

Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients

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Abstract

This paper aims at giving an overview of estimates in general Besov spaces for the Cauchy problem on $t = 0$ related to the vector field $\partial_t + v \cdot \nabla$. The emphasis is on the conservation or loss of regularity for the initial data.

When ∇v belongs to $L^1(0, T; L^\infty)$ (plus some convenient conditions depending on the functional space considered for the data), the initial regularity is preserved. On the other hand, if ∇v is slightly less regular (e.g. ∇v belongs to some limit space for which the embedding in L^∞ fails), the regularity may coarsen with time. Different scenarios are possible going from linear to arbitrarily small loss of regularity. This latter result will be used in a forthcoming paper to prove global well-posedness for two-dimensional incompressible density-dependent viscous fluids (see [11]).

Besides, our techniques enable us to get estimates uniformly in $\nu \geq 0$ when adding a diffusion term $-\nu \Delta u$ to the transport equation.

Introduction

This paper is concerned with estimates in Besov spaces for transport-diffusion equations:

$$\begin{cases} \partial_t f + v \cdot \nabla f - \nu \Delta f = g, \\ f_{t=0} = f_0, \end{cases} \quad (\mathcal{T}_\nu)$$

where $\nu \geq 0$ stands for a constant diffusion parameter.

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Such equations appear as the result of linearization in a number of PDE's coming from fluid mechanics and have been extensively studied. In the case $\nu = 0$, existence and uniqueness theory in L^∞ has been studied under very weak assumptions on v (roughly $v \in W^{1,1}$ (or even BV) with $\operatorname{div} v \in L^\infty$, see e.g [13], [7], [8] and the references therein).

As our work is motivated by the study of nonlinear models, we aim at estimating the fractional derivatives of the solutions to (\mathcal{T}_ν) . It is well known that such estimates are available when v has enough regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as ∇v belongs to $L^1(0, T; L^\infty)$ (plus some convenient assumption depending on the number of derivatives to be transported). When $\nu = 0$, this qualitative result has been proved in a number of functional frameworks:

- Sobolev spaces H^s with $0 \leq s < \frac{N}{2}$ provided that $\nabla v \in L^1(0, T; H^{\frac{N}{2}} \cap L^\infty)$,
- Hölder spaces C^r if $|r| \leq 1$, $\nabla v \in L^1(0, T; L^\infty)$ and $\operatorname{div} v = 0$ (see e.g. [3]),
- General Besov spaces $B_{p,r}^s$ with $1 \leq p, r \leq +\infty$ and $-\frac{N}{p} < s < \frac{N}{p} + 1$ (or $|s| < \frac{N}{p} + 1$ if $\operatorname{div} v = 0$) if $\nabla v \in L^1(0, T; B_{p,r}^{\frac{N}{p}} \cap L^\infty)$ (see e.g [10]).

In section 2 of the present paper, we state estimates in $B_{p,r}^\sigma$ for (\mathcal{T}_ν) whereas ∇v belongs to some different Besov space $B_{p_2,r_2}^{\sigma'}$ and is bounded. The main novelty is that p_2 may differ from p and that the estimates do not depend on ν . The reader is referred to proposition 2.1 for more details.

On the other hand, when ∇v fails to be in $L^1(0, T; L^\infty)$, the initial regularity is unlikely to be preserved. Nevertheless, if v is “almost” lipschitz with respect to the space variables then the solution f may be estimated in spaces whose regularity index coarsens with time. This fact has been observed several times by different authors.

In the case of Besov spaces with third index $r = +\infty$, H. Bahouri and J.-Y. Chemin proved a linear loss of regularity under the hypothesis that v is log-lipschitz:

$$\|v\|_{LL} \stackrel{\text{def}}{=} \sup_{0 < |x-y| < e^{-1}} \frac{|v(y) - v(x)|}{|y - x|(1 - \log|y - x|)} < +\infty.$$

Assuming that $|\sigma| < 1$, they prove the following inequality in [1]:

$$\|f(t)\|_{B_{p,\infty}^{\sigma_t}} \leq 2 \left(\|f_0\|_{B_{p,r}^\sigma} + \int_0^t \|g(\tau)\|_{B_{p,\infty}^{\sigma_\tau}} d\tau \right)$$

whenever $\sigma_t \stackrel{\text{def}}{=} \sigma - C \int_0^t \|v(\tau)\|_{LL} d\tau > -1$.

This result has been improved in [6]. Moreover the regularity assumption on the right-hand side g may be somewhat weakened (see [9]).

Let us mention in passing that results in the same spirit have been used by M. Vishik for solving incompressible Euler equations in limit Besov spaces (see [19]), and by F. Planchon to improve the Beale-Kato-Majda blow-up criterion ([17]).

Section 3.2 of the present paper is devoted to the proof of similar estimates for (\mathcal{T}_ν) uniformly in ν in general Besov spaces (see theorems 3.2, 3.4, 3.9 and 3.10 below.)

It has also been observed that if v is better than log-lipschitz, e.g

$$(0.1) \quad \sup_{0 < |x-y| < \epsilon^{-1}} \frac{|v(y) - v(x)|}{|y - x|(1 - (\log |y - x|)^\alpha)} < +\infty.$$

for some $\alpha \in (0, 1)$, then the loss of regularity is arbitrarily small.

In [5], J.-Y. Chemin and N. Lerner noticed that the flow associated to a vector-field v whose gradient belongs to a space slightly larger than $L^1(0, T; H^{\frac{N}{2}})$ remains in $C^{1-\epsilon}$ on $[0, T]$ for ϵ arbitrarily small despite the fact that $H^{\frac{N}{2}}$ is not embedded in L^∞ . The proof lies on the fact that v satisfies an inequality of type (0.1).

In [15], B. Desjardins stated an even more accurate result in the framework of the two-dimensional torus \mathbb{T}^2 . Here the flow to a vector-field $v \in L^2(0, T; H^2)$ is shown to remain in $W^{2,p}$ for all $p < 2$. In the case of a bounded N -dimensional domain, if the symmetric part of ∇v belongs to $L^p(0, T; W^{\frac{N}{p}, p})$ with $1 < p < +\infty$, and $f_0 \in W^{1,r}$ then $f \in C([0, T]; W^{1,q})$ for all $q < r$ (see [14]).

Theorem 3.12 of the present paper implies the following result:

Theorem 0.1 *Assume that $\nabla v \in L^1(0, T; B^{\frac{N}{p_2}, r_2})$ for some $1 \leq p_2 \leq +\infty$ and $r_2 \in (1, +\infty)$. Let $1 \leq p, r \leq +\infty$. Let p' be the conjugate exponent of p , and σ be such that*

$$\sigma > -N \min\left(\frac{1}{p_2}, \frac{1}{p'}\right) \quad \left(\sigma > -1 - N \min\left(\frac{1}{p_2}, \frac{1}{p'}\right) \quad \text{if} \quad \operatorname{div} v = 0 \right)$$

and $\sigma < 1 + \frac{N}{p_2}$.

Let $\epsilon > 0$ and let f solve (\mathcal{T}_ν) . There exists $C = C(N, \epsilon, p, p_2, r_2, \sigma)$ such that the following inequality holds true on $[0, T]$ uniformly in ν :

$$\|f(t)\|_{B_{p,r}^{\sigma-\epsilon}} \leq C \left(\|f_0\|_{B_{p,r}^\sigma} + \int_0^t \|g(\tau)\|_{B_{p,r}^\sigma} d\tau \right) \exp \left\{ C \left(\int_0^t \|\nabla v(\tau)\|_{B_{p_2, r_2}^{\frac{N}{p_2}}} d\tau \right)^{r_2} \right\}.$$

We intend to use the above result to prove global well-posedness for the inhomogeneous incompressible Navier-Stokes equations in \mathbb{T}^2 or \mathbb{R}^2 (see [11]).

Remark 0.2 *As our results strongly rely on the use of Fourier analysis (namely the Littlewood-Paley decomposition), we restricted ourselves to the case of \mathbb{T}^N or \mathbb{R}^N with $N \geq 1$. In the case $\nu = 0$ however, we expect our results to be true in any smooth domain Ω if v is tangent to the boundary. As a matter of fact, in the case of a Lipschitz vector-field v , estimates in certain Besov spaces have been proved in [16].*

Remark 0.3 *In order to simplify the statements, we did not track systematically the gain of derivatives induced by the diffusion operator $-\nu\Delta$. This work has been done in proposition 2.1 and leads to uniform estimates for νf in a space very close to $L^1(0, t; B_{p,r}^{\sigma+2})$ whereas f_0 is in $B_{p,r}^\sigma$. A careful reading of our proofs should enable us to get a similar gain of two derivatives in the theorems of section 3.*

Our paper is structured as follows. The first section is devoted to a basic presentation of Littlewood-Paley decomposition and Besov spaces. In section 2, we state general estimates in Besov spaces in the case when v is Lipschitz. Section 3 is devoted to the study of the case when v is not Lipschitz. We focus on the study of “abstract” coupled inequalities from which general uniform (with respect to ν) estimates for (\mathcal{T}_ν) may be inferred. We then give several examples with either linear loss of derivatives or arbitrarily small loss of derivatives depending on the assumption made on ∇v . An appendix is devoted to the proof of technical estimates pertaining to a commutator.

Notation: In order to have more concise statements, we shall adopt the convention that $\sigma > [-1] - \alpha_1$ means that $\sigma > -\alpha_1$ has to be satisfied for a general vector-field v , and that this condition may be weakened into $\sigma > -\alpha_1 - 1$ if $\operatorname{div} v = 0$.

1. Besov spaces and Littlewood-Paley decomposition

The proof of the results presented in the paper is based on a dyadic decomposition in Fourier variables, the so-called (*inhomogeneous*) *Littlewood-Paley decomposition*. For the sake of conciseness, we only treat the case of \mathbb{R}^N . The reader is referred to [10] for a similar construction in \mathbb{T}^N .

