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On coincidence of p-module of a family of curves and p-capacity on the Carnot group

Irina Markina

Abstract

The notion of the extremal length and the module of families of curves has been studied extensively and has given rise to a lot of applications to complex analysis and the potential theory. In particular, the coincidence of the p-module and the p-capacity plays an important role. We consider this problem on the Carnot group. The Carnot group $\mathbb{G}$ is a simply connected nilpotent Lie group equipped with an appropriate family of dilations. Let Ω be a bounded domain on $\mathbb{G}$ and K_0, K_1 be disjoint non-empty compact sets in the closure of Ω. We consider two quantities, associated with this geometrical structure $(K_0, K_1; \Omega)$. Let $M_p(\Gamma(K_0, K_1; \Omega))$ stand for the p-module of a family of curves which connect K_0 and K_1 in Ω. Denoting by $\mathrm{cap}_p(K_0, K_1; \Omega)$ the p-capacity of K_0 and K_1 relatively to Ω, we show that

$$M_p(\Gamma(K_0, K_1; \Omega)) = \mathrm{cap}_p(K_0, K_1; \Omega).$$

Introduction

Let D be a domain (an open, connected set) in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, and let K_0, K_1 be disjoint non-empty compact sets in the closure of D. We denote by $M_p(\Gamma(K_0, K_1; D))$ the p-module of a family of curves which connect K_0 and K_1 in D. Next we use the notation $\mathrm{cap}_p(K_0, K_1; D)$ for the p-capacity of the condenser $(K_0, K_1; D)$ relatively to D. The question about coincidence of the p-module of a family of curves and the p-capacity for various geometric configuration has been studied by many authors. For example, in the case

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when K_0 and K_1 do not intersect the boundary of D and either K_0 or K_1 contains the complement to an open n-ball the problem has been solved affirmatively by Ziemer in [23]. Hesse in [10] has generalized this result requiring only $(K_0 \cup K_1) \cap \partial D = \emptyset$. In the series of papers [2, 3, 4] Caraman has been studying the problem under various conditions on the tangency geometry of the sets K_0 and K_1 with the boundary of D, $D \in \overline{\mathbb{R}}^n$. In 1993 Shlyk [16] proved, that the coincidence of the p-module and p-capacity is valid for an arbitrary condenser $(K_0, K_1; D)$, K_0, $K_1 \in \overline{D}$, $D \in \overline{\mathbb{R}}^n$, $(K_0 \cup K_1) \cap \partial D \neq \emptyset$.

A stratified nilpotent group (of which $\mathbb{R}^n$ is the simplest example) is a Lie group equipped with an appropriate family of dilations. Thus, this group forms a natural habitat for extensions of many of the objects studied in the Euclidean space. The fundamental role of such groups in analysis was envisaged by Stein [17, 18]. There has been since a wide development in the analysis of the so-called stratified nilpotent Lie groups, nowadays, also known as Carnot groups. In the present article we are studying the problem of the coincidence between the p-module of a family of curves and the p-capacity of an arbitrary condenser $(K_0, K_1; \Omega)$, Ω is a bounded domain on the Carnot group. In [12] the identity $M_p(\Gamma(K_0, K_1; \Omega)) = \mathrm{cap}_p(K_0, K_1; \Omega)$ was proved for the condenser $(K_0, K_1; \Omega)$ on the Heisenberg group, which is a two-step Carnot group, requiring that the compacts K_0 and K_1 are strictly inside the domain Ω. We would like to mention the result by Heinonen and Koskela [8] which states that on every general metric spaces the p-capacity coincides with the p-module but in comparison with our paper they used different definitions. The use of this general result [8] for the Carnot groups requires the fact that the smallest very weak upper gradient of a Lipschitz function is given by the horizontal gradient (see for instance [9]). However it is not clear that the result of [8] covers the case when the intersection of the compacts K_i, $i = 0, 1$, with the boundary of Ω is not empty. Moreover the case when Ω is not φ-convex is not obtained from [8].

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1. Notation and definitions

The Carnot group is a connected, simply connected nilpotent Lie group $\mathbb{G}$, whose Lie algebra $\mathcal{G}$ splits into the direct sum of vector spaces $V_1 \oplus V_2 \oplus \ldots \oplus V_m$ which satisfy the following relations

$$\begin{aligned} [V_1, V_k] &= V_{k+1}, \qquad 1 \leq k < m, \\ [V_1, V_m] &= \{0\}. \end{aligned}$$

We identify the Lie algebra $\mathcal{G}$ with the space of left-invariant vector fields. Let $X_{11}, \ldots, X_{1n_1}$ be a bases of V_1, $n_1 = \dim V_1$, and let $\langle \cdot, \cdot \rangle$ be a left-invariant Riemannian metric on V_1 such that

$$\langle X_{1i}, X_{1j} \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then, V_1 determines a subbundle HT of the tangent bundle $T\mathbb{G}$ with fibers

$$HT_x = span\,\{X_{11}(x), \ldots, X_{1n_1}(x)\}, \quad x \in \mathbb{G}.$$

We call HT the *horizontal tangent bundle* of $\mathbb{G}$ with HT_x as the *horizontal tangent space* at $x \in \mathbb{G}$. Respectively, the vector fields X_{1j}, $j = 1, \ldots, n_1$, we will call *the horizontal vector fields*.

Next, we extend $X_{11}, \ldots, X_{1n_1}$ to an orthonormal basis

$$X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}, \ldots, X_{m1}, \ldots, X_{mn_m}$$

of $\mathcal{G}$. Here each vector field X_{ij}, $2 \leq i \leq m$, $1 \leq j \leq n_i = \dim V_i$, is a commutator

$$X_{ij} = [\ldots[[X_{1k_1}, X_{1k_2}]X_{1k_3}]\ldots X_{1k_i}]$$

of length $i - 1$ generated by the basis vector fields of the space V_1.

