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Translation averages of dyadic weights are not always good weights

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Abstract

The process of translation averaging is known to improve dyadic BMO to the space BMO of functions of bounded mean oscillation, in the sense that the translation average of a family of dyadic BMO functions is necessarily a BMO function. The present work investigates the effect of translation averaging in other dyadic settings. We show that translation averages of dyadic doubling measures need not be doubling measures, translation averages of dyadic Muckenhoupt weights need not be Muckenhoupt weights, and translation averages of dyadic reverse Hölder weights need not be reverse Hölder weights. All three results are proved using the same construction.

1. Introduction.

Several important function spaces on the real line, such as the space BMO of functions of bounded mean oscillation, Muckenhoupt's spaces of A_p weights, and the spaces of reverse Hölder weights, are defined in terms of properties that must hold on every real interval. Such function spaces have less restrictive *dyadic versions*, in which the defining property is only

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required to hold on dyadic intervals. These dyadic versions then contain the original function spaces. Similarly, the space of doubling measures, defined by requiring the ratio $\mu(I)/\mu(J)$ to be uniformly bounded for all pairs of adjacent intervals I and J of equal length, has a dyadic version, in which the doubling property need only hold for dyadic sister intervals.

Averaging a collection of functions tends to smooth out irregularities, producing a better-behaved function. In this paper we consider the following averaging process. The *translation average* φ of a collection of functions $\{\varphi_t\}$ on the unit interval with endpoints identified, indexed by $t \in [0,1]$, is defined by

$$\varphi(x) = \int_0^1 \varphi_t(x+t)\,dx\,. \tag{1}$$

The translation average of a collection of measures is defined analogously.

Garnett and Jones showed in [GJ] that the translation average of a collection of dyadic BMO functions is necessarily a BMO function. (This is not true for the usual average.) Specifically, if $t \longrightarrow \varphi_t$ is a measurable mapping from $\mathbb{R}^m$ to the space of dyadic BMO functions such that all φ_t are supported on a fixed dyadic cube, the dyadic BMO constant of each φ_t is bounded by 1, and each φ_t has mean zero, then the function

$$\varphi^N(x) = \frac{1}{(2N)^m}\int_{|t_j|\le N} \varphi_t(x+t)\,dt \tag{2}$$

is in BMO for each N, with uniform BMO constant. They exploit this idea to give new proofs of theorems on the structure of BMO functions and on the distance to L^∞ in BMO, and of the factorization theorem for A_p weights, by first obtaining each theorem in the easier dyadic case, and then averaging the results of the dyadic decomposition over translations. In this paper we investigate whether the translation averaging process has the same "improving" effect in other settings.

We prove that translation averaging does not always improve dyadic spaces. First, there are families of dyadic doubling measures, with uniform and arbitrarily small dyadic doubling constant, whose translation averages are *not* doubling measures.

Theorem 1.1. *Given $C > 1$, there is a family of dyadic doubling measures $\{\mu_t\}_{t\in[0,1]}$ on $[0,1]$, with dyadic doubling constant at most C for all $t \in$*

$[0,1]$, *such that the translation average*

$$\mu(\cdot) = \int_0^1 \mu_t(\cdot + t)\, dt \tag{3}$$

is not a doubling measure.

Second, for each of Muckenhoupt's A_p spaces, there are families of dyadic A_p weights on the unit interval, with uniform and arbitrarily small dyadic A_p constant, whose translation averages are *not* A_p weights.

Theorem 1.2. *For each real p with $1 \leq p \leq \infty$ and for each $C > 1$, there is a family of A_p^d weights $\{w_t\}_{t\in[0,1]}$ on $[0,1]$, with $\|w_t\|_{A_p^d} \leq C$ for all $t \in [0,1]$, such that the translation average*

$$w(x) = \int_0^1 w_t(x+t)\, dt \tag{4}$$

is not an A_p weight.

Third, for each reverse Hölder space RH_q, there are families of dyadic reverse Hölder-q weights on the unit interval, with uniform and arbitrarily small dyadic reverse Hölder-q constant, whose translation averages are *not* reverse Hölder weights.

Theorem 1.3. *For each real q with $1 < q < \infty$ and for each $C > 1$, there is a family of RH_q^d weights $\{w_t\}_{t\in[0,1]}$ on $[0,1]$, with $\|w_t\|_{RH_q^d} \leq C$ for all $t \in [0,1]$, such that the translation average*

$$w(x) = \int_0^1 w_t(x+t)\, dt \tag{5}$$

is not a RH_q weight.

Doubling measures, Muckenhoupt's classes of A_p weights, reverse Hölder weights, and their dyadic versions are defined in Section 2.

All three results are proved by the same construction, which is developed in Theorem 1.4 below. We construct a family of weights $\{w_t\}$, $0 \leq t \leq 1$, depending on a parameter α, $0 < \alpha < 1$, with the following properties. The weights are the densities of dyadic doubling measures μ_t. The weights lie in every A_p^d space, and in every RH_q^d space for which

$q < 1/\alpha$, and so by decreasing the parameter α we can ensure that the weights lie in any given RH_q^d space. Moreover, by decreasing α, we can also make the dyadic doubling constant, the A_1^d constant, the A_p^d constant, and the RH_q^d constant arbitrarily close to 1. However, the measure μ whose density is the translation average $w(x) = \int_0^1 w_t(x+t)\,dt$ of the weights is *not* a doubling measure, which proves Theorem 1.1.

On the other hand, measures whose densities are A_p weights or reverse Hölder weights are necessarily doubling; this follows from the reverse Hölder property and $\cup_{p\geq 1} A_p = \cup_{q>1} RH_q$. Therefore $w(x)$ is not in A_p for any p, nor in RH_q for any q, establishing Theorems 1.2 and 1.3.

In order to state the main theorem precisely, we first make some definitions.

Let α be a real number with $0 < \alpha < 1$. Let β be a rational number such that

$$0 < \beta < \frac{\alpha}{\alpha+1} \qquad \text{and} \qquad \beta + \alpha < 1\,. \tag{6}$$

The first condition implies that $\alpha - \beta\,\alpha - \beta > 0$. Let $\{N_j\}_{j=1}^{\infty}$ be a sequence of rapidly increasing integers, chosen so that $N_j\beta$ is an integer for all j, so that $N_1 \geq 1/\beta$ and $N_2 \geq 2/\beta$, and so that

$$N_j \geq \frac{1}{\alpha - \beta\,\alpha - \beta}\,(N_1 + \cdots + N_{j-1})\,, \qquad \text{for } j = 2, 3, \ldots \tag{7}$$

For example, when $\alpha = 1/2$ and $\beta = 1/4$, the sequence $N_1 = 4$, $N_j = 8\,(9)^{j-2}$ for $j \geq 2$ satisfies these conditions. In general, a sequence satisfying $N_j \geq C\,(N_1 + \cdots + N_{j-1})$ must grow exponentially, since $N_j \geq C\,(1+C)^{j-1}\,N_1$.

We identify the unit circle with the interval $[0,1]$. The *dyadic intervals* in $[0,1]$ comprise the set

$$\mathcal{D} = \Big\{ I = \Big[\frac{j}{2^k}, \frac{j+1}{2^k}\Big) : \ j, k \text{ integers},\ 0 \leq j \leq 2^k - 1,\ k = 0, 1, 2, \ldots \Big\} \tag{8}$$

of half-open dyadic subintervals of $[0,1]$. Let $\mathcal{D}_N$ denote the collection of intervals in $\mathcal{D}$ of length 2^{-N}, for $N = 0,\, 1,\, 2,\, \ldots$ We use the term *dyadic sisters* for the two halves or *daughters* of a single dyadic interval, their *parent*. Any two dyadic intervals are either nested or disjoint.

Define the *generational distance* $d(I,J)$ between two dyadic intervals I and J of equal length by

$$d(I,J) = \log_2\Big(\frac{|K|}{|I|}\Big)\,, \tag{9}$$

where K is the smallest dyadic interval containing both I and J. Thus $d(I,I)=0$, and in general $d(I,J)$ is the number of generations back to the first dyadic common ancestor of I and J.

Given a point $t\in[0,1]$, and $j\geq 1$, let I_t^j be the unique dyadic interval in $\mathcal{D}_{N_j}$ which contains t. Label the dyadic intervals in $\mathcal{D}_{N_j}$ immediately to the right of I_t^j as J_t^j, K_t^j, L_t^j, and M_t^j, reading from left to right. When t is very near the right end of $[0,1]$, some or all of J_t^j, K_t^j, L_t^j, and M_t^j are wrapped around to the left end of $[0,1]$.

On the unit circle, K_t^j is the translation of I_t^j to the right by $2\,|I_t^j|$. We define a nested, decreasing sequence $\{\mathcal{G}_j\}_{j=1}^{\infty}$ of sets in $[0,1]$ by letting $\mathcal{G}_0=[0,1]$, and for $j\geq 1$ letting

$$
\begin{aligned}
\mathcal{G}_j &= \{t\in\mathcal{G}_{j-1}:\ d(I_t^j,K_t^j)=N_j\,\beta\}\\
&= \{t\in[0,1]:\ d(I_t^1,K_t^1)=N_1\,\beta,\dots,d(I_t^j,K_t^j)=N_j\,\beta\}\,.
\end{aligned}
\tag{10}
$$

For instance, $t\in\mathcal{G}_2$ if the generational distance from the dyadic interval I_t^1 of length 2^{-N_1} containing t, to the dyadic interval K_t^1 which is the translation of I_t^1 to the right by twice the length of I_t^1, is exactly $N_1\,\beta$, and if in addition $d(I_t^2,K_t^2)$ is exactly $N_2\,\beta$.

