

Quasiconformal groups of compact type

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Abstract

We establish that a quasiconformal group is of compact type if and only if its limit set is purely conical and find that the limit set of a quasiconformal group of compact type is uniformly perfect. A key tool is the result of Bowditch–Tukia on compact-type convergence groups. These results provide crucial tools for studying the deformations of quasiconformal groups and in establishing isomorphisms between such groups and conformal groups.

1. Introduction

A *Kleinian group* Γ is a discrete non-elementary group of isometries of hyperbolic n -space which we identify as the unit ball \mathbb{B}^n endowed with the metric $ds = |dx|/(1 - |x|^2)$ of constant negative curvature. The orbit spaces \mathbb{B}^n/Γ of Kleinian groups are hyperbolic n -manifolds if Γ is torsion free and hyperbolic n -orbifolds otherwise.

One of the more important properties a Kleinian group may possess is that its orbit space admits a natural compactification. This property finds its expression in the concept of groups of compact type.

A Kleinian group Γ acting on \mathbb{B}^n naturally extends to $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ and there are several well-known equivalent notions for Γ to be *convex co-compact*. One of these definitions is for $(\overline{\mathbb{B}^n} \setminus \Lambda(\Gamma))/\Gamma$ to be compact, where $\Lambda(\Gamma)$ is the limit set of Γ . Denoting by $Con(\Lambda(\Gamma))$ the convex hull of $\Lambda(\Gamma)$ (in \mathbb{B}^n), an equivalent condition is for $Con(\Lambda(\Gamma))/\Gamma$ to be compact. Furthermore, both of these conditions are equivalent to the limit set of Γ being purely conical.

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There is a natural association between conformal groups acting on \mathbb{S}^{n-1} and Kleinian groups (acting on \mathbb{B}^n) via the Poincaré extension [1]. For quasiconformal groups acting on \mathbb{S}^{n-1} it is unknown, and an important question, as to whether such an extension may be possible. Thus two of the above potential definitions cannot be used without modifications. Even if the quasiconformal group action does extend to \mathbb{B}^n , in general the convex hull of the limit set is not invariant under the group action so that one cannot readily divide by the group action.

However the various notions of compactness play a crucial role in our investigations of the algebraic and topological rigidity of quasiconformal groups [4, 5]. The central question here is to establish whether or not a discrete uniformly quasiconformal group is canonically isomorphic to a conformal group and if so, to establish a topological conjugacy on the respective limit sets. These questions provide the major motivation for the current paper, however we shall see that there are other interesting consequences as well.

In [8] and [18], Bowditch and Tukia address these questions relating the various notions of compactness for the class of convergence groups, topological generalizations of conformal and quasiconformal groups [10]. They replace hyperbolic space by the triple space of an underlying metric space on which the group acts; this metric space is typically the limit set or boundary of the group. It is immediate that a convergence group G acting on a compact metric space M induces a properly discontinuous action on the triple space $T(M) = \{(x, y, z) \mid x, y, z \in M \text{ distinct}\}$ by $g((x, y, z)) := (g(x), g(y), g(z))$. Here, $T(M)$ inherits the product topology from $M \times M \times M$, and can be naturally compactified by adjoining a copy of M .

Independently, Bowditch and Tukia show:

Theorem 1.1 (Bowditch, Tukia) *Let G be a discrete convergence group acting on a perfect compact metric space M . Then $T(M)/G$ is compact if and only if every point of M is a conical limit point.*

Here, a *conical limit point* of a discrete convergence group is a point $x \in M$ for which there exists a sequence $\{g_j\}$ in G and distinct points $a, b \in M$ so that $g_j(x) \rightarrow a$ but $g_j(y) \rightarrow b$ for all $y \in M \setminus \{x\}$. This notion first appears in work of Beardon and Maskit [3] and was called *point of approximation*. They showed the equivalence between this notion and that of being a conical limit point and related these concepts, in three dimensions, to geometrical finiteness (the existence of a finite sided fundamental polyhedron). This definition of conical limit point has shown itself to be of considerable use in the study of convergence groups ([14, 15]).

If we restrict the action of G to the perfect, compact metric space $\Lambda(G)$, then Theorem 1.1 gives that $T(\Lambda(G))/G$ is compact if and only if $\Lambda(G)$ is purely conical.

Returning to discrete quasiconformal groups acting on $\overline{\mathbb{B}^n}$, we see that we can recover two of the above mentioned definitions for the group to be convex co-compact. One is for the limit set to be purely conical, and the other is for the triple space of the limit set divided by the group to be compact (this latter formulation corresponds to the convex hull of the limit set divided by the group to be compact in the Kleinian case). We choose the third notion of being convex co-compact for our definition in the case of quasiconformal groups and then show equivalence to the other two notions later.

Definition 1.2 *Let G be a non-elementary discrete quasiconformal group acting on $\overline{\mathbb{B}^n}$. Then G is of compact type if $(\overline{\mathbb{B}^n} \setminus \Lambda(G))/G$ is compact.*

Our main theorem is:

Theorem 1.3 *Let G be a non-elementary discrete quasiconformal group acting on $\overline{\mathbb{B}^n}$. Then G is of compact type if and only if the limit set of G is purely conical.*

In order to prove Theorem 1.3 we show a result which is of independent interest, see also Theorem 1.6 for a more general version:

Theorem 1.4 *Let G be a non-elementary discrete quasiconformal group acting on $\overline{\mathbb{B}^n}$. If $\Lambda(G)$ is purely conical, then $\Lambda(G)$ is uniformly perfect.*

The definition of uniformly perfect appears below in one of its many equivalent forms. The notion was first introduced by Pommerenke ([16]). A result of Järvi and Vuorinen [12] asserts that uniformly perfect sets have positive Hausdorff dimension.

We note that the converse of the last theorem is of course not true. For example, there are Kleinian groups acting on \mathbb{B}^2 with limit set the whole circle \mathbb{S}^1 , but the groups contain parabolic elements. Thus the limit set is uniformly perfect, but not purely conical.

