RESEARCH

Real Analysis Exchange
Vol. 40(1), 2014/2015, pp. 193–208

John C. Georgiou, Metsovo 44200, Epirus, Greece.
email: johnchrygeorgiou@gmail.com

EXTREME RESULTS ON CERTAIN GENERALIZED RIEMANN DERIVATIVES

In Memory of Professor Jan Mařik[1]

Abstract

In this paper the following question is investigated. Given a natural number r and numbers α_j, β_j for $j = 0, 1, \ldots, r$ satisfying $\alpha_0 < \alpha_1 < \cdots < \alpha_r$ and

$$\sum_{j=0}^{r} \beta_j \alpha_j^k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, r-1 \\ r! & \text{if } k = r \end{cases}$$

is there a 2π-periodic, $r-1$ times continuously differentiable function f such that

$$\limsup_{h \nearrow 0} h^{-r}\Big(\sum_{j=0}^{r} \beta_j f(x+\alpha_j h)\Big) = \limsup_{h \searrow 0} h^{-r}\Big(\sum_{j=0}^{r} \beta_j f(x+\alpha_j h)\Big) = \infty,$$

$$\liminf_{h \nearrow 0} h^{-r}\Big(\sum_{j=0}^{r} \beta_j f(x+\alpha_j h)\Big) = \liminf_{h \searrow 0} h^{-r}\Big(\sum_{j=0}^{r} \beta_j f(x+\alpha_j h)\Big) = -\infty$$

for every $x \in \mathbb{R}$?

1 Introduction

1.1 Acknowledgement

It is essential to specify that all of the main results presented in this work were obtained jointly with Professor Jan Mařik and were intended for publication jointly as a research paper.

Mathematical Reviews subject classification: Primary: 26A27, 26A51, 26A24, 26B08; Secondary: 40A30, 54C50, 11A55
Key words: non-differentiability, convexity, divided differences, generalized derivatives
Received by the editors December 9, 2011
Communicated by: Brian S. Thomson

[1] Professor Jan Mařik died on Jan. 6, 1994.

1.2 Reader's Motivation

Let r be a natural number and let $\alpha_0 < \alpha_1 < \cdots < \alpha_r$. There are numbers $\beta_j \neq 0, j = 0, 1, \ldots, r$ (see Theorem 1) such that

$$\sum_{j=0}^{r} \beta_j \alpha_j^k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, r-1 \\ r! & \text{if } k = r \end{cases}$$

(we denote $0^0 = 1$). In this paper the following question is investigated.
Is there a 2π-periodic, $r-1$ times continuously differentiable function f such that

$$\limsup_{h \nearrow 0} h^{-r} \Big(\sum_{j=0}^{r} \beta_j f(x + \alpha_j h) \Big) = \limsup_{h \searrow 0} h^{-r} \Big(\sum_{j=0}^{r} \beta_j f(x + \alpha_j h) \Big) = \infty,$$

$$\liminf_{h \nearrow 0} h^{-r} \Big(\sum_{j=0}^{r} \beta_j f(x + \alpha_j h) \Big) = \liminf_{h \searrow 0} h^{-r} \Big(\sum_{j=0}^{r} \beta_j f(x + \alpha_j h) \Big) = -\infty$$

for every $x \in \mathbb{R}$?

For each finite real function f on $\mathbb{R} = (-\infty, \infty)$ and for each pair of real numbers x, h set (see Notation 1)

$$L_g(f, x, h) = \sum_{j=0}^{r} \beta_j f(x + \alpha_j h).$$

As a particular case let

$$L_c(f, x, h) = \frac{1}{2^r} \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} f\big(x + (2j - r)h\big).$$

The question posed is answered affirmatively for $L_c(f, x, h)$ provided r is odd (see Corollary 3).
Also let

$$L_p(f, x, h) = \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} f(x + jh).$$

In this case the answer to the question is also positive provided $r \geq 3$ (see Corollary 4).

