The bidual of a tensor product of Banach spaces

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Abstract

This paper studies the relationship between the bidual of the (projective) tensor product of Banach spaces and the tensor product of their biduals.

1. Definition of the main operator and organization of the paper

In this paper we investigate the relationship between the bidual of the (projective) tensor product of Banach spaces and the tensor product of their biduals.

First of all, let us show that given Banach spaces $X_1, \ldots, X_k$, there is an 'intertwining' linear operator $\alpha : X_1'' \widehat{\otimes} \cdots \widehat{\otimes} X_k'' \longrightarrow (X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k)''$. This mapping is 'natural', up to a permutation of $\{1, \ldots, k\}$.

For background on extension of multilinear operators we refer the reader to [2, 3, 10, 7]. Here, we only recall the Davie-Gamelin description of the so-called Aron-Berner extension method. Suppose

$$T : X_1 \times \cdots \times X_k \longrightarrow Z$$

is a multilinear operator acting between Banach spaces. Then we can extend T to a multilinear operator $\epsilon(T) : X_1'' \times \cdots \times X_k'' \longrightarrow Z''$ taking

$$(1.1) \qquad \epsilon(T)(x_1'', \ldots, x_k'') = w^* -\lim_{x_1 \to x_1''} \cdots \ w^* -\lim_{x_k \to x_k''} T(x_1, \ldots, x_k),$$

where the iterated limit is taken in the weak* topology of Z'', as $x_i \in X_i$ converges to x_i'' in the weak* topology of X_i''. It is easily seen that $\|\epsilon(T)\| = \|T\|$.

Note that we could have chosen any other ordering of the variables and we would have got another extension, in general different from the given in (1.1).

Now, fix Banach spaces $X_1, \ldots, X_k$, and consider the obvious multilinear operator $\otimes : X_1 \times \cdots \times X_k \longrightarrow X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k$ sending $(x_1, \ldots, x_k)$ into the elementary tensor $x_1 \otimes \cdots \otimes x_k$. Applying the Aron-Berner procedure, we obtain a multilinear operator $\epsilon(\otimes) : X_1'' \times \cdots \times X_k'' \longrightarrow (X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k)''$. By the universal property of the (projective) tensor product, there is a unique linear operator $\alpha : X_1'' \widehat{\otimes} \cdots \widehat{\otimes} X_k'' \longrightarrow (X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k)''$ which linearizes $\epsilon(\otimes)$, that is, such that

$$\alpha(x_1'' \otimes \cdots \otimes x_k'') = \epsilon(\otimes)(x_1'', \ldots, x_k'').$$

It is clear from (1.1) that

$$\alpha(x_1'' \otimes \cdots \otimes x_k'') = w^*\text{--}\lim_{x_1 \to x_1''} \cdots \; w^*\text{--}\lim_{x_k \to x_k''} (x_1 \otimes \cdots \otimes x_k),$$

and also that $\|\alpha\| = \|\epsilon(\otimes)\| = \|\otimes\| = 1$.

We do not know if, in general, α is one-to-one, let alone an isomorphic embedding. As often happens when dealing with tensor products, approximation properties will play a crucial rôle in our proofs.

We now explain the organization of the paper and summarize the main results. Section 2 contains some preparatory material on the approximation property. Mainly, that the bounded approximation property (BAP in short) is nicely stable by tensor products and a simple test for local complementation of subspaces having the BAP.

In Section 3 we prove our main result: if $X_1, \ldots, X_k$ are Banach spaces whose biduals have the BAP, then α embeds $X_1'' \widehat{\otimes} \cdots \widehat{\otimes} X_k''$ as a locally complemented subspace of $(X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k)''$.

An almost straightforward consequence is that if X'' has the BAP, then $\mathcal{L}(^k X'')$ is a complemented subspace of $\mathcal{L}(^k X)''$ for all $k \geq 1$.

In Section 4 we show that sometimes $\alpha : X'' \widehat{\otimes} X'' \longrightarrow (X \widehat{\otimes} X)''$ is still an isomorphic embedding even if X (hence X'') lacks the approximation property. Thus, for instance, the 'bidimensional' Varopoulos algebra $X'' \widehat{\otimes} X''$ is a closed subspace of $(X \widehat{\otimes} X)''$ for all C*-algebras X. This also applies to Pisier's creature having no uniformly complemented finite dimensional subspaces.

In Section 5 we present a more concrete application, namely that if K_1 and K_2 are infinite compact Hausdorff spaces, then $(C(K_1) \widehat{\otimes} C(K_2))''$ lacks the Dunford-Pettis property. In particular, the bidual of $c_0 \widehat{\otimes} c_0$ does not have it. Incidentally, this was the original motivation of our work.

The remainder of the paper presents some applications to holomorphic functions. First, we prove a 'symmetric' version of the main Theorem. Then,

by using Schauder decompositions, we obtain that if X a Banach space whose bidual has the BAP, then $\mathcal{H}_b(X'')$ is a complemented subspace of the strong bidual $\mathcal{H}_b(X)''$. The same result holds for holomorphic functions of bounded type on the ball of X.

2. BAP in tensor products and locally complemented subspaces

Definition 1 *A Banach space X is said to have the λ-approximation property (λ-AP, for short) if there is a net T_γ of finite rank operators, with $\|T_\gamma\| \leq \lambda$, such that $\lim_\gamma \|T_\gamma(x) - x\| = 0$ for all $x \in X$. A Banach space X having the λ-AP for some finite λ is said to have the BAP.*

The following two lemmata are surely well known. We include their proofs for the sake of completeness and to fix some notations.

Lemma 1 (BAP in tensor products) *If $X_1, \ldots, X_k$ have the BAP, then so does $X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k$.*

Proof. Since taking tensor products is associative it suffices to consider the case $k = 2$. Suppose that X and Y have the BAP, with constant λ_X and λ_Y, respectively. Let $\{L_\gamma : \gamma \in \Gamma\}$ and $\{R_\delta : \delta \in \Delta\}$ be the corresponding bounded nets of finite rank operators converging to the identity of X and Y, respectively. Consider the set $\Gamma \times \Delta$, with the product order and let $T_{(\gamma, \delta)} = L_\gamma \otimes R_\delta$. Clearly, $\|T_{(\gamma, \delta)}\| \leq \lambda_X \cdot \lambda_Y$. It remains to show that

$$(2.1) \qquad \lim_{(\gamma, \delta)} \|T_{(\gamma, \delta)}(u) - u\| = 0 \qquad ((\gamma, \delta) \in \Gamma \times \Delta, u \in X \widehat{\otimes} Y).$$