A consequence of Theorem 1.4 is that for a discrete quasiconformal group with purely conical limit set, acting on $\overline{\mathbb{B}^n}$, the limit set is “uniformly conical” in the following sense.

Theorem 1.5 *Let G be a non-elementary discrete K -quasiconformal group acting on $\overline{\mathbb{B}^n}$ with purely conical limit set. Then there exist constants $M > 0$ and $\beta > 0$ with the following property: For every $\omega_0 \in \Lambda(G)$ there exist $g_j \in G$ so that $g_j(0) \rightarrow \omega_0$, $g_0 = id$, and furthermore*

$$\rho(g_j(0), [0, \omega_0]) \leq \beta \quad \text{and} \quad \rho(g_j(0), g_{j+1}(0)) \leq M$$

for all j .

In particular this implies that for each point x on the radial segment $[0, \omega_0]$ there exists $g \in G$ so that $\rho(x, g(0)) \leq 3\beta + M$.

Before beginning the proofs of these results, we wish to make a couple of remarks.

Remark 1. Extending the action of G by reflection to all of $\overline{\mathbb{R}}^n$, defining $\Omega = \overline{\mathbb{R}}^n \setminus \Lambda(G)$ and letting q_Ω be the quasihyperbolic metric in Ω (see Section 2) we also obtain that the analog of Theorem 1.5 holds in the quasihyperbolic metric (use Lemma 2.7 and the fact that $q_\Omega(a, b) \leq 2\rho(a, b)$ for all $a, b \in \mathbb{B}^n$).

Remark 2. As we have mentioned, it is probably not true in general that quasiconformal groups of \mathbb{S}^{n-1} extend to quasiconformal groups of \mathbb{B}^n , except possibly in low dimensions. Our discussion below is limited only to quasiconformal groups on \mathbb{B}^n and it would be nice if we were to establish the results in question for quasiconformal groups of \mathbb{S}^{n-1} as well. This we outline below.

Every K -quasiconformal map $f : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$ extends to a $K' = K'(K, n)$ -quasiconformal map $\tilde{f} : \overline{\mathbb{B}}^n \rightarrow \overline{\mathbb{B}}^n$ ([19]). Any two K' -quasiconformal extensions of a map f differ by a bounded amount in the hyperbolic metric that is independent of the extensions and only depends on K' and n . This is proved in [6], we sketch a proof here for the reader's convenience. If f_1 and f_2 are two K' -quasiconformal extensions of a K -quasiconformal map $f : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$, then $g = f_1^{-1} \circ f_2 : \overline{\mathbb{B}}^n \rightarrow \overline{\mathbb{B}}^n$ is K'^2 -quasiconformal and has the identity as boundary values. Suppose there is a sequence $\{x_j\}$ in \mathbb{B}^n such that $\rho(x_j, g(x_j)) \rightarrow \infty$. Choose hyperbolic isometries ϕ_j with $\phi_j(0) = x_j$. We evidently have

$$\rho(g(x_j), x_j) = \rho(g(\phi_j(0)), \phi_j(0)) = \rho((\phi_j^{-1} \circ g \circ \phi_j)(0), 0) \rightarrow \infty$$

But $h_j = \phi_j^{-1} \circ g \circ \phi_j$ is K'^2 -quasiconformal and restricts to the identity on \mathbb{S}^{n-1} . Compactness of the family of all such maps provides us with a contradiction. Thus there is a constant C depending only on K' such that for each $x \in \mathbb{B}^n$ we have $\rho(x, g(x)) \leq C$. Each K' -quasiconformal mapping is uniformly Hölder continuous in the hyperbolic metric, see Lemma 2.1 below. Hence $\rho(f_1(x), f_1(g(x))) \leq C' \rho(x, g(x))^\alpha \leq C''$. Thus there is a constant $C = C(n, K)$ such that for each $x \in \mathbb{B}^n$ we have

$$(1.1) \quad \rho(f_1(x), f_2(x)) \leq C.$$

This has the consequence that a K -quasiconformal group Γ admits an extension as a family $\tilde{\Gamma}$ of K' -quasiconformal maps $\overline{\mathbb{B}}^n \rightarrow \overline{\mathbb{B}}^n$ such that for $f, g \in \Gamma$, $\tilde{f} \circ \tilde{g}$ and $\tilde{f} \circ g$ differ by a uniformly bounded amount in the hyperbolic metric. In other words these extensions, while not forming a group, are in a sense a uniformly bounded distance from being so. The reader will see in the sequel that this is all that is needed to carry through our arguments in greater generality. We have not done so because the added complication

significantly lengthens the presentation and obscures the general ideas, and of course we have to redefine compact type as it is not true, even for conformal groups that compactness of $(\mathbb{S}^{n-1} \setminus \Lambda(\Gamma))/\Gamma$ implies that Γ is convex cocompact (for instance if $\mathbb{S}^{n-1} = \Lambda(\Gamma)$). We claim the following consequence of our discussion and the proof of Theorem 1.4.

Theorem 1.6 *Let G be a non-elementary discrete quasiconformal group acting on \mathbb{S}^{n-1} . If $\Lambda(G)$ is purely conical, then $\Lambda(G)$ is uniformly perfect.*

As in the Kleinian case it is probably true that the limit set of a finitely generated discrete non-elementary quasiconformal group is uniformly perfect. However our ideas here do not seem to stretch to this generality. It is easy to prove that every quasiconformal group contains a subgroup which is of compact type. Simply take the group generated by suitably high powers of a pair of loxodromic elements with different fixed points. (In fact a result of Freedman [9] implies that such a group will be quasiconformally conjugate to a conformal group). As mentioned above, uniformly perfect sets have positive Hausdorff dimension [12] and so we have a generalization of a result of Beardon [2] for Kleinian groups. See also Theorem 3.3 in [7].

