$(n, 2)$-sets have full Hausdorff dimension

Themis Mitsis

Abstract

We prove that a set containing translates of every 2-plane must have full Hausdorff dimension.

1. Introduction

This is a continuation of [4] where a partial result on the problem under investigation was obtained. Since that paper is unpublished work, we will reproduce certain parts of it for the sake of completeness.

An $(n, 2)$-set in $\mathbb{R}^n$ is a subset $E \subset \mathbb{R}^n$ containing a translate of every 2-dimensional plane.

The natural question that arises is whether E must have positive Lebesgue measure. This turns out to be true in low dimensions. Marstrand [3] proved that $(3, 2)$-sets have positive measure. Bourgain [1] showed the same for $(4, 2)$-sets and made a connection with the Kakeya conjecture.

In higher dimensions the question is open. However, it has been known for some time that if $n > 4$ then $\dim_H(E) \geq (2n + 2)/3$, where $\dim_H$ denotes Hausdorff dimension. This follows from the estimates for the 2-plane transform due to Christ [2]. Recent work by the author [4] has led to the mild improvement $\dim_H(E) \geq (2n + 3)/3$. In the present paper we modify the argument in [4], which in turn is based on geometric-combinatorial ideas very much in the spirit of Wolff [6], to obtain full dimension.

Namely we prove the following.

Theorem 1.1 *Suppose $n > 4$ and let $E \subset \mathbb{R}^n$ be an $(n, 2)$-set. Then*

$$dim_H(E) = n \,.$$

2. Terminology and notation

$S^{n-1} \subset \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere.

$B(a,r)$ is the closed ball of radius r centered at the point a.

For $X \subset \mathbb{R}^n$, $X^\perp$ denotes its orthogonal complement.

If $e \in S^{n-1}$, $a \in \mathbb{R}^n$ then $L_e(a) = \{a+te : t \in \mathbb{R}\}$ is the line in the e-direction passing through the point a.

If $e \in S^{n-1}$, $a \in \mathbb{R}^n$, $\beta > 0$ then $T_e^\beta(a) = \{x \in \mathbb{R}^n : \text{dist}(x, L_e(a)) \leq \beta\}$ is the infinite tube with axis $L_e(a)$ and cross-section radius β.

$\mathcal{L}^k$ denotes k-dimensional Lebesgue measure and $\mathcal{L}^0$ counting measure. When the context is clear we will use the notation $|\cdot|$ for all these measures.

Let $\mathcal{G}_n$ be the Grassmannian manifold of all 2-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the unique probability measure $\gamma_{n,2}$ which is invariant under the action of the orthogonal group. The elements of $\mathcal{G}_n$ will be refered to as *direction planes*.

If $P_1, P_2 \in \mathcal{G}_n$, then their distance is defined by

$$d(P_1, P_2) = \|\text{proj}_{P_1} - \text{proj}_{P_2}\|$$

where $\text{proj}_P : \mathbb{R}^n \to P$ is the orthogonal projection onto P.

A set of points or direction planes is called ρ-*separated* if the distance between any two of its elements is at least ρ.

If $P \in \mathcal{G}_n$, $1 \leq l \leq 4$, $\delta > 0$ then $P^{l,\delta}$ is a rectangle of dimensions

$$l \times l \times \underbrace{\delta \times \cdots \times \delta}_{n-2},$$

that is, the image of $[0,l] \times [0,l] \times [0,\delta] \times \cdots \times [0,\delta]$ under a rotation and a translation, such that its faces with dimensions $l \times l$ are parallel to P. Such a set will be referred to as a δ-*plate* or simply as a *plate*. When $l = 1$ the superscript l will be suppressed.

If $P_1^{l,\delta} \cap P_2^{l,\delta} \neq \emptyset$ and $d(P_1, P_2) = r$ we will say that the plates *intersect at angle* $\arcsin r$.

The letter C will denote various positive constants whose values may change from line to line. Similarly, C_ϵ will denote constants depending on ϵ. If we need to keep track of the value of a constant through a calculation we will use subscripted letters $C_1, C_2, \ldots$ or the notation $\tilde{C}$. $x \lesssim y$ means $x \leq Cx$ and $x \simeq y$ means $(x \lesssim y \ \& \ y \lesssim x)$.