Let (χ, φ) be a couple of C^∞ functions with

$$\operatorname{Supp} \chi \subset \left\{ |\xi| \leq \frac{4}{3} \right\}, \quad \operatorname{Supp} \varphi \subset \left\{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\},$$

$$\forall \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1.$$

Denoting $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}\varphi_q$ and $\check{h} = \mathcal{F}^{-1}\chi$, we define the dyadic blocks as

$$\Delta_q u \stackrel{\text{def}}{=} 0 \quad \text{if } q \leq -1, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}^N} \check{h}(y)u(x-y) dy,$$

$$\Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y) dy \quad \text{if } q \geq 0.$$

We shall also use the following low-frequency cut-off:

$$S_q u \stackrel{\text{def}}{=} \sum_{k \leq q-1} \Delta_k u = \chi(2^{-q}D)u.$$

One can easily prove that

$$(1.1) \quad \forall u \in \mathcal{S}'(\mathbb{R}^N), \quad u = \sum_{q \in \mathbb{Z}} \Delta_q u.$$

Littlewood-Paley decomposition has nice properties of quasi-orthogonality:

$$(1.2) \quad \Delta_k \Delta_q u \equiv 0 \quad \text{if } |k-q| \geq 2 \quad \text{and} \quad \Delta_k (S_{q-1} u \Delta_q u) \equiv 0 \quad \text{if } |k-q| \geq 5.$$

Let us now define the (non-homogeneous) Besov spaces:

Definition 1.1 For $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^N)$, we set

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left(\sum_{q \geq -1} 2^{rsq} \|\Delta_q u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < +\infty \quad \text{and} \quad \|u\|_{B_{p,\infty}^s} \stackrel{\text{def}}{=} \sup_{q \geq -1} 2^{sq} \|\Delta_q u\|_{L^p}.$$

We then define the Besov space $B_{p,r}^s$ as the set of temperate distributions with finite $\|\cdot\|_{B_{p,r}^s}$ norm.

The definition of $B_{p,r}^s$ does not depend on the choice of the couple (χ, φ) . One can further remark that H^s coincide with $B_{2,2}^s$, and that $C^r = B_{\infty,\infty}^r$ if $r \in \mathbb{R}^+ \setminus \mathbb{N}$.

The reader is referred to [18] for a complete study of Besov spaces. Let us just recall some of their most basic properties.

Proposition 1.2 The following properties hold:

- i) *Derivatives:* we have $\|\nabla u\|_{B_{p,r}^{s-1}} \lesssim \|u\|_{B_{p,r}^s}$.
- ii) *Sobolev embeddings:*

- If $p_1 \leq p_2$ and $r_1 \leq r_2$ then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}$.

- If $s_1 > s_2$ and $1 \leq p, r_1, r_2 \leq +\infty$, then $B_{p,r_1}^{s_1} \hookrightarrow B_{p,r_2}^{s_2}$.
- If $1 \leq p \leq +\infty$ then $B_{p,1}^{\frac{N}{p}} \hookrightarrow L^\infty$.

iii) Algebraic properties: for $s > 0$, $B_{p,r}^s \cap L^\infty$ is an algebra.

iv) Real interpolation: $(B_{p,r}^{s_1}, B_{p,r}^{s_2})_{\theta,r'} = B_{p,r'}^{\theta s_2 + (1-\theta)s_1}$.

We aim at proving estimates for (\mathcal{T}_ν) in spaces $L^\rho(0, T; B_{p,r}^\sigma)$. Taking into account the definition of Besov spaces, it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain bounds in spaces which are not of type $L^\rho(0, T; B_{p,r}^s)$. That remark naturally leads to the following definition (introduced in [4]):

Definition 1.3 Let $s \in \mathbb{R}$, $1 \leq p, r, \rho \leq +\infty$ and $T \in [0, +\infty]$. We set

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \stackrel{\text{def}}{=} \left(\left(\int_0^T \|\Delta_{-1}u(t)\|_{L^p}^\rho dt \right)^{\frac{r}{\rho}} + \sum_{q \in \mathbb{N}} 2^{rqs} \left(\int_0^T \|\Delta_q u(t)\|_{L^p}^\rho dt \right)^{\frac{r}{\rho}} \right)^{\frac{1}{r}}$$

and denote by $\tilde{L}_T^\rho(B_{p,r}^s)$ the set of distributions of $\mathcal{S}'(0, T \times \mathbb{R}^N)$ with finite $\|\cdot\|_{\tilde{L}_T^\rho(B_{p,r}^s)}$ norm.

Let us remark that by virtue of Minkowski inequality, we have

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq \|u\|_{L_T^\rho(B_{p,r}^s)} \quad \text{if } \rho \leq r$$

and

$$\|u\|_{L_T^\rho(B_{p,r}^s)} \leq \|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \quad \text{if } \rho \geq r.$$

2. A priori estimates with no loss of regularity

We first concentrate on the case when ∇v belongs to $L^1(0, T; L^\infty)$. In this case, the initial Besov regularity is conserved:

Proposition 2.1 Let $1 \leq p, p_2, r \leq +\infty$ and $p' \stackrel{\text{def}}{=} (1 - 1/p)^{-1}$. Assume that

$$\sigma > [-1] - N \min\left(\frac{1}{p_2}, \frac{1}{p'}\right).$$

Denote $f^{HF} \stackrel{\text{def}}{=} f - \Delta_{-1}f$. There exists a constant C depending on N, p, p_2 and σ but not on ν and a universal constant $\kappa > 0$ such that the following

estimates hold:

$$(2.1) \quad \|f\|_{\tilde{L}_t^\infty(B_{p,r}^\sigma)} + \kappa\nu \left(\frac{p-1}{p^2}\right) \|f^{HF}\|_{\tilde{L}_t^1(B_{p,r}^{\sigma+2})} \leq \left(\|f_0\|_{B_{p,r}^\sigma} + \int_0^t e^{-CZ(\tau)} \|g(\tau)\|_{B_{p,r}^\sigma} d\tau\right) e^{CZ(t)},$$

$$(2.2) \quad \|f\|_{\tilde{L}_t^\infty(B_{p,r}^\sigma)} + \kappa\nu \left(\frac{p-1}{p^2}\right) \|f^{HF}\|_{\tilde{L}_t^1(B_{p,r}^{\sigma+2})} \leq \left(\|f_0\|_{B_{p,r}^\sigma} + \|g\|_{\tilde{L}_t^1(B_{p,r}^\sigma)}\right) e^{CZ(t)},$$

with

$$\begin{cases} Z(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p_2, \infty}^{\frac{N}{p_2}} \cap L^\infty} d\tau & \text{if } \sigma < 1 + \frac{N}{p_2}, \\ Z(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p_2, r}^{\sigma-1}} d\tau & \text{if } \sigma > 1 + \frac{N}{p_2} \text{ or } \left\{ \sigma = 1 + \frac{N}{p_2} \text{ and } r = 1 \right\}. \end{cases}$$

If $f = v$ then for all $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$) estimates (2.1) and (2.2) hold with

$$Z(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

Proof: Applying the operator Δ_q to (\mathcal{T}_ν) yields

$$\begin{cases} (\partial_t + S_{q+1}v \cdot \nabla)\Delta_q f - \nu \Delta \Delta_q f = \Delta_q F + R_q, \\ \Delta_q f|_{t=0} = \Delta_q f_0, \end{cases} \quad (\mathcal{T}_q)$$

with $R_q \stackrel{\text{def}}{=} S_{q+1}v \cdot \nabla \Delta_q f - \Delta_q(v \cdot \nabla f)$.

As $S_{q+1}v$ is smooth, one readily gets in the case $\nu = 0$:

$$(2.3) \quad \|\Delta_q f(t)\|_{L^p} \leq \|\Delta_q f_0\|_{L^p} + \int_0^t \left(\|R_q(\tau)\|_{L^p} + \frac{1}{p} \|\operatorname{div} S_{q+1}v(\tau)\|_{L^\infty} \|\Delta_q f(\tau)\|_{L^p} \right) d\tau + \int_0^t \|\Delta_q F(\tau)\|_{L^p} d\tau,$$

where it is understood that $1/p = 0$ if $p = +\infty$.

Let us now focus on the case $\nu > 0$ and $q \geq 0$.

1. Case $1 < p < +\infty$.

After multiplying (\mathcal{T}_q) by $|\Delta_q f|^{p-1} \operatorname{sgn}(\Delta_q f)$, the new term we have to deal with is

$$-\nu \int \Delta \Delta_q f |\Delta_q f|^{p-1} \operatorname{sgn}(\Delta_q f) dx.$$

As $q \geq 0$ (hence $\mathcal{F}\Delta_q f$ is supported in $C(0, \frac{3}{4}2^q, \frac{8}{3}2^q)$), lemma A.5 in [12] ensures that, for some universal constant $\kappa > 0$,

$$(2.4) \quad - \int \Delta \Delta_q f |\Delta_q f|^{p-1} \operatorname{sgn}(\Delta_q f) dx \geq \kappa \left(\frac{p-1}{p^2} \right) 2^{2q} \int |\Delta_q f|^p dx,$$

hence

$$(2.5) \quad \begin{aligned} & \|\Delta_q f(t)\|_{L^p} + \kappa \nu \left(\frac{p-1}{p^2} \right) 2^{2q} \int_0^t \|\Delta_q f(\tau)\|_{L^p} d\tau \leq \\ & \leq \|\Delta_q f_0\|_{L^p} + \int_0^t \left(\|R_q(\tau)\|_{L^p} + \frac{1}{p} \|\operatorname{div} S_{q+1} v(\tau)\|_{L^\infty} \|\Delta_q f(\tau)\|_{L^p} \right) d\tau \\ & \quad + \int_0^t \|\Delta_q F(\tau)\|_{L^p} d\tau. \end{aligned}$$

2. Case $p = 1$.

The same estimate holds true. Indeed, introduce the function

$$T_\epsilon(x) \stackrel{\text{def}}{=} x/\sqrt{\epsilon^2 + x^2}.$$

From a straightforward integration by parts, the following inequality is inferred:

$$- \int \Delta \Delta_q u T'_\epsilon(\Delta_q u) dx = \int \frac{\epsilon^2 |\nabla \Delta_q u|^2}{\epsilon^2 + |\Delta_q u|^2} dx \geq 0.$$

Hence, multiplying equation (\mathcal{T}_q) by $T_\epsilon(\Delta_q f)$, integrating in space and in time and using Lebesgue dominated convergence theorem to pass to the limit $\epsilon \rightarrow 0$ eventually yields inequality (2.5).