Theorem 1.7 *Let G be a non-elementary discrete quasiconformal group. Then the Hausdorff dimension of $\Lambda(G)$ is positive.*

Other approaches to the question of uniformly perfect limit sets and Theorem 1.7 can be found in [13].

2. Some basic results

In this section we provide some basic facts concerning the distortion of the hyperbolic and quasihyperbolic metrics under quasiconformal mappings. The first lemma describes how the hyperbolic distance is distorted under quasiconformal mappings preserving \mathbb{B}^n . This lemma is a special case of the more general Theorem 2.5 and Corollary 2.6 below.

Lemma 2.1 *For each $n \in \mathbb{N}$ and $K \geq 1$ there exists a homeomorphism $\Phi_{K,n} : [0, \infty) \rightarrow [0, \infty)$ so that any K -quasiconformal mapping g preserving \mathbb{B}^n satisfies*

$$\rho(g(x), g(y)) \leq \Phi_{K,n}(\rho(x, y))$$

for all $x, y \in \mathbb{B}^n$. Here, ρ is the hyperbolic metric on \mathbb{B}^n . Furthermore, there exists a constant $L_{K,n}$ depending only on n and K , so that

$$\frac{1}{L_{K,n}}\rho(x, y) \leq \rho(g(x), g(y)) \leq L_{K,n}\rho(x, y)$$

holds for all $x, y \in \mathbb{B}^n$ with $\rho(x, y) \geq 1$.

Let $T(\mathbb{S}^{n-1})$ be the triple space of \mathbb{S}^{n-1} , that is $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ minus the large diagonal. Denote by

$$p : T(\mathbb{S}^{n-1}) \rightarrow \mathbb{B}^n$$

the projection that maps a triple $(a, b, c) \in T(\mathbb{S}^{n-1})$ to the hyperbolic projection of c onto the hyperbolic line $L_{a,b}$ with endpoints a and b . For a homeomorphism $g : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$ let its extension to triple space be given by

$$g(a, b, c) = (g(a), g(b), g(c)).$$

Furthermore, for two distinct points $a, b \in \mathbb{S}^{n-1}$ denote by $\pi_{a,b} : \mathbb{B}^n \rightarrow L_{a,b}$ the projection onto the hyperbolic geodesic $L_{a,b}$. The following lemma is a fairly straightforward consequence of Lemma 2.1 and the local compactness of the family of K -quasiconformal mappings.

Lemma 2.2 *For each $n \in \mathbb{N}$ and for each $K \geq 1$ there exists a constant $C_{K,n}$ so that for any K -quasiconformal mapping $g : \mathbb{B}^n \rightarrow \mathbb{B}^n$ (naturally extended to \mathbb{S}^{n-1}) and any triple $(a, b, c) \in T(\mathbb{S}^{n-1})$ we have*

$$\begin{aligned} \rho(g(p(a, b, c)), p((g(a), g(b), g(c)))) &\leq C_{K,n}, \text{ and} \\ \rho(g(p(a, b, c)), \pi_{g(a),g(b)}(g(p(a, b, c)))) &\leq C_{K,n}, \text{ and} \\ \rho(p(g(a), g(b), g(c)), \pi_{g(a),g(b)}(g(p(a, b, c)))) &\leq C_{K,n}. \end{aligned}$$

Definition 2.3 *Let $L_{a,b}$ be a hyperbolic geodesic in \mathbb{B}^n with distinct endpoints $a, b \in \mathbb{S}^{n-1}$. We say that $x \in L_{a,b}$ is closer to a than $y \in L_{a,b}$ if x lies on the hyperbolic ray starting at y and ending at a .*

The next result quantifies the fact that the order on a line $L_{a,b}$ given by “closer” is preserved “in the large” under quasiconformal mappings:

Lemma 2.4 *For each $n \in \mathbb{N}$ and each $K \geq 1$ there exists a constant $D_{K,n}$ that only depends on n and K such that for any K -quasiconformal mapping $g : \mathbb{B}^n \rightarrow \mathbb{B}^n$ (naturally extended to \mathbb{S}^{n-1}) and any two distinct points $a, b \in \mathbb{S}^{n-1}$ the following is true: If $x = p(a, b, c) \in L_{a,b}$ is closer to a than $y = p(a, b, d) \in L_{a,b}$, and if $\rho(x, y) \geq D_{K,n}$, then $p(g(a), g(b), g(c)) \in L_{g(a),g(b)}$ is closer to $g(a)$ than $p(g(a), g(b), g(d)) \in L_{g(a),g(b)}$.*

Proof. Let $x = p(a, b, c)$ and $y = p(a, b, d)$ be two points on $L_{a,b}$ with $\rho(x, y) \geq \max\{1, 2L_{K,n}C_{K,n} + 2\}$, where $L_{K,n}$ and $C_{K,n}$ are the constants from Lemmas 2.1 and 2.2. Then these lemmas imply that

$$\begin{aligned} (2.1) \quad \rho(\pi_{g(a),g(b)}(g(x)), \pi_{g(a),g(b)}(g(y))) &\geq \rho(g(x), g(y)) - 2C_{K,n} \\ &\geq \frac{1}{L_{K,n}}\rho(x, y) - 2C_{K,n}. \end{aligned}$$

