{2H_l}} 2^{-\frac{r(1+B_l)}{2H_l}}. \tag{3.3}$$

By the choice of r we also have

$$2^{j/2} \geq 2^{\frac{r}{2(a_{l+1}-a_l)}}. \tag{3.4}$$

By combining (3.3) and (3.4) we obtain

$$2^{\frac{r}{2(a_{l+1}-a_l)}} \leq \lambda^{\frac{1}{2(1+a_l+A_l+a_l B_l)}} \, 2^{-\frac{r(1+B_l)}{2(1+a_l+A_l+a_l B_l)}},$$

which implies

$$2^{\frac{r}{2}} \leq \lambda^{\frac{1}{2}\cdot\frac{a_{l+1}-a_l}{1+a_{l+1}+A_l+a_{l+1}B_l}}. \tag{3.5}$$

By the definition of $A_l(\eta)$, $B_l(\eta)$, $C_l(\eta)$, $D_l(\eta)$ and $a_l(\eta)$ it is easy to see that

$$A_l(\eta) + a_{l+1}(\eta)B_l(\eta) = A_{l+1}(\eta) + a_{l+1}(\eta)B_{l+1}(\eta), \tag{3.6}$$

$$C_l(\eta) + a_{l+1}(\eta)D_l(\eta) = C_{l+1}(\eta) + a_{l+1}(\eta)D_{l+1}(\eta). \tag{3.7}$$

Applying (3.6) with $\eta = \mathbf{id}$ to (3.5) we obtain

$$2^{\frac{r}{2}} \leq \lambda^{\frac{1}{2}\cdot\frac{a_{l+1}-a_l}{H_{l+1}}}. \tag{3.8}$$

Here we separately treat two cases: $E_l \geq \epsilon F_l$; $E_l < \epsilon F_l$.

Subcase 1: $E_l \geq \epsilon F_l$

In this case we use (3.3) to obtain

$$\sum_j ||T_{jk}|| \leq \lambda^{-1/2}\lambda^{\frac{E_l-\epsilon F_l}{2H_l}} 2^{\frac{r}{2}I} \tag{3.9}$$

where

$$I = (B_l - \epsilon D_l) - \frac{(1+B_l)(E_l-\epsilon F_l)}{H_l}.$$

If $I < 0$, then the summation of (3.9) in r yields

$$\sum_{(j,k);\, a_l j \ll k \ll a_{l+1} j} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon F_l}{H_l}}.$$

If $I \geq 0$, then we use (3.8) to make a summation of (3.9) in r and obtain

$$\begin{aligned} \sum_{(j,k);\, a_l j \ll k \ll a_{l+1} j} ||T_{jk}|| &\leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon F_l}{H_l}} \lambda^{\frac{1}{2}\cdot\frac{(a_{l+1}-a_l)I}{H_{l+1}}} \\ &\leq \lambda^{-\frac{1}{2}\left[\frac{1+a_l+\epsilon F_l}{H_l} - \frac{(a_{l+1}-a_l)I}{H_{l+1}}\right]}. \end{aligned}$$

We claim that

$$\frac{1+a_l+\epsilon F_l}{H_l} - \frac{(a_{l+1}-a_l)I}{H_{l+1}} = \frac{1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)}{1+a_{l+1}+A_l+a_{l+1}B_l}. \tag{3.10}$$

By rewriting (3.10) we have to show

$$\begin{aligned}&[1+a_l+\epsilon(C_l+a_lD_l)][1+a_{l+1}+A_l+a_{l+1}B_l]-(a_{l+1}-a_l)\\ &\quad\times[(B_l-\epsilon D_l)(1+a_l+A_l+a_lB_l)-(B_l+1)\{A_l+a_lB_l-\epsilon(C_l+a_lD_l)\}]\\ &\quad=[1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)][1+a_l+A_l+a_lB_l].\end{aligned}$$

Now we take derivatives of the left and right hand sides with respect to a_{l+1}:

$$\begin{aligned}\frac{d}{da_{l+1}}(\text{LHS}) &= (1+B_l)[1+a_l+\epsilon(C_l+a_lD_l)]\\ &\quad -[(B_l-\epsilon D_l)(1+a_l+A_l+a_lB_l)-(B_l+1)\times\\ &\qquad \times\{A_l+a_lB_l-\epsilon(C_l+a_lD_l)\}]\\ &= (1+B_l)[1+a_l+\epsilon(C_l+a_lD_l)+A_l+a_lB_l-\epsilon(C_l+a_lD_l)]\\ &\quad -(B_l-\epsilon D_l)(1+a_l+A_l+a_lB_l)\\ &= (1+\epsilon D_l)(1+a_l+A_l+a_lB_l),\\ \frac{d}{da_{l+1}}(\text{RHS}) &= (1+\epsilon D_l)(1+a_l+A_l+a_lB_l).\end{aligned}$$