It was proved (see, for instance, [6]) that the exponential map $\exp : \mathcal{G} \to \mathbb{G}$ from the Lie algebra $\mathcal{G}$ into the Lie group $\mathbb{G}$ is a global diffeomorphism. We can identify the points $q \in \mathbb{G}$ with the points $x \in \mathbb{R}^N$, $N = \sum_{i=1}^m \dim V_i$, by the rule $q = \exp(\sum_{i,j} x_{ij} X_{ij})$. The collection $\{x_{ij}\}$ is called the *coordinates* of $q \in \mathbb{G}$. The number $N = \sum_{i=1}^m \dim V_i$ is the topological dimension of the Carnot group. The biinvariant Haar measure on $\mathbb{G}$ is denoted by dx; this is the push-forward of the Lebesgue measure in $\mathbb{R}^N$ under the exponential map. *The family of dilations* $\{\delta_\lambda(x) : \lambda > 0\}$ on the Carnot group is defined as

$$\delta_\lambda x = \delta_\lambda(x_{ij}) = (\lambda x_1, \lambda^2 x_2, \ldots, \lambda^m x_m),$$

where $x_i = (x_{i1}, \ldots, x_{in_i})$. Moreover, $d(\delta_\lambda x) = \lambda^Q dx$ and the quantity $Q = \sum_{i=1}^m i \dim V_i$ is called *the homogeneous dimension* of $\mathbb{G}$.

The Euclidean space $\mathbb{R}^n$ with the standard structure is an example of the Abelian Carnot group: the exponential map is the identity and the vector fields $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$ have only trivial commutators and form the basis of the corresponding Lie algebra.

The simplest example of a non-abelian Carnot group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),$$

where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, $t\in\mathbb{R}$, and the left translation $L_p(q)=pq$ is defined. The left-invariant vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t}, \quad i=1,\dots,n,\\ Y_i &= \frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t}, \quad i=1,\dots,n,\\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

form the basis of the Lie algebra of the Heisenberg group. All non-trivial relations are of the form $[X_i,Y_i]=-4T$, $i=1,\dots,n$, and all other commutators vanish. Thus, the Heisenberg algebra has the dimension $2n+1$ and splits into the direct sum $\mathcal{G}=V_1\oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i, $i=1,\dots,n$, and the space V_2 is the one-dimensional center which is spanned by the vector field T.

We use the Carnot-Carathéodory metric based on the length of horizontal curves. An absolutely continuous map $\gamma:[0,b]\to\mathbb{G}$ is called a curve. A curve $\gamma:[0,b]\to\mathbb{G}$ is said to be *horizontal* if its tangent vector $\dot\gamma(s)$ lies in the horizontal tangent space $HT_{\gamma(t)}$, i.e. there exist functions $a_j(s)$, $s\in[0,b]$, such that $\sum_{j=1}^{n_1}a_j^2\le 1$ and

$$\dot\gamma(s)=\sum_{j=1}^{n_1}a_j(s)X_{1j}(\gamma(s)).$$

The result of [5] implies that one can connect two arbitrary points x, $y\in\mathbb{G}$ by a horizontal curve. We fix on HT_x a quadratic form $\langle\cdot,\cdot\rangle$, so that the vector fields $X_{11}(x),\dots,X_{1n_1}(x)$ are orthonormal with respect to this form at every $x\in\mathbb{G}$. Then the length of the horizontal curve $l(\gamma)$ is defined by the formula

$$l(\gamma)=\int_0^b\langle\dot\gamma(s),\dot\gamma(s)\rangle^{1/2}ds=\int_0^b\left(\sum_{j=1}^{n_1}|a_j(s)|^2\right)^{1/2}ds.$$

The Carnot-Carathéodory distance $d_c(x,y)$ is the infimum of the length over all horizontal curves connecting x and $y\in\mathbb{G}$. Since the quadratic form is left-invariant, the Carnot-Carathéodory metric is also left-invariant. The group $\mathbb{G}$ is connected, therefore, the metric $d_c(x,y)$ is finite (see [19]). A homogeneous norm on $\mathbb{G}$ is, by definition, a continuous non-negative function $|\cdot|$ on $\mathbb{G}$, such that $|x|=|x^{-1}|$, $|\delta_\lambda(x)|=\lambda|x|$, and $|x|=0$, if and only if $x=0$. Since all homogeneous norms are equivalent, we choose one that satisfies the triangle inequality: $|x^{-1}y| \leq |x| + |y|$. By $B(x,r)$ we denote the open ball of radius $r > 0$ centered at x in the metric d_c. Note that $B(x,r) = \{y \in \mathbb{G} : |x^{-1}y| < r\}$ is the left translation of the ball $B(0,r)$ by x which is the image of the "unit ball" $B(0,1)$ under δ_r. The Hausdorff dimension of the metric space $(\mathbb{G}, d_c)$ coincides with its homogeneous dimension Q. By $|E|$ we denote the measure of the set E. Our normalizing condition is such that the balls of radius one have measure one:

$$|B(0,1)| = \int_{B(0,1)} dx = 1.$$

We have that $|B(0,r)| = r^Q$ because the Jacobian of the dilation δ_r is r^Q.

A curve $\gamma : [0,b] \to \mathbb{G}$ is called rectifiable if the supremum

$$\sup \left\{ \sum_{k=1}^{p-1} d_c\big(\gamma(t_k), \gamma(t_{k+1})\big) \right\}$$

is finite where the supremum ranges over all partitions $0 = t_1 \leq t_2 \leq \ldots \leq t_p = b$ of the segment $[0,b]$. We remark that the definition of a rectifiable curve is based on the Carnot-Carathéodory metric. That is why a curve is not rectifiable if it is not horizontal [11, 13]. Thus, from now on, we work only with horizontal curves. A horizontal curve γ, which is rectifiable with respect to the Carnot-Carathéodory metric, is differentiable almost everywhere and the tangent vector $\dot{\gamma}$ belongs to V_1 (see [14]).

Let us define the p-module $M_p(\Gamma(K_0, K_1; \Omega))$ of the family of curves $\Gamma(K_0, K_1; \Omega)$ and the p-capacity $\mathrm{cap}_p(K_0, K_1; \Omega)$ on the Carnot group.

Our assumption is the following. Let $\langle a, b\rangle$ be an interval of one of the following types: $[a,b], [a,b), (a,b]$, or (a,b). From now on, we suppose that a curve $\gamma : \langle a,b\rangle \to \mathbb{G}$ is parameterized by the length element. Let Ω be an open connected set (domain) on $\mathbb{G}$, K_0 and K_1 be closed nonempty disjoint sets in the closure $\overline{\Omega}$ of Ω. We will call $(K_0, K_1; \Omega)$ the *condenser*. We will use the symbol $\Gamma(K_0, K_1; \Omega)$ to denote the family of curves $\gamma : \langle a,b\rangle \to \Omega \subset \mathbb{G}$ which connect the sets K_0 and K_1, namely, if $\gamma \in \Gamma(K_0, K_1; \Omega)$, then $\overline{\gamma(\langle a,b\rangle)} \cap K_i \neq \emptyset$, $i = 0, 1$, and $\gamma(t) \in \Omega$, $t \in (a,b)$.