Suppose now that $\pi_{g(a),g(b)}(g(y))$ is closer to $g(a)$ than $\pi_{g(a),g(b)}(g(x))$. Since $g(L_{a,x})$ is connected (here, $L_{a,x}$ is the hyperbolic line segment starting at x and ending at a) and $\pi_{g(a),g(b)} : \mathbb{B}^n \rightarrow L_{g(a),g(b)}$ is continuous, we know that the image $\pi_{g(a),g(b)}(g(L_{a,x}))$ is connected. But this latter line segment contains $\pi_{g(a),g(b)}(g(y))$ by the assumption that $\pi_{g(a),g(b)}(g(y))$ is closer to $g(a)$ than $\pi_{g(a),g(b)}(g(x))$. Hence there exists $z \in L_{a,x}$ so that $\pi_{g(a),g(b)}(g(z)) = \pi_{g(a),g(b)}(g(y))$. Using Lemmas 2.1 and 2.2 we obtain that

$$\begin{aligned} \rho(z, y) &\leq L_{K,n}\rho(g(z), g(y)) + 1 \\ &\leq L_{K,n}[\rho(g(z), \pi_{g(a),g(b)}(g(z))) + \rho(\pi_{g(a),g(b)}(g(y)), g(y))] + 1 \\ &\leq L_{K,n}(2C_{K,n}) + 1. \end{aligned}$$

Since we assumed that $\rho(x, y) \geq 2C_{K,n}L_{K,n} + 2$ we now have that $\rho(z, y) < \rho(x, y)$, which contradicts the fact that z is closer to a than x , and x is closer to a than y .

This contradiction gives that $\pi_{g(a),g(b)}(g(x))$ is closer to $g(a)$ than $\pi_{g(a),g(b)}(g(y))$. Thus, using Lemma 2.2, we can conclude that $p(g(a), g(b), g(c))$ is closer to $g(a)$ than $p(g(a), g(b), g(d))$ if $\rho(\pi_{g(a),g(b)}(g(x)), \pi_{g(a),g(b)}(g(y))) > 2C_{K,n}$. But recall from (2.1) that

$$\rho(\pi_{g(a),g(b)}(g(x)), \pi_{g(a),g(b)}(g(y))) \geq \frac{1}{L_{K,n}}\rho(x, y) - 2C_{K,n}.$$

Thus choosing $D_{K,n} = \max\{1, 2L_{K,n}C_{K,n} + 2, 5C_{K,n}L_{K,n}\}$ we obtain the desired result. ■

Next we record some results concerning the distortion of the quasihyperbolic metric under quasiconformal mappings. Recall that the *chordal metric* d_{chord} is given on $\overline{\mathbb{R}^n}$ by

$$d_{\text{chord}}(x, y) = \begin{cases} \frac{2|x - y|}{(1 + |x|^2)^{(1/2)}(1 + |y|^2)^{(1/2)}} & \text{if } x, y \neq \infty \\ \frac{2}{(1 + |x|^2)^{(1/2)}} & \text{if } y = \infty. \end{cases}$$

For a proper subdomain D of $\overline{\mathbb{R}^n}$ we define the (chordal) *quasihyperbolic metric* q_D by

$$q_D(x_1, x_2) = \inf_C \int_C \frac{1}{\text{dist}_{\text{chord}}(x, \partial D)} |dx|,$$

where the infimum is taken over all rectifiable arcs C joining x_1 and x_2 in D , and $\text{dist}_{\text{chord}}$ denotes the chordal distance. Many of the basic properties of this metric can be found in [11]. Note that in [11] only proper subdomains of \mathbb{R}^n are considered and the Euclidean metric is used instead of the chordal metric.

It is easy to check however that all results translate to the chordal metric. In particular geodesic curves exist for this complete metric. The following theorem is proved in [11]:

Theorem 2.5 (Gehring-Osgood) *For each $n \in \mathbb{N}$ and $K \geq 1$ there exists a constant c only depending on n and K with the following property: If D and D' are proper subdomains of $\overline{\mathbb{R}^n}$ and if f is a K -quasiconformal mapping of D onto D' then*

$$q_{D'}(f(x_1), f(x_2)) \leq c \max(q_D(x_1, x_2), q_D(x_1, x_2)^\alpha), \quad \alpha = K^{1/(1-n)},$$

for all $x_1, x_2 \in D$.

In particular, this theorem implies that a quasiconformal mapping f as in the theorem is bi-Lipschitz “in the large”:

Corollary 2.6 *For each $n \in \mathbb{N}$, each $K \geq 1$ and each $a > 0$ there exists a constant $L > 1$ with the following property: If D and D' are proper subdomains of $\overline{\mathbb{R}^n}$ and if f is a K -quasiconformal mapping of D onto D' then*

$$q_{D'}(f(x_1), f(x_2)) \leq Lq_D(x_1, x_2) \quad \text{for all } x_1, x_2 \in D \text{ with } q_D(x_1, x_2) \geq a.$$

Here, $L \rightarrow \infty$ as $a \rightarrow 0$.

In this paper we will mainly consider the quasihyperbolic metric in the domain $\overline{\mathbb{R}^n} \setminus \Lambda$, where $\Lambda \subset \mathbb{S}^{n-1}$ is a non-empty closed set (in most cases Λ will be the limit set of a quasiconformal group). In the next theorem we will show that in this setting, there exists an absolute positive constant C (independent of Λ) such that an arbitrary point $a \in \mathbb{B}^n$ that lies on a radial segment $[0, w)$ ending at $w \in \Lambda$ has quasihyperbolic distance at least C from any point in \mathbb{S}^{n-1} .

Theorem 2.7 *Let $\Lambda \subset \mathbb{S}^{n-1}$ be a closed non-empty set, let $D = \overline{\mathbb{R}^n} \setminus \Lambda$, and denote by q_D the quasihyperbolic metric in D . If $\omega \in \Lambda$ and $a \in \mathbb{B}^n$ is a point on the radial segment $[0, \omega)$ then for any $b \in \mathbb{S}^{n-1} \setminus \Lambda$ we have that*

$$q_D(a, b) \geq \frac{1}{4}.$$

Furthermore, if $0 < \varepsilon < 1/4$, then the quasihyperbolic ε -neighborhood of the radial segment $[0, \omega)$ contains and is contained in a Euclidean non-tangential cone based at ω .