If $u = x \otimes y$, this follows from

$$\|x \otimes y - T_{(\gamma, \delta)}(x \otimes y)\| = \|x \otimes y - x \otimes R_\delta(y) + x \otimes R_\delta(y) - L_\gamma(x) \otimes R_\delta(y)\|$$
$$\leq \|x\| \cdot \|y - R_\delta(y)\| + \lambda_Y \cdot \|y\| \cdot \|x - L_\gamma(x)\|.$$

Hence (2.1) holds true for $u = \sum_{i=1}^n x_i \otimes y_i$. For arbitrary $u \in X \widehat{\otimes} Y$, fix $\varepsilon > 0$ and take x_i and y_i such that

$$\left\| u - \sum_{i=1}^n x_i \otimes y_i \right\| \leq \varepsilon.$$

Put $v = u - \sum_{i=1}^n x_i \otimes y_i$. One has

$$\|u - T_{(\gamma, \delta)}(u)\| \leq \|u - v\| + \|v - T_{(\gamma, \delta)}(v)\| + \|T_{(\gamma, \delta)}(u - v)\|$$
$$\leq (1 + \lambda_X \lambda_Y)\varepsilon + \|v - T_{(\gamma, \delta)}(v)\|,$$

and the result follows. ∎

Lemma 2 (BAP in duals) *Let X be a Banach space. If X' has the BAP, then the corresponding finite rank operators can be chosen weakly* continuous (without varying the constant).*

Proof. A moment of reflection shows that the λ-AP (of a Banach space X) is equivalent to the following condition:

(*) Given a finite subset F of X and $\varepsilon > 0$ there is a finite rank operator T such that $\|x - T(x)\| < \varepsilon$ for all $x \in F$, with $\|T\| \leq \lambda$.

Of course, the λ-AP implies (*). Conversely, if (*) holds, we can construct the required net of finite rank operators on the index set of all possible pairs (F, ε) directed as follows: $(F, \varepsilon) \leq (F', \varepsilon')$ if and only if $F \subset F'$ and $\varepsilon' \leq \varepsilon$.

Now, assume X' has the λ-AP. Fix a finite subset F of X' and $\varepsilon > 0$. The hypothesis implies the existence of a finite rank operator T on X' such that $\|x' - T(x')\| < \varepsilon$ for all $x' \in F$, with $\|T\| \leq \lambda$. Let E be the range of T and let $T' : E' \to X''$ be the adjoint operator. By Dean's identity $\mathcal{L}(E', X'') = \mathcal{L}(E', X)''$, there is a net of operators $t_\delta : E' \to X$, with $\|t_\delta\| \leq \|T'\| \leq \lambda$, such that t_δ converges to T' in the weak* topology of $\mathcal{L}(E', X)''$ –which is the weak* operator topology of $\mathcal{L}(E', X'')$. Finally, consider the adjoint operators $t'_\delta : X' \to E'' = E$ as finite rank operators on X'. By our choice of (t_δ) and taking into account that E is finite-dimensional, it follows that $t'_\delta(x')$ converges in norm to $T(x')$ for all $x' \in X'$. Hence

$$\|x' - t'_\delta(x')\| < \varepsilon \qquad (x' \in F)$$

for δ large enough. This completes the proof. $\blacksquare$

The meaning of Lemma 2 is that the operators T_γ which obviously admit a representation of the form

$$T_\gamma = \sum_{i=1}^{n} x''_i \otimes x'_i$$

where $x''_i \in X''$ and $x'_i \in X'$ can be replaced by operators of the form

$$\tilde{T}_\gamma = \sum_{i=1}^{n} x_i \otimes x'_i,$$

with $x_i \in X$.

Corollary 1 (BAP in biduals) *If X'' has the λ-AP, then there is a net of finite rank operators $t_\gamma : X \to X''$, with $\|t_\gamma\| \leq \lambda$, such that*

$$\lim_\gamma \|x'' - t''_\gamma(x'')\| = 0 \qquad \text{for all } x'' \in X''.$$

Definition 2 *A linear operator $\kappa : X \to Y$ admits a local left inverse of bound λ if, for each finite-dimensional subspace $E \subset Y$ and each $\varepsilon > 0$, there is an operator $T : E \to X$ such that $T(\kappa(x)) = x$ provided $\kappa(x)$ belongs to E, with $\|T\| \leq \lambda + \varepsilon$.*

It is clear that an operator $\kappa : X \to Y$ admitting a local left inverse is an isomorphic embedding: $\lambda_\kappa^{-1}\|x\| \leq \|\kappa(x)\| \leq \|\kappa\|\|x\|$. Thus we can regard X as a *locally complemented* subspace of Y. The Principle of Local Reflexivity of Lindenstrauss and Rosenthal [23] says that every Banach space is locally complemented in its bidual. Also, it is well-known that every Banach space is locally complemented in its ultrapowers.

Our immediate objective is the following isometric version of a folk result on locally complemented subspaces (see [21, theorem 3.5] or the first section of [20]).

Lemma 3 *Let $\kappa : X \to Y$ be a linear operator acting between Banach spaces. The following are equivalent:*

 (a) *κ has a local left-inverse of bound λ.*
 (b) *$\kappa' : Y' \to X'$ has a right-inverse of norm at most λ.*
 (c) *$\kappa'' : X'' \to Y''$ has a left-inverse of norm at most λ.*
 (d) *For each compact operator K from X into any Banach space Z, there is a compact operator $\tilde{K} : Y \to Z$ such that $K = \tilde{K} \circ \kappa$, with $\|\tilde{K}\| \leq \lambda \|K\|$.*

Proof. The implication (a) $\Rightarrow$ (b) easily follows from 'Lindenstrauss compactness argument', while (b) $\Rightarrow$ (c) is obvious and (c) $\Rightarrow$ (a) is a straightforward consequence of the Principle of Local Reflexivity.

We prove the implication (a) $\Rightarrow$ (d). Consider the set S of all pairs (F, ε), where F is a finite dimensional subspace of Y and $\varepsilon > 0$ directed by

$$(F, \varepsilon) \leq (F', \varepsilon') \Longleftrightarrow F \subset F' \text{ and } \varepsilon' \leq \varepsilon.$$

For each (F, ε), take a linear operator $T_F^\varepsilon : F \to X$ such that $T_F^\varepsilon(\kappa(x)) = x$ for all $x \in \kappa^{-1}(F)$, with $\|T_F^\varepsilon\| \leq \lambda + \varepsilon$. Now, fix any ultrafilter $\mathfrak{V}$ refining the Fréchet filter on S. Let $K : X \to Z$ be a compact operator. Define $\tilde{K} : Y \to Z$ by

$$\tilde{K}(y) = \lim_{\mathfrak{V}(F,\varepsilon)} K(T_F^\varepsilon(y)) \qquad (y \in Y).$$

The definition makes sense because every $y \in Y$ belongs eventually to F, the set $\{T_F^\varepsilon(y) : (F, \varepsilon) \in S, y \in F\}$ is bounded in X and K is compact.