Finally, note that

$$\gamma_{n,2}(\{P \in \mathcal{G}_n : d(P, P_0) \leq \delta\}) \simeq \delta^{2(n-2)} \text{ for all } P_0 \in \mathcal{G}_n, \ \delta \leq 1.$$

So if $\mathcal{A} \subset \mathcal{G}_n$ and $\mathcal{B}$ is a maximal δ-separated subset of $\mathcal{A}$ then

$$\gamma_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}| \delta^{2(n-2)}.$$

Further, if $\mathcal{A} \subset \mathcal{G}_n$ is δ-separated and $\mathcal{B}$ is a maximal η-separated subset of $\mathcal{A}$ with $\eta \geq \delta$ then

$$|\mathcal{B}| \gtrsim |\mathcal{A}|(\delta/\eta)^{2(n-2)}.$$

3. Auxiliary Lemmas

The following technical lemma allows us to control the intersection of two plates (the author is grateful to the referee for suggesting the simple proof below).

Lemma 3.1 *Let $P_1^{l,\eta}, P_2^{l,\eta}$ be two plates such that $d(P_1, P_2) \leq 1/2$. Then there exists a tube $T_e^\beta(a)$ with $\beta = C\eta/d(P_1, P_2)$ such that*

(3.1) $$P_1^{l,\eta} \cap P_2^{l,\eta} \subset T_e^\beta(a).$$

In particular

$$|P_1^{l,\eta} \cap P_2^{l,\eta}| \lesssim \frac{\eta^{n-1}}{d(P_1, P_2)}.$$

Proof. After a suitable rotation, $P_1^{l,\eta} \cap P_2^{l,\eta}$ is contained in the set $(R_1 \cap R_2) \times R$, where R_1 and R_2 are 2-dimensional rectangles of dimensions $l \times \eta$ intersecting at angle $\arcsin(d(P_1, P_2))$, and R is an $(n - 2)$-dimensional rectangle of volume $l\eta^{n-3}$. By elementary geometry,

$$\operatorname{diam}(R_1 \cap R_2) \lesssim \frac{\eta}{d(P_1, P_2)}, \quad \mathcal{L}^2(R_1 \cap R_2) \lesssim \frac{\eta^2}{d(P_1, P_2)}$$

and the lemma follows.   ∎

The proof of Theorem 1.1 will be, essentially, a reduction to the 3-dimensional case via the Radon transform. We give the definitions.

For a function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the appropriate integrability conditions, the Radon transform

$$\mathcal{R}f : S^2 \times \mathbb{R} \to \mathbb{R}$$

is defined by

$$\mathcal{R}f(e, t) = \int_{\langle e, x \rangle = t} f(x) \, d\mathcal{L}^2(x).$$

It is proved in Oberlin and Stein [5] that for any measurable set $E \subset \mathbb{R}^3$ one has the following estimate.

$$\|\mathcal{R}\chi_E\|_{3,\infty} \lesssim \|\chi_E\|_{3/2}$$

where

$$\|\mathcal{R}\chi_E\|_{3,\infty} = \left(\int_{S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \right)^{1/3}$$

and $d\sigma$ is surface measure.

We can discretize this result as follows.

Lemma 3.2 *Suppose E is a set in $\mathbb{R}^3$, $\lambda \leq 1$ and let $\{P_k\}_{k=1}^M$ be a δ-separated set in $\mathcal{G}_3$ such that for each k there is plate $P_k^{l,C\delta}$ satisfying*

$$|P_k^{l,C\delta} \cap E| \gtrsim \lambda\delta.$$

Then

$$|E| \gtrsim \lambda^{3/2} M^{1/2}\delta.$$

Proof. For each $e \in S^2$ let $Q(e)$ be the plane with normal e passing through the origin. Then there is a δ-separated set $\{e_k\}_{k=1}^M$ on S^2 such that $P_k = Q(e_k)$. Note that since $1 \leq l \leq 4$, for each $e \in B(e_k, \delta/2) \cap S^2$ we have

$$\lambda\delta \lesssim |P_k^{l,C\delta} \cap E| \leq \int_{I_e} \mathcal{L}^2((Q(e)+x) \cap E) d\mathcal{L}^1(x),$$

where I_e is an interval on $Q(e)^\perp$ with $\mathcal{L}^1(I_e) \lesssim \delta$. Therefore there exists $x_e \in I_e$ such that

$$\lambda \lesssim \mathcal{L}^2((Q(e)+x_e) \cap E).$$

Hence

$$\lambda \lesssim \sup_t \mathcal{R}\chi_E(e,t).$$

We conclude that

$$\begin{aligned}
\lambda^3\delta^2 M &\lesssim \sum_k \int_{B(e_k,\delta/2)\cap S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \\
&\leq \int_{S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) = \|\mathcal{R}\chi_E\|_{3,\infty}^3 \lesssim \|\chi_E\|_{3/2}^3 = |E|^2.
\end{aligned}$$

∎

This, in turn, gives rise to the following higher dimensional analogue.