3. Case $p = +\infty$. It stems from the maximum principle.

That the term induced by $-\nu \Delta \Delta_q f$ is also non-negative in the case $q = -1$ is left to the reader. This entails that

$$(2.6) \quad \begin{aligned} \|\Delta_{-1} f(t)\|_{L^p} & \leq \|\Delta_{-1} f_0\|_{L^p} + \int_0^t \left(\|R_{-1}(\tau)\|_{L^p} + \frac{1}{p} \|\operatorname{div} S_0 v(\tau)\|_{L^\infty} \|\Delta_{-1} f(\tau)\|_{L^p} \right) d\tau \\ & \quad + \int_0^t \|\Delta_{-1} F(\tau)\|_{L^p} d\tau. \end{aligned}$$

Let us admit the following lemma, the proof of which is postponed in appendix:

Lemma 2.2 *Let $\sigma \in \mathbb{R}$ and $1 \leq p \leq p_2 \leq +\infty$. Let $p_1 = (1/p - 1/p_2)^{-1}$. There exists a constant $K = K(N)$ such that*

$$\begin{aligned}
 (2.7) \quad 2^{q\sigma} \|R_q\|_{L^p} &\leq 4^{|\sigma|} K \left(\sum_{|q'-q|\leq 4} \left(\|\Delta_{-1}\nabla v\|_{L^\infty} + \|S_{q'-1}\nabla v\|_{L^\infty} \right) 2^{q'\sigma} \|\Delta_{q'}f\|_{L^p} \right. \\
 &+ \sum_{q'\geq q-3} 2^{q-q'} 2^{q\sigma} \|\Delta_q f\|_{L^p} \|\Delta_{q'}\nabla v\|_{L^\infty} \\
 &+ \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} 2^{(q-q'')(\sigma-1-\frac{N}{p_2})} 2^{q''\sigma} \|\Delta_{q''}f\|_{L^p} 2^{q'\frac{N}{p_2}} \|\Delta_{q'}\nabla v\|_{L^{p_2}} \\
 &+ \sum_{\substack{q'\geq q-3 \\ |q'-q''|\leq 1}} 2^{(q-q')\left(\sigma+N\min\left(\frac{1}{p_2}, \frac{1}{p'}\right)\right)} \times \\
 &\quad \times 2^{q'\frac{N}{p_2}} \left(2^{q-q'} \|\Delta_{q'}\nabla v\|_{L^{p_2}} + \|\Delta_{q'}\operatorname{div} v\|_{L^{p_2}} \right) 2^{q''\sigma} \|\Delta_{q''}f\|_{L^p} \Big).
 \end{aligned}$$

Besides, the third term in the right-hand side may be replaced by

$$(2.8) \quad 16^{|\sigma|} K \|\nabla f\|_{L^{p_1}} \sum_{|q'-q|\leq 4} 2^{q'(\sigma-1)} \|\Delta_{q'}\nabla v\|_{L^{p_2}}.$$

Let $\kappa_p \stackrel{\text{def}}{=} \kappa(p-1)/p^2$, $f_q \stackrel{\text{def}}{=} 2^{q\sigma} \|\Delta_q f\|_{L^p}$ and $g_q \stackrel{\text{def}}{=} 2^{q\sigma} \|\Delta_q g\|_{L^p}$. Assume that $\sigma < 1 + N/p_2$ and denote for a suitably large $K = K(N)$

$$(2.9) \quad \begin{cases} v_q \stackrel{\text{def}}{=} K \left(\|\Delta_{-1}\nabla v\|_{L^\infty} + \|S_q\nabla v\|_{L^\infty} \right. \\ \quad \left. + \sum_{i\in\mathbb{N}} 2^{-i} \|\Delta_{q+i}\nabla v\|_{L^\infty} + \sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}} \|\Delta_{q'}\nabla v\|_{L^{p_2}} \right), \\ w_q \stackrel{\text{def}}{=} z_q \stackrel{\text{def}}{=} K \sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}} \|\Delta_{q'}\nabla v\|_{L^{p_2}}. \end{cases}$$

Further denote

$$(2.10) \quad \sigma_2 \stackrel{\text{def}}{=} 1 + \frac{N}{p_2} - \sigma \quad \text{and} \quad \sigma_1 \stackrel{\text{def}}{=} \begin{cases} \sigma + \min\left(\frac{N}{p_2}, \frac{N}{p'}\right) & \text{if } \operatorname{div} v \neq 0, \\ 1 + \sigma + \min\left(\frac{N}{p_2}, \frac{N}{p'}\right) & \text{if } \operatorname{div} v = 0. \end{cases}$$

Then inserting inequality (2.7) into (2.5) and (2.6) yields

$$\begin{aligned}
 (2.11) \quad f_q(t) + \kappa_p \delta_q \nu 2^{2q} \int_0^t f_q(\tau) d\tau &\leq f_q(0) + \int_0^t g_q(\tau) d\tau + \sum_{|q'-q|\leq 4} \int_0^t v_{q'}(\tau) f_{q'}(\tau) d\tau \\
 &+ \sum_{q'>q+4} \int_0^t 2^{\sigma_1(q-q')} w_{q'}(\tau) f_{q'}(\tau) d\tau + \sum_{q'<q-4} \int_0^t 2^{\sigma_2(q'-q)} z_q(\tau) f_{q'}(\tau) d\tau,
 \end{aligned}$$

with $\delta_{-1} = 0$ and $\delta_q = 1$ if $q \geq 0$.

With our assumptions on ∇v , we have for a suitably large K :

$$\max(v_q, w_q, z_q) \leq a \stackrel{\text{def}}{=} K \|\nabla v\|_{B^{\frac{N}{p_2}, \infty} \cap L^\infty}.$$

Moreover $f_q(t)$ may be replaced by $\sup_{\tau \in [0, t]} f_q(\tau)$ in the left-hand side.

Hence, taking advantage of $\sigma_1, \sigma_2 > 0$ for using convolution inequalities and applying, where needed, Minkowski inequality, one ends up with

$$\begin{aligned} & \|f\|_{\tilde{L}_t^\infty(B_{p,r}^\sigma)} + \kappa_p \nu \|f^{HF}\|_{\tilde{L}_t^1(B_{p,r}^{\sigma+2})} \leq \\ & \leq \|f_0\|_{B_{p,r}^\sigma} + \int_0^t a(\tau) \|f\|_{\tilde{L}_\tau^\infty(B_{p,r}^\sigma)} d\tau + \left\{ \begin{array}{l} \|g\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} \\ \int_0^t \|g(\tau)\|_{B_{p,r}^\sigma} d\tau \end{array} \right. \end{aligned}$$

so that Gronwall lemma completes the proof of proposition 2.1 in the case $\sigma < 1 + N/p_2$.

The case $\sigma > 1 + N/p_2$ or $\{\sigma = 1 + N/p_2 \text{ and } r = 1\}$ is left to the reader.

It is based on inequality (2.8) and embeddings $B_{p,r}^{\sigma-1} \hookrightarrow B_{p,1}^{\frac{N}{p_2}} \hookrightarrow L^{p_1}$.

The additional estimate in the case $f = v$ is inferred from remark A.1. ■

3. Losing estimates

The key point in the proof of proposition 2.1 is that ∇v belongs to

$$L^1(0, T; B^{\frac{N}{p_2}, \infty} \cap L^\infty).$$

In the present section, we make slightly weaker assumptions on v . The price to pay is a possible loss of derivatives in the estimates.

Throughout this section, the regularity index σ for f satisfies (2.10) (hence in particular $|\sigma| \leq 1 + N$).

3.1. A general statement

Note that no regularity assumptions on v are needed to get (2.11). This induces us to study the following type of coupled inequalities (we set $\kappa_p \nu$ to 0 to simplify the presentation):

$$\begin{aligned} (3.1) \quad f_q(t) & \leq f_q(0) + \int_0^t g_q(\tau) d\tau + \int_0^t \sum_{|q'-q| \leq N_0} v_{q'}(\tau) f_{q'}(\tau) d\tau \\ & + \int_0^t \left(\sum_{q' > q+M} 2^{\sigma_1(q-q')} w_{q'}(\tau) f_{q'}(\tau) \right) d\tau + \int_0^t z_q(\tau) \left(\sum_{q' < q-P} 2^{\sigma_2(q'-q)} f_{q'}(\tau) \right) d\tau, \end{aligned}$$

whenever $q \in \mathbb{Z}$ and $t \in [0, T]$. Above, N_0, M and P belong to \mathbb{N} .

We further assume that

$$v_q(t) \leq v_q^1(t) + v_q^2(t), \quad w_q(t) \leq w_q^1(t) + w_q^2(t), \quad z_q(t) \leq z_q^1(t) + z_q^2(t)$$

for some measurable and nonnegative functions v_q^1, w_q^1 , etc. on $[0, T]$.

Before stating estimates pertaining to (3.1), let us introduce a few notation:

$$V_q^1(t) \stackrel{\text{def}}{=} \int_0^t v_q^1(\tau) d\tau, \quad W_q^1(t) \stackrel{\text{def}}{=} \int_0^t w_q^1(\tau) d\tau, \quad Z_q^1(t) \stackrel{\text{def}}{=} \int_0^t z_q^1(\tau) d\tau.$$

For $R \in \mathbb{Z}$, let $v^R(t) \stackrel{\text{def}}{=} \sup_{q \leq R} v_q^2(t)$ and $V^R(t) \stackrel{\text{def}}{=} \int_0^t v^R(\tau) d\tau$. Further denote $v^\infty(t) \stackrel{\text{def}}{=} \sup_{q \in \mathbb{Z}} v_q^2(t)$ and $V^\infty(t) \stackrel{\text{def}}{=} \int_0^t v^\infty(\tau) d\tau$.

Define in the same way the functions w^R and W^R (resp. z^R and Z^R) pertaining to w^2 (resp. z^2). Finally, let $a \wedge b$ stand for $\max(a, b)$.

For $(f_q)_{q \in \mathbb{Z}}$ satisfying the coupled inequalities (3.1), we have:

Proposition 3.1 *Let $\kappa \geq 0$ and $\lambda > \lambda_0 > 0$. Assume that the following conditions are fulfilled for all $t \in [0, T]$:*

1. *There exist R_1, R_2 and R_3 in $\mathbb{Z} \cup \{+\infty\}$ such that*

$$\begin{cases} \text{if } R_1 < +\infty, & \int_0^t v_q^2(\tau) d\tau \leq \lambda^{-1} \quad \text{for } q > R_1, \\ \text{if } R_1 = +\infty, & V^\infty(t) < +\infty, \end{cases}$$

and similar conditions for w_q^2 with R_2 , and for z_q^2 with R_3 .