Proof. Let $\omega \in \Lambda$, $a \in [0, \omega)$ and $b \in \mathbb{S}^{n-1} \setminus \Lambda$ as in the theorem. Let γ be an arbitrary rectifiable curve in D from a to b . Clearly $l(\gamma) \geq 1 - |a|$, where $l(\gamma)$ denotes the Euclidean length of γ .

Let B be the (Euclidean) n -ball of radius $2(1 - |a|)$, centered at w . We consider two cases:

1. If γ is entirely contained in B then

$$\begin{aligned} q_D(\gamma) &= \int_{\gamma} \frac{1}{\text{dist}_{\text{chord}}(x, \Lambda)} |dx| \geq \int_{\gamma} \frac{1}{2 \text{dist}_{\text{Euc}}(x, \Lambda)} |dx| \\ &\geq \int_{\gamma} \frac{1}{2|x-w|} |dx| \geq \int_{\gamma} \frac{1}{4(1-|a|)} |dx| \\ &\geq \frac{l(\gamma)}{4(1-|a|)} \geq \frac{1}{4}. \end{aligned}$$

2. If γ is not entirely contained in B then let $\tilde{\gamma}$ be the subarc of γ from a to the first point where γ leaves B . Then $l(\tilde{\gamma}) \geq 1 - |a|$ and so $q_D(\tilde{\gamma}) \geq 1/4$ as in (1). Since $q_D(\gamma) \geq q_D(\tilde{\gamma})$ we obtain that $q_D(\gamma) \geq 1/4$.

The second part of the theorem is proved in a similar manner. ■

3. Uniform perfectness of the limit set

In this section we prove uniform perfectness of purely conical limit sets. We then show that purely conical limit sets are “uniformly conical” in the sense of Theorem 1.5.

Proof of Theorem 1.4. Let $T(\Lambda(G))$ be the triple space of $\Lambda(G)$, that is $\Lambda(G) \times \Lambda(G) \times \Lambda(G)$ minus the large diagonal. Then by Theorem 1.1 we have that $T(\Lambda(G))/G$ is compact since $\Lambda(G)$ is purely conical. Hence there exists a compact set $C \subset T(\Lambda(G))$ so that $G(C) = T(\Lambda(G))$.

Suppose now that $\Lambda(G)$ is not uniformly perfect. Then there are distinct annuli $A_m = \text{Ann}(z_m, r_m, R_m) = \{w \in \mathbb{R}^n \mid r_m < |z_m - w| < R_m\}$ separating $\Lambda(G)$ so that

$$\text{mod}(A_m) \rightarrow \infty \text{ as } m \rightarrow \infty$$

and $z_m \in \mathbb{S}^{n-1}$, $0 < r_m < R_m < 1$ for all m . Without loss of generality we have that $z_m \rightarrow z_0$ as $m \rightarrow \infty$, and since $\Lambda(G)$ does not contain isolated points it is easy to see that $R_m \rightarrow 0$ and $r_m \rightarrow 0$ as $m \rightarrow \infty$. By slightly changing A_m (without changing the fact that $\text{mod}(A_m) \rightarrow \infty$) we can also assume that

$$B(z_m, \frac{r_m}{2}) \cap \Lambda(G) \neq \emptyset \quad \text{and} \quad (\overline{\mathbb{R}^{n-1}} \setminus \overline{B(z_m, 2R_m)}) \cap \Lambda(G) \neq \emptyset.$$

Here $B(z, r)$ denotes the Euclidean n -ball in \mathbb{R}^n centered at z and of radius r .

Hence there are $a_m \in B(z_m, \frac{r_m}{2}) \cap \Lambda(G)$ and $b_m \in (\mathbb{R}^n \setminus \overline{B(z_m, 2R_m)}) \cap \Lambda(G)$ so that the hyperbolic line L_{a_m, b_m} in \mathbb{B}^n with endpoints a_m, b_m has a “hole”, that is a set which is not covered by $\{p(a_m, b_m, x) \mid x \in \Lambda(G)\}$, and the hole has hyperbolic length growing to ∞ as $m \rightarrow \infty$. Hence there are

$$x_m = p(a_m, b_m, c_m) \in p(T(\Lambda(G))) \text{ and } y_m = p(a_m, b_m, d_m) \in p(T(\Lambda(G)))$$

so that $\rho(x_m, y_m) \rightarrow \infty$, and

$$L_{x_m, y_m} \cap \{p(a_m, b_m, x) \mid x \in \Lambda(G)\} = \emptyset,$$

where L_{x_m, y_m} denotes the hyperbolic geodesic starting at x_m and ending at y_m (note that here $x_m, y_m \in \mathbb{B}^n$). We can also assume that the order of points on the line L_{a_m, b_m} is so that x_m lies on the ray starting at y_m and ending at a_m (and hence y_m lies on the ray starting at x_m and ending at b_m). Choose $g_m \in G$ so that $(g_m(a_m, b_m, c_m)) \in C$, and let $v_m = p(g_m(a_m, b_m, c_m))$, see Figure 1.

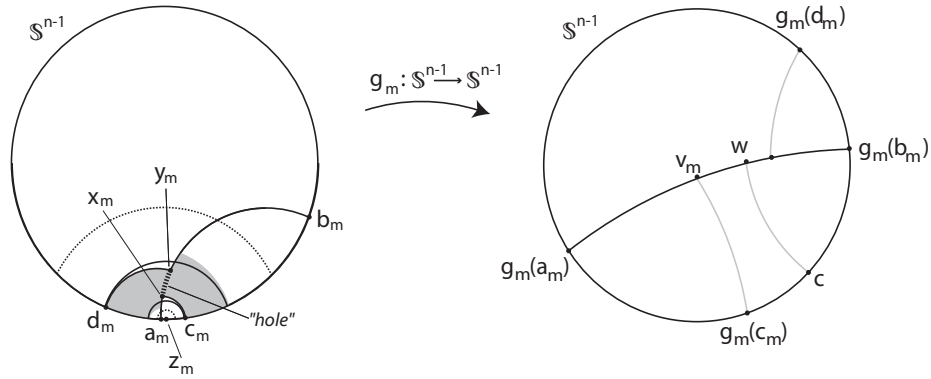


Figure 1: The proof of uniform perfectness

We will now show that the line $L_{g_m(a_m), g_m(b_m)}$ has an increasingly large hole starting near v_m , that is a set that is not covered by