Also if $a_{l+1}=a_l$ then it is easy to see that the left hand side is same to the right hand side. Thus (3.10) has been proved, which implies

$$\sum_{(j,k);\ a_lj\ll k\ll a_{l+1}j} ||T_{jk}|| \le \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon F_l}{H_l}}\lambda^{\frac{1}{2}\cdot\frac{(a_{l+1}-a_l)I}{H_{l+1}}} \le \lambda^{-\frac{1}{2}\frac{1+a_{l+1}+\epsilon F_{l+1}}{H_{l+1}}}.$$

Subcase 2: $E_l<\epsilon F_l$

(3.4), (3.6), and (3.7) yield

$$\sum_j ||T_{jk}|| \le \lambda^{-\frac{1}{2}}2^{\frac{r}{2}\frac{E_l-\epsilon F_l}{a_{l+1}-a_l}}2^{\frac{r}{2}(B_l-\epsilon D_l)} \le \lambda^{-\frac{1}{2}}2^{\frac{r}{2}\frac{E_{l+1}-\epsilon F_{l+1}}{a_{l+1}-a_l}}.$$

If $E_{l+1}\ge\epsilon F_{l+1}$, then (3.5) yields

$$\sum_{(j,k);\ a_lj\ll k\ll a_{l+1}j} ||T_{jk}|| \le \lambda^{-\frac{1}{2}}\lambda^{\frac{1}{2}\cdot\frac{E_{l+1}-\epsilon F_{l+1}}{H_{l+1}}} \le \lambda^{-\frac{1}{2}\cdot\frac{1+a_{l+1}+\epsilon F_{l+1}}{H_{l+1}}}.$$

If $E_{l+1}<\epsilon F_{l+1}$, then

$$\sum_{(j,k);\ a_lj\ll k\ll a_{l+1}j} ||T_{jk}|| \le \lambda^{-\frac{1}{2}}.$$

Now we consider the case where

$$2^{j/2} \geq \lambda^{\frac{1}{2H_l}} 2^{-\frac{r(1+B_l)}{2H_l}}. \tag{3.11}$$

We note that (3.4) still holds true in this case. We consider two cases:

$$\lambda^{\frac{1}{2H_l}} 2^{-\frac{r(1+B_l)}{2H_l}} \geq 2^{\frac{r}{2(a_{l+1}-a_l)}}; \tag{3.12}$$

$$\lambda^{\frac{1}{2H_l}} 2^{-\frac{r(1+B_l)}{2H_l}} < 2^{\frac{r}{2(a_{l+1}-a_l)}}. \tag{3.13}$$

We rewrite (3.12) to obtain

$$2^{\frac{r}{2}} \leq \lambda^{\frac{a_{l+1}-a_l}{2H_{l+1}}}. \tag{3.14}$$

By using (3.11) we obtain

$$\sum_j ||T_{jk}|| \leq \lambda^{-\frac{1}{2}} \lambda^{\frac{E_l-\epsilon F_l}{2H_l}} 2^{\frac{r}{2}I}.$$

If $I < 0$ then we have a convergent geometric series which we sum to obtain

$$\sum_{j,k} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon F_l}{H_l}}.$$

If $I \geq 0$ then we use (3.14) and (3.10) to obtain

$$\sum_{j,k} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_{l+1}+\epsilon F_{l+1}}{H_{l+1}}}.$$

Now we rewrite (3.13) to obtain

$$2^{\frac{r}{2}} > \lambda^{\frac{a_{l+1}-a_l}{2H_{l+1}}}. \tag{3.15}$$

By using (3.4) we obtain

$$\sum_j ||T_{jk}|| \leq 2^{-\frac{r}{2}\cdot\frac{1+a_l+\epsilon F_l}{a_{l+1}-a_l}} 2^{-\frac{r}{2}\cdot(1+\epsilon D_l)}.$$

We then use (3.15) to get

$$\sum_{jk} ||T_{jk}|| \leq \lambda^{-\frac{1+a_{l+1}+\epsilon F_{l+1}}{2H_{l+1}}},$$

which is the desired estimate.