On the other hand, the image under $\tilde{K}$ of the unit ball of Y is contained in the closure of the image under K of the ball of radius λ of X. This shows

at once that $\tilde{K}$ is compact and that $\|\tilde{K}\| \leq \lambda\|K\|$. It remains to see that $\tilde{K}$ extends K. Take $x \in X$. Then

$$\tilde{K}(\kappa(x)) = \lim_{\mathfrak{V}(F,\,\varepsilon)} K(T_F^\varepsilon(\kappa(x))) = K(x),$$

as desired.

Finally, we prove that (d) implies (b). Let Γ be the net of all finite dimensional subspaces of X' ordered by inclusion. For each $E \in \Gamma$, let $K_E : X \to E'$ be the preadjoint of the inclusion map $E \to X'$. The hypothesis yields $\tilde{K}_E : Y \to E'$ such that $\tilde{K}_E \circ \kappa = K_E$, with $\|\tilde{K}_E\| \leq \lambda$. Let $\mathfrak{V}$ be an ultrafilter (refining the Fréchet filter) on Γ and define $S : X' \to Y'$ by

$$S(x') = w^* - \lim_{\mathfrak{V}(E)} \tilde{K}_E'(x') \qquad (x' \in X')$$

The definition is correct since for each $x' \in X'$ one eventually has $x' \in E$. Obviously, $\|S\| \leq \lambda$. That S is a right inverse for κ' follows from

$$\langle \kappa \circ S(x'), x \rangle = \lim_{\mathfrak{V}(E)} \langle \tilde{K}_E'(x'), \kappa(x) \rangle = \lim_{\mathfrak{V}(E)} \langle x', \tilde{K}_E(\kappa(x)) \rangle = \langle x', x \rangle.$$

This completes the proof. ∎

The fourth condition in the preceding Lemma is often called the compact extension property (CEP for short). We now prove the following test for local complementation of subspaces having the BAP.

Lemma 4 *Let $\kappa : X \to Y$ be a linear operator, where X is a Banach space with the BAP given by the net T_γ. Suppose that for every γ there is $\tilde{T}_\gamma : Y \to X$ such that $\tilde{T}_\gamma \circ \kappa = T_\gamma$, with $\|\tilde{T}_\gamma\| \leq \lambda$. Then κ admits a local inverse with bound λ.*

Proof. We show that X has the CEP in Y with constant λ. Let K be a compact operator from Y into any Banach space Z. Choose any ultrafilter $\mathfrak{V}$ refining the Fréchet (=order) filter on Γ and define $\tilde{K} : Y \to Z$ by

$$\tilde{K}(y) = \lim_{\mathfrak{V}(\gamma)} K(\tilde{T}_\gamma(y)) \qquad (y \in Y).$$

The definition makes sense because K is compact and the set $\{\tilde{T}_\gamma(y) : \gamma \in \Gamma\}$ is contained in the ball of radius $\lambda\|y\|$ of X. This implies that $\tilde{K}$ is compact and also that $\|\tilde{K}\| \leq \lambda\|K\|$.

It remains to see that $\tilde{K} \circ \kappa = K$. Take $x \in X$. Then,

$$\tilde{K}(\kappa(x)) = \lim_{\mathfrak{V}(\gamma)} K(\tilde{T}_\gamma(x)) = \lim_{\mathfrak{V}(\gamma)} K(T_\gamma(x)) = K(x),$$

which completes the proof. ∎

3. The main result

We are now ready to prove the main result of the paper.

Theorem 1 *Let $X_1, \ldots, X_k$ be Banach spaces whose biduals have the BAP. Then α embeds $X_1'' \widehat{\otimes} \cdots \widehat{\otimes} X_k''$ as a locally complemented subspace of*

$$(X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_k)''.$$

Proof. We write the proof for two Banach spaces X and Y. The verification of the general case is left to the reader. Suppose X'' and Y'' have the BAP, with constants $\lambda_{X''}$ and $\lambda_{Y''}$, respectively. By Corollary 1, there exist nets of finite rank operators $l_\gamma : X \to X''$ and $r_\delta : Y \to Y''$, with $\|l_\gamma\| \leq \lambda_{X''}$ and $\|r_\delta\| \leq \lambda_{Y''}$ in such a way that

$$\lim_\gamma \|x'' - l_\gamma''(x'')\| = 0 \qquad (x'' \in X'')$$

$$\lim_\delta \|y'' - r_\delta''(y'')\| = 0 \qquad (y'' \in Y'').$$

By Lemma 1, the net $T_{(\gamma, \delta)} = l_\gamma'' \otimes r_\delta''$ transfers the BAP to $X'' \widehat{\otimes} Y''$, with constant $\lambda_{X''} \lambda_{Y''}$. In view of Lemma 4, the proof will be complete if we show that for each (γ, δ) there is a an operator $\widetilde{T}_{(\gamma, \delta)}$ making commute the diagram

$$X'' \widehat{\otimes} Y'' \quad \xrightarrow{\ \alpha\ } \quad (X \widehat{\otimes} Y)''$$

(3.1) $$T_{(\gamma, \delta)} \searrow \qquad \nearrow \widetilde{T}_{(\gamma, \delta)}$$

$$X'' \widehat{\otimes} Y''$$

with $\|\widetilde{T}_{(\gamma, \delta)}\|$ uniformly bounded.

Fix (γ, δ) and consider the linear operator

$$l_\gamma \otimes r_\delta : X \widehat{\otimes} Y \longrightarrow X'' \widehat{\otimes} Y''$$

and the bitranspose map

$$(r_\gamma \otimes l_\delta)'' : (X \widehat{\otimes} Y)'' \longrightarrow X'' \widehat{\otimes} Y''$$

(recall that $l_\gamma \otimes r_\delta$ has finite dimensional range). Set

$$\widetilde{T}_{(\gamma, \delta)} = (l_\gamma \otimes r_\delta)''.$$

Obviously, $\|\widetilde{T}_{(\gamma, \delta)}\| = \|l_\gamma\| \cdot \|r_\delta\| \leq \lambda_{X''} \lambda_{Y''}$. We end the proof by showing that (3.1) commutes. It suffices to see that

$$(l_\gamma \otimes r_\delta)''(\alpha(x'' \otimes y'')) = l_\gamma''(x'') \otimes r_\delta''(y'')$$

holds for all $x'' \in X'', y'' \in Y''$.