Let $\mathcal{F}(\Gamma(K_0, K_1; \Omega))$ denote the set of Borel functions $\rho : \Omega \to [0; \infty]$, such that for every locally rectifiable $\gamma \in \Gamma(K_0, K_1; \Omega)$ we have

$$\sup \int_{\gamma'} \rho\, ds = \sup \int_0^{l(\gamma')} \rho(\gamma'(t))\, dt \geq 1.$$

The supremum is taken over all arcs γ', such that $\gamma' = \gamma|_{[\alpha,\beta]} \to \Omega$, $[\alpha, \beta] \subset \langle a,b\rangle$ and $l(\gamma')$ is the length of γ'. The quantity $\mathcal{F}(\Gamma(K_0, K_1; \Omega))$ is called the set of *admissible densities* for $\Gamma(K_0, K_1; \Omega)$.

Definition 1.1 Let $p \in (1,\infty)$. The quantity

$$M_p(\Gamma(K_0,K_1;\Omega)) = \inf \int_\Omega \rho^p \, dx$$

is called the *p-module of the family of curves* $\Gamma(K_0,K_1;\Omega)$. The infimum is taken over all functions $\rho \in \mathcal{F}(\Gamma(K_0,K_1;\Omega))$.

The Vitali–Carathéodory theorem [15] about approximation of a function from L_p implies that the set of admissible densities can be reduced to the set of Borel lower semicontinuous functions. Hence, without loss of generality, we can assume that $\rho \in \mathcal{F}(\Gamma(K_0,K_1;\Omega))$ is semicontinuous in $\mathbb{G}$.

The properties of the module of a family of curves in the case of $\mathbb{G} = \mathbb{R}^n$ one can find, for instance, in [7].

If there exists a constant L such that $|\varphi(x) - \varphi(y)| \le L d_c(x,y)$ for all $x,y \in D$, $D \subset\subset \Omega$, then the function $\varphi : \Omega \to \mathbb{R}$ is called locally Lipschitz continuous in $\Omega \subset \mathbb{G}$. The Sobolev space $L_p^1(\Omega)$ over the domain Ω is defined as a completion of the class of locally Lipschitz continuous function with respect to the seminorm

$$\big\|\varphi \mid L_p^1(\Omega)\big\| = \Big(\int_\Omega \|X\varphi\|^p \, dx\Big)^{1/p} < \infty.$$

Here $X\varphi = (X_{11}\varphi, \dots, X_{1n_1}\varphi)$ is called *the horizontal gradient of* φ and $\|X\varphi\| = \Big(\sum_{j=1}^{n_1} |X_{1j}\varphi|^2\Big)^{1/2}$. Thus, if u is a smooth function, then Xu is a horizontal component of the usual Riemannian gradient of u.

Let $\mathcal{A}(K_0,K_1;\Omega)$ denote the set of non-negative real valued, locally Lipschitz continuous functions $\varphi \in L_p^1(\Omega) \cap C(\Omega)$, such that $\varphi(x) = 0$ ($\varphi(x) \ge 1$) in a neighborhood of K_0 (K_1).

Definition 1.2 For $p \in (1,\infty)$ we define the p-capacity of $(K_0,K_1;\Omega)$ by

$$\operatorname{cap}_p(K_0,K_1;\Omega) = \inf \int_\Omega \|X\varphi\|^p \, dx,$$

where the infimum is taken over all $\varphi \in \mathcal{A}(K_0,K_1;\Omega)$.

Our main result is the following theorem.

Theorem. *Let Ω be a bounded domain in the Carnot group $\mathbb{G}$. Suppose that K_0 and K_1 are disjoint non-empty compact sets in the closure of Ω. Then*

$$M_p(\Gamma(K_0,K_1;\Omega)) = \operatorname{cap}_p(K_0,K_1;\Omega).$$

2. Preliminary results

We define an absolutely continuous function on curves of the horizontal fibration. For this we consider a family of horizontal curves $\mathcal{Y}$ which form a smooth fibration of an open set $D \subset \mathbb{G}$. Usually, a curve $\gamma \in \mathcal{Y}$ is an orbit of a smooth horizontal vector field $Y \in V_1$. If we denote by φ_s the flow associated with this vector field, then the fiber is of the form $\gamma(s) = \varphi_s(p)$. Here the point p belongs to the surface S which is transversal to the vector field Y. The parameter s ranges over an open interval $J \in \mathbb{R}$. One can assume that there is a measure $d\gamma$ on the fibration $\mathcal{Y}$ of the set $D \subset \mathbb{G}$. The measure $d\gamma$ satisfies the inequalities

$$k_0 |B(x,R)|^{\frac{Q-1}{Q}} \leq \int_{\gamma \in \mathcal{Y},\, \gamma \cap B(x,R) \neq \emptyset} d\gamma \leq k_1 |B(x,R)|^{\frac{Q-1}{Q}}$$

for a sufficiently small ball $B(x,R) \subset D$ with constants k_0, k_1 which do not depend on the ball $B(x,R)$ (see [12, 20, 22]).

Definition 2.1 A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, is said to be *absolutely continuous on lines* ($u \in ACL(\Omega)$) if for any domain U, $\overline{U} \subset \Omega$ and any fibration $\mathcal{X}$ defined by a left-invariant vector field X_{1j}, $j = 1, \dots, n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the $\mathcal{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathcal{X}$.

For such a function u the derivatives $X_{1j}u$, $j = 1, \dots, n_1$, exist almost everywhere in Ω. If they belong to $L_p(\Omega)$ for all $X_{1j} \in V_1$, then u is said to be in $ACL^p(\Omega)$. If the function f belongs to $L^1_p(\Omega)$, then there exists a function $u \in ACL^p(\Omega)$ such that $f = u$ almost everywhere.

The following lemma and theorem are reformulations of the well known result by Fuglede [7] (see also [21]) for the Carnot group.

Lemma 2.2 *Suppose E is a Borel set on the Carnot group $\mathbb{G}$ and $g_k : E \to \mathbb{R}$ is a sequence of Borel functions which converges to a Borel function $g : E \to \mathbb{R}$ in $L_p(E)$. There is a subsequence $\{g_{k_j}\}$, such that the equality*

$$\lim_{j \to \infty} \int_\gamma |g_{k_j} - g|\, ds = 0$$

holds for all rectifiable horizontal curves $\gamma \subset E$ except for some family whose p-module vanishes.