Define a family $\{w_t\}_{t\in[0,1]}$ of weights on the unit circle as follows. For $t\in\mathcal{G}_0\setminus\mathcal{G}_1$, let $w_t\equiv 1$. For $t\in\mathcal{G}_j\setminus\mathcal{G}_{j+1}$, $j\geq 1$, let

$$
w_t(x)=\begin{cases} (M_d(c\,\delta_t(x)))^{\alpha}\,, & x\notin I_t^j\,,\\ 2^{N_j\alpha}\,, & x\in I_t^j\,. \end{cases}
\tag{11}
$$

Here $c=(2-2^{\alpha})^{1/\alpha}$, δ_t is the Dirac delta function centred at t, I_t^j is the unique dyadic interval of length 2^{-N_j} that contains t, and M_d is the dyadic maximal operator, defined by

$$
M_d(f)(x)=\sup_{\substack{I\ni x\\ I\in\mathcal{D}}}\frac{1}{|I|}\int_I |f(y)|\,dy\,.
\tag{12}
$$

The weight w_t is constant on each dyadic interval of length 2^{N_j}, and decays away from its maximum, taken on the dyadic interval I_t^j containing t, as $|x-t|$ increases.

For each j, let A^j be the interval of length 2^{-N_j} centred at 0, and let B^j be the interval of the same length obtained by translating A^j to the right by the distance $3\,|A^j|$. These intervals A^j and B^j are not dyadic.

Theorem 1.4. *Define the numbers* α, β, *and* $\{N_j\}_{j=1}^{\infty}$, *the sets* $\{\mathcal{G}_j\}_{j=1}^{\infty}$, *the weights* $\{w_t\}_{t\in[0,1]}$, *and the intervals* $\{A^j\}_{j=1}^{\infty}$ *and* $\{B^j\}_{j=1}^{\infty}$ *as above. Then the weights* w_t *are dyadic* A_p *weights, with uniform* A_p^d *constant, for* $1 \le p \le \infty$, *and they are dyadic reverse Hölder weights, with uniform* RH_q^d *constant, for* $1 < q < 1/\alpha$. *Also, the associated measures* μ_t *with densities* w_t *are dyadic doubling measures with uniform dyadic doubling constant. All these constants converge to* 1 *as* α *decreases to* 0. *Finally, the translation average* $\mu(\cdot) = \int_0^1 \mu_t(\cdot + t)\,dt$ *of the measures* μ_t *satisfies*

$$\frac{\mu(A^j)}{\mu(B^j)} \ge C\, 2^{N_j \beta\alpha} \longrightarrow \infty\,, \qquad \text{as } j \longrightarrow \infty\,, \tag{13}$$

and therefore μ *is not a doubling measure.*

Remark. We mention that it is possible to prove Theorem 1.1 using a completely different family of examples of dyadic doubling measures on the unit interval. One fixes r, s such that $0 < s < r < 1$ and $r + s = 1$, and defines the measures μ_t recursively, first assigning mass r to either the left or right half of $[0, 1]$ and mass s to the other half. Next, one assigns the fraction r of the mass of $[0, 1/2)$ to either the left or right half of $[0, 1/2)$ and the fraction s to the other half, and similarly for $[1/2, 1]$, and so on at smaller and smaller scales, producing dyadic doubling measures with uniform dyadic doubling constant r/s. However, the details of making the ensemble of left/right choices so that the resulting translation average μ is not doubling make the construction quite intricate.

I am grateful to a referee and to David Cruz-Uribe for pointing out related work of Petermichl, Nazarov, Treil, and Volberg. In [P], Petermichl obtains the kernel of the one-dimensional Hilbert transform as the result of an averaging process over kernels of dyadic shift operators. Her averaging process involves both dilations and translations of the standard dyadic grid. She applies the decomposition to show that the commutator of the Hilbert transform with matrix multiplication by a BMO matrix of size $n\times n$ is bounded by a multiple of $\log n$ times the BMO-norm of the matrix, and, with Pott [PP], to prove an analogue of Burkholder's theorem for operator-weighted spaces. Namely, for an operator weight function W taking values in the bounded linear operators on a Hilbert space $\mathcal{H}$, if the dyadic martingale transforms are uniformly bounded on $L^2_{\mathbb{R}}(W)$ for every dilated and translated dyadic grid in $\mathbb{R}$, then the Hilbert transform is bounded on $L^2_{\mathbb{R}}(W)$.

Nazarov, Treil, and Volberg [NTV] use dyadic martingale techniques and an averaging process on dyadic lattices to extend the following well known result to the case where μ is not doubling. Let T be an operator of Calderón-Zygmund type associated to a one-dimensional standard kernel K in $\mathbb{R}^2$, satisfying the antisymmetry condition $K(x,y) = -K(y,x)$, for all x, $y \in \mathbb{R}^2$. Let μ be a positive Radon measure such that $\mu(Q) \leq C\,\ell(Q)$ for all squares Q, where $\ell(Q)$ is the side-length of Q. Then $T : L^2(\mu) \longrightarrow L^2(\mu)$ is bounded if and only if the L^2 estimate holds for characteristic functions of squares. They generalize to operators T which are not antisymmetric, and to higher dimensions.

In Section 2 we recall the definitions of doubling measures, Muckenhoupt's A_p spaces, the reverse Hölder spaces, and their dyadic versions. In Section 3 we define the weights $v_t = (M_d(c\,\delta_t(x)))^\alpha$ in terms of dyadic maximal functions of Dirac delta functions, and compute the dyadic A_p constants and the dyadic reverse Hölder constants of these weights and the dyadic doubling constants for the associated measures. We also compute the analogous constants for the truncated weights w_t defined in (11). In Section 4 we prove our main result, Theorem 1.4.

2. Background.

A positive measure μ on the real line is called a *dyadic doubling measure* if there is a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{\mu(I)}{\mu(J)} \leq C \tag{14}$$

for all pairs I, J of dyadic sister intervals. The smallest such constant C is called the dyadic doubling constant of μ. Equivalently, there is a constant $C' \geq 1$ such that for every dyadic interval I, $\mu(\widetilde{I}) \leq C'\mu(I)$, where the interval $\widetilde{I}$ is the dyadic parent of I. When there is a constant $C \geq 1$ such that (14) holds for every pair I, J of adjacent intervals of equal length, not only for dyadic sisters, μ is called a *doubling measure*.

We say that a positive locally integrable function w on the real line is a *dyadic A_p weight*, written $w \in A_p^d$, for real p with $1 < p < \infty$, if

$$\sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|}\int_I w\Big)\Big(\frac{1}{|I|}\int_I \Big(\frac{1}{w}\Big)^{1/(p-1)}\Big)^{p-1} < \infty\,. \tag{15}$$

We say w is a *dyadic A_1 weight*, written $w \in A_1^d$, if

$$\sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|}\int_I w\Big)\Big(\text{ess sup}_I \frac{1}{w}\Big) < \infty\,. \tag{16}$$

The suprema in equations (15) and (16) are called the dyadic A_p constant and the dyadic A_1 constant, respectively, of the weight w, and are denoted by $\|w\|_{A_p^d}$ and $\|w\|_{A_1^d}$. When the suprema in equations (15) and (16) are taken over all real intervals, not just over the dyadic intervals, we recover the definitions of Muckenhoupt's original A_p and A_1 weights.

We say w is a *dyadic reverse Hölder-q weight*, written $w \in RH_q^d$, for real q with $1 < q < \infty$, if the measure μ whose density is w is a dyadic doubling measure, and if there is a constant $C \geq 1$ such that for all dyadic intervals $I \in \mathcal{D}$,

$$\left(\frac{1}{|I|}\int_I w^q\right)^{1/q} \leq C\,\frac{1}{|I|}\int_I w\,. \tag{17}$$

The smallest such constant C is called the dyadic reverse Hölder-q constant of the weight w, and is denoted by $\|w\|_{RH_q^d}$. The original reverse Hölder-q spaces RH_q are defined by requiring (17) to hold for all real intervals I, not only for dyadic intervals. In the non-dyadic setting RH_q, (17) implies that the associated measure μ is doubling [CF], [GC-RF]. In the dyadic setting RH_q^d, this is not true [B], and we impose the additional condition that μ is a dyadic doubling measure so that the dyadic and non-dyadic theories will be parallel.

The dyadic doubling constant and the constants $\|w\|_{A_p^d}$, $\|w\|_{A_1^d}$, and $\|w\|_{RH_q^d}$ are necessarily greater than or equal to 1.

The canonical examples of weights are the functions $w(x) = |x|^\alpha$, for which $w \in A_\infty^d$ if and only if $\alpha > -1$, $w \in A_p^d$ if and only if $-1 < \alpha < p-1$, $w \in A_1^d$ if and only if $-1 < \alpha \leq 0$, and $w \in RH_q^d$ if and only if $\alpha > -1/p$.