2. *There exist Q_1, Q_2 and Q_3 in \mathbb{Z} such that*

$$\begin{aligned} v_q^1(t) &\leq v_{Q_1}^1(t) \quad \text{for } q \leq Q_1, \\ w_q^1(t) &\leq w_{Q_2}^1(t) \quad \text{for } q \leq Q_2, \\ z_q^1(t) &\leq z_{Q_3}^1(t) \quad \text{for } q \leq Q_3. \end{aligned}$$

3. *Let $A_m \stackrel{\text{def}}{=} V_{m \wedge Q_1}^1 + W_{m \wedge Q_2}^1 + Z_{m \wedge Q_3}^1$, $A_{q,q'}^\lambda(t) \stackrel{\text{def}}{=} \sup_{\tau \in [0,t]} \left(e^{\lambda(A_{q'} - A_q)(\tau)} \right)$,*

$$\begin{aligned} K_\lambda^1(t) &= \max \left(\sup_q \sum_{q' > q+M} 2^{\sigma_1(q-q')} A_{q,q'}^\lambda(t), \sup_{q'} \sum_{q < q'-M} 2^{\sigma_1(q-q')} A_{q,q'}^\lambda(t) \right), \\ K_\lambda^2(t) &= \max \left(\sup_q \sum_{q' < q-P} 2^{\sigma_2(q'-q)} A_{q,q'}^\lambda(t), \sup_{q'} \sum_{q > q'+P} 2^{\sigma_2(q'-q)} A_{q,q'}^\lambda(t) \right). \end{aligned}$$

Then we assume that

$$(3.2) \quad 2 \left(K_\lambda^1(t) + K_\lambda^2(t) + 2^\kappa (2N_0 + 1) \right) \leq \lambda_0.$$

4. For all (q, q') such that $|q - q'| \leq N_0$, we have

$$\lambda|A_{q'}(t) - A_q(t)| \leq \kappa \log 2.$$

Let

$$B \stackrel{\text{def}}{=} V^{R_1} + W^{R_2} + Z^{R_3}, \quad \tilde{f}_q^\lambda(\tau) \stackrel{\text{def}}{=} e^{-\lambda(A_q+B)(\tau)} f_q(\tau)$$

and

$$\tilde{g}_q^\lambda(\tau) \stackrel{\text{def}}{=} e^{-\lambda(A_q+B)(\tau)} g_q(\tau).$$

The following estimate holds true for all $r \in [1, +\infty]$ and $t \in [0, T]$:

$$\left(\sum_q \left(\sup_{\tau \in [0, t]} \tilde{f}_q^\lambda(\tau) \right)^r \right)^{\frac{1}{r}} \leq \frac{\lambda}{\lambda - \lambda_0} \left(\left(\sum_q (f_q(0))^r \right)^{\frac{1}{r}} + \left(\sum_q \left(\int_0^t \tilde{g}_q^\lambda(\tau) d\tau \right)^r \right)^{\frac{1}{r}} \right).$$

Proof: According to (3.1), we have for all $t \in [0, T]$:

$$\begin{aligned} (3.3) \quad & e^{\lambda(A_q+B)(t)} \tilde{f}_q^\lambda(t) \leq f_q(0) + \int_0^t e^{\lambda(A_q+B)(\tau)} \tilde{g}_q^\lambda(\tau) d\tau \\ & + \sum_{|q'-q| \leq N_0} \left(\int_0^t v_{q'}^1(\tau) e^{\lambda(A_{q'}+B)(\tau)} \tilde{f}_{q'}^\lambda(\tau) d\tau + \int_0^t v_{q'}^2(\tau) e^{\lambda(A_{q'}+B)(\tau)} \tilde{f}_{q'}^\lambda(\tau) d\tau \right) \\ & + \sum_{q' > q+M} 2^{\sigma_1(q'-q)} \left(\int_0^t w_{q'}^1(\tau) e^{\lambda(A_{q'}+B)(\tau)} \tilde{f}_{q'}^\lambda(\tau) d\tau + \int_0^t w_{q'}^2(\tau) e^{\lambda(A_{q'}+B)(\tau)} \tilde{f}_{q'}^\lambda(\tau) d\tau \right) \\ & + \sum_{q' < q-P} 2^{\sigma_2(q'-q)} \left(\int_0^t z_q^1(\tau) e^{\lambda(A_{q'}+B)(\tau)} \tilde{f}_{q'}^\lambda(\tau) d\tau + \int_0^t z_q^2(\tau) e^{\lambda(A_{q'}+B)(\tau)} \tilde{f}_{q'}^\lambda(\tau) d\tau \right). \end{aligned}$$

According to assumption 2, we have for all $q \in \mathbb{Z}$ and $t \in [0, T]$,

$$v_q^1(t) \leq v_{q \wedge Q_1}^1(t), \quad w_q^1(t) \leq w_{q \wedge Q_2}^1(t) \quad \text{and} \quad z_q^1(t) \leq z_{q \wedge Q_3}^1(t),$$

whence

$$(3.4) \quad \begin{cases} \int_0^t v_{q'}^1(\tau) e^{\lambda(A_{q'}+B)(\tau)} d\tau \leq \frac{e^{\lambda(A_{q'}+B)(t)}}{\lambda}, \\ \int_0^t w_{q'}^1(\tau) e^{\lambda(A_{q'}+B)(\tau)} d\tau \leq \frac{e^{\lambda(A_{q'}+B)(t)}}{\lambda}. \end{cases}$$

On the other hand, we have for $q' < q - P$,

$$\begin{aligned} (3.5) \quad \int_0^t z_q^1(\tau) e^{\lambda(A_{q'}+B)(\tau)} d\tau & \leq \int_0^t z_{q \wedge Q_3}^1(\tau) e^{\lambda(A_q+B)(\tau)} e^{\lambda(A_{q'}-A_q)(\tau)} d\tau, \\ & \leq \lambda^{-1} e^{\lambda(A_q+B)(t)} A_{q,q'}^\lambda(t). \end{aligned}$$

By mean of explicit integration, we get

$$(3.6) \quad \int_0^t v_{q'}^2(\tau) e^{\lambda(A_{q'}(\tau)+B(\tau))} d\tau \leq \begin{cases} \lambda^{-1} e^{\lambda(A_{q'}(t)+B(t))} & \text{if } q' \leq R_1, \\ \left(\int_0^t v_{q'}^2(\tau) d\tau\right) e^{\lambda(A_{q'}(t)+B(t))} & \text{if } q' > R_1. \end{cases}$$

Similar inequalities hold for $\int_0^t w_{q'}^2(\tau) \exp(\lambda(A_q(\tau) + B(\tau))) d\tau$, and

$$(3.7) \quad \int_0^t z_q^2(\tau) e^{\lambda(A_{q'}(\tau)+B(\tau))} d\tau \leq \begin{cases} \lambda^{-1} e^{\lambda(A_{q'}(t)+B(t))} & \text{if } q \leq R_3, \\ \left(\int_0^t z_q^2(\tau) d\tau\right) e^{\lambda(A_{q'}(t)+B(t))} & \text{if } q > R_3. \end{cases}$$

Let $\tilde{F}_q^\lambda(t) \stackrel{\text{def}}{=} \sup_{\tau \in [0,t]} \tilde{f}_q^\lambda(\tau)$. Plugging (3.4), (3.5), (3.6) and (3.7) in (3.3) and using 1 yields

$$\begin{aligned} \tilde{f}_q^\lambda(t) &\leq f_q(0) + \int_0^t \tilde{g}_q^\lambda(\tau) d\tau + \frac{2}{\lambda} \sum_{|q'-q| \leq N_0} A_{q,q'}^\lambda(t) \tilde{F}_{q'}^\lambda(t) \\ &\quad + \frac{2}{\lambda} \sum_{q' > q+M} 2^{\sigma_1(q-q')} A_{q,q'}^\lambda(t) \tilde{F}_{q'}^\lambda(t) + \frac{2}{\lambda} \sum_{q' < q-P} 2^{\sigma_2(q'-q)} \tilde{F}_{q'}^\lambda(t) A_{q,q'}^\lambda(t). \end{aligned}$$

Clearly, t may be replaced by any $t' \in [0, t]$ in the left-hand side so that $\tilde{f}_q^\lambda(t)$ may be replaced by $\tilde{F}_q^\lambda(t)$. Now, Schur's lemma yields

$$\begin{aligned} \left(\sum_q \left(\tilde{F}_q^\lambda(t)\right)^r\right)^{\frac{1}{r}} &\leq \left(\sum_q \left(f_q(0)\right)^r\right)^{\frac{1}{r}} + \left(\sum_q \left(\int_0^t \tilde{g}_q^\lambda(\tau) d\tau\right)^r\right)^{\frac{1}{r}} + \\ &\quad \frac{2}{\lambda} \left((2N_0 + 1)2^\kappa + K_\lambda^1(t) + K_\lambda^2(t)\right) \left(\sum_q \left(\tilde{F}_q^\lambda(t)\right)^r\right)^{\frac{1}{r}}, \end{aligned}$$

which entails the desired inequality. ■

Proposition 3.1 may be applied to (2.11). We end up with the following general losing estimates:

Theorem 3.2 *Let $1 \leq p, p_2, r \leq +\infty$ and*

$$\sigma \in \left([-1] - N \min\left(\frac{1}{p_2}, \frac{1}{p'}\right), 1 + \frac{N}{p_2} \right).$$

There exists a $\lambda_0 > 0$ and $K = K(N)$ such that if v_q, w_q and z_q are defined as in (2.9) and satisfy

$$v_q \leq v_q^1 + v_q^2, \quad w_q \leq w_q^1 + w_q^2, \quad z_q \leq z_q^1 + z_q^2,$$

for some sequences of functions

$$(v_q^i)_{q \in \mathbb{Z}}, \quad (w_q^i)_{q \in \mathbb{Z}} \quad \text{and} \quad (z_q^i)_{q \in \mathbb{Z}} \quad (i \in \{1, 2\})$$

verifying conditions 1, 2, 3 and 4 of proposition 3.1 with $M = N_0 = P = 4$, σ_1, σ_2 defined in (2.10), some $\kappa \geq 0$ and some $\lambda > \lambda_0$, then we have

$$\begin{aligned} & \left[\sum_q \left(\sup_{\tau \in [0, t]} \left(e^{-\lambda(A_q(\tau) + B(\tau))} 2^{q\sigma} \|\Delta_q f(\tau)\|_{L^p} \right) \right)^r \right]^{\frac{1}{r}} \\ & \leq \frac{\lambda}{\lambda - \lambda_0} \left(\|f_0\|_{B_{p,r}^\sigma} + \left[\sum_q \left(\int_0^t e^{-\lambda(A_q(\tau) + B(\tau))} 2^{q\sigma} \|\Delta_q g(\tau)\|_{L^p} d\tau \right)^r \right]^{\frac{1}{r}} \right). \end{aligned}$$

Remark 3.3 Let us mention in passing that in the case where

$$\nabla v \in L^1(0, T; B_{p_2, \infty}^{\frac{N}{p_2}} \cap L^\infty) \quad \text{and} \quad \sigma < 1 + N/p_2$$

then theorem 3.2 leads back to the inequalities of proposition 2.1 (up to a multiplicative constant in the right-hand side).