We will prove the next theorem for completeness.

Theorem 2.1 *Let Ω be an open subset of $\mathbb{G}$, and $u : \Omega \to \mathbb{R}$ be a function from $ACL^p(\Omega)$, $p \in (1, \infty)$. The function u is absolutely continuous on rectifiable closed parts of horizontal curves, except for a family of horizontal curves whose p-module vanishes.*

Proof: Let U_l be a sequence of open sets, such that $\overline{U}_0 \subset \cdots \subset \overline{U}_l \subset \cdots \subset \Omega$, $\bigcup_{l=0}^{\infty} U_l = \Omega$. Denote by Γ the family of locally rectifiable horizontal curves whose trace lies in Ω, and such that the function u is not absolutely continuous on each curve of Γ. By Γ_l we denote the family of closed arcs of the curves $\gamma \in \Gamma$ which intersect U_l. By the property of monotonicity of the p-module we deduce that

$$M_p(\Gamma) \leq \sum_{l=1}^{\infty} M_p(\Gamma_l).$$

The proof will be complete if we establish that $M_p(\Gamma_l) = 0$ for any arbitrary index l. For a function u satisfying the assertion of Theorem 2.1 there exists a sequence of the C^∞-functions $u^{(i)}$, $i \in \mathbb{N}$, which converges to u uniformly in $\overline{U}_l$ [6]. Moreover, the sequence $X_{1k}u^{(i)}$ converges to $X_{1k}u$ in $L_p(\Omega)$, $k = 1, \ldots, n_1$. By Lemma 2.2 we choose a subsequence (which we denote by the same symbol) $u^{(i)}$, such that

$$\int_\gamma \|X_{1k}u^{(i)} - X_{1k}u\| \, ds \;\to\; 0 \qquad \forall \quad k = 1, \ldots, n_1 \tag{2.1}$$

for all rectifiable horizontal curves $\gamma : [0, b] \to U_l$ except for a family $\widetilde{\Gamma}$ whose p-module $M_p(\widetilde{\Gamma})$ vanishes. We show that $\Gamma_l \subset \widetilde{\Gamma}$. Suppose that there exists a rectifiable horizontal curve $\gamma \in \Gamma_l \setminus \widetilde{\Gamma}$. It is assumed that this curve is parameterized by its length element. Since the functions $u^i(\gamma(s))$ are absolutely continuous, the sequence of functions

$$u^{(i)}(\gamma(s)) = u^{(i)}(\gamma(0)) + \int_0^s \left(\sum_{k=1}^{n_1} a_k(t) X_{1k} u^{(i)}(\gamma(t)) \right) dt,$$

is defined for any $s \in [0, b]$.

The sequence $u^{(i)}(\gamma(s))$ converges uniformly to the function $u(\gamma(s))$ as $i \to \infty$. Moreover, from (2.1) we deduce that

$$u(\gamma(s)) = u(\gamma(0)) + \int_0^s \left(\sum_{k=1}^{n_1} a_k(t) X_{1k} u(\gamma(t)) \right) dt.$$

Hence, the function u is absolutely continuous, and we derive that u is absolutely continuous on $\gamma(s)$. This contradicts $\gamma \in \Gamma_l \setminus \widetilde{\Gamma}$. ∎

In our next step we establish an approximation property for functions $f \in L_p(D)$ defined on an open set $D \neq \mathbb{G}$.

Lemma 2.3 *Suppose that D is a bounded domain on the Carnot group $\mathbb{G}$. Let $f \in L_p(D)$ and $\varepsilon > 0$. Then there exists a continuous function $\tilde{f}$ such that*

$$\|f - \tilde{f} \mid L_p(D)\| < \varepsilon.$$

Proof: Making use of the Whitney lemma we can find points $x_1, x_2, \dots$ in D and positive numbers $r_1, r_2, \dots$, such that

(i) $B(x_i, r_i) \subset D$,

(ii) $D \subset \bigcup\limits_i B(x_i, r_i)$,

(iii) $B(x_i, 2r_i) \subset D$,

(iv) $\sum\limits_i \chi_{B(x_i, 2r_i)} \leq M$, with some number M independent of the choice of the set D and of the point $x \in D$.

Also we can suppose, that the radii of the balls do not exceed $1/2$.

Let $\{h_1(x), h_2(x), \dots\}$ be a partition of unity on D subordinate to the cover $\{B(x_1, r_1), B(x_2, r_2), \dots\}$: $h_i(x) \geq 0$, $\operatorname{supp}(h_i(x)) \subset B(x_i, r_i)$, and $\sum_{i=1}^{\infty} h_i(x) = 1$ for $x \in D$. Set $f_i(x) = h_i f(x)$. Then f_i, $i = 1, 2, \dots$, satisfy the following condition: $\operatorname{supp} f_i \subset B(x_i, r_i)$, $f_i \in L_p(\mathbb{G})$ and $f(x) = \sum_{i=1}^{\infty} f_i(x)$ for $x \in D$.

We write φ^i for the continuous function supported in the ball $B(x_i, 2r_i)$ such that $\int_{B(x_i, 2r_i)} \varphi^i(x)\, dx = 1$. Let us consider the convolution

$$\tilde{f}_i(x) = f_i \star \varphi_t^i(x) = \int_{\mathbb{G}} f_i(y) \varphi_t^i(y^{-1}x)\, dy = \int_{\mathbb{G}} f_i(xy^{-1}) \varphi_t^i(y)\, dy,$$

where $\varphi_t^i(x) = t^{-Q} \varphi^i(\delta_{1/t} x)$. It is known [6] that in this case the inequality

$$\|\tilde{f}_i - f_i \mid L_p(\mathbb{G})\| < 2^{-i}\varepsilon$$

holds as $t \to 0$ for arbitrary $\varepsilon > 0$.

Let us define $\tilde{f}(x) = \sum\limits_{i=1}^{\infty} \tilde{f}_i(x)$. The continuity of $\tilde{f}$ and the inequality

$$\int_D |\tilde{f} - f|^p\, dx \leq \sum_{i=1}^{\infty} \Big\{ \int_{\mathbb{G}} |\tilde{f}_i - f_i|^p\, dx \Big\}^{1/p} < \varepsilon$$

yield the required approximation. ∎

Using similar argumentation as in [10], we prove the next lemma.