$$\{p(g_m(a_m), g_m(b_m), c) \mid c \in \Lambda(G)\}.$$

To do so suppose that $w = p(g_m(a_m), g_m(b_m), c)$ for some $c \in \Lambda(G)$. Then, using first the triangle inequality and then Lemmas 2.2 and 2.1 we obtain:

$$\begin{aligned} \rho(x_m, p(a_m, b_m, g_m^{-1}(c))) &\leq \rho(x_m, g_m^{-1}(v_m)) + \rho(g_m^{-1}(v_m), g_m^{-1}(w)) \\ &\quad + \rho(g_m^{-1}(w), p(a_m, b_m, g_m^{-1}(c))) \\ &\leq C_{K,n} + \Phi_{K,n}(\rho(v_m, w)) + C_{K,n}, \end{aligned}$$

and this last quantity is smaller than $\rho(x_m, y_m)$ if

$$\rho(v_m, w) < \Phi_{K,n}^{-1}(\rho(x_m, y_m) - 2C_{K,n})$$

We now use Lemma 2.4 to see that if $\rho(v_m, w) \geq D_{K,n}$ then x_m is closer to a than $p(a_m, b_m, g_m^{-1}(c))$, that is $p(a_m, b_m, g_m^{-1}(c))$ lies on the ray starting at x_m and ending at b_m . But such a point $w = p(g_m(a_m), g_m(b_m), c)$ cannot exist unless $\rho(v_m, w) \geq \Phi_{K,n}^{-1}(\rho(x_m, y_m) - 2C_{K,n})$, and this means that there is a big “hole” on $L_{g_m(a_m), g_m(b_m)}$, starting roughly at v_m , and $v_m \in p(C)$.

Since p is continuous we have that $p(C)$ is compact. Passing to a subsequence we may assume that $v_m \rightarrow v \in p(C)$, $g_m(a_m) \rightarrow a \in \Lambda(G)$, and $g_m(b_m) \rightarrow b \in \Lambda(G)$. Since $v \in L_{a,b}$ and $v \in p(C) \subset \mathbb{B}^n$ we furthermore have that $a \neq b$. But this implies that b is an isolated point in $\Lambda(G)$, and this is a contradiction. ■

Proof of Theorem 1.5. Recall that G is a discrete quasiconformal group acting on $\overline{\mathbb{B}^n}$ with purely conical limit set. Let $T(\Lambda(G))$ be the triple space over $\Lambda(G)$, and denote by $p : T(\Lambda(G)) \rightarrow \mathbb{B}^n$ the projection map that maps the triple (a, b, c) onto the projection of c onto the hyperbolic line $L_{a,b}$ (see Section 2). Then $p(T(\Lambda(G)))$ is contained in the set of all hyperbolic lines with endpoints in $\Lambda(G)$. We will show:

Claim 1: There exists $\varepsilon > 0$ so that $p(T(\Lambda(G)))$ is ε -dense (in the hyperbolic metric) in the set of all hyperbolic lines with endpoints in $\Lambda(G)$.

To prove the claim, suppose in the contrary that such an ε does not exist. Then there are $a_m, b_m \in \Lambda(G)$ and $z_m \in L_{a_m, b_m}$ so that $B_\rho(z_m, m) \cap p(T(\Lambda(G))) = \emptyset$, where $B_\rho(z_m, m)$ denotes the hyperbolic ball of radius m , centered at z_m . In particular, there exists an annulus A_m in \mathbb{S}^{n-1} separating a_m and b_m , that does not contain any points of $\Lambda(G)$. In order to determine the modulus of this annulus, we use a Möbius transformation ϕ_m to map \mathbb{S}^{n-1} onto $\overline{\mathbb{R}^{n-1}}$, \mathbb{B}^n onto \mathbb{H}^n , with $\phi_m(a_m) = 0$, $\phi_m(b_m) = \infty$. Then $\phi_m(L_{a_m, b_m})$ is the hyperbolic line in \mathbb{H}^n with endpoints 0 and ∞ , and $\phi_m(z_m)$ lies on that line. Furthermore, the hyperbolic ball centered at $\phi_m(z_m)$ of radius m does not contain any points of $\phi_m(p(T(\Lambda(G)))) = p(\phi_m(T(\Lambda(G))))$, and thus the Euclidean annulus in \mathbb{R}^{n-1} centered at 0 and of inner radius $e^{-m}|z_m|$ and outer radius $e^m|z_m|$ (which is $\phi_m(A_m)$) does not contain any points from $\phi_m(\Lambda(G))$. Hence

$$\text{mod}(A_m) = \text{mod}(\phi_m(A_m)) = \log e^{2m} = 2m.$$

We have thus shown that there are annuli of arbitrarily large modulus in \mathbb{S}^{n-1} that separate $\Lambda(G)$, and this implies that $\Lambda(G)$ is not uniformly perfect. This contradicts Theorem 1.4, and this contradiction proves Claim 1.

Thus there exists $\varepsilon > 0$ so that $p(T(\Lambda(G)))$ is ε -dense in the set of all hyperbolic lines with endpoints in $\Lambda(G)$. Since $\Lambda(G)$ is purely conical, by Theorem 1.1, $T(\Lambda(G))/G$ is compact. Hence there exists a compact set

$C^* \subset T(\Lambda(G))$ so that $G(C^*) = T(\Lambda(G))$. By continuity of the projection map $p : T(\Lambda(G)) \rightarrow \mathbb{B}^n$, the set $\tilde{C} = p(C^*)$ is compact in \mathbb{B}^n . Let now

$$C = \{x \in \mathbb{B}^n \mid \rho(x, \tilde{C}) \leq \Phi_{K,n}(\varepsilon) + C_{K,n}\},$$

where $\Phi_{K,n}$ is the function from Lemma 2.1 and $C_{K,n}$ is the constant in Lemma 2.2. Clearly, C is compact in \mathbb{B}^n since \tilde{C} is compact. We show:

Claim 2: $G(C)$ contains all hyperbolic lines with endpoints in $\Lambda(G)$.