2 Notation and Elementary Theorems.

Theorem 1. *Let r be a natural number and let $\alpha_0 < \alpha_1 < \cdots < \alpha_r$. There are β_j such that*

$$\sum_{j=0}^{r} \beta_j \alpha_j^k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, r-1 \\ r! & \text{if } k = r. \end{cases}$$

Furthermore $\beta_0 \cdots \beta_r \neq 0$.
(If we have $k = 0$ and $\alpha_j = 0$ for some j then we denote by $\alpha_j^k = 1$.)

PROOF. The matrix $\left(\alpha_j^k\right)$ is a Van der Monde matrix and hence invertible. More precisely

$$\beta_j = \frac{r!}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{r} (\alpha_j - \alpha_i)}. \tag{1}$$

□

Notation 1. *Let $r, \alpha_j, \beta_j, j = 0, 1, \ldots, r$ be as in Theorem 1. For each finite real function f on $\mathbb{R} = (-\infty, \infty)$ and for each pair of real numbers x, h set*

$$L_g(f, x, h) = \sum_{j=0}^{r} \beta_j f(x + \alpha_j h).$$

Theorem 2. *Let $M \in (0, \infty)$ and let f be a function such that $\left|f^{(r)}(x)\right| \leq M$ for each $x \in \mathbb{R}$. Set $\mu = \dfrac{M}{r!} \sum_{j=0}^{r} \left|\alpha_j^r \beta_j\right|$. Then, $|L_g(f, x, h)| \leq \mu\, |h^r| \quad (x, h \in \mathbb{R})$.*

PROOF. Let $x, h \in \mathbb{R}$. Set $a_k = \dfrac{f^{(k)}(x)}{k!} \quad (k = 0, \ldots, r-1)$. There are ξ_j such that

$$f(x + \alpha_j h) = \sum_{k=0}^{r-1} a_k \alpha_j^k h^k + \alpha_j^r h^r f^{(r)}(\xi_j)/r!.$$

Therefore

$$L_g(f, x, h) = \frac{h^r}{r!} \sum_{j=0}^{r} \alpha_j^r \beta_j f^{(r)}(\xi_j)$$

which easily implies our assertion. □

The proof of the following assertion is straight forward.

Theorem 3. *Let d be real number. Let ω be a bounded function on $\mathbb{R}$, which is continuous at* 0 *and such that*

$$\omega(2x) = d\omega(x) \quad \text{for each } x \in \mathbb{R}.$$

Then, ω is constant.

Note that if $d \neq 1$, then $\omega = 0$ on $\mathbb{R}$ and if $d = 1$, then $\omega = \omega(0)$ on $\mathbb{R}$. The continuity of ω at 0 is needed only in the case $|d| = 1$.

3 Auxiliary Theorems.

The proof of the following assertion is easy.

Theorem 4. *Let q be a natural number, $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_q$ and let $a_0, a_1, b_1, \ldots, a_q, b_q$ be real numbers. Set*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{q} (a_k \cos \gamma_k x + b_k \sin \gamma_k x).$$

Then for every $x_0 \in \mathbb{R}$

$$a_j = 2 \lim_{x\to\infty} \{(x - x_0)^{-1} \int_{x_0}^{x} f(t) \cos(\gamma_j t) dt\} \quad (j = 0, \ldots, q)$$

$$b_j = 2 \lim_{x\to\infty} \{(x - x_0)^{-1} \int_{x_0}^{x} f(t) \sin(\gamma_j t) dt\} \quad (j = 1, \ldots, q).$$

If $f \geq 0$ on (x_0, ∞), then $|a_j| \leq a_0, \quad |b_j| \leq a_0 \quad (j = 1, \ldots, q)$.