Lemma 3.3 *Suppose E is a set in $\mathbb{R}^n$, $\lambda \leq 1$, $\Pi \subset \mathbb{R}^n$ is a 3-plane and $\{P_k\}_{k=1}^M$ is a δ-separated set in $\mathcal{G}_n$ such that for each k the plate P_k^δ satisfies*

$$P_k^\delta \subset \Pi^{\widetilde{C}\delta} \quad and \quad |P_k^\delta \cap E| \geq \lambda|P_k^\delta|$$

where $\Pi^{\widetilde{C}\delta} = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Pi) \leq \widetilde{C}\delta\}$ is the $\widetilde{C}\delta$-neighborhood of Π. Then

$$|E \cap \Pi^{\widetilde{C}\delta}| \gtrsim \lambda^3 M^{1/2}\delta^{n-2}.$$

Proof. Whithout loss of generality we may asssume that Π is the $x_1 x_2 x_3$-plane. Since $P_k^\delta \subset \Pi^{\widetilde{C}\delta}$ there is a direction plane $Q_k \subset \Pi$ such that $d(P_k, Q_k) \lesssim \delta$. Therefore we can find a plate $Q_k^{2,C_1\delta}$ with $P_k^\delta \subset Q_k^{2,C_1\delta}$. It follows that

$$|Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}| \gtrsim \lambda\delta^{n-2}.$$

Let $\mathcal{B}$ be a maximal $C_2\delta$-separated subset of $\{P_k\}_{k=1}^M$ and put $\mathcal{B}' = \{Q_k : P_k \in \mathcal{B}\}$. Then for $Q_j, Q_k \in \mathcal{B}'$, $j \neq k$, we have

$$d(Q_j, Q_k) \geq d(P_j, P_k) - d(P_j, Q_j) - d(P_k, Q_k) \geq (C_2 - C)\delta \geq \delta$$

for C_2 sufficiently large.

Now for each $Q_k \in \mathcal{B}'$ let

$$L_k = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \leq \frac{\lambda\delta}{C_3} \right\},$$

$$H_k = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \geq \frac{\lambda\delta}{C_3} \right\}.$$

Note that

$$\mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \lesssim \delta, \text{ for all } x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp.$$

Hence

$$\lambda\delta^{n-2} \lesssim |Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}|$$

$$= \int_{B(0,\widetilde{C}\delta) \cap \Pi^\perp} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x)$$

$$= \int_{L_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x)$$

$$\quad + \int_{H_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x)$$

$$\leq \frac{\lambda\delta}{C_3} C\delta^{n-3} + C\delta\mathcal{L}^{n-3}(H_k).$$

Therefore, $\mathcal{L}^{n-3}(H_k) \gtrsim \lambda\delta^{n-3}$ for C_3 sufficiently large.

Next, notice that $|\mathcal{B}'| \simeq M$ and define

$$L = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : |\{k : x \in H_k\}| < \frac{\lambda M}{C_4} \right\},$$

$$H = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : |\{k : x \in H_k\}| \geq \frac{\lambda M}{C_4} \right\}.$$

Then

$$\lambda\delta^{n-3}M \;\lesssim\; \sum_k \int \chi_{H_k} = \int_H \sum_k \chi_{H_k} + \int_L \sum_k \chi_{H_k}$$

$$\leq\; M\mathcal{L}^{n-3}(H) + \frac{\lambda M}{C_4}\mathcal{L}^{n-3}(L) \leq M\mathcal{L}^{n-3}(H) + \frac{\lambda M}{C_4}C\delta^{n-3}.$$

Therefore $\mathcal{L}^{n-3}(H) \gtrsim \lambda\delta^{n-3}$ for C_4 sufficiently large.