Muckenhoupt's spaces of A_p weights are well known in connection with two important operators in harmonic analysis. The Hilbert transform and the maximal function are both bounded operators from L^p to itself, for $1 < p < \infty$. They are bounded from $L^p(d\mu)$ to itself if and only if μ is absolutely continuous with respect to Lebesgue measure and the density of μ is an A_p weight [M], [HMW]. The spaces of A_p weights are nested and increasing with p, and their union is denoted by A_∞. The reverse Hölder spaces are nested and decreasing as q increases, $1 < q < \infty$, and their union is also A_∞. The reverse Hölder spaces are also known as the B_p spaces. In [FKP] and [B], the spaces A_∞, A_p, RH_q and their dyadic versions are characterized by summation conditions. The A_p spaces are also closely related to BMO; if $w \in A_\infty$ then $\log w$ is in BMO, and if $\log w$ is in BMO then there is a $\beta > 0$ such that $w^\beta \in A_\infty$. See [GC-RF] for more on the theory of weights.

3. Weights from dyadic maximal functions of Dirac deltas.

In this section we define the particular weights w_t used in our construction, and compute their dyadic A_p constants, their dyadic reverse Hölder constants, and the dyadic doubling constants of the associated measures μ_t.

For most values of $t \in [0,1]$, our weights w_t are identically 1. For the remaining values of t, our weights are modified versions of powers of dyadic maximal functions of Dirac delta functions, following the characterization of A_1 weights by Coifman and Rochberg [CR]. The underlying idea is to produce a large drop in the value of w_t at a dyadic point as close as possible to t, so that w_t is large on the interval $A^j + t$ and small on $B^j + t$.

Fix a number α with $0 < \alpha < 1$, and define a normalizing constant $c = (2 - 2^\alpha)^{1/\alpha}$. Fix $t \in [0,1]$. Let δ_t denote the Dirac delta function at t. Define a weight v_t by

$$v_t = (M_d(c\,\delta_t(x)))^\alpha = \Big(\sup_{\substack{I \ni x \\ I \in \mathcal{D}}} \frac{1}{|I|} \int_I c\,\delta_t(y)\,dy \Big)^\alpha . \tag{18}$$

See Figure 1. We will see below that the weight $v_t(x)$ has a peak at $x = t$, decaying away as $|x - t|$ increases, and that the requirement $\alpha < 1$ ensures that v_t is integrable. The constant $c = (2 - 2^\alpha)^{1/\alpha}$ is chosen so that v_t has integral 1.

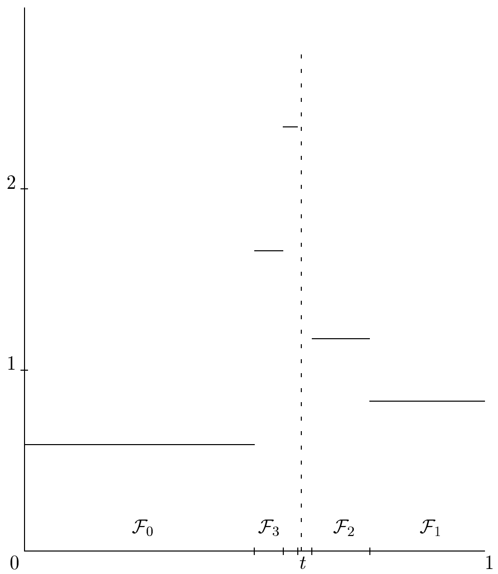

Figure 1. The weight $v_t(x) = (M_d(c\,\delta_t(x)))^\alpha$, with $\alpha = 1/2$.

We partition $[0,1]$ into the intervals $\mathcal{F}_k$ on which v_t is constant. The supremum in (18) is achieved when I is the smallest dyadic interval containing both x and t. For $k = 0, 1, 2, \ldots$ let

$$(19)\qquad \begin{aligned}\mathcal{F}_k = \{x \in [0,1] :\ |\widehat{I}| = 2^{-k}\,,\ &\text{where } \widehat{I} \text{ is the smallest}\\ &\text{dyadic interval containing both } x \text{ and } t\}\,.\end{aligned}$$

Then $|\mathcal{F}_k| = 2^{-k-1}$. The set $\mathcal{F}_0$ is the half of $[0,1]$ that does not contain t, $\mathcal{F}_1$ is the half of $[0,1] \setminus \mathcal{F}_0$ that does not contain t, and so on. The weight v_t is constant on each $\mathcal{F}_k$: for $x \in \mathcal{F}_k$,

$$(20)\qquad v_t(x) = (M_d(c\,\delta_t(x)))^\alpha = \Big(\frac{1}{|\widehat{I}|}\int_{\widehat{I}} c\,\delta_t(y)\,dy\Big)^\alpha = 2^{k\alpha}\,c^\alpha\,.$$

Given a weight v_t, we denote by $\nu_t(\cdot)$ the associated measure whose density is v_t. We use I_t to denote dyadic intervals containing the point t, and J to denote dyadic intervals not containing t. In particular, for the unique dyadic interval $I_t \in \mathcal{D}_N$ that contains t, we have $I_t = \cup_{k=N}^\infty \mathcal{F}_k$.

We collect here some useful properties of the weights v_t.

Lemma 3.1. *Suppose $0 < \alpha < 1$, and let $c = (2 - 2^\alpha)^{1/\alpha}$. Fix $t \in [0,1]$. The weight $v_t(x) = (M_d(c\,\delta_t(x)))^\alpha$ and its associated measure $\nu_t(\cdot) = \int_\cdot v_t(x)\,dx$ satisfy the following properties.*

i) *The total mass of ν_t on $[0,1]$ is 1.*

ii) *The dyadic interval $I_t \in \mathcal{D}_N$ containing the point t has mass*

$$\nu_t(I_t) = 2^{N\alpha - N}\,,$$

and so the average value of v_t on I_t is $2^{N\alpha}$.

iii) *Any dyadic interval $J \in \mathcal{D}_N$ not containing the point t has mass*

$$\nu_t(J) = 2^{N\alpha - d(I_t,J)\alpha - N}\,c^\alpha\,,$$

and so the average value of v_t on J is $2^{N\alpha - d(I_t,J)\alpha}\,c^\alpha$.

iv) *The measure ν_t associated to v_t is a dyadic doubling measure, with dyadic doubling constant $2^\alpha/(2-2^\alpha)$.*

v) *The weight v_t lies in every dyadic A_p space. Its dyadic A_1 constant is $(2-2^\alpha)^{-1}$. Its dyadic A_p constant, for $1 < p < \infty$, is*

$$(2-2^\alpha)^{-1}\,(2-2^{\alpha/(p-1)})^{-(p-1)}\,.$$

vi) *The weight v_t lies in all dyadic reverse Hölder-q spaces such that $1 < q < 1/\alpha$. Its dyadic reverse Hölder-q constant, for $1 < q < 1/\alpha$, is*

$$(2 - 2^{\alpha})\,(2 - 2^{\alpha q})^{-1/q}\,.$$

vii) *The dyadic doubling constant, the A_1^d constant, the A_p^d constant, and the RH_q^d constant of v_t decrease to 1 as α decreases to 0.*

Before proving Lemma 3.1 we define the truncated version w_t of the weight v_t, and state its properties in Lemma 3.3.

Definition 3.2. *Let L be an integer greater than 0. Let I_t^L denote the unique dyadic interval of length 2^{-L} containing t. The truncation of v_t at level L is the new weight w_t defined by*

$$w_t(x) = \begin{cases} (M_d(c\,\delta_t(x)))^{\alpha}\,, & x \notin I_t^L\,, \\ 2^{L\alpha}\,, & x \in I_t^L\,. \end{cases} \tag{21}$$

Denote by μ_t the measure whose density is w_t; so $\mu_t(\cdot) = \int_{\cdot} w_t(x)\,dx$.

The only difference between v_t and w_t is that v_t has been replaced on I_t^L by its average value on I_t^L. In other words, the "peak" of v_t around t has been lopped off, and replaced by the constant value $2^{L\alpha}$ on I_t^L. The truncated weight w_t has the same dyadic doubling constant and A_1^d constant as v_t, and smaller A_p^d and RH_q^d constants than v_t.

Lemma 3.3. *Suppose $0 < \alpha < 1$, and let $c = (2 - 2^{\alpha})^{1/\alpha}$. Fix $t \in [0, 1]$. Let w_t be the weight given by truncating v_t at level L,*

$$w_t(x) = \begin{cases} (M_d(c\,\delta_t(x)))^{\alpha}\,, & x \notin I_t^L\,, \\ 2^{L\alpha}\,, & x \in I_t^L\,, \end{cases}$$

where $t \in I_t^L \in \mathcal{D}_L$. Then w_t and its associated measure $\mu_t(\cdot) = \int_{\cdot} w_t(x)\,dx$ satisfy the following properties.

i) *The total mass of μ_t on $[0, 1]$ is 1.*

ii) *The dyadic interval $I_t \in \mathcal{D}_N$ containing the point t has mass*

$$\mu_t(I_t) = \begin{cases} 2^{N\alpha - N}\,, & \text{if } 0 \le N \le L \text{ (so } I_t \supseteq I_t^L)\,, \\ 2^{L\alpha - N}\,, & \text{if } N > L \text{ (so } I_t \subset I_t^L)\,. \end{cases}$$

Therefore the average value of w_t *on* I_t *is* $2^{N\alpha}$ *if* $0 \le N \le L$, *and* $2^{L\alpha}$ *if* $N > L$.

iii) *Any dyadic interval* $J \in \mathcal{D}_N$ *not containing the point* t *has mass*

$$\mu_t(J) = \begin{cases} 2^{N\alpha - d(I_t,J)\alpha - N}\, c^{\alpha}\,, & \text{if } J \not\subset I_t^L\,, \\ 2^{L\alpha - N}\,, & \text{if } J \subset I_t^L\,. \end{cases}$$