Indeed, it is only a matter of taking

$$v_q^1 = w_q^1 = z_q^1 = 0, \quad v_q^2 = K \|\nabla v\|_{B_{p_2, \infty}^{\frac{N}{p_2}} \cap L^\infty}, \quad \text{and} \quad w_q^2 = z_q^2 = K \|\nabla v\|_{B_{p_2, \infty}^{\frac{N}{p_2}}}.$$

Assumption 1 is fulfilled with $R_1 = R_2 = R_3 = +\infty$, one can take $\kappa = 0$ in 4 and we clearly have $K_\lambda^1(t) = K_\lambda^2(t) = 1$.

3.2. Linear loss of regularity

In this part, we aim at extending the results [1] and [6] to general Besov spaces. In the former paper, the vector field v is only log-lipschitz in the space variable (which amounts to assuming that $\|S_q \nabla v(t)\|_{L^\infty} \leq (q + 2)u(t)$ for some $u \in L^1(0, T)$) and estimates in Hölder spaces C^σ with $\sigma \in (0, 1)$ are investigated. The authors point out a loss of derivatives of order $\int_0^t u(\tau) d\tau$ at time t .

In the latter paper, the functional framework is more general: Besov spaces $B_{p, \infty}^\sigma$ with $\sigma \in (0, 1)$, and the assumption on v is somewhat weaker:

$$\exists C \geq 0, \quad \forall q \in \mathbb{N}, \quad \frac{1}{q + 2} \int_0^T \|S_q \nabla v(\tau)\|_{L^\infty} d\tau < C.$$

We here aim at getting estimates in the same spirit in a more general framework. Our most general result reads:

Theorem 3.4 *Let $\alpha, \beta > 0$ and $\sigma \in ([-1] - \min(N/p_2, N/p') + \alpha, 1 + N/p_2 - \beta)$. Let σ_1, σ_2 be defined as in (2.10). Assume that the following two conditions are satisfied on $[0, T]$ for some nondecreasing bounded functions V and W and all $q, q' \geq -1$:*

1. $\left| \sum_{i=0}^{+\infty} 2^{-i} \int_0^t \left(\|S_{q+i} \nabla v(\tau)\|_{L^\infty} - \|S_{q'+i} \nabla v(\tau)\|_{L^\infty} \right) d\tau \right| \leq |q - q'|V(t),$
2. $\left| \int_0^t \left(2^{\frac{qN}{p_2}} \|\Delta_q \nabla v(\tau)\|_{L^{p_2}} - 2^{\frac{q'N}{p_2}} \|\Delta_{q'} \nabla v(\tau)\|_{L^{p_2}} \right) d\tau \right| \leq |q - q'|W(t).$

There exists $K = K(N)$ and $\lambda_0 = \lambda_0(\alpha, \beta, N, p, p_2)$ such that the following inequality holds true for (\mathcal{T}_ν) uniformly in ν with $\sigma_t \stackrel{\text{def}}{=} \sigma - K\lambda(V + W)(t)$:

$$(3.8) \quad \left(\sum_q \left(\sup_{\tau \in [0, t]} 2^{q\sigma_\tau} \|\Delta_q f(\tau)\|_{L^p} \right)^r \right)^{\frac{1}{r}} \leq \frac{\lambda}{\lambda - \lambda_0} \left[\|f_0\|_{B_{p,r}^\sigma} + \left(\sum_q \left(\int_0^t 2^{q\sigma_\tau} \|\Delta_q g(\tau)\|_{L^p} d\tau \right)^r \right)^{\frac{1}{r}} \right]$$

whenever

$$(3.9) \quad \sigma_1 - \lambda K(V + W)(t) \geq \alpha \quad \text{and} \quad \sigma_2 - \lambda K(V + W)(t) \geq \beta.$$

Proof: It stems from theorem 3.2 with an appropriate choice of v_q^i, w_q^i and z_q^i : let

$$v_q^1 \stackrel{\text{def}}{=} K \left(\sum_{i=0}^{+\infty} 2^{-i} \|S_{q+i} \nabla v\|_{L^\infty} + \sum_{|q'-q| \leq 4} 2^{q' \frac{N}{p_2}} \|\Delta_{q'} \nabla v\|_{L^{p_2}} \right),$$

$$w_q^1 \stackrel{\text{def}}{=} z_q^1 \stackrel{\text{def}}{=} K \sum_{|q'-q| \leq 4} 2^{q' \frac{N}{p_2}} \|\Delta_{q'} \nabla v\|_{L^{p_2}} \quad \text{and} \quad v_q^2 = w_q^2 = z_q^2 = 0.$$

Choosing large enough $K = K(N)$ yields $v_q \leq v_q^1 + v_q^2, w_q \leq w_q^1 + w_q^2$ and $z_q \leq z_q^1 + z_q^2$.

Now, condition 1 of proposition 3.1 is trivially satisfied. So does 2 with $Q_1 = Q_2 = Q_3 = -1$. Next, defining A_q as in proposition 3.1 and taking advantage of hypotheses 1 and 2, we have (up to a change of K),

$$|A_q(t) - A_{q'}(t)| \leq K \log 2 |q - q'| (V(t) + W(t)).$$

Hence,

$$K_\lambda^1(t) \leq \sum_{m>4} 2^{-m(\sigma_1 - \lambda K(V+W)(t))} \quad \text{and} \quad K_\lambda^2(t) \leq \sum_{m>4} 2^{-m(\sigma_2 - \lambda K(V+W)(t))}.$$

Choosing $\kappa = 4 \min(\sigma_1 - \alpha, \sigma_2 - \alpha) / \log 2$, hypothesis (3.9) ensures that conditions 3 and 4 of proposition 3.1 are satisfied for some $\lambda_0 = \lambda_0(\alpha, \beta, N, p, p_2)$.

Then theorem 3.2 yields the desired inequality. ■

Remark 3.5 Hypothesis 1 above may be replaced by the stronger following one:

$$\left| \int_0^t \left(\|S_q \nabla v(\tau)\|_{L^\infty} - \|S_{q'} \nabla v(\tau)\|_{L^\infty} \right) d\tau \right| \leq |q' - q|V(t).$$

Corollary 3.6 Assume that $\nabla v \in \tilde{L}_T^1(B_{p_2, \infty}^{\frac{N}{p_2}})$. Then inequality (3.8) holds true with

$$\sigma_t = \sigma - K\lambda \|\nabla v\|_{\tilde{L}_t^1(B_{p_2, \infty}^{\frac{N}{p_2}})}$$

whenever

$$\sigma_1 - \lambda K \|\nabla v\|_{\tilde{L}_T^1(B_{p_2, \infty}^{\frac{N}{p_2}})} \geq \alpha \quad \text{and} \quad \sigma_2 - \lambda K \|\nabla v\|_{\tilde{L}_T^1(B_{p_2, \infty}^{\frac{N}{p_2}})} \geq \beta.$$

Proof: We just have to use the following inequalities (if $q' > q$):

$$\begin{aligned} & \left| \sum_{i=0}^{+\infty} 2^{-i} \int_0^t \left(\|S_{q+i} \nabla v(\tau)\|_{L^\infty} - \|S_{q'+i} \nabla v(\tau)\|_{L^\infty} \right) d\tau \right| \\ & \leq \sum_{i=0}^{+\infty} 2^{-i} \int_0^t \sum_{p=q+i}^{q'+i-1} \|\Delta_p \nabla v(\tau)\|_{L^\infty} d\tau \leq \sum_{i=0}^{+\infty} 2^{-i} (q' - q) \|\nabla v\|_{\tilde{L}_t^1(B_{\infty, \infty}^0)}, \\ & \leq 2(q' - q) \|\nabla v\|_{\tilde{L}_t^1(B_{\infty, \infty}^0)}, \end{aligned}$$

and, of course, for all $q, q' \geq -1$,

$$\left| \int_0^t \left(2^{\frac{qN}{p_2}} \|\Delta_q \nabla v(\tau)\|_{L^{p_2}} - 2^{\frac{q'N}{p_2}} \|\Delta_{q'} \nabla v(\tau)\|_{L^{p_2}} \right) d\tau \right| \leq \|\nabla v\|_{\tilde{L}_t^1(B_{p_2, \infty}^{\frac{N}{p_2}})}.$$

■

Remark 3.7 In the case $p_2 = +\infty$, theorem 3.4 entails the result stated in [6].

Remark 3.8 One can also allow for a linear growth of the dyadic blocks of the source term g (like in [9], lemma 2.5). This growth will entail an additional (linear) loss of regularity. The details are left to the reader.