Theorem 5. *Let J, K be intervals such that J is compact. Let f be a continuous function on $J \times K$, and suppose that for each $x \in J$ there is an $h \in K$ such that $f(x, h) > 0$. Then there is a compact interval $M \subset K$ such that*

$$\inf \Big\{ \max \big\{ f(x, h);\ h \in M \big\}; x \in J \Big\} > 0.$$

PROOF. Let $K_1, K_2, \ldots$ be compact intervals,

$$K_1 \subset K_2 \subset \cdots, \ \bigcup_{n=1}^{\infty} K_n = K.$$

For each $x \in J$ and for each $m \in \mathbb{N}$ there is a $h_m(x) \in K_m$ such that $f(x, h_m(x)) = \max\{f(x,h); h \in K_m\}$. Let $G_m = \{x \in J; f(x, h_m(x)) > 0\}$. The sets G_m are easily seen to be open in J. Since J is compact and since the sets G_m are increasing with m with union equal to J, there is an m_0 such that $J \subset G_{m_0}$. Then set $M = K_{m_0}$. It is easy to show that this choice satisfies the assertion. $\square$

Theorem 6. *Let $\alpha_j, \beta_j, j = 0, 1, \ldots, r$ be as in Theorem 1. Define*

$$\phi(h) = \sum_{j=0}^{r} \beta_j \cos(\alpha_j h), \quad \psi(h) = \sum_{j=0}^{r} \beta_j \sin(\alpha_j h).$$

There is a natural number q and real numbers a_j, b_j, γ_j such that $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_q$ and

$$\phi(h) = \frac{a_0}{2} + \sum_{j=1}^{q} a_j \cos(\gamma_j h), \qquad \psi(h) = \sum_{j=1}^{q} b_j \sin(\gamma_j h). \tag{2}$$

Proof. The conclusion of the theorem is clear if either $\alpha_0 > 0$ or $\alpha_0 = 0$. Assume $\alpha_0 < 0$ and let $\Gamma = \{|\alpha_j|; j = 0, 1, \ldots, r\}$. Order the positive elements of Γ as $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_q$. Let $\gamma_k \in \Gamma$. If there are $i < j$ such that $\gamma_k = -\alpha_i = \alpha_j$, then combine the two corresponding terms in ϕ into $(\beta_i + \beta_j)\cos(\gamma_k h)$ and in ψ into $(-\beta_i + \beta_j)\sin(\gamma_k h)$. That's the most difficult case. The other cases are if $\gamma_k = -\alpha_j$ or if $\gamma_k = \alpha_j$. The final case to consider is if there is a j with $\alpha_j = 0$. Then the corresponding term in ϕ is $\beta_j \cos(\alpha_j h) = \beta_j$ and in ψ is $\beta_j \sin(\alpha_j h) = 0$. $\square$

It should be mentioned here that, as a consequence of Theorem 4, the set of functions

$$\{1, \sin(\gamma_k h), \cos(\gamma_k h); k = 1, 2, \ldots, q\}$$

is a linearly independent set of functions, which is why the representations of ϕ and ψ in (2) are more useful than the original definitions.

Theorem 7. *Let $\alpha_j, \beta_j, j = 0, 1, \ldots, r,\ q, a_0, a_j, b_j, j = 1, 2, \ldots, q$ and ϕ, ψ be as in Theorem 6. For each finite real function f on $\mathbb{R}$ and for each pair of real numbers x, h let L_g be as in Notation 1.*

1) If ψ is constant, then r is even, $r = 2q, a_j = 2\beta_{q-j} = 2\beta_{q+j} \neq 0, b_j = 0, \gamma_j = -\alpha_{q-j} = \alpha_{q+j}, j = 1, 2, \ldots, q, \quad \alpha_q = 0$ and

$$L_g(f, x, h) = A_0 f(x) + \sum_{j=1}^{q} A_j \big(f(x + \gamma_j h) + f(x - \gamma_j h)\big)$$

for some $A_j \in \mathbb{R}, j = 0, 1, \ldots, q$.

2) If ϕ *is constant, then* r *is odd,* $r = 2q-1, a_j = 0, j = 0, 1, \ldots, q, b_j = -2\beta_{q-j} = 2\beta_{q+j-1}, \gamma_j = -\alpha_{q-j} = \alpha_{q+j-1}, j = 1, 2, \ldots, q$ *and*

$$L_g(f, x, h) = \sum_{j=1}^{q} B_j \big(f(x + \gamma_j h) - f(x - \gamma_j h)\big)$$

for some $B_j \in \mathbb{R}, j = 1, 2, \ldots q$.