To see this, let x be a point on a hyperbolic line with endpoints in $\Lambda(G)$. Then there are $a, b \in \Lambda(G)$ so that $x \in L_{a,b}$. We have to show that $x \in G(C)$. Since $p(T(\Lambda(G)))$ is ε -dense in the set of all hyperbolic lines with endpoints in $\Lambda(G)$, there are $a', b', c' \in \Lambda(G)$ so that $\rho(p(a', b', c'), x) \leq \varepsilon$. Furthermore, since $G(C^*) = T(\Lambda(G))$, there exists $g \in G$ so that $g(a', b', c') \in C^*$. Hence $p(g(a'), g(b'), g(c')) \in \tilde{C}$ by definition of \tilde{C} , and since

$$\rho(p(g(a'), g(b'), g(c')), g(p(a', b', c'))) \leq C_{K,n}$$

by Lemma 2.2, we conclude:

$$\begin{aligned} \rho(g(x), \tilde{C}) &\leq \rho(g(x), p(g(a'), g(b'), g(c'))) \\ &\leq \rho(g(x), g(p(a', b', c'))) + \rho(g(p(a', b', c')), p(g(a'), g(b'), g(c'))) \\ &\leq \Phi_{K,n}(\rho(x, p(a', b', c'))) + C_{K,n} \leq \Phi_{K,n}(\varepsilon) + C_{K,n}. \end{aligned}$$

Hence $g(x) \in C$, so that $x \in G(C)$, and this proves Claim 2.

Since $C \subset \mathbb{B}^n$ is compact, there exists $d > 0$ so that C is contained in the hyperbolic ball centered at 0 and of radius d .

Let $\omega_0 \in \Lambda(G)$ be arbitrary. Choose $\omega^* \in \Lambda(G)$ with $|\omega_0 - \omega^*| \geq \text{diam}_{\text{Euc}} \Lambda(G)$. Let $y_1 \in L_{\omega_0, \omega^*}$ be the point closest to 0, and for $j \geq 2$ let $y_j \in L_{\omega_0, y_{j-1}}$ so that $\rho(y_{j-1}, y_j) = 3\Phi_{K,n}(d)$. Then $y_j \rightarrow \omega_0$, and using Claim 2, for each $j \geq 1$ we find $g_j \in G$ so that $g_j^{-1}(y_j) \in C$. Define $x_j = g_j(0)$. Then

$$\rho(x_j, y_j) \leq \Phi_{K,n}(\rho(0, g_j^{-1}(y_j))) \leq \Phi_{K,n}(d),$$

so that $x_j \rightarrow \omega_0$ within bounded hyperbolic distance from the radial segment $[0, \omega_0)$, where the bound only depends on d and $|\omega_0 - \omega^*|$ (and the latter only depends on $\text{diam}_{\text{Euc}} \Lambda(G)$). Furthermore, for all $j \geq 1$ we have that

$$\rho(x_j, x_{j+1}) \leq \rho(x_j, y_j) + \rho(y_j, y_{j+1}) + \rho(y_{j+1}, x_{j+1}) \leq 5\Phi_{K,n}(d),$$

and also

$$\rho(x_j, x_{j+1}) \geq \rho(y_j, y_{j+1}) - \rho(x_j, y_j) - \rho(x_{j+1}, y_{j+1}) \geq \Phi_{K,n}(d)$$

so that all x_j are distinct.

Letting $x_0 = 0$ we have that $\rho(x_0, x_1) \leq \rho(0, y_1) + \rho(y_1, x_1) \leq \rho(0, y_1) + \Phi_{K,n}(d)$, and $\rho(0, y_1)$ is bounded above in terms of $|\omega_0 - \omega_*|$, and hence has an upper bound only depending on $\text{diam}_{\text{Euc}} \Lambda(G)$, but not depending on ω_0 . Hence there exists $M < \infty$ so that

$$\rho(x_j, x_{j+1}) \leq M \quad \text{for all } j \geq 0,$$

and furthermore,

$$\rho(x_j, [0, \omega_0]) \leq \beta \quad \text{for all } j \geq 0,$$

where β only depends on d and $\text{diam}_{\text{Euc}} \Lambda(G)$ (and n, K), but not on ω_0 . This proves the first part of the theorem.

To see the second part, let z_j be the projection (in the hyperbolic metric) of x_j onto the radial segment $[0, \omega_0)$. Then $\rho(z_j, z_{j+1}) \leq \rho(z_j, x_j) + \rho(x_j, x_{j+1}) + \rho(x_{j+1}, z_{j+1}) \leq 2\beta + M$ for all $j \geq 0$. If $x \in [0, \omega_0)$ is arbitrary, then $\rho(x, z_j) \leq 2\beta + M$ for some j , and hence $\rho(x, x_j) \leq 3\beta + M$. ■

4. Proof of the Main Theorem

Proof of Theorem 1.3. Recall that G is a discrete K -quasiconformal group acting on $\overline{\mathbb{B}^n}$. Extend the action of G by reflection to all of $\overline{\mathbb{R}^n}$, let $D = \overline{\mathbb{R}^n} \setminus \Lambda(G)$, and denote by q_D the quasihyperbolic metric in D .

1. Suppose first that G is of compact type, that is $(\overline{\mathbb{B}^n} \setminus \Lambda(G))/G$ is compact. We have to show that $\Lambda(G)$ is purely conical. Let Ω be a relatively compact fundamental set for the action of G on $\overline{\mathbb{R}^n} \setminus \Lambda(G)$.