Note that for each $x \in H$ there are at least $\lambda M/C_4$ plates in $\Pi + x$, that is, plates in a copy of $\mathbb{R}^3$, with δ-separated direction planes and such that the 3-dimensional measure of their intersection with $E \cap (\Pi + x)$ is at least $C^{-1}\lambda\delta$. Hence, by Lemma 3.2

$$\mathcal{L}^3(E \cap (\Pi + x)) \gtrsim \lambda^{3/2}(\lambda M)^{1/2}\delta.$$

We conclude that

$$|E \cap \Pi^{\widetilde{C}\delta}| \geq \int_H \mathcal{L}^3(E\cap(\Pi+x))d\mathcal{L}^{n-3}(x) \gtrsim \lambda\delta^{n-3}\lambda^{3/2}(\lambda M)^{1/2}\delta = \lambda^3 M^{1/2}\delta^{n-2}.$$

∎

4. The main argument

Theorem 1.1 will be a consequence of the following.

Proposition 4.1 *Suppose E is a set in $\mathbb{R}^n$, $\lambda \leq 1$ and $\{P_j\}_{j=1}^M$ is a $\delta-$separated set in $\mathcal{G}_n$ with $\mathrm{diam}(\{P_j\}_{j=1}^M) \leq 1/2$, such that for each j there is plate P_j^δ satisfying*

$$|P_j^\delta \cap E| \geq \lambda|P_j^\delta|.$$

Then

$$|E| \geq C_\epsilon \delta^\epsilon \lambda^{(n+2)/2} M^{1/2}\delta^{n-2}$$

Proof. We say that a point $x \in E$ has multiplicity μ if it belongs to exactly μ plates P_j^δ. We claim that there exists a set $P_{j_0}^\delta \cap E$ such that the measure of the set of its points with multiplicity at least $\frac{1}{2}M\lambda\delta^{n-2}/|E|$ is at least $\frac{1}{2}|P_j^\delta \cap E|$, because otherwise we would have

$$|E| \geq \Big|\bigcup_{j=1}^M P_j^\delta \cap E\Big| > \frac{2|E|}{M\lambda\delta^{n-2}}\sum_{j=1}^M \frac{1}{2}|P_j^\delta \cap E| \geq |E|.$$

So letting

$$(4.1) \qquad\qquad \mu_0 = \frac{1}{2}\frac{M}{|E|}\lambda\delta^{n-2},$$

we see that there is a plate $P^\delta := P^\delta_{j_0}$ such that

$$|\{x \in P^\delta \cap E : |\{k : x \in P^\delta_k\}| \geq \mu_0\}| \geq \frac{\lambda}{2}\delta^{n-2}.$$

Note that for each $x \in P^\delta \cap E$ with $|\{k : x \in P^\delta_k\}| \geq \mu_0$ we have

$$\{k : x \in P^\delta_k\} = \bigcup_{i=1}^{\log(C/\delta)} \{k : x \in P^\delta_k \text{ and } \delta 2^{i-1} \leq d(P_k, P) < \delta 2^i\}.$$

Therefore, by the pigeonhole principle, there is an integer $i(x)$ with $1 \leq i(x) \leq \log(C/\delta)$ such that

$$|\{k : x \in P^\delta_k \text{ and } \delta 2^{i(x)-1} \leq d(P_k, P) < \delta 2^{i(x)}\}| \geq (\log(C/\delta))^{-1}\mu_0.$$

And so,

$$\{x \in P^\delta \cap E : |\{k : x \in P^\delta_k\}| \geq \mu_0\}$$
$$\subset \bigcup_{i=1}^{\log(C/\delta)} \Big\{x \in P^\delta \cap E : |\{k : x \in P^\delta_k \text{ and } \delta 2^{i-1} \leq d(P_k, P) < \delta 2^i\}|$$
$$\geq (\log(C/\delta))^{-1}\mu_0\Big\}.$$

Applying the pigeonhole principle again, we see that there exists a number $\rho := \delta 2^{i_0-1}$ and a set $A \subset P^\delta \cap E$ of measure

$$(4.2) \qquad\qquad |A| \gtrsim |\log \delta|^{-1}\lambda\delta^{n-2}$$

such that for every $x \in A$

$$(4.3) \qquad |\{k : x \in P^\delta_k \text{ and } \rho \leq d(P_k, P) < 2\rho\}| \gtrsim |\log \delta|^{-1}\mu_0.$$

Heuristically, (4.2) and (4.3) tell us that a large number of plates intersect P^δ at approximately the same angle. We are going to estimate this number using the bound for the measure of their pairwise intersections. To do this, define

$$\mathcal{D} = \{P^\delta_k : P^\delta_k \cap P^\delta \neq \emptyset \text{ and } \rho \leq d(P_k, P) < 2\rho\}.$$