Therefore the average value of w_t *on* J *is* $2^{N\alpha - d(I_t,J)\alpha}\, c^{\alpha}$ *if* $J \not\subset I_t^L$, *and* $2^{L\alpha}$ *if* $J \subset I_t^L$.

iv) *The measure* μ_t *associated to* w_t *is a dyadic doubling measure, with dyadic doubling constant* $2^{\alpha}/(2 - 2^{\alpha})$.

v) *The weight* w_t *lies in every dyadic* A_p *space. Its dyadic* A_1 *constant is* $(2 - 2^{\alpha})^{-1}$. *Its dyadic* A_p *constant, for* $1 < p < \infty$, *is*

$$\frac{(2-2^{\alpha})^{-1}}{(2-2^{-\alpha/(p-1)})^{p-1}}\Big(1 + (2^{-1-\alpha/(p-1)})^{L}\Big(\frac{2-2^{-1-\alpha/(p-1)}}{(2-2^{\alpha})^{-1/(p-1)}-1}\Big)\Big)^{p-1},$$

which is less than the dyadic A_p *constant of the untruncated weight* v_t.

vi) *The weight* w_t *lies in all dyadic reverse Hölder-q spaces such that* $1 < q < 1/\alpha$. *Its dyadic reverse Hölder-q constant, for* $1 < q < 1/\alpha$, *is*

$$\frac{2-2^{\alpha}}{(2-2^{\alpha q})^{1/q}}\Big(1 + (2^{\alpha q-1})^{L}\Big(\frac{2-2^{\alpha q}}{(2-2^{\alpha})^{q}} - 1\Big)\Big)^{1/q},$$

which is less than the dyadic reverse Hölder-q constant of v_t.

vii) *The dyadic doubling constant, the* A_1^d *constant, the* A_p^d *constant, and the* RH_q^d *constant of* w_t *converge to* 1 *as* α *decreases to* 0.

Next we establish the above properties, first for the untruncated weight $v_t(x) = (M_d(c\,\delta_t(x)))^{\alpha}$, and then for the weight $w_t(x)$ which has been truncated at level L.

Proof of Lemma 3.1. i) We verify that the total mass of ν_t is 1

$$\begin{aligned}\nu_t([0,1]) &= \int_0^1 v_t(x)\,dx \\ &= \sum_{k=0}^{\infty}\int_{\mathcal{F}_k} v_t(x)\,dx\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} 2^{-k-1}\, 2^{k\alpha} c^{\alpha} \qquad (22)\\
&= \frac{c^{\alpha}}{2} \sum_{k=0}^{\infty} (2^{\alpha-1})^k \\
&= \frac{2-2^{\alpha}}{2}\, \frac{1}{1-2^{\alpha-1}} \\
&= 1\,.
\end{aligned}
$$

ii) If $I_t \in \mathcal{D}_N$ is the unique dyadic interval of length 2^{-N} which contains t, then $I_t = \cup_{k=N}^{\infty} \mathcal{F}_k$, and so the total mass of I_t is

$$
\begin{aligned}
\nu_t(I_t) &= \int_{I_t} v_t(x)\, dx \\
&= \sum_{k=N}^{\infty} \int_{\mathcal{F}_k} v_t(x)\, dx \qquad (23)\\
&= \frac{2-2^{\alpha}}{2}\, \frac{(2^{\alpha-1})^N}{1-2^{\alpha-1}} \\
&= 2^{N\alpha-N}\,.
\end{aligned}
$$

iii) Now suppose $J \in \mathcal{D}_N$ does not contain t. The smallest dyadic interval containing both I_t and J is $d(I_t, J)$ generations back. Here $1 \leq d(I_t, J) \leq N$. Then $J \subset \mathcal{F}_{N-d(I_t,J)}$. By (20), $v_t \equiv 2^{(N-d(I_t,J))\alpha} c^{\alpha}$ on J, and so

$$
\nu_t(J) = \int_J v_t = 2^{-N}\, 2^{(N-d(I_t,J))\alpha}\, c^{\alpha} = 2^{N\alpha - d(I_t,J)\alpha - N}\, c^{\alpha}\,. \qquad (24)
$$

Thus the interval I_t containing t has the largest mass in $\mathcal{D}_N$, and the mass of $J \in \mathcal{D}_N$ decreases as the generational distance of J from I_t increases.

iv) The weight v_t is doubling on dyadic sister intervals, with maximal ratio $2^{\alpha}/(2-2^{\alpha})$. For if $I, J \in \mathcal{D}_N$ have the same parent, and if neither I nor J contains t, then I and J are at the same generational distance from the $I_t \in \mathcal{D}_N$ which does contain t, and so by (24)

$$
\nu_t(I) = 2^{N\alpha - d(I_t,I)\alpha - N}\, c^{\alpha} = 2^{N\alpha - d(I_t,J)\alpha - N}\, c^{\alpha} = \nu_t(J)\,. \qquad (25)
$$

On the other hand, if $I_t \ni t$, and I_t and J have the same parent, then $d(I_t, J) = 1$ and so the worst possible ratio is

$$
\frac{\nu_t(I_t)}{\nu_t(J)} = \frac{2^{N\alpha-N}}{2^{N\alpha-\alpha-N}\, c^{\alpha}} = \frac{2^{\alpha}}{c^{\alpha}} = \frac{2^{\alpha}}{2-2^{\alpha}}\,, \qquad (26)
$$

as required. Further, $2^\alpha/(2-2^\alpha)$ decreases to 1 as α decreases to 0.

v) Next we show that the A_1^d constant of v_t is $(2-2^\alpha)^{-1}$. If $J \in \mathcal{D}_N$ does not contain t, then v_t is constant on J and so

$$\frac{1}{|J|}\int_J v_t = \text{ess inf}_J\, v_t\ .$$

For the interval $I_t \in \mathcal{D}_N$ which contains t, the mean value is

$$\frac{1}{|I_t|}\int_{I_t} v_t = 2^N\, 2^{N\alpha-N} = 2^{N\alpha}\ .$$

The interval I_t can be written as $\cup_{k=N}^{\infty}\mathcal{F}_k$, and on each $\mathcal{F}_k$ with $k \geq N$, $v_t \equiv 2^{k\alpha}\,c^\alpha \geq 2^{N\alpha}\,c^\alpha$, by (20). So the essential infimum of v_t on I_t is $2^{N\alpha}\,c^\alpha$, and we have

$$\frac{\dfrac{1}{|I_t|}\displaystyle\int_{I_t} v_t}{\text{ess inf}_{I_t} v_t} = \frac{2^{N\alpha}}{2^{N\alpha}\,c^\alpha} = \frac{1}{2-2^\alpha}\ . \tag{27}$$

Hence v_t is in A_1^d, with A_1^d constant equal to $(2-2^\alpha)^{-1}$, as claimed. The A_1^d constant $(2-2^\alpha)^{-1}$ decreases to 1 as α decreases to 0.

Since the A_p^d spaces are nested and increasing as $p \longrightarrow \infty$, the weight v_t is actually in every A_p^d. We compute the A_p^d constant of v_t.

First, if $J \in \mathcal{D}_N$, and J does not contain t, then v_t is constant on J and so the product in the A_p^d condition (15) is 1. Second, if $I_t \in \mathcal{D}_N$ contains t, then $I_t = \cup_{k=N}^{\infty}\mathcal{F}_k$, and on each $\mathcal{F}_k$, $v_t \equiv 2^{k\alpha}\,c^\alpha$. Now by (23) the mean value of v_t on I_t is $2^{N\alpha}$. So the product in (15) is

$$\begin{aligned}
&\Big(\frac{1}{|I_t|}\int_{I_t} v_t\Big)\Big(\frac{1}{|I_t|}\int_{I_t}\Big(\frac{1}{v_t}\Big)^{1/(p-1)}\Big)^{p-1}\\
&= 2^{N\alpha}\Big(2^N\sum_{k=N}^{\infty}\int_{\mathcal{F}_k}\Big(\frac{1}{v_t}\Big)^{1/(p-1)}\Big)^{p-1}\\
&= 2^{N\alpha}\Big(2^N\sum_{k=N}^{\infty}2^{-k-1}(2^{-\alpha k}\,c^{-\alpha})^{1/(p-1)}\Big)^{p-1}\\
&= 2^{N\alpha}\,c^{-\alpha}\Big(2^{N-1}\sum_{k=N}^{\infty}(2^{-1-\alpha/(p-1)})^k\Big)^{p-1}\\
&= 2^{N\alpha}\,c^{-\alpha}\Big(2^N\,\frac{(2^{-1-\alpha/(p-1)})^N}{2-2^{-\alpha/(p-1)}}\Big)^{p-1}\\
&= \frac{1}{2-2^\alpha}\,\frac{1}{(2-2^{-\alpha/(p-1)})^{p-1}}\ .
\end{aligned} \tag{28}$$

The last expression is greater than 1, and so it is the A_p^d constant of the weight v_t. Further, the A_p^d constant decreases to 1 as α decreases to 0, since both its factors decrease to 1.