Lemma 2.4 *Let $\mathcal{B} \subset \mathcal{F}(\Gamma(K_0, K_1; \Omega))$ consist of continuous functions on $\Omega \setminus (K_0 \cup K_1)$. Then,*

$$M = \inf_{\rho \in \mathcal{B}} \int_{\Omega} \rho^p(x)\, dx = M_p(\Gamma(K_0, K_1; \Omega)). \tag{2.2}$$

Proof: Let $\{B(x_i, r_i)\}$ be a cover of the domain $D = \Omega \setminus (K_0 \cup K_1)$ chosen as in the previous lemma. We also use the notation $\rho = \sum_{i=1}^{\infty} \rho_i = \sum_{i=1}^{\infty} h_i \rho$, where $\{h_i\}$ is a partition of unity subordinate to $\{B(x_i, r_i)\}$. Then by Lemma 2.3 for $\varepsilon > 0$ and $\rho \in \mathcal{F}(\Gamma(K_0, K_1; \Omega))$ we find continuous function $\tilde{\rho}$, such that

$$\int_{\Omega \setminus (K_0 \cup K_1)} \tilde{\rho}^p(x)\, dx < \varepsilon + M_p(\Gamma(K_0, K_1; \Omega)). \tag{2.3}$$

We claim that $(1+\varepsilon)\tilde{\rho}$ is an admissible density for $\Gamma(K_0, K_1; \Omega)$. If γ belongs to $\Gamma(K_0, K_1; \Omega)$, then

$$1 \le \int_{\gamma} \rho\, ds = \int_{\gamma} \sum_{i=1}^{\infty} \rho_i\, ds \le \sum_{i=1}^{\infty} \int_{\gamma \cap B(x_i, 2r_i)} \rho_i\, ds.$$

Making use of the construction of approximation from Lemma 2.3 with parameters $t_i < \varepsilon$, $i = 1, 2, \ldots$, we get

$$\begin{aligned}
\int_{\gamma} \tilde{\rho}\, ds &= \int_{\gamma} \sum_{i=1}^{\infty} \tilde{\rho}_i\, ds = \int_{\gamma} \sum_{i=1}^{\infty} \int_{\mathbb{G}} \rho_i(xy^{-1}) \varphi^i_{t_i}(y)\, dy\, ds \\
&= \int_{\gamma} \sum_{i=1}^{\infty} \int_{\mathbb{G}} \rho(x(\delta_{t_i} z)^{-1}) \varphi^i(z)\, dz\, ds \\
&= \sum_{i=1}^{\infty} \int_{B(x_i, 2r_i)} \varphi^i(z)\, dz \int_{\gamma \cap B(x_i, r_i)} \rho_i(x(\delta_{t_i} z)^{-1})\, ds(x).
\end{aligned} \tag{2.4}$$

We note that $\int_{B(x_i, 2r_i)} \varphi^i(z)\, dz = 1$ by definition. Let us denote by $\tilde{\gamma}$ the image of the curve γ under the map $\gamma \to \gamma \cdot (\delta_t z)^{-1}$. We can choose a sufficiently small t_i, such that the image $\gamma \cap B(x_i, r_i)$ is contained in the ball $B(x_i, 2r_i)$. Moreover, $|\dot{\tilde{\gamma}}| = |\dot{\gamma} \cdot (\delta_{t_i} z)^{-1})|$. Changing variables in the last integral of (2.4), we obtain

$$\begin{aligned}
\int_{\gamma} \tilde{\rho}\, ds &= \sum_{i=1}^{\infty} \int_{\tilde{\gamma} \cap B(x_i, 2r_i)} \rho_i(y) |\dot{\tilde{\gamma}}|^{-1}\, ds(y) \\
&\ge \sum_{i=1}^{\infty} \int_{\tilde{\gamma} \cap B(x_i, 2r_i)} \rho_i(y) \big(|\dot{\gamma}| + t_i |z| \big)^{-1}\, ds(y) \ge \big(1 + \varepsilon\big)^{-1}.
\end{aligned}$$

In the latter we used the inequalities

$$|\dot{\tilde{\gamma}}| \le |\dot{\gamma}| + t_i|z|\,, \qquad |\dot{\gamma}| \le 1\,, \qquad t_i < \varepsilon, \qquad |z| < 2r_i < 1, \quad i = 1, 2, \ldots .$$

Since ε and $\rho \in \Gamma(K_0, K_1; \Omega)$ are arbitrary, we get from (2.3) that

$$M = \inf_{\rho \in \mathcal{B}} \int_\Omega \rho^p(x)\, dx \le M_p(\Gamma(K_0, K_1; \Omega)).$$

The reverse inequality is obvious and we have (2.2) as desired. ■

3. Proof of the main result

In this section we will be working under the assumption that K_0 and K_1 are disjoint non-empty compacts in the closure $\overline{\Omega}$ of a bounded domain $\Omega \subset \mathbb{G}$. Moreover, let K_0^j and K_1^j be a sequence of closed sets, such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap_{j=0}^\infty K_0^j$, and $K_1 = \bigcap_{j=0}^\infty K_1^j$.

The next lemma in the particular case $\mathbb{G} = \mathbb{R}^n$ goes back to the work [16] and, then is digested by Ohtsuka (see for instance [1]).

Lemma 3.1 *Let $\rho \in L_p(\mathbb{G})$ be a positive function which is continuous in $\Omega \setminus (K_0 \cup K_1)$. For each $\varepsilon > 0$ we can construct a function ρ' on Ω, $\rho' \ge \rho$, with the following properties:*

(i) $\int_\Omega \rho'^p\, dx \le \int_\Omega \rho^p\, dx + \varepsilon$.

(ii) Suppose that for each j there is $\gamma_j \in \Gamma(K_0^j, K_1^j; \Omega)$ such that $\int_{\gamma_j} \rho'\, ds \le \alpha$. Then there exists $\tilde{\gamma} \in \Gamma(K_0, K_1; \Omega)$, such that $\int_{\tilde{\gamma}} \rho\, ds \le \alpha + \varepsilon$.

Proof: The most difficult part of the lemma is the existence of $\tilde{\gamma}$ inside Ω. It is rather easy to find a curve in $\overline{\Omega}$, but such a curve is not necessarily from $\Gamma(K_0, K_1; \Omega)$.