3) At most one of the functions ϕ, ψ *is constant.*

PROOF. 1) If $\psi = c$ on $\mathbb{R}$ for some $c \in \mathbb{R}$, since $\psi(0) = 0$, we get $c = 0$. Therefore $\psi(h) = \sum_{j=0}^{r} \beta_j \sin(\alpha_j h) = 0$ for every $h \in \mathbb{R}$. By differentiation we get

$$\sum_{j=0}^{r} \beta_j \alpha_j^{2i-1} \cos(\alpha_j h) = 0,\ i = 1, 2, \ldots \textit{ for every } h \in \mathbb{R}.$$

For $h = 0$

$$\sum_{j=0}^{r} \beta_j \alpha_j^{2i-1} = 0,\ i = 1, 2, \ldots.$$

If r is odd, then $\sum_{j=0}^{r} \beta_j \alpha_j^r = 0$ which contradicts $\sum_{j=0}^{r} \beta_j \alpha_j^r = r!$. Thus r is even. Since $\psi(h) = 0$ for every h, the linear independence shows that all the coefficients in the representation of ψ in equation (2) are 0. The only case of Theorem 6, which does not contradict $\beta_j \neq 0,\ j = 0, 1, \ldots, r$ with $r = 2q$, is $\gamma_j = \alpha_{q+j} = -\alpha_{q-j},\ j = 1, 2, \ldots, q$ and $\alpha_q = 0$. For each $j = 1, 2, \ldots, q$, the corresponding term in ϕ is $(\beta_{q+j} + \beta_{q-j})\cos(\gamma_j h)$ and in ψ is $(\beta_{q+j} - \beta_{q-j})\sin(\gamma_j h)$. Consequently $b_j = \beta_{q+j} - \beta_{q-j} = 0,\ j = 1, 2, \ldots, q$, $a_0 = 2\beta_q,\ a_j = 2\beta_{q+j} = 2\beta_{q-j}$, and $\phi(h) = \beta_q + 2\sum_{j=1}^{q} \beta_{q+j} \cos(\gamma_j h))$.

Consider now

$$\begin{aligned} L_g(f,x,h) &= \sum_{j=0}^{2q} \beta_j f(x+\alpha_j h) \\ &= \sum_{j=1}^{q} \beta_{q-j} f(x-\gamma_j h) + \beta_q f(x) + \sum_{j=1}^{q} \beta_{q+j} f(x+\gamma_j h) \\ &= A_0 f(x) + \sum_{j=1}^{q} A_j \big(f(x+\gamma_j h) + f(x-\gamma_j h) \big) \end{aligned}$$

where $A_0 = \beta_q, A_j = \beta_{q+j} = \beta_{q-j}, j = 1, 2, \ldots, q$.

2) The proof is similar to the proof of part 1) with minor modifications.

3) r can't be both odd and even. □

4 Main Results.

In this section Theorem 10 (also see Notation 2) indicates what is sufficient to obtain a positive answer to the major question dealt with in the paper and Theorem 8 is the first result, which refers to the methods of succeeding it (see Corollary 1). Theorem 11 (see Notation 2 and Theorem 9) may also have some independent interest.

Notation 2. *Let $r, \alpha_j, \beta_j, j = 0, 1, \ldots, r$ be as in Theorem 1. Let V be the system of all continuous 2π-periodic functions f such that for each $x \in \mathbb{R}$ there are $h_1, h_2 \in (-\infty, 0)$ and $h_3, h_4 \in (0, \infty)$ with*

$$(-1)^i L_g(f,x,h_i) > 0 \quad (i = 1, 2, 3, 4). \tag{3}$$

Let V^* be the system of all 2π-periodic functions f such that $f^{(r-1)}$ is continuous on $\mathbb{R}$ and that, for each $x \in \mathbb{R}$,

$$\begin{gathered} \limsup_{h \nearrow 0} h^{-r} L_g(f,x,h) = \limsup_{h \searrow 0} h^{-r} L_g(f,x,h) = \infty \\ \text{and} \\ \liminf_{h \nearrow 0} h^{-r} L_g(f,x,h) = \liminf_{h \searrow 0} h^{-r} L_g(f,x,h) = -\infty. \end{gathered} \tag{4}$$