vi) We now show that our weights v_t also lie in the dyadic reverse Hölder-q space RH_q^d, as long as $q < 1/\alpha$, and we compute their dyadic reverse Hölder-q constants. For dyadic intervals J which do not contain t, the weight v_t is constant on J and so the reverse Hölder condition (17) holds with constant 1. For the dyadic interval I_t of length 2^{-N} which contains t, we begin by computing the average value of v_t^q on $I_t = \cup_{k=N}^{\infty} \mathcal{F}_k$

$$
\begin{aligned}
\frac{1}{|I_t|}\int_{I_t} v_t^q &= 2^N \sum_{k=N}^{\infty} \int_{\mathcal{F}_k} v_t^q \\
&= 2^N \sum_{k=N}^{\infty} 2^{k\alpha q}\, c^{\alpha q}\, 2^{-k-1} \\
&= 2^N \, \frac{c^{\alpha q}}{2} \sum_{k=N}^{\infty} (2^{\alpha q-1})^k \\
&= 2^N \, \frac{c^{\alpha q}}{2} \, \frac{(2^{\alpha q-1})^N}{1-2^{\alpha q-1}} \\
&= \frac{(2-2^{\alpha})^q}{2-2^{\alpha q}}\, 2^{\alpha q N}\,,
\end{aligned} \tag{29}
$$

assuming that $2^{\alpha q-1} < 1$, in other words that $q < 1/\alpha$. Therefore, since the mean value of v_t on I_t is $2^{\alpha N}$,

$$
\Big(\frac{1}{|I_t|}\int_{I_t} v_t^q\Big)^{1/q} = \frac{2-2^{\alpha}}{(2-2^{\alpha q})^{1/q}}\,\frac{1}{|I_t|}\int_{I_t} v_t\,. \tag{30}
$$

Since the expression $(2-2^{\alpha})/(2-2^{\alpha q})^{1/q}$ is greater than 1, it is the dyadic reverse Hölder-q constant of v_t.

vii) We have already observed that the dyadic doubling constant, the A_1^d constant, and the A_p^d constant of v_t decrease to 1 as α decreases to 0. The numerator and denominator of the RH_q^d constant $(2-2^{\alpha})/(2-2^{q\alpha})^{1/q}$ of v_t both increase to 1 as α decreases to 0. The function $f(\alpha) = (2 - 2^{\alpha})^q - (2-2^{q\alpha})$ has $f(0) = 0$ and has positive derivative for $0 < \alpha < 1/q$. It follows that the RH_q^d constant of v_t decreases to 1 as α decreases to 0.

In particular, we have shown that $v_t \in RH_q^d$ for all $q < 1/\alpha$, and that by decreasing α towards 0 we can both make the dyadic reverse Hölder-q

constant arbitrarily small, and make our weights lie in RH_q^d spaces for arbitrarily large q.

This completes the proof of Lemma 3.1.

We establish the corresponding properties for the truncated weight w_t.

Proof of Lemma 3.3. i) The truncated weight w_t has the same total mass 1 on $[0,1]$ as v_t does, since the only change is to replace v_t on the subinterval I_t^L by its average value on I_t^L.

ii) The mass of $I_t \in \mathcal{D}_N$ is unchanged by truncation of the weight at level L, if $I_t \supseteq I_t^L$. However if $I_t \subset I_t^L$, then $w_t \equiv 2^{L\alpha}$ on I_t, so the mass of I_t is $2^{L\alpha - N}$.

iii) If $t \notin J \in \mathcal{D}_N$ and $J \not\subset I_t^L$, then the mass of J is unchanged by truncation of v_t at level L. If $J \subset I_t^L$, then the mass of J becomes $2^{L\alpha-N}$.

iv) The ratio $\mu_t(I_t)/\mu(J) = 2^\alpha/(2-2^\alpha)$ is achieved whenever I_t and J are dyadic sisters of length 2^{-N}, $I_t \ni t$, and $0 \le N < L$. For all other pairs of dyadic sisters, the ratio of the masses is 1. Therefore the dyadic doubling constant of w_t is still $2^\alpha/(2-2^\alpha)$, as for v_t.

v) We need only consider the dyadic intervals $I_t \ni t$ of length $|I_t| = 2^{-N}$, where $0 \le N < L$, since the truncated weight w_t is constant on all other dyadic intervals.

The product in the A_1^d condition (16) achieves the value $(2-2^\alpha)^{-1}$ for each such interval I_t, so the A_1^d constant of w_t is still $(2-2^\alpha)^{-1}$. The first factor in the product in the A_p^d condition (15) is the mean value of w_t on I_t, which is still $2^{N\alpha}$. Integrating $w_t^{-1/(p-1)}$ on I_t,

$$
\begin{aligned}
&\int_{I_t} \left(\frac{1}{w_t}\right)^{1/(p-1)} \\
&= \sum_{k=N}^{L-1} \int_{\mathcal{F}_k} \left(\frac{1}{2^{k\alpha}\, c^\alpha}\right)^{1/(p-1)} dx + \int_{I_t^L} \left(\frac{1}{2^{L\alpha}}\right)^{1/(p-1)} dx \\
&= \left(\frac{c^{-\alpha/(p-1)}}{2} \sum_{k=N}^{L-1} (2^{-1-\alpha/(p-1)})^k\right) + (2^{-1-\alpha/(p-1)})^L \\
&= \frac{c^{-\alpha/(p-1)}}{2} \, \frac{(2^{-1-\alpha/(p-1)})^N - (2^{-1-\alpha/(p-1)})^L}{1 - 2^{-1-\alpha/(p-1)}} + (2^{-1-\alpha/(p-1)})^L \\
&= \frac{(2-2^\alpha)^{-1/(p-1)}}{2 - 2^{-\alpha/(p-1)}} \left((2^{-1-\alpha/(p-1)})^N - (2^{-1-\alpha/(p-1)})^L\right)
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&+ (2^{-1-\alpha/(p-1)})^L \\
&= \frac{(2-2^{\alpha})^{-1/(p-1)}}{2-2^{-\alpha/(p-1)}} \\
&\quad \cdot \Big((2^{-1-\alpha/(p-1)})^N + (2^{-1-\alpha/(p-1)})^L \Big(\frac{2-2^{-1-\alpha/(p-1)}}{(2-2^{\alpha})^{-1/(p-1)}} - 1 \Big) \Big) .
\end{aligned}
$$

Hence the product in the A_p^d condition (15) is

$$
\begin{aligned}
&\Big(\frac{1}{|I_t|} \int_{I_t} w_t \Big) \Big(\frac{1}{|I_t|} \int_{I_t} \Big(\frac{1}{w_t} \Big)^{1/(p-1)} \Big)^{p-1} \\
&\qquad = \frac{(2-2^{\alpha})^{-1}}{(2-2^{-\alpha/(p-1)})^{p-1}} \, 2^{N\alpha + N(p-1)} \\
&\qquad\quad \cdot \Big((2^{-1-\alpha/(p-1)})^N + (2^{-1-\alpha/(p-1)})^L \Big(\frac{2-2^{-1-\alpha/(p-1)}}{(2-2^{\alpha})^{-1/(p-1)}} - 1 \Big) \Big) \\
&\qquad = \frac{(2-2^{\alpha})^{-1}}{(2-2^{-\alpha/(p-1)})^{p-1}} \\
&\qquad\quad \cdot \Big(1 + (2^{-1-\alpha/(p-1)})^{L-N} \Big(\frac{2-2^{-1-\alpha/(p-1)}}{(2-2^{\alpha})^{-1/(p-1)}} - 1 \Big) \Big)^{p-1} .
\end{aligned} \tag{32}
$$

The first factor on the right hand side is the old A_p^d constant of the untruncated weight v_t.

The second factor is positive but less than 1, because $2-2^{-1-\alpha/(p-1)} < (2-2^{\alpha})^{-1/(p-1)}$ for $p > 1$.

Further, since $2^{-1-\alpha/(p-1)} < 1$, the second factor achieves its maximum when $N = 0$. Therefore the dyadic A_p constant of the truncated weight w_t is

$$
\frac{(2-2^{\alpha})^{-1}}{(2-2^{-\alpha/(p-1)})^{p-1}} \Big(1 + (2^{-1-\alpha/(p-1)})^L \Big(\frac{2-2^{-1-\alpha/(p-1)}}{(2-2^{\alpha})^{-1/(p-1)}} - 1 \Big) \Big)^{p-1} , \tag{33}
$$

which is less than the dyadic A_p constant $(2-2^{\alpha})^{-1} \, (2-2^{\alpha/(p-1)})^{-(p-1)}$ of the untruncated weight v_t, as required.