For the beginning let us construct the function ρ'. Let $K^j = K_0^j \cup K_1^j$, $W^j = K^{j-1} \setminus \operatorname{int} K^j$, and $d_j = \operatorname{dist}(\partial K^{j-1}, \partial K^j) > 0$. Since the function ρ is strictly positive, we can find a sequence $\varepsilon_j \to 0$, such that

$$\sum_{j=1}^\infty \big(1 + \varepsilon_j^{-1}\big)^p \varepsilon_j < \varepsilon, \tag{3.1}$$

$$(1 + \varepsilon_j^{-1}) d_j \inf_{x \in W^j \cap \Omega} \rho(x) > \alpha. \tag{3.2}$$

We can find a sequence of compact subsets $\Omega_j \subset \Omega$ increasing to Ω, such that

$$\int_{\Omega \setminus \Omega_j} \rho^p\, dx < \varepsilon_j.$$

Let $V^j = (\Omega \setminus \Omega_j) \cap W^j$, and set

$$\rho'(x) = \begin{cases} (1+\varepsilon_j^{-1})\rho(x) & \text{if } x \in V^j, \\ \rho(x) & \text{if } x \in \Omega \setminus (\cup V^j). \end{cases}$$

Now, applying (3.1), we obtain

$$\begin{aligned} \int_\Omega \rho'^p\,dx &= \sum_j \int_{V^j} \Big((1+\varepsilon_j^{-1})\rho(x)\Big)^p\,dx + \int_{\Omega\setminus(\cup V^j)} \rho^p\,dx \\ &\le \sum_j (1+\varepsilon_j^{-1})^p \int_{V^j} \rho^p\,dx + \int_\Omega \rho^p\,dx \\ &\le \sum_j (1+\varepsilon_j^{-1})^p \varepsilon_j + \int_\Omega \rho^p\,dx < \varepsilon + \int_\Omega \rho^p\,dx. \end{aligned}$$

We see that (i) holds. Now let us show (ii). Fix $j \ge 1$. The curve γ_k is from $\Gamma(K_0^j, K_1^j; \Omega)$ for the $k \ge j$ by definition. Hence, γ_k contains two arcs: γ_k' such that γ_k' connects ∂K_0^j and ∂K_0^{j-1}; and γ_k'' which connects ∂K_1^j and ∂K_1^{j-1}. Let us show that γ_k' and γ_k'' are not included in V^j. On the contrary, let us suppose that $\gamma_k' \subset V^j$. Then, using (3.2), we deduce the inequality

$$\alpha \ge \int_{\gamma_k} \rho'\,ds \ge \int_{\gamma_k'} \rho'\,ds \ge \big(1+\varepsilon_j^{-1}\big) \int_{\gamma_k'} \rho\,ds \ge (1+\varepsilon_j^{-1}) \inf_{x\in W^j\cap\Omega} \rho(x) \int_{\gamma_k'} ds > \alpha,$$

which is false. In the same way γ_k'' is not included in V^j , therefore,

$$\gamma_k \bigcap \Big(\Omega_j \cap \big(K_i^{j-1} \setminus \operatorname{int} K_i^j\big)\Big) \ne \emptyset \quad \text{for} \quad i = 0,1 \quad \text{and} \quad k \ge j.$$

Observe that the sets $\Omega_j \cap \big(K_i^{j-1} \setminus \operatorname{int} K_i^j\big)$, $i = 0,1$, are compacts. For a fixed j let us consider a sequence $\{\gamma_k^j\}_{k=j}^\infty$. We can extract a subsequence (we use the same notation $\{\gamma_k^j\}_{k\to\infty}$) which converges to a curve γ_0^j, such that

$$\gamma_0^j \bigcap \Big(\Omega_j \cap \big(K_0^{j-1} \setminus \operatorname{int} K_0^j\big)\Big) \ne \emptyset.$$

Further, we fix a point $x_0^j \in \Omega_j \cap \big(K_0^{j-1} \setminus \operatorname{int} K_0^j\big)$ on it. Since ρ is continuous at $x_0^j \in \Omega_j \cap \big(K_0^{j-1} \setminus \operatorname{int} K_0^j\big)$, we can choose a ball $B\big(x_0^j, r(x_0^j)\big) \subset \Omega$ so small that

$$\int_l \rho\,ds \le \varepsilon/2^{j+3} \tag{3.3}$$

for any shortest curve $l \subset B\big(x_0^j, r(x_0^j)\big)$ which connects the center x_0^j with the boundary of $B\big(x_0^j, r(x_0^j)\big)$. Renumbering the subsequence, we may assume that each member of the subsequence $\{\gamma_k^j\}$ intersects $B\big(x_0^j, r(x_0^j)\big)$.

In the same way we can find a small closed ball $B\big(x_1^j, r(x_1^j)\big) \subset \Omega$, $x_1^j \in \Omega_j \cap \big(K_1^{j-1} \setminus \operatorname{int} K_1^j\big)$, so that $\{\gamma_k^j\}$ intersects $B\big(x_1^j, r(x_1^j)\big)$ and an analogue of (3.3) holds. We start this process from $j = 1$ and extract a subsequence $\{\gamma_k^j\}$ from the sequence constructed in the previous step, such that the new subsequence $\{\gamma_k^j\}$ intersects $B\big(x_0^j, r(x_0^j)\big)$ and $B\big(x_1^j, r(x_1^j)\big)$.

Now let us consider the diagonal $\{\gamma_k^k\}$. Then $\{\gamma_k^k\}$ intersects $B\big(x_0^j, r(x_0^j)\big)$ and $B\big(x_1^j, r(x_1^j)\big)$ for $1 \leq j \leq k$. In each ball $B\big(x_i^j, r(x_i^j)\big)$, $i = 0, 1$, we add two shortest curves to $\{\gamma_k^k\}$ connecting x_i^j with the points of intersections of $\{\gamma_k^k\}$ with $\partial B\big(x_i^j, r(x_i^j)\big)$, $i = 0, 1$, $1 \leq j \leq k$. Thus, we have a connected horizontal curve $\tilde{\gamma}_k \in \Gamma(K_0^k, K_1^k; \Omega)$ passing through all pairs $\{x_0^j, x_1^j\}_{j=1}^k$. We have by (3.3) that

$$\int_{\tilde{\gamma}_k} \rho\, ds \leq \int_{\gamma_k^k} \rho\, ds + 2\sum_{j=1}^{k} \frac{\varepsilon}{2^{j+3}} \leq \alpha + \frac{\varepsilon}{4}.$$