Case 2: $k \approx a_l j$

In this case the dyadic rectangle is close to roots $y = r(x)$ of $f''_{xy}(x,y) = 0$ or $g(x,y) = 0$ of the form $cx^{a_l} + \cdots$ $(c \neq 0)$. If c is a complex number, then $|y - r(x)| \sim 2^{-a_l j}$ so further resolution of singularities is not necessary. Therefore we may assume that c is a positive real number. We set $t(x) = cx^{a_l}$ and $\eta : (x,y) \mapsto (x, y - t(x))$. Let

$$a_1(\eta) < a_2(\eta) < \cdots < a_k(\eta) < \cdots$$

be leading exponents of $\{r_\nu(x) - t(x) \mid \nu \in I(f''_{xy})\} \cup \{s_\mu(x) - t(x) \mid \mu \in I(g)\}$. Since we consider a dyadic rectangle close to $y = cx^{a_l}$, we may assume that $a_1(\eta) \geq a_l$. If $a_{l'}(\eta) j \ll m \ll a_{l'+1} j$ then we have

$$|f''_{xy}(x,y)| \sim 2^{-A_k(\eta) j} 2^{-B_k(\eta) m}; \quad |g(x,y)| \sim 2^{-C_k(\eta) j} 2^{-D_k(\eta) m}.$$

We write

$$T^\lambda_{j,k,m}\varphi(x) = \int_{\mathbb{R}} e^{i\lambda f(x,y)} |g(x,y)|^{\epsilon/2} \chi(x,y)\varphi(y)\chi_j(x)\chi_k(y)\chi_m(y - t(x))dy.$$

By applying the operator Van der Corput lemma and Schur's lemma again we obtain

$$\text{(3.16)} \quad ||T^\lambda_{j,k,m}|| \leq C(\lambda 2^{-(A_{l'}(\eta) j + B_{l'}(\eta) m)})^{-1/2} (2^{-(C_{l'}(\eta) j + D_{l'}(\eta) m)})^{\epsilon/2},$$

$$\text{(3.17)} \quad ||T^\lambda_{j,k,m}|| \leq 2^{-m} 2^{j(a_l - 1)/2} (2^{-(C_{l'}(\eta) j + D_{l'}(\eta) m)})^{\epsilon/2}$$

since $\Delta y \leq 2^{-m}$ and $\Delta x \leq 2^{-m} 2^{a_l - 1}$, where Δy is the maximal variation in y for a fixed x in the region under consideration and Δx is defined in a similar way. Now we follow the same procedure in Case 1 to prove the desired estimate. Since arguments are parallel to those in Case 1, we omit detailed calculations. By putting $m = a_{l'}(\eta) j + r$ with $0 \ll r \ll (a_{l'+1}(\eta) - a_{l'}(\eta)) j$, we obtain

$$||T^\lambda_{j,k,m}|| \leq \min\left\{\lambda^{-1/2} 2^{j(E_{l'}(\eta) - \epsilon F_{l'}(\eta))/2} 2^{r(B_{l'}(\eta) - \epsilon D_{l'}(\eta))/2},\right.$$
$$\left. 2^{-j[(1 + 2a_{l'}(\eta) - a_l) + \epsilon F_{l'}(\eta)]/2} 2^{-r(2 + \epsilon D_{l'}(\eta))/2}\right\}.$$

First we consider the case where

$$\lambda^{-1/2}\, 2^{j(E_{l'}(\eta) - \epsilon F_{l'}(\eta))/2}\, 2^{r(B_{l'}(\eta) - \epsilon D_{l'}(\eta))/2} \leq$$
$$\leq 2^{-j[(1 + 2a_{l'}(\eta) - a_l) + \epsilon F_{l'}(\eta))]/2}\, 2^{-r(2 + \epsilon D_{l'}(\eta))/2},$$

that is,

$$\text{(3.18)} \qquad 2^{j/2} \leq \lambda^{\frac{1}{2H_{l,l'}(\eta)}} 2^{-\frac{r}{2} \cdot \frac{B_{l'}(\eta) + 2}{H_{l,l'}(\eta)}}.$$

By the choice of r we also have

$$2^{j/2} \geq 2^{\frac{r}{2(a_{l'+1}(\eta)-a_{l'}(\eta))}}. \tag{3.19}$$

(3.18) and (3.19) yield

$$2^{\frac{r}{2}} \leq \lambda^{\frac{1}{2}\frac{a_{l'+1}(\eta)-a_{l'}(\eta)}{H_{l,l'}(\eta)}}.$$

We therefore have

$$\sum_j ||T^\lambda_{j,k,m}|| \leq \lambda^{-\frac{1}{2}\frac{2a_{l'}(\eta)-a_l+1+\epsilon F_{l'}(\eta)}{H_{l,l'}(\eta)}} 2^{\frac{r}{2}J},$$

where

$$J = (B_{l'}(\eta) - \epsilon D_{l'}(\eta)) - \frac{(B_{l'}(\eta)+2)(E_{l'}(\eta) - \epsilon F_{l'}(\eta))}{H_{l,l'}(\eta)}.$$

If $J < 0$, then

$$\sum_{j,k,m;\, a_{l'}(\eta)j \ll m \ll a_{l'+1}(\eta)j} ||T^\lambda_{j,k,m}|| \leq \lambda^{-\frac{1}{2}\frac{2a_{l'}(\eta)-a_l+1+\epsilon F_{l'}(\eta)}{H_{l,l'}(\eta)}}.$$

If $J \geq 0$, then

$$\sum_{j,k,m;\, a_{l'}(\eta)j \ll m \ll a_{l'+1}(\eta)j} ||T^\lambda_{j,k,m}|| \leq \lambda^{-\frac{1}{2}\frac{2a_{l'+1}(\eta)-a_l+1+\epsilon F_{l'+1}(\eta))}{H_{l,l'+1}}}.$$

To treat the case where

$$2^{j/2} > \lambda^{\frac{1}{2H_{l,l'}(\eta)}} 2^{-\frac{r}{2}\cdot\frac{B_{l'}(\eta)+2}{H_{l,l'}(\eta)}}$$

we can use the same argument for (3.11). We omit the detail here.