vi) When we truncate the weight v_t at level L, the dyadic reverse Hölder-q constant also decreases. Again, we need only consider the dyadic intervals $I_t \ni t$ of length $|I_t| = 2^{-N}$, where $0 \leq N < L$, since the truncated weight w_t is constant on all other dyadic intervals. The average value of w_t on I_t is still $2^{N\alpha}$. Now we compute the average value of w_t^q on I_t

$$\begin{aligned}
\frac{1}{|I_t|}\int_{I_t} w_t^q &= 2^N\Big(\sum_{k=N}^{L-1}\int_{\mathcal{F}_k}(2^{k\alpha}\,c^\alpha)^q\,dx + \int_{I_t^L}(2^{L\alpha})^q\,dx\Big)\\
&= 2^N\Big(\Big(\frac{c^{\alpha q}}{2}\sum_{k=N}^{L-1}(2^{\alpha q-1})^k\Big) + (2^{\alpha q-1})^L\Big)\\
&= 2^N\Big(\frac{c^{\alpha q}}{2}\,\frac{(2^{\alpha q-1})^N-(2^{\alpha q-1})^L}{1-2^{\alpha q-1}} + (2^{\alpha q-1})^L\Big) \qquad (34)\\
&= 2^N\Big(\frac{(2-2^\alpha)^q}{2-2^{\alpha q}}\,((2^{\alpha q-1})^N-(2^{\alpha q-1})^L) + (2^{\alpha q-1})^L\Big)\\
&= 2^N\,\frac{(2-2^\alpha)^q}{2-2^{\alpha q}}\Big((2^{\alpha q-1})^N + (2^{\alpha q-1})^L\Big(\frac{2-2^{\alpha q}}{(2-2^\alpha)^q}-1\Big)\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\Big(\frac{1}{|I_t|}\int_{I_t} w_t^q\Big)^{1/q}}{\frac{1}{|I_t|}\int_{I_t} w_t} &= \frac{2-2^\alpha}{(2-2^{\alpha q})^{1/q}}\,2^{(N/q)-N\alpha}\\
&\quad\cdot\Big((2^{\alpha q-1})^N + (2^{\alpha q-1})^L\Big(\frac{2-2^{\alpha q}}{(2-2^\alpha)^q}-1\Big)\Big)^{1/q} \qquad (35)\\
&= \frac{2-2^\alpha}{(2-2^{\alpha q})^{1/q}}\Big(1+(2^{\alpha q-1})^{L-N}\Big(\frac{2-2^{\alpha q}}{(2-2^\alpha)^q}-1\Big)\Big)^{1/q}.
\end{aligned}$$

The first factor on the right hand side is the old RH_q^d constant of the untruncated weight v_t. The second factor is positive but less than 1, because $2-2^{\alpha q}<(2-2^\alpha)^q$ for $q>1$. Further, since $2^{\alpha q-1}<1$, the second factor achieves its maximum when $N=0$. Therefore the dyadic reverse Hölder-q constant of the truncated weight w_t is

$$\frac{2-2^\alpha}{(2-2^{\alpha q})^{1/q}}\Big(1+(2^{\alpha q-1})^{L}\Big(\frac{2-2^{\alpha q}}{(2-2^\alpha)^q}-1\Big)\Big)^{1/q}, \qquad (36)$$

and this is less than $(2-2^\alpha)\,(2-2^{\alpha q})^{-1/q}$, as required.

vii) The dyadic doubling constant and the A_1^d constant of w_t are the same as those of v_t, so they decrease to 1 as α decreases to 0. The A_p^d

constant, for $1 < p < \infty$, and the RH_q^d constant of w_t are less than those of v_t, and so they also converge to 1 as α decreases to 0. This completes the proof of Lemma 3.3.

A weight w is in A_∞^d if and only if there is a constant C such that

$$\frac{1}{|I|}\int_I w \le C \exp\Big(\frac{1}{|I|}\int_I \log w\Big), \tag{37}$$

for all dyadic intervals I; the smallest such C is the A_∞^d constant of w. Short computations, similar to those for A_p^d above, show that the A_∞^d constant of the untruncated weight v_t is $c^{-\alpha}\,2^{-\alpha}$, which decreases to 1 as α decreases to 0, and that the A_∞^d constant of the weight w_t truncated at level L is $(c^{-\alpha}\,2^{-\alpha})^{1-2^{-L}}$, which is less than that of v_t.

4. Proof of Theorem 1.4.

In this section, we prove the main result of our paper, Theorem 1.4. Here C denotes a constant that may change from line to line.

We must prove that, with the definitions given in the introduction, the translation average $\mu(x)$ is not a doubling measure. Specifically, we show that for our sequence of pairs A^j, B^j of non-dyadic intervals in $[0,1]$ of length $|A^j| = |B^j| = 2^{-N_j}$, we have $\mu(A^j)/\mu(B^j) \longrightarrow \infty$ as $j \longrightarrow \infty$, where $\mu(\cdot) = \int_\cdot w(x)\,dx$ as usual. Although our intervals A^j, B^j are not adjacent, they are always separated by exactly twice the length of A^j: for each j our B^j is the translation of A^j to the right by $3\,|A^j|$. To show that this is sufficient, note that if μ is doubling, there is a constant C such that for each pair of adjacent intervals of equal length, the mass of one is at most C times the mass of the other. Let the intervals E^j and F^j be the translations of A^j to the right by $|A^j|$ and $2\,|A^j|$ respectively. Then $\mu(A^j) \le C\mu(E^j) \le C^2\mu(F^j) \le C^3\mu(B^j)$, contradicting the fact that $\mu(A^j)/\mu(B^j) \longrightarrow \infty$ as $j \longrightarrow \infty$.

Given a point $t \in [0,1]$, let I_t^j denote the unique dyadic interval in $\mathcal{D}_{N_j}$ which contains t, dropping the N in $I_t^{N_j}$ for ease of notation. Label the dyadic intervals in $\mathcal{D}_{N_j}$ immediately to the right of I_t^j as J_t^j, K_t^j, L_t^j, and M_t^j, reading from left to right. When t is very near the right end of $[0,1]$, some or all of J_t^j, K_t^j, L_t^j, and M_t^j are wrapped around to the left end of $[0,1]$. The nested, decreasing subsets $\mathcal{G}_j$ of $[0,1]$ are defined by

$$\mathcal{G}_0 = [0,1]\,,$$

and

$$
(38)\qquad \begin{aligned} \mathcal{G}_j &= \{t \in \mathcal{G}_{j-1} :\ d(I_t^j, K_t^j) = N_j\beta\} \\ &= \{t \in [0,1] :\ d(I_t^1, K_t^1) = N_1\beta, \ldots, d(I_t^j, K_t^j) = N_j\beta\} \end{aligned}
$$

for $j \geq 1$. We will see that each $\mathcal{G}_j$ is a non-empty, finite union of dyadic intervals.

For $t \in \mathcal{G}_0 \setminus \mathcal{G}_1$, let $w_t \equiv 1$, so that μ_t is Lebesgue measure on $[0,1]$. For $t \in \mathcal{G}_j \setminus \mathcal{G}_{j+1}$, $j \geq 1$, let w_t be the truncation at level N_j of the weight $v_t(x) = (M_d(c\,\delta_t(x)))^\alpha$, and let μ_t be the measure whose density is $w_t(x)$. In Lemma 3.3, we have explicitly computed the μ_t-masses of dyadic intervals such as I_t^j and K_t^j. The key idea is that since A^j is centred at 0, the interval $A^j + t$ almost coincides with the dyadic interval $I_t^j \ni t$, so we can bound $\mu(A^j) = \int_0^1 \mu_t(A^j + t)\,dt$ below via $\mu_t(I_t^j)$. Similarly, $B^j + t$ almost coincides with K_t^j, so we can bound $\mu(B^j)$ above via $\mu_t(K_t^j)$. Moreover, for $t \in \mathcal{G}_j \setminus \mathcal{G}_{j+1}$, the generational distance $d(I_t^j, K_t^j) = N_j\beta$ is very large, and so the definition of w_t ensures that $\mu_t(I_t^j)$ is much larger than $\mu_t(K_t^j)$.

Lemma 4.1. *For $j = 0, 1, 2, \ldots$, the measure of $\mathcal{G}_j$ is given by*

$$
(39)\qquad |\mathcal{G}_j| = 2^{-N_1\beta - N_2\beta - \cdots - N_j\beta + j}\,.
$$

Proof. Fix j, and consider the subset S_l^j of $[0,1]$ defined by

$$
(40)\qquad S_l^j = \{t :\ d(I_t^j, K_t^j) = l\}\,.
$$

The dyadic intervals I_t^j and K_t^j have length 2^{-N_j}, and K_t^j is the translation of I_t^j to the right by $2\,|I_t^j|$. So I_t^j and K_t^j are never dyadic sisters.

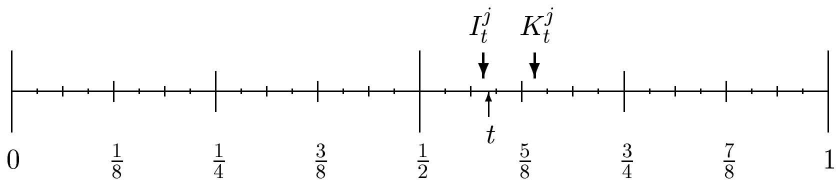

Figure 2. The collection $\mathcal{D}_5$ of dyadic intervals of length 2^{-5}, showing intervals $I_t^j \ni t$ and K_t^j with $d(I_t^j, K_t^j) = 3$.

The values taken by the generational distance $d(I_t^j, K_t^j)$ are the integers l such that $2 \leq l \leq N_j$.