Then, by Lemma 3.1, we have

$$|\mathcal{D}| \gtrsim \sum_{P^\delta_k \in \mathcal{D}} |P^\delta_k \cap P^\delta| \frac{\rho}{\delta^{n-1}} = \frac{\rho}{\delta^{n-1}} \int_{P^\delta} \sum_{P^\delta_k \in \mathcal{D}} \chi_{P^\delta_k} \geq \frac{\rho}{\delta^{n-1}} \int_A \sum_{P^\delta_k \in \mathcal{D}} \chi_{P^\delta_k}$$

$$(4.4) \qquad \gtrsim \frac{\rho}{\delta^{n-1}} |A| |\log \delta|^{-1}\mu_0 \gtrsim |\log \delta|^{-2}\lambda^2 \frac{\rho}{\delta} \frac{M}{|E|} \delta^{n-2}.$$

Where the last inequality follows from (4.1) and (4.2) and the one before last from (4.3).

We are now in a position to carry out a geometric construction, in the spirit of [6], which will allow us to use Lemma 3.3. In order to help the reader understand our strategy, we first give an informal description.

We know that the Radon transform estimate due to Oberlin and Stein is sharp in $\mathbb{R}^3$. So, we would like to slice our set with a number of thin neighborhoods of $\mathbb{R}^3$ and then apply the higher dimensional discretized version of that estimate (as given by Lemma 3.3) to each of these neighborhoods. To this end, we pass $(\rho/\delta)^{n-3}$ 3-dimensional planes (these are the sets Π_i to be defined below) through the 2-dimensional plane which is parallel to the direction plane P and passes through the center c of the plate P^δ. We do that in a "radial", so to speak, fashion (see Figure 1).
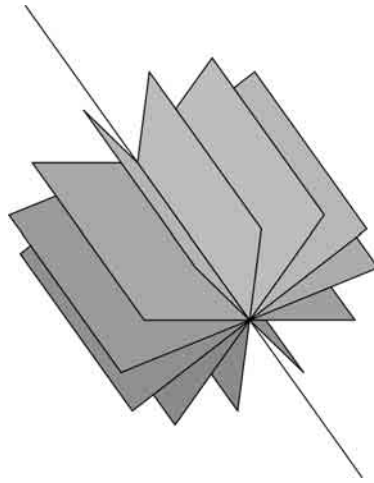

FIGURE 1: In this picture, the planes represent the 3-planes Π_i and the line represents the 2-plane $c + P$.

That is, each 3-plane Π_i is the translate of a 3-dimensional subspace spanned by P and a certain vector in the orthogonal complement of P. This ensures that every plate in $\mathcal{D}$ belongs to some $\Pi_i^{\widetilde{C}\delta}$, where $\Pi_i^{\widetilde{C}\delta}$ is the $\widetilde{C}\delta$-neighborhood of Π_i. Our goal is to use Lemma 3.3 to estimate the measure of $\Pi_i^{\widetilde{C}\delta} \cap E$, and then sum up these individual estimates to get a lower bound on the measure of our set. However, in order to do this efficiently, we have to take into account the overlap of the sets $\Pi_i^{\widetilde{C}\delta}$. If there are "too many" $\Pi_i^{\widetilde{C}\delta}$'s, that is, if $\rho/\delta \geq \lambda^{-1}|\log \delta|$ (this is case I below), we observe that their overlap increases as we approach the plane $c + P$. So we choose a suitable neighborhood $\mathcal{X}$ of $c + P$ in such a way that:

- The overlap of the sets $\Pi_i^{\widetilde{C}\delta} \cap \mathcal{X}^{\complement}$ is smaller.
- The measure of the intersection of every plate in $\mathcal{D}$ with the reduced set $E \cap \mathcal{X}^{\complement}$ is still large.

Then, we work with this reduced set $E \cap \mathcal{X}^{\complement}$. On the other hand, if there are not "too many" $\Pi_i^{\widetilde{C}\delta}$'s, that is, if $\rho/\delta \leq \lambda^{-1}|\log \delta|$ (this is case II below), we just estimate their overlap with their number $(\rho/\delta)^{n-3}$.

We now proceed with the formal argument.