If $m \approx a_{l'}(\eta)j$, then there exists $\tilde{t}$ such that $y - \tilde{t}(x)$ is "small". Put $y - \tilde{t}(x) \sim 2^{-p}$ and repeat the same arguments as before until we completely resolve the singularities. By putting things together we conclude

$$||T_\lambda|| \leq C\lambda^{-\delta/2}$$

where

$$\delta = \min\left(\frac{1}{2}, \frac{1}{2}\cdot\frac{1+a_l+\epsilon(C_l+a_lD_l)}{1+a_l+A_l+a_lB_l}, \frac{1}{2}\cdot\frac{1+2a_{l'}(\eta)-a_l+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{1+2a_{l'}(\eta)-a_l+(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta))}\right),$$

which is the desired estimate for $p = 2$. ■

4. Proof of Theorem 1.5

In this section we will prove Theorem 1.5. We construct an analytic family of operators T_λ^β so that when $\mathrm{Re}(\beta) = 1/2$, T_λ^β is a damped oscillatory integral operator of the form

$$T_\lambda^{1/2}\varphi(x) = \int e^{i\lambda f(x,y)} |f''_{xy}(x,y)|^{1/2} \chi(x,y)\varphi(y)dy,$$

whose L^2 decay estimate we know of. When $\mathrm{Re}(\beta) = -\alpha/(1-2\alpha)$, we shall prove T_λ^β is bounded on $L^{\frac{p(1-2\alpha)}{1-p\alpha}}$, which yields Theorem 1.5 by complex interpolation in [18].

Proof of Theorem 1.5. We consider an analytic family of operators

$$T_\lambda^\beta\varphi(x) = \int e^{i\lambda f(x,y)} |g(x,y)|^{\epsilon(1/2-\beta)} |f''_{xy}(x,y)|^{\beta} \chi(x,y)\varphi(y)dy. \tag{4.1}$$

We note that $T_\lambda^0 = T_\lambda$.

Theorem 4.1 ([13]) *If* $Re(\beta) = 1/2$ *then*

$$||T_\lambda^\beta||_{L^2 \to L^2} = O(\lambda^{-1/2}).$$

When $\mathrm{Re}(\beta) = -\alpha/(1-2\alpha)$, T_λ^β is a form of fractional integration and we want to obtain estimate without any decay rate. To do this we shall use the following lemma.

Lemma 4.2 *If* $K(x,y) \geq 0$ *be the kernel of an operator* T *and* $K(x,y)$ *satisfies the following,*

$$\int K(x,y)y^{-\frac{1}{p}}dy \leq Cx^{-\frac{1}{p}}, \quad \int K(x,y)x^{-\frac{1}{q}}dx \leq Cy^{-\frac{1}{q}},$$

where $1/p + 1/q = 1$, *then*

$$T\varphi(x) = \int K(x,y)\varphi(y)dy$$

is bounded in L^p.

Proof of Lemma 4.2. For $\varphi \in L^p$ and $\psi \in L^q$ ($\frac{1}{p} + \frac{1}{q} = 1$) with $||\varphi||_p = ||\psi||_q = 1$, we have

$$\begin{aligned} |\varphi(y)\psi(x)| &= |\varphi(y)x^{-\frac{1}{pq}}y^{\frac{1}{pq}}\psi(x)y^{-\frac{1}{pq}}x^{\frac{1}{pq}}| \\ &\leq \frac{1}{p}|\varphi(y)|^p x^{-\frac{1}{q}}y^{\frac{1}{q}} + \frac{1}{q}|\psi(x)|^q y^{-\frac{1}{p}}x^{\frac{1}{p}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|\int\int K(x,y)\varphi(y)\psi(x)dydx\right|\\ \leq\ & \int\int K(x,y)\frac{1}{p}|\varphi(y)|^p x^{-\frac{1}{q}}y^{\frac{1}{q}}dydx + \int\int K(x,y)\frac{1}{q}|\psi(x)|^q y^{-\frac{1}{p}}x^{\frac{1}{p}}dydx\\ \leq\ & C/p + C/q.\end{aligned}$$

This completes the proof. ■

Now we shall prove the following lemma.