Let t run from 0 to 1. Half of the 2^{N_j} dyadic intervals I_t^j in $\mathcal{D}_{N_j}$ have $d(I_t^j, K_t^j) = 2$. Reading from left to right, these intervals are the first and second, fifth and sixth, ninth and tenth, and so on, up to the fourth last and third last. Half of the remaining dyadic intervals I_t^j in $\mathcal{D}_{N_j}$ have $d(I_t^j, K_t^j) = 3$; these are the third and fourth, eleventh and twelfth, and so on. Continuing in this way we see that the total length of S_l^j is

$$|S_l^j| = |\{t: \ d(I_t^j, K_t^j) = l\}| = 2^{-l+1}, \qquad \text{for } 2 \leq l \leq N_j - 1. \tag{41}$$

The set S_l^j consists of $2^{N_j - l+1}$ dyadic intervals of length 2^{-N_j}, arranged in $2^{N_j - l}$ blocks of two adjacent intervals. The blocks are equally spaced around the unit circle, and each block has length 2^{-N_j+1}. Therefore the distance between consecutive blocks is

$$\frac{1 - 2^{-l+1}}{2^{N_j - l}} = 2^{-N_j}\,(2^l - 2). \tag{42}$$

When $l = N_j$, we have instead

$$|S_{N_j}^j| = |\{t: \ d(I_t^j, K_t^j) = N_j\}| = 2^{-l+2} = 2^{-N_j+2}, \qquad \text{for } l = N_j. \tag{43}$$

This is because there are four intervals rather than two with $d(I_t^j, K_t^j) = N_j$: the two intervals I_t^j immediately to the left of 1 as well as the two immediately to the left of 1/2.

We prove the lemma by induction. First, $\mathcal{G}_1 = \{t : \ d(I_t^1, K_t^1) = N_1\beta\} = S_{N_1\beta}^1$, and the hypotheses (7) and $\beta < 1$ imply that $2 \leq N_1\beta < N_1$. Hence, by equation (41),

$$|\mathcal{G}_1| = 2^{-N_1\beta+1}. \tag{44}$$

Next, fix j and assume that

$$|\mathcal{G}_j| = 2^{-N_1\beta - \cdots - N_j\beta + j}. \tag{45}$$

Recall that

$$\begin{aligned} \mathcal{G}_j &= \{t \in [0,1]: \ d(I_t^1, K_t^1) = N_1\beta, \ldots, d(I_t^j, K_t^j) = N_j\beta\} \\ &= S_{N_1\beta}^1 \cap \cdots \cap S_{N_j\beta}^j, \end{aligned} \tag{46}$$

and define

$$\widetilde{\mathcal{G}_{j+1}} = \{t \in [0,1] : \ d(I_t^{j+1}, K_t^{j+1}) = N_{j+1}\beta\}\,. \tag{47}$$

So $\widetilde{\mathcal{G}_{j+1}}$ contains $\mathcal{G}_{j+1}$, and in fact $\mathcal{G}_{j+1}$ is the intersection of $\widetilde{\mathcal{G}_{j+1}}$ with $\mathcal{G}_j$. Since $\widetilde{\mathcal{G}_{j+1}} = S^{j+1}_{N_{j+1}\beta}$, and $2 \le N_j\beta < N_j$, the blocks of $\widetilde{\mathcal{G}_{j+1}}$ are equally spaced around the circle, and equation (42) shows that the distance between consecutive blocks of $\widetilde{\mathcal{G}_{j+1}}$ is

$$2^{-N_{j+1}}\,(2^{N_{j+1}\beta} - 2)\,. \tag{48}$$

The blocks of $\mathcal{G}_j$ are of length 2^{-N_j+1}, and each of them contains two dyadic sister intervals of length 2^{-N_j}.

The set $\mathcal{G}_{j+1} = \widetilde{\mathcal{G}_{j+1}} \cap \mathcal{G}_j$ will be nonempty as soon as the gaps in $\widetilde{\mathcal{G}_{j+1}}$ are shorter than the blocks of $\mathcal{G}_j$, and this happens as soon as N_{j+1} is sufficiently large. Note that the two dyadic intervals in each block are dyadic sisters, and so the block itself is also a dyadic interval. Therefore, since dyadic intervals are either nested or disjoint, each block of $\widetilde{\mathcal{G}_{j+1}}$ that meets a block of $\mathcal{G}_j$ must be wholly contained in it.

Now, our hypothesis (7) implies that

$$N_{j+1} \ge \frac{N_1 + \cdots + N_j}{\alpha - \beta\,\alpha - \beta} > \frac{N_j}{1-\beta}\,, \tag{49}$$

since $\alpha - \beta\,\alpha < 1$. Therefore

$$2^{-N_{j+1}}\,(2^{N_{j+1}\beta} - 2) < 2^{-N_{j+1}(1-\beta)} < 2^{-N_j} < 2^{-N_j+1}\,. \tag{50}$$

We have shown that the distance between consecutive blocks of $\widetilde{\mathcal{G}_{j+1}}$ is shorter than the length of the blocks in $\mathcal{G}_j$, and so $\mathcal{G}_{j+1}$ is nonempty.

Finally,

$$\frac{|\mathcal{G}_{j+1}|}{2^{-N_1\beta - \cdots - N_j\beta + j}} = \frac{|\widetilde{\mathcal{G}_{j+1}} \cap \mathcal{G}_j|}{|\mathcal{G}_j|} = \frac{|\widetilde{\mathcal{G}_{j+1}}|}{|[0,1]|} = 2^{-N_{j+1}\beta + 1}\,, \tag{51}$$

and so

$$|\mathcal{G}_{j+1}| = 2^{-N_1\beta - \cdots - N_j\beta - N_{j+1}\beta + j + 1}\,, \tag{52}$$

as required.

Lemma 4.2. *The μ-mass of the interval A^j is bounded below as follows*

$$\mu(A^j) \geq C\, 2^{-N_1\beta-\cdots-N_j\beta+N_j\alpha-N_j+j}\,, \tag{53}$$

where C is independent of j.

Proof. The key point is that $A^j + t$ contains at least one of the halves of I_t^j, and so $\mu_t(A^j+t) \geq C\mu_t(I_t)$ with a constant C depending only on the dyadic doubling constant of μ_t. Hence, using lemmas 3.3.ii) and 4.1 in the fourth line,

$$\begin{aligned}
\mu(A^j) &= \int_0^1 \mu_t(A^j+t)\,dt \\
&\geq C\int_0^1 \mu_t(I_t^j)\,dt \\
&\geq C\int_{\mathcal{G}_j\setminus\mathcal{G}_{j+1}} \mu_t(I_t^j)\,dt \\
&\geq C\, 2^{-N_1\beta-\cdots-N_j\beta+j-1}\, 2^{N_j\alpha-N_j} \\
&= C\, 2^{-N_1\beta-\cdots-N_j\beta+N_j\alpha-N_j+j}\,,
\end{aligned} \tag{54}$$

as required.

Lemma 4.3. *The μ-mass of the interval B^j is bounded above as follows*

$$\mu(B^j) \leq C\, 2^{-N_1\beta-\cdots-N_j\beta+N_j\alpha-N_j\beta\alpha-N_j+j}\,, \tag{55}$$

where C is independent of j.

Proof. Fix j. We first bound $\mu(B^j)$ by integrals over the sets $\mathcal{G}_k\setminus\mathcal{G}_{k+1}$. Noting that B^j+t lies in $K_t^j\cup L_t^j\cup M_t^j$, and that $\mu_t(K_t^j)\geq\mu_t(L_t^j)\geq\mu_t(M_t^j)$ because w_t decays away from its peak at $t\in I_t^j$, we obtain

$$\begin{aligned}
\mu(B^j) &= \int_0^1 \mu_t(B^j+t)\,dt \\
&\leq \int_0^1 \mu_t(K_t^j)+\mu_t(L_t^j)+\mu_t(M_t^j)\,dt \\
&\leq 3\int_0^1 \mu_t(K_t^j)\,dt
\end{aligned} \tag{56}$$

$$= 3\int_{\mathcal{G}_0\setminus\mathcal{G}_1} \mu_t(K_t^j)\,dt + 3\sum_{k=1}^{\infty}\int_{\mathcal{G}_k\setminus\mathcal{G}_{k+1}} \mu_t(K_t^j)\,dt$$

$$\leq 3\cdot 2^{-N_j} + 3\sum_{k=1}^{\infty}\int_{\mathcal{G}_k\setminus\mathcal{G}_{k+1}} \mu_t(K_t^j)\,dt\,.$$

The last inequality holds because, for $t \in \mathcal{G}_0 \setminus \mathcal{G}_1$, we have $w_t \equiv 1$ and so $\mu_t(K_t^j) = |K_t^j| = 2^{-N_j}$.

Next, we bound $\mu_t(K_t^j)$ for $t \in \mathcal{G}_k\setminus\mathcal{G}_{k+1}$, $k > 0$. We need a preliminary lemma.

Lemma 4.4. *Fix j. For $t \in \mathcal{G}_k$, $k = 1, 2, \dots$, the μ_t-mass of the interval K_t^j is bounded above as follows*

$$\mu_t(K_t^j) \leq \begin{cases} 2^{N_j\alpha - N_j\beta\alpha - N_j}\,, & \text{if } k \geq j\,, \\ 2^{N_k\alpha - N_j}\,, & \text{if } 1 \leq k < j\,. \end{cases} \tag{57}$$

The quantity $2^{-N_j\beta\alpha}$ which appears here is the key to the whole paper. It leads to the result $\mu(A^j)/\mu(B^j) \geq C\,2^{N_j\beta\alpha}$.

Proof. The truncated weight w_t is identically equal to $2^{N_k\alpha}$ on I_t^k. Since $t \in \mathcal{G}_k$, we have

$$d(I_t^1, K_t^1) = N_1\beta, \dots, d(I_t^k, K_t^k) = N_k\beta\,. \tag{58}$$

If $k \geq j$, then $d(I_t^j, K_t^j) = N_j\beta$, and K_t^j is disjoint from I_t^k since $I_t^k \subset I_t^j$.