Let $\{e_i\}_i$ be a maximal δ/ρ-separated set of points on the $(n-3)$-dimensional unit sphere $S^{n-1} \cap P^{\perp}$ and let

$$\Pi_i = c + \Pi_i'$$

where c is the center of P^{δ} and Π_i' is the 3-dimensional space spanned by e_i and P. Then for each $P_k^{\delta} \in \mathcal{D}$ there exists an i such that

$$P_k^{\delta} \subset \Pi_i^{\widetilde{C}\delta}, \qquad \text{where } \Pi_i^{\widetilde{C}\delta} \text{ is the } \widetilde{C}\delta\text{-neighborhood of } \Pi_i.$$

To see this, let $y \in P_k^{\delta}$, and pick $w \in P_k^{\delta} \cap P^{\delta}$. Then $|y - w|$ is bounded by the diameter of P_k^{δ} and belongs to a $C\delta$-neighborhood of the direction plane P_k. So, there exists a point $z \in P_k$ (just take z to be the projection of $y-w$ onto P_k) with $|z| \lesssim 1$ and $|y-w-z| \lesssim \delta$. Now write

$$
\begin{aligned}
z &= z_1 + z_2 \in P \oplus P^{\perp}, \\
c - w &= w_1 + w_2 \in P \oplus P^{\perp}.
\end{aligned}
$$

Since $d(P, P_k) \simeq \rho$, we have $|z_2| \lesssim \rho$, and since $c - w$ belongs to a $C\delta$-neighborhood of the direction plane P we get $|w_2| \lesssim \delta$. Now $z_2/|z_2|$ belongs to the unit sphere of $P^{\perp}$, so we can find an e_i such that $|z_2/|z_2| - e_i| \leq \delta/\rho$. Therefore,

$$|z_2 - |z_2|e_i| \leq \frac{\delta}{\rho}|z_2| \lesssim \delta.$$

Finally, notice that

$$y = [(y - w - z) + (z_2 - |z_2|e_i) - w_2] + [z_1 - w_1 + |z_2|e_i] + c,$$

where the vector in the first square bracket has length at most $C\delta$ and the vector in the second square bracket belongs to Π_i'. We conclude that $y \in \Pi_i^{\widetilde{C}\delta}$.

Therefore, if we let

$$\mathcal{D}_i = \left\{ P_k^{\delta} \in \mathcal{D} : P_k^{\delta} \subset \Pi_i^{\widetilde{C}\delta} \right\}$$

then

$$\mathcal{D} = \bigcup_i \mathcal{D}_i.$$

Now let $\gamma = \lambda|\log \delta|^{-1}$ and consider two cases.

CASE I. $\delta \leq \gamma\rho$.

Let
$$\mathcal{X} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, c + P) \leq \gamma\rho\}.$$

First, we show that each $P_k^\delta \in \mathcal{D}$ has large intersection with $E \cap \mathcal{X}^\complement$. Indeed, notice that
$$P_k^\delta \cap \mathcal{X} \subset P_k^{2\gamma\rho} \cap \mathcal{X}.$$

Hence, by (3.1) in Lemma 3.1, $P_k^{2\gamma\rho} \cap \mathcal{X}$ is contained in a tube of cross-section radius $C\gamma$. Now, the intersection of a tube of cross-section radius $C\gamma$ with the plate P_k^δ is contained in the intersection of two rectangles of dimensions $\infty \times C\gamma \times C\gamma \cdots \times C\gamma$ and $1 \times 1 \times \delta \times \cdots \times \delta$, and therefore has volume at most $C\gamma\delta^{n-2}$ (recall that $\delta \leq \gamma\rho \leq \gamma$). We conclude that the volume of $P_k^\delta \cap \mathcal{X}$ is at most $C\lambda|\log\delta|^{-1}\delta^{n-2}$. Consequently
$$|P_k^\delta \cap (E \cap \mathcal{X}^\complement)| = |P_k^\delta \cap E| - |P_k^\delta \cap E \cap \mathcal{X}| \geq |P_k^\delta \cap E| - |P_k^\delta \cap \mathcal{X}|$$
$$\geq \lambda\delta^{n-2} - C\lambda|\log\delta|^{-1}\delta^{n-2} \geq \frac{\lambda}{2}\delta^{n-2}$$

for δ sufficiently small.