Let Γ_0 be the union of all horizontal curves in $\Omega \setminus (K_0 \cup K_1)$ connecting x_0^1 and x_1^1. For $i = 0, 1$, let Γ_i^j be the collection of all horizontal curves in $\Omega \setminus (K_0 \cup K_1)$ connecting x_i^j and x_i^{j+1}. Then,

$$\inf_{\gamma \in \Gamma_0} \int_{\gamma} \rho\, ds + \sum_{j=1}^{k} \inf_{\gamma \in \Gamma_0^j} \int_{\gamma} \rho\, ds + \sum_{j=1}^{k} \inf_{\gamma \in \Gamma_1^j} \int_{\gamma} \rho\, ds \leq \int_{\tilde{\gamma}_k} \rho\, ds \leq \alpha + \frac{\varepsilon}{4}.$$

Therefore, we can choose $C_0 \in \Gamma_0$ and $C_i^j \in \Gamma_i^j$, such that

$$\int_{C_0} \rho\, ds < \inf_{\gamma \in \Gamma_0} \int_{\gamma} \rho\, ds + \frac{\varepsilon}{2},$$
$$\int_{C_i^j} \rho\, ds < \inf_{\gamma \in \Gamma_i^j} \int_{\gamma} \rho\, ds + \frac{\varepsilon}{2^{j+3}}.$$

Let

$$\tilde{\gamma} = \ldots + C_0^1 + C_0 + C_1^1 + \ldots.$$

Then, $\tilde{\gamma} \in \Gamma(K_0, K_1; \Omega)$ and

$$\int_{\tilde{\gamma}} \rho\, ds \leq \alpha + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + 2\sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+3}} = \alpha + \varepsilon.$$

The lemma is proved. ■

Theorem 3.1 *Let Ω be a bounded domain in the Carnot group $\mathbb{G}$. Suppose K_0 and K_1 to be disjoint non-empty compacts in the closure of Ω. Then,*

$$M_p(\Gamma(K_0, K_1; \Omega)) = \operatorname{cap}_p(K_0, K_1; \Omega).$$

Proof: Our proof falls into three steps.

Step 1. We start proving the inequality

$$M_p(\Gamma(K_0, K_1; \Omega)) \leq \operatorname{cap}_p(K_0, K_1; \Omega). \tag{3.4}$$

Let $u \in \mathcal{A}(K_0, K_1; \Omega)$. Let Γ_0 be the locally rectifiable horizontal curves $\gamma \in \Gamma(K_0, K_1; \Omega)$, such that u is absolutely continuous on every rectifiable closed part of γ. Define $\rho : \Omega \to [0, \infty]$ by

$$\rho(x) = \begin{cases} \|Xu\| & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Suppose that $\gamma \in \Gamma_0$ and $\gamma : (a, b) \to \Omega$ is parameterized by the length element. If $a < t_1 < t_2 < b$, then making use of the inequality $|\dot{\gamma}(t)| \leq 1$, we get

$$\begin{aligned} \int_\gamma \rho\, ds &= \int_a^b \rho(\gamma(t))\, dt \geq \int_{t_1}^{t_2} \|Xu(\gamma(t))\|\, dt \\ &\geq \left| \int_{t_1}^{t_2} \langle Xu(\gamma(t)) \cdot \dot{\gamma}(t) \rangle\, dt \right| = |u(\gamma(t_2)) - u(\gamma(t_1))|. \end{aligned} \tag{3.5}$$

Since t_1 and t_2 are arbitrary, (3.5) implies the inequality $\int_\gamma \rho\, ds \geq 1$. Hence, ρ is admissible for the family of curves $\Gamma(K_0, K_1; \Omega)$. Therefore,

$$M_p(\Gamma_0) \leq \int_\Omega \rho^p(x)\, dx = \int_\Omega \|Xu(x)\|^p\, dx.$$

Taking infimum over all $u \in \mathcal{A}(K_0, K_1; \Omega)$ we get $M_p(\Gamma_0) \leq \operatorname{cap}_p(K_0, K_1; \Omega)$. Theorem 2.1 implies $M_p(\Gamma_0) = M_p(\Gamma(K_0, K_1; \Omega))$, and (3.4) follows from the above.

Step 2. Now we prove the reverse inequality

$$M_p(\Gamma(K_0, K_1; \Omega)) \geq \operatorname{cap}_p(K_0, K_1; \Omega) \tag{3.6}$$

for the case $(K_0 \cup K_1) \cap \partial\Omega = \emptyset$. Lemma 2.4 allows us to assume that $\rho \in \mathcal{F}(\Gamma(K_0, K_1; \Omega))$ is continuous in $\Omega \setminus (K_0 \cup K_1)$. Let us define $u : \Omega \to [0, \infty]$ by $u(x) = \min(1, \inf \int_{\beta_x} \rho\, ds)$ where the infimum is taken over all locally rectifiable horizontal curves $\beta_x \in \Omega$ connecting K_0 and x. We claim that $u \in \mathcal{A}(K_0, K_1; \Omega)$ and $\|Xu\| \leq \rho$ almost everywhere in Ω. If $u \equiv 1$, then there is nothing to prove.

Let $u \not\equiv 1$, and let α_{x_1,x_2} be a shortest curve which connect x_1 and x_2, and β_{x_1} be a rectifiable curve connecting K_0 and x_1. Then,

$$u(x_2) \leq \int_{\beta_{x_1}} \rho\, ds + \int_{\alpha_{x_1,x_2}} \rho\, ds \leq \int_{\beta_{x_1}} \rho\, ds + \max_{x \in \alpha_{x_1,x_2}} \rho(x) d_c(x_1, x_2).$$

Since β_{x_1} is arbitrary, we obtain

$$u(x_2) \leq u(x_1) + \max_{x \in \alpha_{x_1,x_2}} \rho(x) d_c(x_1, x_2).$$

Similarly, we have

$$u(x_1) \leq u(x_2) + \max_{x \in \alpha_{x_1,x_2}} \rho(x) d_c(x_1, x_2).$$

These two inequalities prove that

$$|u(x_1) - u(x_2)| \leq \max_{x \in \alpha_{x_1,x_2}} \rho(x) d_c(x_1, x_2). \tag{3.7}$$

If u satisfies (3.7), then u is locally Lipschitz continuous in Ω. Therefore, u has the derivative $X_{1j}u$, $j = 1, \dots, n_1$, almost everywhere in Ω by [14]. Suppose now that $x_0 \in \Omega$ is a point where the derivatives $X_{1j}u$, $j = 1, \dots, n_1$ exist, then we get

$$|u(x_0 h) - u(x_0)| = |h| \|Xu(x_0)\| + o(|h|) \leq \max_{x \in \alpha_{x_0, x_0 h}} \rho(x) |h|.$$

Letting $|h| \to 0$, we obtain $\|Xu(x_0)\| \leq \rho(x_0)$. Therefore,

$$\mathrm{cap}_p(K_0, K_1; \Omega) \leq \int_\Omega \|Xu\|^p\, dx \leq \int_\Omega \rho^p\, dx$$

and (3.6) holds.