Actually, in the case where ∇v belongs to $\tilde{L}_T^1(B_{p_2, \infty}^{\frac{N}{p_2}})$ and

$$2^{q\frac{N}{p_2}} \|\Delta_q \nabla v\|_{L_T^1(L^{p_2})}$$

is suitably small for large q , only the growth of $\|S_q \nabla v\|_{L^\infty}$ is responsible for the loss of regularity:

Theorem 3.9 *Let α, β and σ satisfy the assumptions of theorem 3.4. Assume that ∇v belongs to $\tilde{L}_T^1(B_{p_2, \infty}^{\frac{N}{p_2}})$. There exists $\lambda_0 = \lambda_0(\alpha, \beta, N, p, p_2)$ and $K = K(N)$ such that if $\lambda > \lambda_0$ and if there exists some $R \in \mathbb{N}$ such that*

$$(3.10) \quad K2^{q\frac{N}{p_2}} \|\Delta_q \nabla v\|_{L_T^1(L^{p_2})} \leq \lambda^{-1} \quad \text{for } q > R,$$

then the following inequality holds true for (\mathcal{T}_ν) uniformly in ν :

$$\begin{aligned} & \left(\sum_q \left(\sup_{\tau \in [0, t]} e^{-\lambda W^R(\tau)} 2^{q\sigma\tau} \|\Delta_q f(\tau)\|_{L^p} \right)^r \right)^{\frac{1}{r}} \\ & \leq \frac{\lambda}{\lambda - \lambda_0} \left(\|f_0\|_{B_{p, r}^\sigma} + \left(\sum_q \left(\int_0^t e^{-\lambda W^R(\tau)} 2^{q\sigma\tau} \|\Delta_q g(\tau)\|_{L^p} d\tau \right)^r \right)^{\frac{1}{r}} \right), \end{aligned}$$

with

$$W^R(t) \stackrel{\text{def}}{=} \int_0^t \sup_{q \leq R+3} 2^{q\frac{N}{p_2}} \|\Delta_q \nabla v(\tau)\|_{L_T^1(L^{p_2})} d\tau$$

and

$$\sigma_t \stackrel{\text{def}}{=} \sigma - K\lambda \|\nabla v\|_{\tilde{L}_t^1(B_{\infty, \infty}^0)} \geq \alpha$$

whenever

$$(3.11) \quad \sigma_1 - \lambda K \|\nabla v\|_{\tilde{L}_T^1(B_{\infty, \infty}^0)} \geq \alpha \quad \text{and} \quad \sigma_2 - \lambda K \|\nabla v\|_{\tilde{L}_T^1(B_{\infty, \infty}^0)} \geq \beta.$$

If $\nabla v \in L^1(0, T; B_{p_2, \infty}^{\frac{N}{p_2}})$ then condition (3.10) is useless provided that W^R has been replaced by $\|\nabla v\|_{L_t^1(B_{p_2, \infty}^{\frac{N}{p_2}})}$.

Proof: Choose $w_q^1 = z_q^1 = 0$,

$$v_q^1 = K \left(\sum_{q' \leq q} \|\Delta_{q'} \nabla v\|_{L^\infty} + \sum_{i \in \mathbb{N}} 2^{-i} \|\Delta_{q+i} \nabla v\|_{L^\infty} \right)$$

and

$$v_q^2 = w_q^2 = z_q^2 = K \sum_{|q'-q| \leq 4} 2^{q'\frac{N}{p_2}} \|\Delta_{q'} \nabla v\|_{L^{p_2}}.$$

Taking advantage of the above assumptions and choosing $K = K(N)$ large enough, one can easily check that

$$v_q \leq v_q^1 + v_q^2, \quad w_q \leq w_q^1 + w_q^2 \quad \text{and} \quad z_q \leq z_q^1 + z_q^2.$$

Besides, condition 1 of proposition 3.1 is satisfied with $R_1 = R_2 = R_3 = R+3$ (with the convention that $R = +\infty$ if $\nabla v \in L^1(0, T; B_{p_2}^{\frac{N}{p_2}, \infty})$). So does condition 2 with $Q_1 = Q_2 = Q_3 = -1$. Moreover,

$$\forall q \geq q', \int_0^t (v_q^1 - v_{q'}^1)(\tau) d\tau \leq K(q - q') \|\nabla v\|_{\tilde{L}_t^1(B_{\infty, \infty}^0)}.$$

so that choosing $\kappa = 4 \min(\sigma_1 - \alpha, \sigma_2 - \beta) / \log 2$ and using (3.11) ensures conditions 3 and 4. Applying theorem 3.2 completes the proof. ■

In the case where the growth assumptions on the blocks are made *before* time integration, one can exhibit more explicit sufficient conditions for having losing estimates:

Theorem 3.10 *Let $\alpha > 0$ and $\sigma \in ([-1] - \min(N/p_2, N/p') + \alpha, 1 + N/p_2)$. Assume that there exist two integrable functions u and w such that the following conditions are satisfied on $[0, T]$:*

1. $\forall q \geq -1, \|S_q \nabla v(t)\|_{L^\infty} \leq (q + 1)u(t),$
2. $\forall q \geq -1, 2^{q\frac{N}{p_2}} \|\Delta_q \nabla v(t)\|_{L^{p_2}} \leq (q + 2)w(t),$

Let $V = \int_0^t u(\tau) d\tau$ and $W = \int_0^t w(\tau) d\tau$. There exist $\lambda_0 = \lambda_0(\alpha, \sigma, N, p, p_2)$ and $K = K(N)$ such that whenever $\lambda > \lambda_0$ and $\sigma_1 - \lambda K(V + W)(t) \geq \alpha$, we have:

$$\begin{aligned} & \left(\sum_q \left(\sup_{\tau \in [0, t]} 2^{q\sigma_\tau} \|\Delta_q f(\tau)\|_{L^p} \right)^r \right)^{\frac{1}{r}} \\ & \leq \frac{\lambda}{\lambda - \lambda_0} \left[\|f_0\|_{B_{p, r}^\sigma} + \left(\sum_q \left(\int_0^t 2^{q\sigma_\tau} \|\Delta_q g(\tau)\|_{L^p} d\tau \right)^r \right)^{\frac{1}{r}} \right], \end{aligned}$$

with $\sigma_t = \sigma - K\lambda(V + W)(t)$.

Proof: Take $v_q^2 = w_q^2 = z_q^2 = 0$ and, for conveniently large $K = K(N)$,

$$v_q^1 = K(q + 2)(u + w) \quad \text{and} \quad w_q^1 = z_q^1 = K(q + 2)w.$$

Then one can easily check that with the hypotheses above, we have $v_q \leq v_q^1 + v_q^2, w_q \leq w_q^1 + w_q^2$ and $z_q \leq z_q^1 + z_q^2$. Now, conditions of proposition 3.1 are satisfied with $Q_i = -1$ and $\kappa = 4(\sigma_1 - \alpha) / \log 2$ and theorem 3.2 may be applied once again while $\sigma_1 - \lambda K(V + W)(t) \geq \alpha$ (note that A_q is a nondecreasing sequence so that $A_{q, q'}^\lambda(t) \equiv 1$ for $q \geq q'$). ■

Remark 3.11 *Note that the case $p = p_2 = r = +\infty$ leads back to result treated in [1].*

3.3. Limited loss of regularity

We now make the additional assumption that $\nabla v \in L^1(0, T; B_{\infty, r_2}^0)$ for some $r_2 \in (1, +\infty)$. The new result we get in this case is the following one:

Theorem 3.12 *Assume that ∇v belongs to $\tilde{L}_T^1(B_{p_2, \infty}^{\frac{N}{p_2}}) \cap L^1(0, T; B_{\infty, r_2}^0)$ for some $r_2 \in (1, +\infty)$. Let $\sigma \in [-1] - \min(N/p_2, N/p'), 1 + N/p_2)$ and $\epsilon \in (0, \sigma_1/2)$ where σ_1 has been defined in (2.10). There exist $\lambda_0 = \lambda_0(\sigma_1, \sigma_2)$ and $K = K(N)$ such that if $\lambda > \lambda_0$ and if there exists some $R \in \mathbb{N}$ such that*

$$(3.12) \quad K2^{q\frac{N}{p_2}} \|\Delta_q \nabla v\|_{L_T^1(L^{p_2})} \leq \lambda^{-1} \quad \text{for } q > R,$$

then the following inequality holds true:

$$(3.13) \quad \|f\|_{\tilde{L}_T^\infty(B_{p, r}^{\sigma-\epsilon})} \leq \frac{4^\epsilon \lambda e^{\lambda KW^R(T)}}{\lambda - \lambda_0} e^{\frac{(K\lambda)r_2}{\epsilon r_2 - 1} \left(\int_0^T \|\nabla v(\tau)\|_{B_{\infty, r_2}^0} d\tau \right)^{r_2}} \left(\|f_0\|_{B_{p, r}^\sigma} + \|g\|_{\tilde{L}_T^1(B_{p, r}^{\sigma, \tau})} \right)$$

with $W^R(t) \stackrel{\text{def}}{=} \int_0^t \sup_{q \leq R+3} 2^{q\frac{N}{p_2}} \|\Delta_q \nabla v(\tau)\|_{L_T^1(L^{p_2})} d\tau$.

If $\nabla v \in L^1(0, T; B_{p_2, \infty}^{\frac{N}{p_2}})$ then condition (3.12) may be removed and $W^R(T)$ has to be replaced by $\|\nabla v\|_{L_T^1(B_{p_2, \infty}^{\frac{N}{p_2}})}$.

Proof: Once again, this is a corollary of theorem 3.2. Indeed, according to Hölder inequality,

$$\|S_q \nabla v\|_{L^\infty} \leq (q + 1)^\eta \|\nabla v\|_{B_{\infty, r_2}^0} \quad \text{with } \eta = 1 - 1/r_2$$

so that one can choose

$$v_q^1 = K(q+1)^\eta \|\nabla v\|_{B_{\infty, r_2}^0}, \quad v_q^2 = K \left(\sum_{q' \geq q} 2^{q-q'} \|\Delta_{q'} \nabla v\|_{L^\infty} + \sum_{|q'-q| \leq 4} 2^{q'\frac{N}{p_2}} \|\Delta_{q'} \nabla v\|_{L^{p_2}} \right),$$

$$w_q^1 = z_q^1 = 0, \quad w_q^2 = z_q^2 = K \sum_{|q'-q| \leq 4} 2^{q'\frac{N}{p_2}} \|\Delta_{q'} \nabla v\|_{L^{p_2}}.$$

Condition 1 is satisfied with $R_1 = R_2 = R_3 = R + 3$. As $(v_q^1)_{q \geq -1}$ is a nondecreasing sequence of functions, condition 2 is fulfilled for any Q_1 (that we shall merely denote by Q). Next, as $\eta \in (0, 1)$, for all $Q \in \mathbb{N}$ and $q' \geq q$, the following inequality holds:

$$(3.14) \quad (q' \wedge Q + 1)^\eta - (q \wedge Q + 1)^\eta \leq \eta(q' - q)(Q + 1)^{\eta-1}.$$

Let $Y(t) \stackrel{\text{def}}{=} \int_0^t \|\nabla v(\tau)\|_{B_{\infty, r_2}^0} d\tau$. Up to a change of K , $A_q(t) = K \log 2 Y(t) (q \wedge Q + 1)^\eta$. Hence, according to inequality (3.14), we have for $q' \geq q$,

$$A_{q, q'}^\lambda(t) \leq 2^{\lambda \eta K Y(t) (q' - q) (Q + 1)^{\eta - 1}},$$

hence

$$(3.15) \quad K_\lambda^1(t) \leq \sum_{q' > q+4} 2^{(q - q')(\sigma_1 - \eta \lambda K Y(t) (Q + 1)^{\eta - 1})}.$$

From now on, assume that

$$(3.16) \quad Q + 1 \geq \left(\frac{2\eta \lambda K Y(t)}{\sigma_1} \right)^{\frac{1}{1 - \eta}}.$$

Then easy computations show that $K_\lambda^1(t) \leq \sum_{m > 4} 2^{-p \frac{\sigma_1}{2}} = 2^{-5 \frac{\sigma_1}{2}} / (1 - 2^{-\frac{\sigma_1}{2}})$. As A_q is nondecreasing, we readily get $K_\lambda^2(t) \leq 2^{-5 \sigma_2} / (1 - 2^{-\sigma_2})$.