One has (all limits are taken as $y \in Y$ converges to y'' in the weak*
topology of Y'' and $x \in X$ converges to x'' in the weak* topology of X''):

$$
\begin{aligned}
\tilde{T}_{(\gamma,\delta)}(\alpha(x'' \otimes y'')) &= (l_\gamma \otimes r_\delta)''(\alpha(x'' \otimes y'')) \\
&= (l_\gamma \otimes r_\delta)''\Big(w^*\!-\!\lim_{x \to x''} \big(w^*\!-\!\lim_{y \to y''} x \otimes y\big)\Big) \\
&= \lim_{x \to x''} \Big((l_\gamma \otimes r_\delta)''\big(w^*\!-\!\lim_{y \to y''} x \otimes y\big)\Big) \\
&= \lim_{x \to x''} \Big(\lim_{y \to y''} (l_\gamma \otimes r_\delta)''(x \otimes y)\Big) \\
&= \lim_{x \to x''} \Big(\lim_{y \to y''} (l_\gamma \otimes r_\delta)(x \otimes y)\Big) \\
&= \lim_{x \to x''} \Big(\lim_{y \to y''} l_\gamma(x) \otimes r_\delta(y)\Big) \\
&= \lim_{x \to x''} l_\gamma(x) \otimes r_\delta''(y'') \\
&= l_\gamma''(x'') \otimes r_\delta''(y'') \\
&= T_{(\gamma,\delta)}(x'' \otimes y'').
\end{aligned}
$$

Which completes the proof. ∎

The proof shows that the local complementation constant of $X''\widehat{\otimes}Y''$ in
$(X\widehat{\otimes}Y)''$ is at most $\lambda_{X''}\lambda_{Y''}$. For an arbitrary number of Banach spaces
$X_1, \ldots, X_k$, the local complementation constant of $X_1''\widehat{\otimes}\cdots\widehat{\otimes}X_k''$ in
$(X_1\widehat{\otimes}\cdots\widehat{\otimes}X_k)''$ is at most $\lambda_{X_1''}\cdots\lambda_{X_k''}$. In particular, if each X_i'' has the
metric approximation property (that is, the 1-AP; MAP in short), then α
is an isometry and $X_1''\widehat{\otimes}\cdots\widehat{\otimes}X_k''$ is a locally 1-complemented subspace of
$(X_1\widehat{\otimes}\cdots\widehat{\otimes}X_k)''$.

An almost straightforward consequence is the following.

Corollary 2 *If X'' has the BAP, then $\mathcal{L}(^kX'')$ is a complemented subspace
of $\mathcal{L}(^kX)''$ for all $k \geq 1$.*

Proof. Since the space of multilinear forms on $X_1 \times \cdots \times X_k$ is naturally
isomorphic to $(X_1\widehat{\otimes}\cdots\widehat{\otimes}X_k)'$ this obviously follows from Theorem 1 and
Lemma 3. ∎

Notice that if X'' has the MAP, then $\alpha' : \mathcal{L}(^kX)'' \to \mathcal{L}(^kX'')$ is an
isometric quotient map and admits a right inverse of norm 1. In this case
$\mathcal{L}(^kX'')$ is isometric to a 1-complemented subspace of $\mathcal{L}(^kX)''$.

There are some connections between α, injective tensors and the Borel
transform. Given two Banach spaces X and Y, we write $X\check{\otimes}Y$ for the
injective tensor product of X and Y, that is, the completion of the algebraic
tensor product $X \otimes Y$ under the least reasonable crossnorm [12].

There is a natural map $a : X'' \widehat{\otimes} Y'' \to (X' \check{\otimes} Y')'$ which identifies elements of $X'' \widehat{\otimes} Y''$ with integral (actually nuclear) bilinear forms on $X' \times Y'$. Approximation properties can be easily described by means of a: X'' has the AP (respectively, the BAP and the MAP) if and only if, for all Banach spaces Y, the map a is injective (respectively, an isomorphic embedding and an isometry). On the other hand there is an obvious operator $b : X' \check{\otimes} Y' \to (X \widehat{\otimes} Y)'$ (which is always an isometric embedding) whose adjoint $b' : (X \widehat{\otimes} Y)'' \to (X' \check{\otimes} Y')'$ is often called the Borel transform.

It is easily seen that the following diagram commutes

(3.2)
$$
\begin{array}{ccc}
X'' \widehat{\otimes} Y'' & \overset{\alpha}{\longrightarrow} & (X \widehat{\otimes} Y)'' \\
 & a \searrow \quad \swarrow b' & \\
 & (X' \check{\otimes} Y')' &
\end{array}
$$

Thus, we have proved the following:

Proposition 1 *If X'' has the BAP, then $\alpha : X'' \widehat{\otimes} Y'' \to (X \widehat{\otimes} Y)''$ is an isomorphic embedding for all Banach spaces Y.*

A more general result shall be proved in the next Section.

To some extent, this paper is the "predual" of Jaramillo, Prieto and Zalduendo's [19]. There, it is proved that if X'' has the BAP, then there exists a linear surjection $\beta : \mathcal{L}(^k X)'' \to \mathcal{L}(^k X'')$ for all $k \geq 1$. It is not hard to verify that β is (essentially) the adjoint of "our" α. Hence, β is not only a surjection, but a projection (provided X'' has the BAP). As noted in [19, theorem 4], there exist (even non-reflexive) Banach spaces X for which the map β is an isomorphism for all k. This is easier to verify for α:

Corollary 3 (Jaramillo-Prieto-Zalduendo) *Suppose X'' has the approximation property and the Radon-Nikodým property. Then*

$$
\alpha : (\widehat{\otimes}^k X'') \longrightarrow (\widehat{\otimes}^k X)''
$$

is an isomorphism if and only if every multilinear form on X^k is nuclear.

Proof. Consider the diagram

(3.3)
$$
\begin{array}{ccc}
(\widehat{\otimes}^k X'') & \overset{\alpha}{\longrightarrow} & (\widehat{\otimes}^k X)'' \\
 & a \searrow \quad \swarrow b' & \\
 & (\check{\otimes}^k X')' &
\end{array}
$$

The approximation property of X'' means that a is injective. The RNP implies that a is surjective [12]. The hypothesis about the multilinear forms on X^k means that $\mathcal{L}(^k X) = (\check{\otimes}^k X')$ through the preadjoint of the Borel transform. $\blacksquare$

One space X satisfying the three relevant conditions of the above result is the original Tsirelson space T^*. The non-reflexive example is the Tsirelson-James type space presented in [4]. See [1, 17] and [13, chapter 2, section 4] for details.