Next, we show that the sets $\Pi_i^{\widetilde{C}\delta} \cap \mathcal{X}^\complement$ have small overlap. Namely, we claim that if $\operatorname{dist}(x, c+P) \geq \gamma\rho$, then x belongs to at most $C\gamma^{-(n-3)}$ sets $\Pi_i^{\widetilde{C}\delta}$. To see this, we can clearly assume that $c = \mathbf{0}$. Now suppose that $x \in \Pi_i^{\widetilde{C}\delta}$ and write $x = u + w \in P \oplus P^\perp$. Then $|w - \langle w, e_i\rangle e_i| \lesssim \delta$. Therefore, by simple algebra, either $|w - |w|e_i| \lesssim \delta$, or $|w + |w|e_i| \lesssim \delta$. On the other hand, $\operatorname{dist}(x, P) \geq \gamma\rho$ implies that $|w| \geq \gamma\rho$. Consequently we have either $|e_i - w/|w|| \lesssim \delta/(\gamma\rho)$, or $|e_i + w/|w|| \lesssim \delta/(\gamma\rho)$. It follows that
$$\{e_i : x \in \Pi_i^{\widetilde{C}\delta}\} \subset B(w/|w|, C\delta/(\gamma\rho)) \cup B(-w/|w|, C\delta/(\gamma\rho)).$$

Since the e_i's are δ/ρ-separated points on an $(n-3)$-dimensional unit sphere, we conclude that
$$\operatorname{card}(\{e_i : x \in \Pi_i^{\widetilde{C}\delta}\}) \lesssim \left(\frac{\delta/(\gamma\rho)}{\delta/\rho}\right)^{n-3} = \gamma^{-(n-3)}.$$

Hence
$$|E| \geq \left|\bigcup_i (E \cap \mathcal{X}^\complement) \cap \Pi_i^{\widetilde{C}\delta}\right| \gtrsim \gamma^{n-3}\sum_i \left|(E \cap \mathcal{X}^\complement) \cap \Pi_i^{\widetilde{C}\delta}\right| \gtrsim \gamma^{n-3}\lambda^3\delta^{n-2}\sum_i |\mathcal{D}_i|^{1/2}$$

where the last inequality follows from Lemma 3.3 applied, for each i, to the set $E \cap \mathcal{X}^\complement$, the plates in $\mathcal{D}_i$ and the 3-plane Π_i.

CASE II. $\delta \geq \gamma\rho$.

Since $|\{\Pi_i\}_i| \lesssim (\rho/\delta)^{n-3}$, we have

$$|E| \geq \left| \bigcup_i E \cap \Pi_i^{\widetilde{C}\delta} \right| \gtrsim (\delta/\rho)^{n-3} \sum_i |E \cap \Pi_i^{\widetilde{C}\delta}|$$

$$\geq \gamma^{n-3} \sum_i |E \cap \Pi_i^{\widetilde{C}\delta}| \gtrsim \gamma^{n-3} \lambda^3 \delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}$$

with the last inequality true by Lemma 3.3 applied, for each i, to the set E, the plates in $\mathcal{D}_i$ and the 3-plane Π_i.

We conclude that in either case

(4.5) $$|E| \gtrsim \gamma^{n-3} \lambda^3 \delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}.$$

To estimate the sum above, note that $\Pi_i^{\widetilde{C}\delta}$, being the $\widetilde{C}\delta$-neighborhood of a copy of $\mathbb{R}^3$, can contain at most $C(\rho/\delta)^2$ plates whose direction planes are δ-separated and at distance approximately ρ from P. Therefore

(4.6) $$|\mathcal{D}| \leq \sum_i |\mathcal{D}_i| \lesssim \frac{\rho}{\delta} \sum_i |\mathcal{D}_i|^{1/2}.$$

Combining (4.4), (4.5) and (4.6) we obtain

$$|E| \geq C_\epsilon \delta^{2\epsilon} \lambda^{n+2} \frac{M}{|E|} \delta^{2(n-2)}$$

where the logarithmic factors have been absorbed into $C_\epsilon \delta^{2\epsilon}$. Consequently

$$|E| \geq C_\epsilon \delta^\epsilon \lambda^{(n+2)/2} M^{1/2} \delta^{n-2}$$

proving the proposition. ∎

5. Proof of Theorem 1.1

This follows by a standard argument in [1]. We give a sketch for the convenience of the reader. Let E be an $(n, 2)$-set, and $\mathcal{A} \subset \mathcal{G}_n$ any set with $\text{diam}(\mathcal{A}) < 1/2$ and $\gamma_{n,2}(\mathcal{A}) > 0$. Then for every $P \in \mathcal{A}$ there is a square S_P of unit area such that $S_P \subset E$. Fix a covering (not necessarily finite) $\{B(x_i, r_i)\}$ of E. For every $\epsilon > 0$, we will bound $\sum_i r_i^{n-2\epsilon}$ from below by a constant depending only on ϵ. Let

$$I_k = \left\{ i : 2^{-k} \leq r_i \leq 2^{-(k-1)} \right\},$$

$$E_k = E \cap \bigcup_{i \in I_k} B(x_i, r_i), \quad \widetilde{E}_k = \bigcup_{i \in I_k} B(x_i, 2r_i).$$