Now, if we take $\kappa = 2\sigma_1 / \log 2$, inequality (3.14) and the above computations show that conditions 3 and 4 are fulfilled for some $\lambda_0 = \lambda_0(\sigma_1, \sigma_2)$. Hence, theorem 3.2 yields

$$\begin{aligned} & \left[\sum_q \left(\sup_{[0, t]} \left(e^{-K \lambda W^R(\tau)} 2^{q\epsilon - [(q \wedge Q) + 1]^\eta K \lambda Y(\tau)} 2^{q(\sigma - \epsilon)} \|\Delta_q f(\tau)\|_{L^p} \right) \right)^r \right]^{\frac{1}{r}} \\ & \leq \frac{\lambda}{\lambda - \lambda_0} \left(\|f_0\|_{B_{p, r}^\sigma} + \left[\sum_q \left(\int_0^t e^{-K \lambda W^R(\tau)} 2^{-[(q \wedge Q) + 1]^\eta K \lambda Y(\tau)} 2^{q\sigma} \|\Delta_q g(\tau)\|_{L^p} d\tau \right)^r \right]^{\frac{1}{r}} \right). \end{aligned}$$

Now, getting the desired inequality in theorem 3.12 amounts to making a convenient choice of Q . Indeed, denoting $C = \eta \lambda K \epsilon^{-1} Y(T)$, we have

$$\forall x > 0, \epsilon(x + 1) - Cx^\eta \geq \epsilon + \left(\frac{C}{\epsilon^\eta} \right)^{\frac{1}{1 - \eta}} \left(\eta^{\frac{1}{1 - \eta}} - \eta^{\frac{\eta}{1 - \eta}} \right) \geq \epsilon - \left(\frac{C \eta^\eta}{\epsilon^\eta} \right)^{\frac{1}{1 - \eta}}.$$

Therefore

$$(3.17) \quad \begin{aligned} q\epsilon - K \lambda Y(t) (1 + q \wedge Q)^\eta & \geq \\ & \geq \begin{cases} \epsilon - \left(\frac{K \lambda Y(t) \eta^\eta}{\epsilon^\eta} \right)^{\frac{1}{1 - \eta}} & \text{if } q \geq Q, \\ -\epsilon - K \lambda Y(t) (1 + Q)^\eta & \text{if } -1 \leq q < Q. \end{cases} \end{aligned}$$

Assuming that $0 < \epsilon \leq \sigma_1 / 2$ and choosing

$$Q \stackrel{\text{def}}{=} \left[\left(\frac{\eta K \lambda Y(T)}{\epsilon} \right)^{\frac{1}{1 - \eta}} \right],$$

condition (3.16) is fulfilled on $[0, T]$, and we have, according to (3.17)

$$2^{q\epsilon - [(q \wedge Q) + 1] \eta K \lambda Y(T)} \leq 2^{2\epsilon \left(1 + (\lambda \epsilon^{-1} Y(T))^{1-\frac{1}{\eta}}\right)}.$$

This implies inequality (3.13). ■

Actually, estimates with limited loss of regularity may be proved under the weaker additional assumption that $\nabla v \in \tilde{L}^1_T(B^0_{\infty, r_2})$ for some $r_2 \in (1, +\infty)$:

Theorem 3.13 *Assume that ∇v belongs to $\tilde{L}^1_T(B^{\frac{N}{p_2}, \infty}) \cap \tilde{L}^1_T(B^0_{\infty, r_2})$ for some $r_2 \in (1, +\infty)$. Let $\sigma \in [-1] - \min(N/p_2, N/p')$, $1 + N/p_2$ and $\epsilon \in (0, \sigma_1)$. There exist $\lambda_0 = \lambda_0(\sigma_1, \sigma_2)$ and $K = K(N)$ such that if $\lambda > \lambda_0$ and if there exists some $R \in \mathbb{N}$ such that (3.12) is fulfilled, then the following inequality holds true:*

(3.18)

$$\|f\|_{\tilde{L}^\infty_T(B^{\sigma-\epsilon}_{p,r})} \leq \frac{2^\epsilon \lambda e^{\lambda K W^R(T)}}{\lambda - \lambda_0} e^{K \lambda (Q+1)^{1-\frac{1}{r_2}} \|\nabla v\|_{\tilde{L}^1_T(B^0_{\infty, r_2})}} \left(\|f_0\|_{B^{\sigma}_{p,r}} + \|g\|_{\tilde{L}^1_T(B^{\sigma}_{p,r})} \right)$$

with Q such that $\sup_{q \geq Q} \|\nabla \Delta_q v\|_{L^1_T(L^\infty)} \leq \epsilon / K \lambda$ and $W^R(t)$ defined as in theorem 3.12 (with the usual change if $\nabla v \in L^1(0, T; B^{\frac{N}{p_2}, \infty})$).

Proof: Define $v_q^2, w_q^2, z_q^2, w_q^1, z_q^1$ as previously, and $v_q^1 \stackrel{\text{def}}{=} K \sum_{q' \leq q-1} \|\Delta_{q'} \nabla v\|_{L^\infty}$.

As for $q' \geq q$, we have (up to a change of K)

$$\int_0^T \left(v_{q'}^1(t) - v_q^1(t) \right) dt \leq K \log 2 (q' - q) \sup_{q'' \geq q} \|\Delta_{q''} \nabla v\|_{L^1_T(L^\infty)},$$

we easily get by choosing $Q_1 = Q$,

$$\forall q' \geq q, A_{q, q'}^\lambda \leq 2^{\epsilon(q' - q)}.$$

Hence condition 3 is satisfied. So does 4 with $\kappa = 2\sigma_1 / \log 2$. Therefore the general inequality stated in theorem 3.2 holds true with $B(t) = W^R(t)$ and $A_q(t) = K \int_0^t \sum_{p \leq q \wedge Q - 1} \|\Delta_p \nabla v(\tau)\|_{L^\infty} d\tau$. Now, according to Hölder inequality and the definition of Q , we have

$$\begin{aligned} K \log 2 \lambda \int_0^t \sum_{p \leq q \wedge Q - 1} \|\Delta_p \nabla v(\tau)\|_{L^\infty} d\tau \\ \leq K \log 2 \lambda (Q+1)^{1-\frac{1}{r_2}} \|\nabla v\|_{\tilde{L}^1_T(B^0_{\infty, r_2})} + \epsilon \max(0, q - Q), \end{aligned}$$

whence for all $t \in [0, T]$ and $q \geq -1$,

$$e^{\lambda A_q(t)} 2^{-q\epsilon} \leq e^{K \lambda (Q+1)^{1-\frac{1}{r_2}} \|\nabla v\|_{\tilde{L}^1_T(B^0_{\infty, r_2})}}.$$

This yields inequality (3.18). ■

A. Appendix

We here give the proof of lemma 2.2. In order to show that only the gradient part of v is involved in the estimates, it is convenient to split v into low and high frequencies: $v = \Delta_{-1}v + \tilde{v}$. Obviously, there exists a constant C such that

$$(A.1) \quad \forall r \in [1, +\infty], \|\Delta_{-1}\nabla v\|_{L^r} \leq C \|\nabla v\|_{L^r} \quad \text{and} \quad \|\nabla \tilde{v}\|_{L^r} \leq C \|\nabla v\|_{L^r}.$$

Since there exists a $R > 0$ so that for all $t \in [0, T]$, $\text{Supp } \mathcal{F}\tilde{v}(t) \cap B(0, R) = \emptyset$, Bernstein inequality holds true for $\Delta_q\tilde{v}$ even for $q = -1$, namely

$$(A.2) \quad \forall q \geq -1, \|\Delta_q\nabla\tilde{v}\|_{L^p} \approx 2^q \|\Delta_q\tilde{v}\|_{L^p}.$$

Let us define the paraproduct between two distributions according to J.-M. Bony in [2]:

$$T_f g \stackrel{\text{def}}{=} \sum_{q \in \mathbb{N}} S_{q-1} f \Delta_q g.$$

Denoting

$$R(f, g) \stackrel{\text{def}}{=} \sum_{q \geq -1} \Delta_q f \tilde{\Delta}_q g \quad \text{with} \quad \tilde{\Delta}_q g \stackrel{\text{def}}{=} (\Delta_{q-1} + \Delta_q + \Delta_{q+1})g,$$

we have the following so-called Bony’s decomposition:

$$fg = T_f g + T_g f + R(f, g).$$

Now, we have

$$\begin{aligned} R_q &= S_{q+1}v \cdot \nabla \Delta_q f - \Delta_q(v \cdot \nabla f), \\ &= [\tilde{v}^j, \Delta_q] \partial_j f + [\Delta_{-1}v^j, \Delta_q] \partial_j f + (S_{q+1}v - v) \cdot \nabla \Delta_q f, \end{aligned}$$

where the summation convention on repeated indices is understood.