4. Banach spaces failing the AP

In this Section we study the map $\alpha : X''\widehat{\otimes}Y'' \to (X\widehat{\otimes}Y)''$ for some spaces without the AP. Roughly speaking, a Banach space X has a certain approximation property if the tensor product $X\widehat{\otimes}Y$ is nice (in a certain sense) for all Banach spaces Y, or equivalently, for $Y = X'$. Obviously, $X\widehat{\otimes}Y$ may be nice for some choices of Y, even if $X\widehat{\otimes}X'$ is bad. An illustrative example is provided by Theorem 2 below.

Lemma 5 *Let X and Y be Banach spaces. Suppose there are constants C and λ such that, for every operator $T : X \to Y'$ there is a Banach space Z having the λ-AP and a factorization $T = R \circ L$, where $L : X \to Z$ and $R : Z \to Y'$ satisfy $\|R\| \cdot \|L\| \leq C\|T\|$. Then, for each $u \in X\widehat{\otimes}Y$, one has*

$$\|u\| \leq C\lambda \sup_{\|B\|\leq 1} |B(u)|,$$

where B runs over the bilinear forms of finite type on $X \times Y$.

Proof. We may and do assume that $u = \sum_{i=1}^{n} x_i \otimes y_i$. Let B be a norm-one bilinear form on $X \times Y$ such that $|B(u)| = \|u\|$ and let $B_1 : X \to Y'$ be the (first) associated linear operator. Take operators $L : X \to Z$ and $R : Z \to Y'$ such that $B_1 = R \circ L$, with $\|R\|\|L\| \leq C$.

Now, fix $\varepsilon > 0$ and use the λ-AP of Z to get a finite rank $F : Z \to Z$ such that $\|F\| \leq \lambda$ and

$$\|F(L(x_i)) - L(x_i)\| < \varepsilon \qquad (1 \leq i \leq n).$$

Let $\tilde{B}_1 = R \circ F \circ L$. It is clear that $\tilde{B}_1$ is a finite rank operator of norm at most $C\lambda$. Hence the associated bilinear form $\tilde{B}$ is of finite type, and $\|\tilde{B}\| \leq C\lambda$. One has,

$$|B(u) - \tilde{B}(u)| = \left| B\left(\sum_{i=1}^{n} x_i \otimes y_i \right) - \tilde{B}\left(\sum_{i=1}^{n} x_i \otimes y_i \right) \right|$$

$$\leq \sum_{i=1}^{n} |B(x_i \otimes y_i) - \tilde{B}(x_i \otimes y_i)| = \sum_{i=1}^{n} |\langle R(L(x_i)) - R(F(L(x_i))), y_i\rangle|$$

$$\leq \sum_{i=1}^{n} \|R\|\|L(x_i) - F(L(x_i))\|\|y_i\| \leq \left(\sum_{i=1}^{n} \|R\|\|y_i\| \right) \varepsilon.$$

Since ε was arbitrary the proof is complete. ∎

Lemma 6 *Let X and Y be Banach spaces. Then for each $u \in X''\widehat{\otimes}Y''$ the norm of u as a linear functional on $X'''\check{\otimes}Y'''$ coincides with the norm of u regarded as a functional on $X'\check{\otimes}Y'$.*

Proof. Clearly,

$$(4.1) \qquad \|u : X'''\check{\otimes}Y''' \to \mathbb{K}\| = \sup_{\|B\| \leq 1} |B(u)|,$$

where B runs over all bilinear forms of finite type on $X'' \times Y''$, that is,

$$B = \sum_{k=1}^{m} f_k''' \otimes g_k''' \qquad (f_k''' \in X''', \ g_k''' \in Y''').$$

Similarly,

$$(4.2) \qquad \|u : X'\check{\otimes}Y' \to \mathbb{K}\| = \sup_{\|b\| \leq 1} |b(u)|,$$

where b runs over all bilinear forms of finite type on $X'' \times Y''$ which are separately weakly* continuous, that is, of the form

$$b = \sum_{k=1}^{m} f_k' \otimes g_k' \qquad (f_k' \in X', \ g_k' \in Y').$$

Of course $\|u : X'''\check{\otimes}Y''' \to \mathbb{K}\| \geq \|u : X'\check{\otimes}Y' \to \mathbb{K}\|$. To prove the reversed inequality we show that for each $u \in X''\widehat{\otimes}Y''$ and each B of finite type one has

$$|B(u)| \leq \sup_{\|b\| \leq \|B\|} |b(u)|,$$

where b runs over the bilinear forms of finite type which are separately weakly* continuous on $X'' \times Y''$.

Again, we may assume

$$u = \sum_{i=1}^{n} x_i'' \otimes y_i''.$$

Let $B : X'' \times Y'' \to \mathbb{K}$ be a bilinear form of finite type, and let $B_1 : X'' \to Y'''$ the associated finite rank operator. Reasoning as in the proof of Lemma 2 one sees that for every $\varepsilon > 0$ there is a weakly* continuous operator $\tilde{B}_1 : X'' \to Y'''$ such that

$$\|B_1(x_i'') - \tilde{B}_1(x_i'')\| < \varepsilon \qquad (1 \leq i \leq n),$$

with $\|\tilde{B}_1\| \leq \|B_1\|$.

Applying the argument given at the end of the proof of Lemma 5 we see that

$$|B(u)| \leq \sup_{\|\tilde{B}\| \leq \|B\|} |\tilde{B}(u)|,$$

where $\tilde{B}$ runs over the bilinear forms of finite type on $X'' \times Y''$ which are weakly* continuous on the first variable.