Figure 3. Here $k \geq j$, and the weight w_t is constant on the small interval I_t^k inside I_t^j.

We estimate the μ_t-mass of the interval K_t^j by applying the first case of Lemma 3.3.iii), with $J = K_t^j$, $I_t = I_t^j$, $I_t^L = I_t^k$, $L = N_k$, and $N = N_j$. We obtain

$$\begin{aligned} \mu_t(K_t^j) &= 2^{N_j\alpha - d(I_t^j, K_t^j)\alpha - N_j}\,c^\alpha \\ &= 2^{N_j\alpha - N_j\beta\alpha - N_j}\,c^\alpha \\ &\leq 2^{N_j\alpha - N_j\beta\alpha - N_j}\,, \end{aligned} \tag{59}$$

since $c^\alpha = 2 - 2^\alpha < 1$.

Now suppose $1 \leq k < j$, so I_t^k contains I_t^j. There are two cases. If I_t^k also contains K_t^j, then applying the second case of Lemma 3.3.iii) we obtain $\mu_t(K_t^j) = 2^{N_k\alpha - N_j}$, as required.

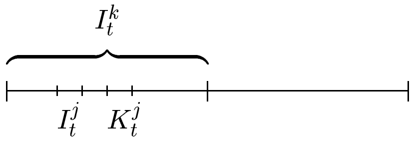

Figure 4. Here $1 \leq k < j$, the weight w_t is constant on the large interval I_t^k, and K_t^j lies in I_t^k.

If, however, t is so close to the right endpoint of I_t^k that K_t^j lies to the right of I_t^k, then the constant value of w_t on K_t^j is less than the value $2^{N_k\alpha}$ of w_t on I_t^k, by the definition of w_t.

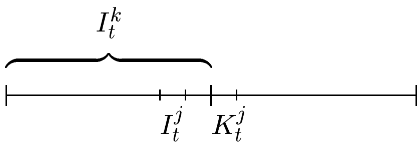

Figure 5. Here $1 \leq k < j$, the weight w_t is constant on the large interval I_t^k, and K_t^j lies outside I_t^k.

Therefore

$$\mu_t(K_t^j) = \int_{K_t^j} w_t(x)\,dx \leq 2^{N_k\alpha - N_j}\,, \tag{60}$$

as required.

Proof of Lemma 4.3, continued. We consider the cases $k = j$, $k > j$, and $1 \leq k < j$ separately. When $k = j$, Lemmas 4.1 and 4.4 imply that

$$\begin{aligned} \int_{\mathcal{G}_j \setminus \mathcal{G}_{j+1}} \mu_t(K_t^j)\,dt &\leq |\mathcal{G}_j|\,2^{N_j\alpha - N_j\beta\alpha - N_j} \\ &= 2^{-N_1\beta - \cdots - N_j\beta + N_j\alpha - N_j\beta\alpha - N_j + j}\,. \end{aligned} \tag{61}$$

Next, when $k > j$, the integral over $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$ is bounded by $2^{-k}C_j$, where

$$C_j = 2^{-N_1\beta - \cdots - N_j\beta + N_j\alpha - N_j\beta\alpha - N_j + j} \tag{62}$$

is the exponential factor on the right hand side of the desired inequality in Lemma 4.3. For by Lemma 4.4,

$$\mu_t(K_t^j) \le 2^{N_j\alpha - N_j\beta\alpha - N_j}\,, \tag{63}$$

and so by Lemma 4.1 on the size of $\mathcal{G}_k$,

$$\begin{aligned}\int_{\mathcal{G}_k\setminus\mathcal{G}_{k+1}} \mu_t(K_t^j)\,dt &\le 2^{-N_1\beta-\cdots-N_k\beta+k+N_j\alpha-N_j\beta\alpha-N_j}\\ &= 2^{-N_{j+1}\beta-\cdots-N_k\beta+k-j}\,C_j\,.\end{aligned} \tag{64}$$

We claim that the first factor in the last expression is at most 2^{-k}. To see this, first note that our hypotheses $N_1 \ge 1/\beta$, $N_2 \ge 2/\beta$, and $N_j \ge (\alpha-\beta\,\alpha-\beta)^{-1}(N_1+\cdots+N_{j-1})$ imply by induction that

$$N_j \ge \frac{j+1}{\beta}\,, \qquad \text{for } j = 1,2,3,\ldots \tag{65}$$

To prove our claim, it is enough to show that

$$2\,k-j \le N_{j+1}\beta+\cdots+N_k\beta\,. \tag{66}$$

The right hand side is

$$\begin{aligned}N_{j+1}\beta+\cdots+N_k\beta &\ge (j+2)+\cdots+(k+1)\\ &= \frac{(k+1)\,(k+2)}{2} - \frac{(j+1)\,(j+2)}{2}\\ &= \frac{k^2+3\,k-j^2-3\,j}{2}\\ &= \frac{k\,(k-1)-(j+1)\,j}{2} + 2\,k-j\\ &\ge 2\,k-j\,,\end{aligned} \tag{67}$$

as required, since $k \ge j+1$.

Now suppose $1 \le k < j$. Again, we show that the integral over $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$ is at most $2^{-k}C_j$. By Lemma 4.4, we know $\mu_t(K_t^j) \le 2^{N_k\alpha-N_j}$. Therefore, by Lemma 4.1,

$$\begin{aligned}\int_{\mathcal{G}_k\setminus\mathcal{G}_{k+1}} \mu_t(K_t^j)\,dt &\le 2^{-N_1\beta-\cdots-N_k\beta+k+N_k\alpha-N_j}\\ &= 2^{N_{k+1}\beta+\cdots+N_j\beta+N_k\alpha-N_j\alpha+N_j\beta\alpha+k-j}\\ &\quad\cdot 2^{-N_1\beta-\cdots-N_j\beta+N_j\alpha-N_j\beta\alpha-N_j+j}\,.\end{aligned} \tag{68}$$

The second factor on the right hand side is C_j. We claim that the first factor is at most 2^{-k}. It suffices to show that

$$(N_{k+1} + \cdots + N_{j-1})\beta + N_k\alpha + 2\,k - j \leq N_j\,(\alpha - \beta\,\alpha - \beta)\,. \tag{69}$$

Note that $k-j<0$, and that $k \leq N_{k+1}\beta < N_{k+1}\alpha$. Also, since $\beta+\alpha<1$, our hypothesis (7) implies that

$$N_j \geq \frac{\beta+\alpha}{\alpha - \beta\,\alpha - \beta}\,(N_1 + \cdots + N_{j-1})\,. \tag{70}$$

Therefore

$$\begin{aligned}(N_{k+1} + \cdots + N_{j-1})\,\beta + N_k\alpha + 2\,k - j \qquad&\\ \leq (N_{k+1} + \cdots + N_{j-1})\,\beta + (N_k + N_{k+1})\,\alpha&\\ \leq (N_1 + \cdots + N_{j-1})\,(\beta+\alpha)&\\ \leq N_j(\alpha - \beta\,\alpha - \beta)\,,&\end{aligned} \tag{71}$$

establishing (69).

We have shown that when $k=j$, the integral of $\mu_t(K_t^j)$ over $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$ is at most C_j, and when $k \neq j$, the integral is at most $2^{-k}C_j$. From (56), we conclude that

$$\begin{aligned}\mu(B^j) &\leq 3 \cdot 2^{-N_j} + 3\sum_{k=1}^{\infty}\int_{\mathcal{G}_k\setminus\mathcal{G}_{k+1}} \mu_t(K_t^j)\,dt\\ &\leq 3 \cdot 2^{-N_j} + 6\,C_j\\ &= 3 \cdot 2^{-N_j} + 6 \cdot 2^{-N_1\beta - \cdots - N_{j-1}\beta + N_j(\alpha-\beta\alpha-\beta)+j}\\ &\leq 9 \cdot 2^{-N_1\beta - \cdots - N_{j-1}\beta + N_j(\alpha-\beta\alpha-\beta)+j}\,.\end{aligned} \tag{72}$$

The last inequality holds because (7) implies that

$$N_j \geq N_1\beta + \cdots + N_{j-1}\beta\,. \tag{73}$$

This proves Lemma 4.3, with constant $C=9$.

Incidentally, truncation of the weight was crucial in controlling the case $1 \leq k < j$. For the original weight $v_t(x) = (M_d(c\,\delta_t(x)))^{\alpha}$, the second inequality $\mu_t(K_t^j) \leq 2^{N_k\alpha - N_j}$ in Lemma 4.4 does not hold, and indeed in Lemma 4.3 the integral of $\mu_t(K_t^j)$ over $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$ is no longer bounded by

$2^{-k} C_j$. The effect of truncation is to increase $\mu_t(I_t^j)$ somewhat and, in most cases, to decrease $\mu_t(K_t^j)$ significantly.

Finally, by Lemmas 4.2 and 4.3,

$$\frac{\mu(A^j)}{\mu(B^j)} \geq C \, \frac{2^{-N_1\beta - \cdots - N_j\beta + N_j\alpha - N_j + j}}{2^{-N_1\beta - \cdots - N_j\beta + N_j\alpha - N_j\beta\alpha - N_j + j}} = C \, 2^{N_j\beta\alpha} \,, \tag{74}$$

which tends to infinity as $j \longrightarrow \infty$. Therefore the translation average μ is *not* a doubling measure. This proves Theorem 1.4, and hence theorems 1.1, 1.2, and 1.3.

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