Hence, taking advantage of Bony’s decomposition, we end up with $R_q = \sum_{i=1}^7 R_q^i$ where

$$\begin{aligned} R_q^1 &= [T_{\tilde{v}^j}, \Delta_q] \partial_j f, \\ R_q^2 &= T_{\partial_j \Delta_q f} \tilde{v}^j, \\ R_q^3 &= -\Delta_q T_{\partial_j f} \tilde{v}^j, \\ R_q^4 &= \partial_j R(\tilde{v}^j, \Delta_q f) - \partial_j \Delta_q R(\tilde{v}^j, f), \\ R_q^5 &= \Delta_q R(\text{div } \tilde{v}, f) - R(\text{div } \tilde{v}, \Delta_q f), \\ R_q^6 &= (S_{q+1}v - v) \cdot \nabla \Delta_q f, \\ R_q^7 &= [\Delta_{-1}v^j, \Delta_q] \partial_j f. \end{aligned}$$

In the following computations, the constant K depends only on N .

Bounds for $2^{q\sigma} \|R_q^1\|_{L^p}$:

By virtue of (1.2), we have

$$R_q^1 = \sum_{|q-q'|\leq 4} [S_{q'-1}\tilde{v}^j, \Delta_q] \partial_j \Delta_{q'} f.$$

On the other hand,

$$[S_{q'-1}\tilde{v}^j, \Delta_q] \partial_j \Delta_{q'} f(x) = \int h(y) [S_{q'-1}\tilde{v}^j(x) - S_{q'-1}\tilde{v}^j(x-2^{-q}y)] \partial_j \Delta_{q'} f(x-2^{-q}y) dy$$

so that applying first order Taylor’s formula, convolution inequalities and (A.1) yields

$$(A.3) \quad 2^{q\sigma} \|R_q^1\|_{L^p} \leq K \sum_{|q'-q|\leq 4} \|S_{q'-1}\nabla v\|_{L^\infty} 2^{q'\sigma} \|\Delta_{q'} f\|_{L^p}.$$

Bounds for $2^{q\sigma} \|R_q^2\|_{L^p}$:

By virtue of (1.2), we have

$$R_q^2 = \sum_{q'\geq q-3} S_{q'-1} \partial_j \Delta_{q'} f \Delta_{q'} \tilde{v}^j.$$

Hence, using inequalities (A.1) and (A.2) yields

$$(A.4) \quad 2^{q\sigma} \|R_q^2\|_{L^p} \leq K \sum_{q'\geq q-3} 2^{q-q'} 2^{q\sigma} \|\Delta_{q'} f\|_{L^p} \|\Delta_{q'} \nabla v\|_{L^\infty}.$$

Bounds for $2^{q\sigma} \|R_q^3\|_{L^p}$:

One proceeds as follows:

$$(A.5) \quad R_q^3 = - \sum_{|q'-q|\leq 4} \Delta_q \left(S_{q'-1} \partial_j f \Delta_{q'} \tilde{v}^j \right),$$

$$(A.6) \quad = - \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} \Delta_q \left(\Delta_{q''} \partial_j f \Delta_{q'} \tilde{v}^j \right).$$

Therefore, denoting $1/p_1=1/p-1/p_2$ and taking advantage of (A.1) and (A.2),

$$(A.7) \quad \begin{aligned} 2^{q\sigma} \|R_q^3\|_{L^p} &\leq K \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} 2^{q\sigma} \|\Delta_{q''} \partial_j f\|_{L^{p_1}} \|\Delta_{q'} \tilde{v}^j\|_{L^{p_2}}, \\ &\leq K \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} 2^{(q-q'')(\sigma-1-\frac{N}{p_2})} 2^{q''\sigma} \|\Delta_{q''} f\|_{L^p} 2^{q'\frac{N}{p_2}} \|\Delta_{q'} \nabla v\|_{L^{p_2}}. \end{aligned}$$

Note that, starting from (A.5), one can alternately get

$$(A.8) \quad 2^{q\sigma} \|R_q^3\|_{L^p} \leq 16^{|\sigma|} K \sum_{|q'-q|\leq 4} \|\nabla S_{q'-1} f\|_{L^{p_1}} 2^{q'(\sigma-1)} \|\Delta_{q'} \nabla v\|_{L^{p_2}}.$$

Bounds for $2^{q\sigma} \|R_q^4\|_{L^p}$:

$$R_q^4 = \underbrace{\sum_{|q'-q|\leq 2} \partial_j(\Delta_{q'}\tilde{v}^j\Delta_q\tilde{\Delta}_{q'}f)}_{R_q^{4,1}} - \underbrace{\sum_{q'\geq q-3} \partial_j\Delta_q(\Delta_{q'}\tilde{v}^j\tilde{\Delta}_{q'}f)}_{R_q^{4,2}}.$$

For the first term, we merely have (by virtue of (A.2)),

$$(A.9) \quad 2^{q\sigma} \|R_q^{4,1}\|_{L^p} \leq 4^{|\sigma|}K \sum_{|q'-q|\leq 2} \|\Delta_{q'}\nabla\tilde{v}\|_{L^\infty} 2^{q'\sigma} \|\tilde{\Delta}_{q'}f\|_{L^p}.$$

For $R_q^{4,2}$, we proceed differently depending on the value of $1/p + 1/p_2$.

First case: $1/p_3 \stackrel{\text{def}}{=} 1/p + 1/p_2 \leq 1$:

$$\begin{aligned} 2^{q\sigma} \|R_q^{4,2}\|_{L^p} &\leq K \sum_{q'\geq q-3} 2^{q(1+\sigma)}2^{q(\frac{N}{p_3}-\frac{N}{p})} \|\Delta_{q'}\tilde{v}\tilde{\Delta}_{q'}f\|_{L^{p_3}}, \\ &\leq K \sum_{q'\geq q-3} 2^{q(1+\sigma)}2^{q\frac{N}{p_2}} \|\Delta_{q'}\tilde{v}\|_{L^{p_2}} \|\tilde{\Delta}_{q'}f\|_{L^p}, \\ &\leq K \sum_{q'\geq q-3} 2^{(q-q')(1+\frac{N}{p_2}+\sigma)}2^{q'\frac{N}{p_2}} \|\nabla\Delta_{q'}\tilde{v}\|_{L^{p_2}} 2^{q'\sigma} \|\tilde{\Delta}_{q'}f\|_{L^p}. \end{aligned}$$

Second case: $1/p + 1/p_2 > 1$:

Taking $p_2 = p'$ in the above computations yields

$$\begin{aligned} 2^{q\sigma} \|R_q^{4,2}\|_{L^p} &\leq K \sum_{q'\geq q-3} 2^{(q-q')(1+\frac{N}{p'}+\sigma)}2^{q'\frac{N}{p'}} \|\nabla\Delta_{q'}\tilde{v}\|_{L^{p'}} 2^{q'\sigma} \|\tilde{\Delta}_{q'}f\|_{L^p}, \\ &\leq K \sum_{q'\geq q-3} 2^{(q-q')(1+\frac{N}{p'}+\sigma)}2^{q'\frac{N}{p_2}} \|\nabla\Delta_{q'}\tilde{v}\|_{L^{p_2}} 2^{q'\sigma} \|\tilde{\Delta}_{q'}f\|_{L^p}. \end{aligned}$$

In view of (A.1) and (A.9), we conclude that

$$(A.10) \quad 2^{q\sigma} \|R_q^4\|_{L^p} \leq 4^{|\sigma|}K \sum_{q'\geq q-3} 2^{(q-q')(1+\sigma+N\min(\frac{1}{p_2},\frac{1}{p'}))}2^{q'\frac{N}{p_2}} \|\Delta_{q'}\nabla v\|_{L^{p_2}} 2^{q'\sigma} \|\tilde{\Delta}_{q'}f\|_{L^p}.$$

Bounds for $2^{q\sigma} \|R_q^5\|_{L^p}$:

Similar computations yield

$$(A.11) \quad 2^{q\sigma} \|R_q^5\|_{L^p} \leq 4^{|\sigma|}K \sum_{q'\geq q-3} 2^{(q-q')(\sigma+N\min(\frac{1}{p_2},\frac{1}{p'}))}2^{q'\frac{N}{p_2}} \|\Delta_{q'}\operatorname{div} v\|_{L^{p_2}} 2^{q'\sigma} \|\tilde{\Delta}_{q'}f\|_{L^p}.$$

Bounds for $2^{q\sigma} \|R_q^6\|_{L^p}$:

Since $R_q^6 = -\sum_{q' \geq q+1} \Delta_{q'} v \cdot \nabla \Delta_q f$, we have, by virtue of Bernstein inequality (note that $q' \geq 0$ in the summation),

$$(A.12) \quad 2^{q\sigma} \|R_q^6\|_{L^p} \leq K \sum_{q' \geq q} 2^{q-q'} \|\Delta_{q'} \nabla v\|_{L^\infty} 2^{q\sigma} \|\Delta_q f\|_{L^p}.$$

Bounds for $2^{q\sigma} \|R_q^7\|_{L^p}$:

As $R_q^7 = \sum_{|q'-q| \leq 1} [\Delta_q, \Delta_{-1} v] \cdot \nabla \Delta_{q'} f$, the first order Taylor formula yields

$$(A.13) \quad 2^{q\sigma} \|R_q^7\|_{L^p} \leq 2^{|\sigma|} K \sum_{|q'-q| \leq 1} \|\nabla \Delta_{-1} v\|_{L^\infty} 2^{q'\sigma} \|\Delta_{q'} f\|_{L^p}.$$

Combining inequalities (A.3), (A.4), (A.7) or (A.8), (A.10), (A.11), (A.12), and (A.13), we end up with the desired estimate for R_q . ■

Remark A.1 *Straightforward modifications in the estimates for R_q^3, R_q^4, R_q^5 show that in the special case where $f = v$, the following estimate holds true:*

$$(A.14) \quad \begin{aligned} 2^{q\sigma} \|R_q\|_{L^p} &\lesssim \sum_{|q'-q| \leq 4} \|S_{q'+1} \nabla v\|_{L^\infty} 2^{q'\sigma} \|\Delta_{q'} v\|_{L^p} + \sum_{q' \geq q-3} 2^{q-q'} 2^{q\sigma} \|\Delta_{q'} v\|_{L^p} \|\Delta_{q'} \nabla v\|_{L^\infty} \\ &+ \sum_{\substack{q' \geq q-3 \\ |q'-q''| \leq 1}} 2^{(q-q')\sigma} \left(2^{q-q'} \|\Delta_{q'} \nabla v\|_{L^\infty} + \|\Delta_{q'} \operatorname{div} v\|_{L^\infty} \right) 2^{q''\sigma} \|\Delta_{q''} v\|_{L^p}. \end{aligned}$$

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