We complete the proof by showing that for each $\tilde{B}$ as above, one has

(4.3)
$$|\tilde{B}(u)| \leq \sup_{\|b\| \leq \|\tilde{B}\|} |b(u)|,$$

where b is as in (4.2). Let $\tilde{B}_2 : Y'' \to X'''$ be the second operator associated to $\tilde{B}$. Note that $\tilde{B}_2$ takes values in X' –instead of X'''. Again, for each $\varepsilon > 0$ there is a weakly* continuous operator $b_2 : Y'' \to X'$ such that $\|\tilde{B}_2(y_i'') - b_2(y_i'')\| < \varepsilon$, with $\|b_2\| \leq \|\tilde{B}_2\|$. This proves (4.3) and completes the proof. ∎

Theorem 2 *Let X and Y be Banach spaces. Suppose there are constants C and λ such that, for every operator $T : X'' \to Y'''$ there is a Banach space Z having the λ-AP and a factorization $T = R \circ L$, where $L : X'' \to Z$ and $R : Z \to Y'''$ satisfy $\|R\| \cdot \|L\| \leq C\|T\|$. Then $\alpha : X'' \widehat{\otimes} Y'' \to (X \widehat{\otimes} Y)''$ is an isomorphic embedding.*

Proof. In view of the diagram (3.2), it suffices to show that $a : X'' \widehat{\otimes} Y'' \to (X' \check{\otimes} Y')'$ is an isomorphic embedding (note that $\|b'\| = \|b\| = 1$). Since

$$\|a(u)\| \overset{\text{def}}{=} \|u : X' \check{\otimes} Y' \to \mathbb{K}\| = \|u : X''' \check{\otimes} Y''' \to \mathbb{K}\|$$

it follows by Lemma 5 that $\|u\| \leq C\lambda \|a(u)\|$. ∎

Corollary 4 *If X has type 2 and X' has cotype 2 then $\alpha : X'' \widehat{\otimes} X'' \longrightarrow (X \widehat{\otimes} X)''$ is an isomorphic embedding.*

Proof. It follows from a result of Pisier [25] that the hypothesis of Theorem 2 can be satisfied with Z a suitable Hilbert space. ∎

Thus, for instance, $X'' \widehat{\otimes} X''$ is a subspace of $(X \widehat{\otimes} X)''$ if X is a C*-algebra, in particular if X is either $\mathcal{K}(\mathcal{H})$ or $\mathcal{L}(\mathcal{H})$ or if X is Pisier's space [24] having no uniformly complemented finite dimensional subspaces. Note that Pisier space and $\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H})''$ both fail the AP [26]. We do not know, however, if in these cases, $X'' \widehat{\otimes} X''$ is locally complemented in $(X \widehat{\otimes} X)''$. It is unclear to us if the results in this Section can be extended for more than two Banach spaces.

5. Applications to the Dunford-Pettis property

Let us recall that a Banach space has the Dunford-Pettis property (DPP) if all weakly compact operators defined on it are completely continuous.

Lemma 7 *Weakly compact operators can be extended from locally complemented subspaces.*

Proof. Suppose X is locally complemented in Y and let $W : X \to Z$ be a weakly compact operator. By an old result of Gantmacher, W'' takes values in Z. Since X'' is complemented in Y'' we can extend W'' to a weakly compact operator $Y'' \to Z$ whose restriction to Y is the required extension. (A direct proof follows replacing 'compact' by 'weakly compact' everywhere in the proof of the implication (a) $\Rightarrow$ (d) in Lemma 3.) ∎

Corollary 5 *A Banach space has the DPP if and only if every locally complemented subspace have it.*

Corollary 6 (Independently obtained by González and Gutiérrez; see [16]) *Let K_1 and K_2 be infinite compact Hausdorff spaces. Then*

$$(C(K_1)\widehat{\otimes}C(K_2))''$$

lacks the DPP. In particular, the bidual of $c_0\widehat{\otimes}c_0$ does not have the DPP.

Proof. In view of the result just proved, this follows from Theorem 1 and the fact (proved by Bombal and Villanueva in [5]) that $C(K_1)\widehat{\otimes}C(K_2)$ lacks the DPP unless both K_1 and K_2 are scattered. ∎

The present authors and Castillo proved in [6] that if X_i and Y_i are Banach spaces such that X_i' is isomorphic to Y_i', then $(X_1\widehat{\otimes}\cdots\widehat{\otimes}X_k)'$ is isomorphic to $(Y_1\widehat{\otimes}\cdots\widehat{\otimes}Y_k)'$. It follows that if X and Y are (infinite dimensional) $\mathcal{L}_\infty$-spaces, then

$$(X\widehat{\otimes}Y)''$$

lacks the DPP.

To the best of our knowledge, $c_0\widehat{\otimes}c_0$ is the "second" Banach space having the DPP and whose bidual lacks it. The first counterexample was $c_0(\ell_2^n)$. It has the DPP because its dual $\ell_1(\ell_2^n)$ has the Schur property. The bidual $\ell_\infty(\ell_2^n)$ lacks the DPP because it contains a complemented subspace isomorphic to ℓ_2 –this was first observed by Stegall who give a rather involved proof; a simpler proof, following [9, Example 4], appears in [6]. We do not know whether or not $(c_0\widehat{\otimes}c_0)''$ contains a reflexive complemented subspace.

6. Symmetrization

This Section is an adaptation of the first one to the symmetric case. Recall that a k-linear operator $T : X \times \overset{k}{\dots} \times X \to Z$ is called symmetric provided $T(x_1, \dots, x_k) = T(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ for every permutation of $\{1, \dots, k\}$ and all $x_i \in X$.

The arguments given in Section 1 cannot be used straightforwardly in the symmetric case because Aron-Berner extensions need not preserve symmetry. This is due to the choice of an specific ordering in the iterated limit (from the last variable to the first). For a multilinear operator $T : X^k \to Z$, and σ in the symmetric group S_k, define $T_\sigma : X^k \to Z$ by

$$T_\sigma(x_1, \dots, x_k) = T(x_{\sigma(1)}, \dots, x_{\sigma(k)}).$$

and $\epsilon^\sigma(T) : (X'')^k \to Z''$ by $\epsilon(T_\sigma)_{\sigma^{-1}}$.

This is the Aron-Berner extension of T, but changing the usual order in the iterated limit by the new ordering given by σ^{-1}; see [7, section 4]. Finally, put

$$\epsilon_s(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon^\sigma(T)$$

Since $\epsilon_s(T)$ is symmetric whenever T is, the problem of symmetry is fixed. Now, we can define another operator

$$\alpha_s : (\widehat{\otimes}^k X'') \longrightarrow (\widehat{\otimes}^k X)''$$

taking $\alpha_s(x_1'' \otimes \cdots \otimes x_k'') = \epsilon_s(\otimes)(x_1'', \dots, x_k'')$.