Lemma 4.3 *Let* $p_0 = \frac{p(1-2\alpha)}{1-p\alpha}$ *and* $\beta_0 = -\frac{\alpha}{1-2\alpha}$. *If* $(1/p, \alpha) \in \mathrm{int}(\mathcal{A})$, *then* $T_\lambda^{\beta_0}$ *is bounded on* L^{p_0} *with the operator norm* $O(1)$.

Proof of Lemma 4.3. Since the oscillation does not play any role, it suffices to obtain L^{p_0} boundedness of the operator

$$D\varphi(x) = \int |g(x,y)|^{\epsilon(1/2-\beta_0)}|f''_{xy}(x,y)|^{\beta_0}\chi(x,y)\varphi(y)dy.$$

Let

$$K(x,y) = |g(x,y)|^{\epsilon(1/2-\beta_0)}|f''_{xy}(x,y)|^{\beta_0}.$$

By Lemma 4.2, it suffices to show that

$$\int_I K(x,y)\frac{1}{y^{1/p_0}}dy \leq \frac{C}{x^{1/p_0}} \tag{4.2}$$

and

$$\int_I K(x,y)\frac{1}{x^{1/q_0}}dx \leq \frac{C}{y^{1/q_0}}, \tag{4.3}$$

where $1/p_0 + 1/q_0 = 1$ and $I = [-|I|, |I|]$ with a sufficiently small $|I|$. Since the argument to prove (4.3) is parallel to the argument for (4.2), we shall only show (4.2). The proof can be divided into finite steps and we shall here show the first two steps. To complete the proof we can repeat the same argument.

Step I: Considering each quadrant separately, we may assume that $x > 0$, $y > 0$ and $I = [0, |I|]$. After reindexing if necessary, we may assume that there exist $c_l > 0$, $d_l > 0$, and $C_l > 0$ such that $c_l < d_l < C_l$, $|r_l(x)| = d_l x^{a_l} + o(x^{a_l})$, and $|s_l(x)| = d_l x^{a_l} + o(x^{a_l})$. We divide I into several subintervals: $0 \leq y \leq c_n x^{a_n}$, $c_l x^{a_l} \leq y \leq C_l x^{a_l}$, $C_{l+1}x^{a_{l+1}} \leq y \leq c_l x^{a_l}$, and $C_1 x^{a_1} \leq y \leq |I|$ and separately treat each cases.

Case 1: $0 \le y \le c_n x^{a_n}$.

If $0 \le y \le c_n x^{a_n}$, then

$$|g(x,y)| \sim x^{C_n} y^{D_n}, \quad \text{and} \quad |f''_{xy}(x,y)| \sim x^{A_n} y^{B_n}.$$

Since $(\alpha, 1/p) \in \mathcal{A} \subset \mathcal{A}_1$, we have

$$\alpha < \frac{\epsilon D_n + 2}{2(B_n+1)} - \frac{1}{B_n+1}\frac{1}{p}, \tag{4.4}$$

which is equivalent to

$$\epsilon D_n(\frac{1}{2} - \beta_0) + B_n\beta_0 - \frac{1}{p_0} > -1.$$

Consequently,

$$\begin{aligned}\int_0^{c_n x^{a_n}} K(x,y) y^{-\frac{1}{p_0}} dy &\sim \int_0^{c_n x^{a_n}} x^{\epsilon C_n(1/2-\beta_0)+A_n\beta_0} y^{\epsilon D_n(1/2-\beta_0)+B_n\beta_0-1/p_0} dy \\ &\le x^{\epsilon F_n(1/2-\beta_0)+E_n\beta_0-a_n/p_0+a_n}.\end{aligned}$$

Since $(\alpha, 1/p) \in \text{int}(\mathcal{A}) \subset \text{int}(\mathcal{A}_1)$, we have

$$\epsilon F_n(\frac{1}{2}-\beta_0)+E_n\beta_0-\frac{a_n}{p_0}+a_n+\frac{1}{p_0} = \frac{H_n}{1-2\alpha}\left[\frac{\epsilon F_n+2a_n}{2H_n}+\frac{1-a_n}{pH_n}-\alpha\right] > 0,$$

which implies

$$\int_0^{c_n x^{a_n}} K(x,y)\frac{1}{y^{1/p_0}} dy \le \frac{C}{x^{1/p_0}}.$$

Case 2: $C_l x^{a_l} \le y \le c_{l+1} x^{a_{l+1}}$.