Let W be the system of all continuous 2π-periodic functions f such that for each $x \in \mathbb{R}$ there is an $h_1 > 0$ and an $h_2 < 0$ with

$$L_g(f,x,h_i) \neq 0 \quad (i = 1, 2).$$

Let W^* be the system of all 2π-periodic functions f such that $f^{(r-1)}$ is continuous on $\mathbb{R}$ and that, for each $x \in \mathbb{R}$,

$$\limsup_{h \nearrow 0} \left| h^{-r} L_g(f, x, h) \right| = \limsup_{h \searrow 0} \left| h^{-r} L_g(f, x, h) \right| = \infty.$$

Theorem 8. *Let $f_1(x) = \cos x$, $f_2(x) = \cos x + \sin(2x)$. Suppose that $\psi(h) \neq 0$ for some h. Let a_j be as in Theorem 6. If $|a_k| > |a_0|$ for some k, then $f_1 \in V$. If $\phi = 0$ on $\mathbb{R}$, then $f_2 \in V$.*

PROOF. It is easy to see that

$$L_g(\cos, x, h) = \phi(h) \cos x - \psi(h) \sin x, \tag{5}$$

$$L_g(\sin, x, h) = \phi(h) \sin x + \psi(h) \cos x. \tag{6}$$

1) Let k be a number such that $|a_k| > |a_0|$ and let $x \in \mathbb{R}$. It follows from (5) that

$$L_g(f_1, x, h) = \frac{a_0 \cos x}{2} + \sum_{j=1}^{q} \big(a_j \cos x \cos(\gamma_j h) - b_j \sin x \sin(\gamma_j h) \big) \quad (h \in \mathbb{R}).$$

a) If $\cos x = 0$, then $|\sin x| = 1$,

$$L_g(f_1, x, h) = -\psi(h) \sin x = -\sum_{j=1}^{q} b_j \sin x \sin(\gamma_j h)$$

and $L_g(f_1, x, -h) = -L_g(f_1, x, h) \quad (h \in \mathbb{R})$.

i) If $L_g(f_1, x, \cdot) \geq 0$ on $(0, +\infty)$, then by Theorem 4 we have $|-b_j \sin x| \leq 0$ and consequently $b_j = 0$ for all $j = 1, \ldots, q$. Hence $\psi(h) = 0$ for every h contrary to $\psi(h) \neq 0$ for some h. Therefore $L_g(f_1, x, h_3) < 0$ for some $h_3 \in (0, +\infty)$. Setting $h_2 = -h_3$, gives $L_g(f_1, x, h_2) > 0$ for some $h_2 \in (-\infty, 0)$.

ii) If $L_g(f_1, x, \cdot) \leq 0$ on $(0, +\infty)$, then

$$-L_g(f_1, x, h) = \sum_{j=1}^{q} b_j \sin x \sin(\gamma_j h) \geq 0 \ \text{ for every } h \in (0, +\infty).$$

As in i), $L_g(f_1, x, h_4) > 0$ for some $h_4 \in (0, +\infty)$ and setting $h_1 = -h_4$ yields $L_g(f_1, x, h_1) < 0$ for some $h_1 \in (-\infty, 0)$.

b) If $\cos x \neq 0$, then from (5) it follows that

$$L_g(f_1, x, h) = \frac{a_0 \cos x}{2} + \sum_{j=1}^{q} \big(a_j \cos x \cos(\gamma_j h) - b_j \sin x \sin(\gamma_j h)\big) \quad (h \in \mathbb{R}).$$

i) If $L_g(f_1, x, \cdot) \geq 0$ on $(0, +\infty)$, then by Theorem 4 we have $a_0 \cos x \geq 0$ and $|a_j \cos x| \leq a_0 \cos x = |a_0|\,|\cos x|$ for $j = 1, 2, \cdots, q$. Because $\cos x \neq 0$, $|a_j| \leq |a_0|$ for $j = 1, 2, \cdots, q$ contrary to the assumption that for some k, $|a_k| > |a_0|$. Therefore $L_g(f_1, x, h_3) < 0$ for some $h_3 \in (0, +\infty)$.