Note that α_s can be regarded as an averaging of maps $\alpha^\sigma = \epsilon^\sigma(\otimes)$, where α corresponds to the choice of σ as the identity of S_k. Moreover, if X'' has the λ-AP, then there is a net of finite rank operators $t_\gamma : X \to X''$, with $\|t_\gamma\| \leq \lambda$ such that $t_\gamma'' \otimes \overset{k}{\dots} \otimes t_\gamma''$ converges to the identity in the strong operator topology of $\mathcal{L}((\widehat{\otimes}^k X''))$. It is easily seen that

$$t_\gamma'' \otimes \overset{k}{\dots} \otimes t_\gamma'' = (t_\gamma \otimes \overset{k}{\dots} \otimes t_\gamma)'' \circ \alpha^\sigma$$

for all $\sigma \in S_k$. Hence,

$$t_\gamma'' \otimes \overset{k}{\dots} \otimes t_\gamma'' = (t_\gamma \otimes \overset{k}{\dots} \otimes t_\gamma)'' \circ \alpha_s.$$

This proves the following:

Theorem 3 *If X'' has the λ-AP, then α_s embeds $(\widehat{\otimes}^k X'')$ as a λ^k-locally complemented subspace of $(\widehat{\otimes}^k X)''$.*

We need the notion of a symmetric tensor product (see [14]). Let X be a Banach space and $\widehat{\otimes}^k X$ its k-fold tensor product. Put

$$\Sigma(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \quad .$$

It is easily seen that Σ defines a contractive projection on $\widehat{\otimes}^k X$: computations are even easier looking at Σ' on $\mathcal{L}(^k X)$.

The range of Σ is, by definition, the k-fold symmetric tensor product of X and it is denoted by $X \widehat{\otimes}_s \overset{k}{\cdots} \widehat{\otimes}_s X$ or $\widehat{\otimes}^k_s X$. We emphasize that the symmetric tensor product is a quotient of the (full) tensor product rather than a subspace (unless you prefer to think the symmetric forms as a quotient of all multilinear forms!). In fact, the symmetric tensor product should be defined as the symmetric k-linear operator

$$X^k \xrightarrow{\ \otimes\ } \widehat{\otimes}^k X \xrightarrow{\ \Sigma\ } \widehat{\otimes}^k_s X.$$

It is pretty obvious that this construction has the following universal property: for every symmetric multilinear operator S from X^k into any Banach space Z there is a unique linear operator $L : \widehat{\otimes}^k_s X \longrightarrow Z$ such that $S = L \circ \Sigma \circ \otimes$, with $\|L\| = \|S\|$.

In particular the dual of $\widehat{\otimes}^k_s X$ is naturally isometric to $\mathcal{L}_s(^k X)$, the space of symmetric k-linear forms on X^k.

Now, consider the composition of α_s with Σ''. This clearly induces a symmetric multilinear map from $(X'')^k$ into $(\widehat{\otimes}^k_s X)''$. By the universal property, there is a norm-one linear map $(\widehat{\otimes}^k_s X'') \longrightarrow (\widehat{\otimes}^k_s X)''$ making commute the following diagram:

$$
\begin{array}{ccc}
\widehat{\otimes}^k X'' & \xrightarrow{\ \alpha_s\ } & (\widehat{\otimes}^k X)'' \\
\Sigma \downarrow & & \downarrow \Sigma'' \\
(\widehat{\otimes}^k_s X'') & \longrightarrow & (\widehat{\otimes}^k_s X)''
\end{array}
$$

Actually, it is not hard to see that the lower map is nothing but the restriction of α_s to the range of Σ. Since "being locally complemented" is a transitive property, we obtain the following.

Corollary 7 *If X'' has the λ-AP, then α_s embeds $(\widehat{\otimes}^k_s X'')$ as a λ^k-locally complemented subspace of $(\widehat{\otimes}^k_s X)''$.*

Corollary 8 *If X'' has the λ-AP, then $\alpha'_s : \mathcal{L}_s(^k X)'' \longrightarrow \mathcal{L}_s(^k X'')$ admits a linear section of norm at most λ^k.*

7. Applications to holomorphic functions

Let U be a balanced open subset of a Banach space X. A U-bounded set is a bounded subset of U whose distance to the boundary of U is strictly positive. If $U = X$, then U-bounded sets are simply bounded sets. We denote by $\mathcal{H}_b(U)$ the Fréchet space of all holomorphic functions on U which are bounded on all U-bounded sets, with the topology τ_b of uniform convergence on U-bounded sets.

Let f be a holomorphic function on U. The Taylor series

$$f(x) \sim \sum_{k=0}^{\infty} \frac{d^k f(0)}{k!}(x, .\overset{k}{.}., x) \qquad (d^k f(0) \in \mathcal{L}_s(^k X))$$

decomposes f as a formal sum $\sum_k T_k$, where $T_k \in \mathcal{L}_s(^k X)$.

We are identifying each symmetric form on X^k with the associated k-homogeneous polynomial $\hat{T}$ given by

$$\hat{T}(x) = T(x, .\overset{r}{.}., x).$$

It is well-known that $(\mathcal{L}_s(^k X))_k$ is then a Schauder decomposition of $\mathcal{H}_b(U)$ for arbitrary open balanced U. Actually this decomposition turns out to be a very special one. Let us recall from [15] that a sequence of Banach subspaces (E_k) of a locally convex space E is said to be an R-decomposition $(0 < R \leq \infty)$ of E if (in addition of being a Schauder decomposition) it satisfies that, given $x_k \in E_k$, the series $\sum_k x_k$ converges in E if and only if

$$\limsup_{k \to \infty} \|x_n\|^{1/k} \leq \frac{1}{R}.$$

For instance, $(\mathcal{L}_s(^k X))_k$ is a 1-decomposition of $\mathcal{H}_b(U)$ if U is the unit ball of X; while it is a ∞-decomposition for $\mathcal{H}_b(X)$. Another remarkable result in [15] is that if (E_k) is an R-decomposition of E, then (E_k'') is an R-decomposition of the strong bidual E''.

We are now ready to prove the main result of the Section. We denote by U'' the norm-interior of the weak*-closure of U in X''. It is clear that U'' is the (norm) interior of the bipolar of U. In particular, if U is the open unit ball of X then U'' is open unit ball of X''.

Theorem 4 *Suppose X'' has the BAP. Then $\mathcal{H}_b(U'')$ is a complemented subspace of the strong bidual $\mathcal{H}_b(U)''$, where U is either a ball of X or X itself.*

Proof. We write the proof for $U = X$. The proof for balls is similar (cf. [15, theorem 9]).