If $C_l x^{a_l} \le y \le c_{l+1} x^{a_{l+1}}$, then

$$|g(x,y)| \sim x^{C_l} y^{D_l}, \quad \text{and} \quad |f''_{xy}(x,y)| \sim x^{A_l} y^{B_l}.$$

By using (3.6) and (3.7) we obtain

$$\begin{aligned}&\int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} K(x,y)\frac{1}{y^{1/p_0}} dy \\ &\sim \int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} x^{\epsilon C_l(1/2-\beta_0)+A_l\beta_0} y^{\epsilon D_l(1/2-\beta_0)+B_l\beta_0-1/p_0} dy \\ &\le Cx^{\epsilon F_l(1/2-\beta_0)+E_l\beta_0-a_l/p_0+a_l}|\ln x| + Cx^{\epsilon F_{l+1}(1/2-\beta_0)+E_{l+1}\beta_0-a_{l+1}/p_0+a_{l+1}}|\ln x|,\end{aligned}$$

where $|\ln x|$ occurs when $\epsilon D_l(1/2-\beta_0)+B_l\beta_0-1/p_0=-1$. Since $(\alpha,1/p)\in \mathrm{int}(\mathcal{A})\subset \mathrm{int}(\mathcal{A}_1)$, we have

$$\epsilon F_l(\frac{1}{2}-\beta_0)+E_l\beta_0-\frac{a_l}{p_0}+a_l+\frac{1}{p_0}=\frac{H_l}{1-2\alpha}\left[\frac{\epsilon F_l+2a_l}{2H_l}+\frac{1-a_l}{pH_l}-\alpha\right]>0,$$

which implies

$$\int_{C_{l+1}x^{a_{l+1}}}^{c_lx^{a_l}} K(x,y)\frac{1}{y^{1/p_0}}dy\le \frac{C}{x^{1/p_0}}.$$

Case 3: $C_1x^{a_1}\le y\le |I|$.

If $C_1x^{a_1}\le y\le |I|$, then

$$|g(x,y)|\sim x^{C_0}y^{D_0},\quad \text{and}\quad |f''_{xy}(x,y)|\sim x^{A_0}y^{B_0}.$$

By using (3.6) and (3.7) again, we obtain

$$\begin{aligned}\int_{C_1x^{a_1}}^{|I|} K(x,y)\frac{1}{y^{1/p_0}}dy &\sim x^{\epsilon C_0(1/2-\beta_0)+A_0\beta_0}\int_{C_1x^{a_1}}^{|I|} y^{\epsilon D_0(1/2-\beta_0)+B_0\beta_0-1/p_0}dy\\ &\le C[x^{\epsilon F_0(1/2-\beta_0)+E_0\beta_0}+x^{\epsilon F_1(1/2-\beta_0)+E_1-a_1/p_0+a_1}]|\ln x|.\end{aligned}$$

By using the fact $(\alpha,1/p)\in \mathrm{int}(\mathcal{A})\subset \mathrm{int}(\mathcal{A}_1)$ again, one can see that the right-hand side is bounded by Cx^{-1/p_0}, which is the desired estimate.

Case 4: $c_lx^{a_l}\le y\le C_lx^{a_l}$.

If $c_lx^{a_l}\le y\le C_lx^{a_l}$,

$$\begin{aligned}|g(x,y)| &\sim x^{C_{l-1}}y^{D_l}\prod_{c_lx^{a_l}\le|s_i(x)|\le C_lx^{a_l}}|y-s_i(x)|,\\ |f''_{xy}(x,y)| &\sim x^{A_{l-1}}y^{B_l}\prod_{c_lx^{a_l}\le|r_i(x)|\le C_lx^{a_l}}|y-r_i(x)|.\end{aligned}$$

To treat this case we need finer decomposition of the domain of integration. Here we start the second step.

Step II: We introduce the following notation:

$$S_l^{\alpha}=\{r_i(x)\mid r_i(x)=c_l^{\alpha}x^{a_l}+o(x^{a_l})\}.$$

We assumed that for all $r_j(x)$ and $s_j(x)$ satisfying

$$c_lx^{a_l}<|r_j(x)|,\ |s_j(x)|<C_lx^{a_l},$$

$|r_j(x)|$ and $|s_j(x)|$ have the same leading term $d_lx^{a_l}$, that is,

$$|r_j(x)|=d_lx^{a_l}+o(x^{a_l})\quad \text{and}\quad |s_j(x)|=d_lx^{a_l}+o(x^{a_l}).$$

If we set $r_j(x) = c_l^\alpha x^{a_l} + o(x^{a_l})$, we have three possible cases: (i) $\mathrm{Im}(c_l^\alpha) \neq 0$, (ii) $c_l^\alpha < 0$, and (iii) $c_l^\alpha > 0$. In (i) and (ii), we have

$$|y - r_j(x)| \sim x^{a_l}$$

if y is in the range $\{c_l x^{a_l} < y < C_l x^{a_l}\}$. Hence we may assume that $c_l^\alpha = d_l > 0$. Now we define a coordinate transformation η so that

$$\eta(x, y) = (x, y + c_l^\alpha x^{a_l}).$$

If we rewrite the integral in terms of y_1, we have

$$\begin{aligned}\int_{c_l x^{a_l}}^{C_l x^{a_l}} K(x,y)\frac{1}{y^{1/p_0}}dy &\leq x^{-a_l/p_0}\int_{-Cx^{a_l}}^{Cx^{a_l}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \\ &= x^{-a_l/p_0}\int_{-Cx^{a_l}}^{0} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 + x^{-a_l/p_0}\int_{0}^{Cx^{a_l}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \\ &= I_{l,-} + I_{l,+}.\end{aligned}$$