By (3.4) and (3.6) we conclude that, if $(K_0 \cup K_1) \cap \partial\Omega = \emptyset$, then

$$\mathrm{cap}_p(K_0, K_1; \Omega) = M_p(\Gamma(K_0, K_1; \Omega)). \tag{3.8}$$

Step 3. Fix $\varepsilon \in (0, 1/2)$ and let $(K_0 \cup K_1) \cap \partial\Omega \neq \emptyset$. Let $\rho \in \mathcal{F}(\Gamma(K_0, K_1; \Omega))$ be a continuous function in $\Omega \setminus (K_0 \cup K_1)$, such that

$$\int_{\Omega \setminus (K_0 \cup K_1)} \rho^p\, dx < \varepsilon + M_p(\Gamma(K_0, K_1; \Omega)).$$

We may assume that ρ is strictly positive on $\Omega \setminus (K_0 \cup K_1)$. If this were not so, we could consider the cut-of-function $\max(\rho, 1/m)$ instead of ρ and suppose that this function satisfies the inequality

$$\int_{\Omega \setminus (K_0 \cup K_1)} \big(\max(\rho, 1/m)\big)^p\, dx < \varepsilon + M_p(\Gamma(K_0, K_1; \Omega))$$

for a sufficiently big $m \in \mathbb{N}$.

Let ρ', $\{K_0^j\}, \{K_1^j\}$, be as in Lemma 3.1. We show that

$$\int_\gamma \rho'\, ds > 1 - 2\varepsilon \quad \text{for all} \quad \gamma \in \Gamma(K_0^j, K_1^j; \Omega)$$

for a sufficiently big $j \in \mathbb{N}$. In fact, if we supposed the contrary, there would be a sequence $\{j_k\}$ and curves $\gamma_k \in \Gamma\big(K_0^{j_k}, K_1^{j_k}; \Omega\big)$, such that

$$\int_{\gamma_k} \rho'\, ds \leq 1 - 2\varepsilon.$$

By Lemma 3.1 we would find $\tilde{\gamma} \in \Gamma(K_0, K_1; \Omega)$, such that

$$\int_{\tilde{\gamma}} \rho\, ds \leq 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon,$$

which contradicts $\rho \in \mathcal{F}(\Gamma(K_0, K_1; \Omega))$.

Next we define the function

$$\tilde{\rho}(x) = \begin{cases} \frac{\rho'}{1-2\varepsilon} & \text{if } x \in \Omega \setminus \big(K_0^j \cup K_1^j\big), \\ 0 & \text{if } x \notin \Omega \setminus \big(K_0^j \cup K_1^j\big). \end{cases}$$

It belongs to $\mathcal{F}\big(\Gamma(K_0, K_1; \Omega \cup K_0^j \cup K_1^j)\big)$. This fact and the equality (3.8) for $(K_0, K_1; \Omega \cup K_0^j \cup K_1^j)$ imply

$$\begin{aligned}(M_p(\Gamma(K_0, K_1; \Omega)) + 2\varepsilon)(1-2\varepsilon)^{1-p} &\geq \int_\Omega \tilde{\rho}^p\, dx \geq M_p\big(\Gamma(K_0, K_1; \Omega \cup K_0^j \cup K_1^j)\big) \\ &= \operatorname{cap}_p\big(K_0, K_1; \Omega \cup K_0^j \cup K_1^j\big) \geq \operatorname{cap}_p(K_0, K_1; \Omega).\end{aligned}$$

Hence, letting $j \to \infty$, and then $\varepsilon \to 0$, we obtain

$$M_p\Gamma((K_0, K_1; \Omega)) \geq \operatorname{cap}_p(K_0, K_1; \Omega)$$

and the theorem is proved. ■

Theorem 3.2 *Let K_0 and K_1 be disjoint non-empty closed sets in the closure $\overline{\Omega}$ of a bounded domain $\Omega \subset \mathbb{G}$. Let K_0^j and K_1^j be sequences of compact sets, such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap_{j=0}^\infty K_0^j$, and $K_1 = \bigcap_{j=0}^\infty K_1^j$. Then,*

$$M_p(\Gamma(K_0, K_1; \Omega)) = \lim_{j\to\infty} M_p\big(\Gamma(K_0^j, K_1^j; \Omega)\big).$$

Proof: Let $\rho \in \mathcal{F}(\Gamma(K_0, K_1; \Omega))$. Lemma 2.4 allows us to assume that ρ is continuous in $\Omega \setminus (K_0 \cup K_1)$. We fix $\varepsilon \in (0,1)$ and choose ρ, such that

$$\int_\Omega \rho^p \, dx \le M_p(\Gamma(K_0, K_1; \Omega)) + \varepsilon.$$

For a function ρ we can construct ρ' as in Lemma 3.1. Moreover, $(1-2\varepsilon)^{-1}\rho' \in \mathcal{F}(\Gamma_j(K_0^j, K_1^j; \Omega))$ as it was shown in the proof of the step 3 of Theorem 3.1. From all these facts we deduce

$$M_p\big(\Gamma(K_0^j, K_1^j; \Omega)\big) \le \int_\Omega \Big((1-2\varepsilon)^{-1}\rho'\Big)^p dx \le (1-2\varepsilon)^{-p}(M_p(\Gamma(K_0, K_1; \Omega)) + \varepsilon).$$

Hence, letting $j \to \infty$, and then $\varepsilon \to 0$, we obtain the desired result. ∎

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Irina Markina
Departamento de Matemática
Universidad Técnica Federico Santa María
Valparaíso, Chile
irina.markina@mat.utfsm.cl

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