Since $\mathcal{L}_s(^kX)''$ (respectively, $\mathcal{L}_s(^kX'')$) is a ∞-decomposition of $\mathcal{H}_b(X)''$ (respectively of $\mathcal{H}_b(X'')$), we can define a mapping $\eta : \mathcal{H}_b(X)'' \to \mathcal{H}_b(X'')$ taking

$$\eta\left(\sum_{k=0}^{\infty} T_k\right) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \alpha_s'(T_k),$$

where

$$\sum_{k=0}^{\infty} T_k \in \mathcal{H}_b(X)'' \quad \text{with} \quad T_k \in \mathcal{L}_s(^kX)''.$$

Clearly,

$$\left\|\alpha_s' : \mathcal{L}_s(^kX)'' \to \mathcal{L}_s(^kX'')\right\| = 1 \quad \text{for all } k.$$

It follows that η is continuous. To end, we show that η admits a continuous right inverse $\psi : \mathcal{H}_b(X'') \to \mathcal{H}_b(X)''$. For each k, let $\psi_k : \mathcal{L}_s(^kX)'' \to \mathcal{L}_s(^kX'')$ be a right inverse for $\alpha_s' : \mathcal{L}_s(^kX'') \to \mathcal{L}_s(^kX)''$ of norm at most λ^k, where λ is the AP constant of X''.

Finally, set

$$\psi(f) = \sum_{k=0}^{\infty} \psi_k(T_k),$$

where

$$f = \sum_{k=0}^{\infty} T_k$$

is the unique decomposition of f with $T_k \in \mathcal{L}_s(^kX'')$. By [15, theorem 9], ψ is continuous and, obviously, $\eta \circ \psi$ is the identity on $\mathcal{H}_b(X'')$, which completes the proof. ∎

We do not know whether or not Theorem 4 is true for every balanced open U. Also it would interesting what happens for other spaces of analytic functions, for instance the space $\mathcal{H}^\infty(B)$, where B is the closed unit ball of X.

Note added in proof. The part of Corollary 4 concerning C*-algebras was obtained earlier in [22]. See Theorem 5.1 in that paper, taking into account that the projective tensor product is denoted $\otimes_\gamma$ there.

Very recently J. Gutiérrez solved the problem stated at the end of Section 5 by showing that $(c_0 \widehat{\otimes} c_0)''$ contains a complemented infinite-dimensional Hilbert subspace [18]. Actually, even $c_0 \widehat{\otimes} \ell_\infty$ contains such a copy. See [8], where it is shown that $c_0 \widehat{\otimes} c_0$ contains a complemented copy of Stegall's counterexample $c_0(\ell_2^n)$.

References

[1] Alencar, R., Aron, R. and Dineen, S.: A reflexive space of holomorphic functions in infinitely many variables. *Proc. Amer. Math. Soc.* **90** (1984), 407–411.

[2] Arens, R.: The adjoint of a bilinear operation. *Proc. Amer. Math. Soc.* **2** (1951), 839–848.

[3] Aron, R. and Berner, P.: A Hahn-Banach extension theorem for analytic mappings. *Bull. Soc. Math. France* **106** (1978), 3–24.

[4] Aron, R. and Dineen, S.: Q-reflexive Banach spaces. *Rocky Mountain J. Math.* **27** (1997), 1009–1025.

[5] Bombal, F. and Villanueva, I.: On the Dunford-Pettis property of tensor product of $C(K)$ spaces. *Proc. Amer. Math. Soc.* **129** (2001), 1359–1363.

[6] Cabello Sánchez, F., Castillo, J. M. F. and García, R.: Polynomials on dual-isomorphic spaces. *Ark. Mat.* **38** (2000), 37–44.

[7] Cabello Sánchez, F. , García, R. and Villanueva, I.: Extension of multilinear operators on Banach space. *Extracta Math.* **15** (2000), 291–334.

[8] Cabello Sánchez, F., Pérez-García, D. and Villanueva, I.: Unexpected subspaces of tensor products, submitted.

[9] Castillo, J. M. F. and González, M.: On the Dunford-Pettis Property in Banach Spaces. *Acta Univ. Carolin. Math. Phys.* **35** (1994), 5–12.

[10] Davie, A. and Gamelin, T. W.: A theorem on polynomial-star approximation. *Proc. Amer. Math. Soc.* **106** (1989), 351–356.

[11] Dean, D.: The equation $L(E, X^{**}) = L(E, X)^{**}$ and the principle of local reflexivity. *Proc. Amer. Math. Soc.* **40** (1973), 146–148.

[12] Diestel, J. and Uhl, J. J. Jr.: *Vector measures.* Mathematical Surveys **15**. Americal Mathematical Society, Providence, R.I., 1977.

[13] Dineen, S.: *Complex analysis on infinite dimensional spaces.* Springer Monographs in Mathematics. Springer-Verlag, London, 1999.

[14] Floret, K.: Natural norms on symmetric tensor products of normed spaces. In: *Proceedings of the Second International Workshop on Functional Analysis at Trier University (1997).* Note Mat. **17** (1997), 153–188.

[15] Galindo, P., Maestre, M. and Rueda, P.: Biduality in spaces of holomorphic functions. *Math. Scand.* **86** (2000), 5–16.

[16] González, M. and Gutiérrez, J. M.: The Dunford-Pettis property on tensor products. *Math. Proc. Cambridge Philos. Soc.* **131** (2001), 185–192.

[17] Gonzalo, R. N.: *Suavidad y polinomios en espacios de Banach.* Ph.D. Thesis. Universidad Complutense, Madrid, 1994.

[18] Gutiérrez, J.: Complemented copies in spaces of integral operators. *Glasgow Math. J.* **47** (2005), 287–290.

[19] Jaramillo, J., Prieto, A. and Zalduendo, I.: The bidual of the space of polynomials on a Banach space. *Math. Proc. Cambridge Philos. Soc.* **122** (1997), 457–471.

[20] Johnson, W. B.: Extensions of c_0. *Positivity* **1** (1997), 55–74.

[21] Kalton, N. J.: Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$. *Math. Nachr.* **115** (1984), 71–97.

[22] Kumar, A. and Sinclair, A. M.: Equivalence of norms on operator space tensor products of C*-algebras. *Trans. Amer. Math. Soc.* **350** (1998), 2033–2048.

[23] Lindenstrauss, J. and Rosenthal, H. P.: The $\mathcal{L}_p$-spaces. *Israel J. Math.* **7** (1969), 325–349.

[24] Pisier, G.: Counterexamples to a conjecture of Grothendieck. *Acta Math.* **151** (1983), 181–208.

[25] Pisier, G.: Factorization of operators through $L_{p\infty}$ or L_{p1} and noncommutative generalizations. *Math. Ann.* **276** (1986), 105–136.

[26] Szankowski, A.: $\mathcal{B}(\mathcal{H})$ does not have the approximation property. *Acta Math.* **147** (1981), 89–108.

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