Since the treatment of $I_{l,+}$ is similar to that of $I_{l,-}$, we only treat $I_{l,+}$. To do this we may assume that we can find $c_{l,l'}$, $d_{l,l'}$, and $C_{l,l'}$ such that $0 < c_{l,l'} < d_{l,l'} < C_{l,l'}$,

$$|r_l(x) - c_l^\alpha x^{a_l}| = d_{l,l'} x^{a_{l'}(\eta)} + o(x^{a_{l'}(\eta)}),$$

and

$$|s_l(x) - c_l^\alpha x^{a_l}| = d_{l,l'} x^{a_{l'}(\eta)} + o(x^{a_{l'}(\eta)}).$$

We decompose the region $\{(x,y) : 0 \leq y \leq Cx^{a_l}\}$ into several subregions: $0 \leq y \leq c_{l,n_1} x^{a_{n_1}(\eta)}$, $C_{l,1} x^{a_1(\eta)} \leq y \leq Cx^{a_l}$, $C_{l,l'+1} x^{a_{l'+1}(\eta)} \leq y \leq c_{l,l'} x^{a_{l'}(\eta)}$, and $c_{l,l'} x^{a_{l'}(\eta)} \leq y \leq C_{l,l'} x^{a_{l'}(\eta)}$. We treat each cases in a separate way. Since the treatment of each case is same to that of each case of Step I, we omit the detailed calculation. Actually one can simply replace a_l, A_l,... with $a_{l'}(\eta)$, $A_{l'}(\eta)$,... in the arguments of Step I.

Case 1: $0 \leq y \leq c_{l,n_1} x^{a_{n_1}(\eta)}$.

By using the same argument for Case 1 of the previous step we obtain

$$\int_0^{c_{l,n_1} x^{a_{n_1}(\eta)}} K(x, y + c_l^\alpha x^{a_l})dy \leq Cx^{\epsilon F_{n_1}(\eta)(1/2-\beta_0) + E_{n_1}(\eta)\beta_0 + a_{n_1}(\eta)}.$$

Since $(1/p, \alpha) \in \mathrm{int}(\mathcal{A}) \subset \mathrm{int}(\mathcal{A}_\in)$, we obtain

$$\begin{aligned}&\epsilon F_{n_1}(\eta)(1/2-\beta_0) + E_{n_1}(\eta)\beta_0 + a_{n_1}(\eta) - \frac{a_l}{p_0} + \frac{1}{p_0} \\ &\qquad = \frac{H_{l,n_1}(\eta)}{1-2\alpha}\left[\frac{\epsilon F_{n_1}(\eta) + 2a_{n_1}(\eta)}{2H_{l,n_1}(\eta)} + \frac{1-a_l}{pH_{l,n_1}(\eta)} - \alpha\right] > 0,\end{aligned}$$

which implies the desired estimate.

Case 2: $C_{l,l'+1}x^{a_{l'+1}(\eta_1)} \leq y \leq c_{l,l'}x^{a_{l'}(\eta_1)}$.

We also use the same idea for Case 2 of the previous step to obtain

$$\begin{aligned}&\int_{C_{l,l'+1}x^{a_{l'+1}(\eta_1)}}^{c_{l,l'}x^{a_{l'}(\eta_1)}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \\ &\quad \leq C[x^{\epsilon F_{l'}(\eta)(1/2-\beta_0)+E_{l'}(\eta)\beta_0+a_{l'}(\eta)} + x^{\epsilon F_{l'+1}(\eta)(1/2-\beta_0)+E_{l'+1}(\eta)\beta_0+a_{l'+1}(\eta)}]|\ln x|.\end{aligned}$$

Since $(1/p, \alpha) \in \mathrm{int}(\mathcal{A}) \subset \mathrm{int}(\mathcal{A}_\in)$, we obtain

$$\begin{aligned}&\epsilon F_{l'}(\eta)(1/2 - \beta_0) + E_{l'}(\eta)\beta_0 + a_{l'}(\eta) - \frac{a_l}{p_0} + \frac{1}{p_0} \\ &\qquad = \frac{H_{l,l'}(\eta)}{1-2\alpha}\left[\frac{\epsilon F_{l'}(\eta) + 2a_{l'}(\eta)}{2H_{l,l'}(\eta)} + \frac{1-a_l}{pH_{l,l'}(\eta)} - \alpha\right] > 0,\end{aligned}$$

which implies the desired estimate.