Let $\omega_0 \in \Lambda(G)$. Our goal is to show that ω_0 is a conical limit point. Let l be the radial line segment from 0 to ω_0 in \mathbb{B}^n . Choose $a_j \in l$ converging to ω_0 . Then there are $x_j \in \Omega \cap \mathbb{B}^n$ and $g_j \in G$ so that $g_j(x_j) = a_j$. Passing to a subsequence we may assume that $\{x_j\}$ converges to a point x_0 both in the quasihyperbolic and the Euclidean metric. By compactness of Ω in $\overline{\mathbb{R}^n} \setminus \Lambda(G)$ we have that $x_0 \in \overline{\mathbb{B}^n} \setminus \Lambda(G)$. We consider two cases:

- (a) If $x_0 \in \mathbb{B}^n$, then let $b_j = g_j(x_0)$. Then by Lemma 2.1 we have that

$$\rho(a_j, b_j) \leq \Phi_{K,n}(\rho(x_j, x_0)) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Hence $\{b_j\}$ converges to ω_0 inside an (arbitrarily small) cone about the radial line $[0, \omega_0)$, so that ω_0 is a conical limit point.

- (b) Otherwise $x_0 \in \mathbb{S}^{n-1} \setminus \Lambda(G)$. Letting c be the constant from Theorem 2.5 and $\alpha = K^{1/(1-n)}$ choose $0 < \varepsilon < 1/2$ with $2\varepsilon \leq (1/(8c))^{1/\alpha}$. Let F be the quasihyperbolic ball centered at x_0

and of (quasihyperbolic) radius ε . Choose $y_0 \in F \cap \mathbb{B}^n$ and let $b_j = g_j(y_0)$. Then using Theorem 2.5 we obtain that

$$q_D(a_j, b_j) \leq c(q_D(x_j, y_0))^\alpha < c(2\varepsilon)^\alpha \leq \frac{1}{8}.$$

Hence by Theorem 2.7 the sequence $\{b_j\}$ is contained in a Euclidean non-tangential cone based at ω_0 , and since $a_j \rightarrow \omega_0 \in \partial D$ and the sequence $\{b_j\}$ stays close to the sequence $\{a_j\}$ in the quasihyperbolic q_D -distance, we conclude that $\{b_j\}$ converges to ω_0 as well. Thus ω_0 is a conical limit point.

2. Suppose now that $\Lambda(G)$ is purely conical. We will show that G is of compact type by constructing a relatively compact fundamental set for the action of G on $\overline{\mathbb{B}^n} \setminus \Lambda(G)$. Define

$$\Omega = \{y \in \overline{\mathbb{B}^n} \mid q_D(0, y) \leq q_D(0, g(y)) \text{ for all } g \in G\}.$$

Then Ω contains a fundamental set for the action of G on $\overline{\mathbb{B}^n} \setminus \Lambda(G)$. We will show that Ω has finite diameter in the quasihyperbolic q_D -metric, so that Ω is bounded away from $\Lambda(G)$. Thus Ω is relatively compact in $\overline{\mathbb{B}^n} \setminus \Lambda(G)$, and we conclude that $(\overline{\mathbb{B}^n} \setminus \Lambda(G))/G$ is compact, which proves the theorem.

Suppose in the contrary that the q_D -diameter of Ω is not finite. Then there are $x_j \in \Omega$ so that $q_D(0, x_j) \rightarrow \infty$. For each $x_j \in \Omega$ let $\omega_j \in \Lambda(G)$ be a point so that $\text{dist}_{\text{Euc}}(x_j, \Lambda(G)) = |x_j - \omega_j|$ (recall that $\Lambda(G)$ is closed). Denote by γ_j the (Euclidean) circular arc from x_j to the radial segment $[0, \omega_j]$, centered at ω_j . Then for each $x \in \gamma_j$ we have that

$$\text{dist}_{\text{chord}}(x, \Lambda(G)) = d_{\text{chord}}(x, \omega(x))$$

for some point $\omega(x) \in \Lambda(G)$, and since $\Lambda(G) \subset \mathbb{S}^{n-1}$ and $x \in \mathbb{B}^n$ we conclude that

$$\text{dist}_{\text{chord}}(x, \Lambda(G)) \geq |x - \omega(x)|.$$

But since ω_j is the closest point (in the Euclidean metric) in $\Lambda(G)$ to x_j and x is on the circular arc γ_j we conclude that $|x - \omega(x)| \geq |x - \omega_j| = |x_j - \omega_j|$, and so

$$\text{dist}_{\text{chord}}(x, \Lambda(G)) \geq |x_j - \omega_j|.$$

But this implies that

$$\begin{aligned} \text{dist}_{q_D}(x_j, [0, \omega_j]) &\leq \int_{\gamma_j} \frac{1}{\text{dist}_{\text{chord}}(x, \Lambda(G))} ds \leq \frac{1}{|x_j - \omega_j|} l(\gamma_j) \\ (4.1) \qquad \qquad \qquad &\leq \frac{1}{|x_j - \omega_j|} \frac{\pi}{2} |x_j - \omega_j| = \frac{\pi}{2}. \end{aligned}$$

Since ω_j is a conical limit point by assumption, we can apply Theorem 1.5 and the remark following it, so that using (4.1) we find that for each x_j there exists $g_j \in G$ so that

$$q_D(x_j, g_j(0)) \leq \frac{\pi}{2} + 2(3\beta + M),$$

where β and M are the constants in Theorem 1.5. But applying Theorem 2.5 we then obtain that then $q_D(g_j^{-1}(x_j), 0)$ is bounded above for all j , and since $q_D(x_j, 0) \rightarrow \infty$ we have that $g_j^{-1}(x_j)$ is closer (in the q_D -metric) to 0 than x_j for large j , and this contradicts the fact that $x_j \in \Omega$. ■

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