We can assume that all the sets I_k are finite, otherwise $\sum_i r_i^{n-2\epsilon} = \infty$ and we are done. So let $\nu_k = |I_k|$. Note that for every $P \in \mathcal{A}$ there exists a k_P such that

$$\mathcal{L}^2(S_P \cap E_{k_P}) \geq \frac{1}{4k_P^2},$$

because otherwise

$$1 = \mathcal{L}^2(S_P \cap E) \leq \sum_k \mathcal{L}^2(S_P \cap E_k) \leq \sum_k \frac{1}{4k^2} < \frac{1}{2}.$$

Now, let

$$\mathcal{A}_k = \{P \in \mathcal{A} : \mathcal{L}^2(S_P \cap E_k) \geq \frac{1}{4k^2}\}.$$

Since $\bigcup_k \mathcal{A}_k = \mathcal{A}$, there exists a k_0 such that

$$\gamma_{n,2}(\mathcal{A}_{k_0}) \geq \frac{\gamma_{n,2}(\mathcal{A})}{2k_0^2},$$

or else

$$\gamma_{n,2}(\mathcal{A}) \leq \sum_k \gamma_{n,2}(\mathcal{A}_k) \leq \sum_k \frac{\gamma_{n,2}(\mathcal{A})}{2k^2} < \gamma_{n,2}(\mathcal{A}).$$

Put $\mathcal{B} = \mathcal{A}_{k_0}$. Then

$$\mathcal{L}^2(S_P \cap E_{k_0}) \gtrsim k_0^{-2}, \text{ for all } P \in \mathcal{B}.$$

Let $\{P_j\}_{j=1}^M$ be a maximal 2^{-k_0}-separated set in $\mathcal{B}$. Then

$$M \gtrsim k_0^{-2} 2^{2k_0(n-2)}$$

and for each P_j there is a plate $P_j^{2^{-k_0}}$ such that

$$|P_j^{2^{-k_0}} \cap \widetilde{E}_{k_0}| \gtrsim k_0^{-2} |P_j^{2^{-k_0}}|.$$

So, by Proposition 4.1

$$|\widetilde{E}_{k_0}| \geq C_\epsilon k_0^{-\alpha} 2^{-k_0\epsilon}$$

where $\alpha = n + 3$. On the other hand

$$|\widetilde{E}_{k_0}| \lesssim \nu_{k_0} 2^{-k_0 n}.$$

Therefore

$$\nu_{k_0} \geq C_\epsilon k_0^{-\alpha} 2^{k_0(n-\epsilon)}.$$

Consequently

$$\sum_i r_i^{n-2\epsilon} \gtrsim \nu_{k_0} 2^{-k_0(n-2\epsilon)} \geq C_\epsilon k_0^{-\alpha} 2^{k_0\epsilon} \geq \widetilde{C}_\epsilon.$$

We conclude that $\dim_H(F) = n$.

References

[1] Bourgain, J.: Besicovitch type maximal operators and applications to Fourier analysis. *Geom. Funct. Anal.* **1** (1991), no. 2, 147–187

[2] Christ, M.: Estimates for the k-plane transform. *Indiana Univ. Math. J.* **33** (1984), no. 6, 891–910.

[3] Marstrand, J.: Packing planes in $\mathbb{R}^3$. *Mathematika* **26** (1979), no. 2, 180–183.

[4] Mitsis, T.: An improved bound for the Hausdorff dimension of $(n, 2)$-sets. Preprint.

[5] Oberlin, D. M., Stein, E. M.: Mapping properties of the Radon transform. *Indiana Univ. Math. J.* **31** (1982), no. 5, 641–650.

[6] Wolff, T.: An improved bound for Kakeya type maximal functions. *Rev. Mat. Iberoamericana* **11** (1995), no. 3, 651–674.

Themis Mitsis
Department of Mathematics
University of Crete
Knossos Ave.
GR-71409, Iraklio (Greece)
mitsis@fourier.math.uoc.gr