Case 3: $C_{l,1}x^{a_1(\eta)} \leq y \leq Cx^{a_l}$.

In this case we have

$$\int_{C_{l,1}x^{a_1(\eta)}}^{Cx^{a_l}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \leq x^{\epsilon F_l(1/2-\beta_0)+E_l\beta_0+a_l} + x^{\epsilon F_1(\eta)(1/2-\beta_0)+E_1(\eta)\beta_0+a_1(\eta)},$$

which gives the desired estimate of this case.

Case 4: $c_{l,l'}x^{a_{l'}(\eta)} \leq y \leq C_{l,l'}x^{a_{l'}(\eta)}$.

It remains to show

$$x^{-a_l/p_0}\int_{c_{l,l'}x^{a_{l'}(\eta)}}^{C_{l,l'}x^{a_{l'}(\eta)}} K(x, y + c_l^\alpha x^{a_l})dy \leq Cx^{-1/p_0}.$$

To prove this inequality we start the third step which has the same argument with the second step. We repeat the same argument until we completely resolve the roots of f''_{xy} and g, that is, until there is at most one root in the range of the integral. If we have only one root $r(x)$ in the range of the integral and if the root is a real root, we have to integrate $|y - r(x)|^{-(2\alpha B_{n(\eta)}(\eta) - \epsilon D_{n(\eta)}(\eta))/2(1-2\alpha)}$ with respect to y near $r(x)$, where η is a coordinate change defined by $\tilde{\eta}(x, y) = (x, y - r(x))$ and $n(\tilde{\eta})$ is the largest index of $a_{l'}(\tilde{\eta})$. The convergence of the integration is guaranteed because by using (1.3) we have

$$\alpha < \frac{\epsilon D_{n(\tilde{\eta})}(\tilde{\eta})) + 2}{2(B_{n(\tilde{\eta})}(\tilde{\eta}) + 2)} \tag{4.5}$$

and (4.5) implies

$$\frac{2\alpha B_{n(\tilde{\eta})}(\tilde{\eta}) - \epsilon D_{n(\tilde{\eta})}(\tilde{\eta})}{2(1-2\alpha)} < 1.$$

If $r(x)$ is not real, we perform the same process with summation of first finite terms of $r(x)$ whose coefficient is real. We can easily see that we have the desired estimates for all integrals which will occur in each step. ■

To finish the proof of Theorem 1.5 we interpolate Lemma 4.3 with Theorem 4.1. ■

Remark 4.4 1. In the proof of Theorem 1.5, we use the strict inequalities at two places (4.4) and (4.5). When we prove (4.3), we have to use one more strict inequality

$$\alpha < \frac{\epsilon C_0}{2(1+A_0)} + \frac{1}{1+A_0}\frac{1}{p}. \tag{4.6}$$

Therefore, Theorem 1.4. can be extended to the boundary of $\mathcal{A}$ when $(1/p, \alpha)$ is not on any of a line which bounds the region in (4.4), (4.5) or (4.6). It would be interesting to obtain L^p decay estimates when $(1/p, \alpha)$ is on one of these lines.

2. Let δ_1 and δ_2 be the weighted Newton distance and the optimal decay rate, respectively. We give an example showing that in general the optimal decay rate for L^2 operator norm of T_λ can be smaller than the weighted Newton distance which has been introduced in [14]. We take f and g such that

$$\begin{aligned} f''_{xy}(x,y) &= (y - x^N)^{R_1}(y - x^N - x^{kN})^{M_1} \\ g(x,y) &= (y - x^N - x^{2N})^{R_2}. \end{aligned}$$

Without any change of variable, we have

$$a_1 = N,\ A_1 = N(R_1 + M_1),\ B_1 = 0,\ C_1 = NR_2, \text{ and } D_1 = 0.$$

One can check that

$$\delta_1 = \frac{1 + N + \epsilon N R_2}{1 + N + N(R_1 + M_1)}.$$

By using the change of variables $\eta : (x,y) \mapsto (x, y - x^N)$, we have

$$a_2(\eta) = kN,\ A_2 = kNM_1,\ B_2 = R,\ C_2 = 2NR_2, \text{ and } D_2 = 0.$$

We then have

$$\delta_2 = \frac{1 + 2kN - N + \epsilon(2NR_2)}{1 + 2kN - N + kN(M_1 + R_1)}.$$

Given N there exists k such that

$$\delta_2 \sim \frac{2N}{2N + N(M_1 + R_1)} = \frac{2}{2 + M_1 + R_1}.$$

For large N, we have

$$\delta_1 \sim \frac{1 + \epsilon R_2}{1 + R_1 + M_1}.$$

Now choosing ϵ and R_2 so that $\epsilon R_2 > 1$, we get $\delta_2 < \delta_1$.

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