ii) If $L_g(f_1, x, \cdot) \leq 0$ on $(0, +\infty)$, then $-L_g(f_1, x, \cdot) \geq 0$ on $(0, +\infty)$. As in i), $L_g(f_1, x, h_4) > 0$ for some $h_4 \in (0, +\infty)$.

iii) If $L_g(f_1, x, \cdot) \geq 0$ on $(-\infty, 0)$, then setting $h' = -h$ we have

$$\frac{a_0 \cos x}{2} + \sum_{j=1}^{q} \big(a_j \cos x \cos(\gamma_j h') + b_j \sin x \sin(\gamma_j h')\big) \geq 0$$

for every $h' \in (0, +\infty)$. As in i), $L_g(f_1, x, h_1) < 0$ for some $h_1 \in (-\infty, 0)$.

iv) If $L_g(f_1, x, \cdot) \leq 0$ on $(-\infty, 0)$, then setting $h' = -h$ we have

$$\frac{-a_0 \cos x}{2} + \sum_{j=1}^{q} \big(- a_j \cos x \cos(\gamma_j h') - b_j \sin x \sin(\gamma_j h')\big) \geq 0$$

for every $h' \in (0, +\infty)$. As in i), $L_g(f_1, x, h_2) > 0$ for some $h_2 \in (-\infty, 0)$.

Now it is easy to see that $f_1 \in V$.

2) If ψ is not constant, then $\psi(x)$, $\psi(2x)$ are linearly independent. Indeed, let for some $c_1, c_2 \in \mathbb{R}, c_1\psi(x) + c_2\psi(2x) = 0$ for each $x \in \mathbb{R}$. If $c_2 \neq 0$, then $\psi(2x) = -\dfrac{c_1}{c_2}\psi(x)$ and by Theorem 3, ψ is constant contrary to the assumption that ψ is not constant. Thus $c_2 = 0$ and $c_1\psi(x) = 0$, but $\psi(x) \neq 0$ for some x. Therefore $c_1 = 0$.

Suppose that $\phi = 0$ on $\mathbb{R}$; let $x \in \mathbb{R}$. First $L_g(f_2, x, h)$ must be written in a form that satisfies the hypothesis of Theorem 4. By (5) and (6) and the property $L_g(f(2\cdot), x, h) = L_g(f, 2x, 2h)$

$$\begin{aligned} L_g(f_2, x, h) &= -\psi(h)\sin x + \psi(2h)\cos(2x) \text{ (Apply Theorem 6)} \\ &= \Big(\sum_{j=1}^{q} -\big(b_j \sin(\gamma_j h)\big)\Big)\sin x + \Big(\sum_{j=1}^{q} b_j \sin(2\gamma_j h)\Big)\cos(2x) \\ &= \sum_{j=1}^{q}(-b_j \sin x)\sin(\gamma_j h) + \sum_{j=1}^{q}\big(b_j\cos(2x)\big)\sin(2\gamma_j h). \end{aligned} \tag{7}$$

Let $0 < \delta_1 < \delta_2 < \cdots < \delta_{q'}$ be the ordering of the set $\{\gamma_j : j = 1, \ldots, q\} \cup \{2\gamma_j : j = 1, \ldots, q\}$. Combining terms from the two sums in (7) having the same angle yields

$$L_g(f_2, x, h) = \sum_{j=1}^{q'} b'_j \sin(\delta_j h) = \frac{a'_0}{2} + \sum_{j=1}^{q'} a'_j \cos(\delta_j h) + \sum_{j=1}^{q'} b'_j \sin(\delta_j h)$$

where $a'_0 = a'_1 = \cdots = a'_{q'} = 0$. Also it is easy to see that

$$L_g(f_2, x, -h) = -L_g(f_2, x, h) \quad (h \in \mathbb{R}).$$

a) If $L_g(f_2, x, \cdot) \geq 0$ on $(0, \infty)$, then by Theorem 4 $\left|b'_j\right| \leq a'_0 = 0 \quad (j = 1, 2, \ldots, q')$ and $L_g(f_2, x, h) = 0$ for each h, but $\psi(h), \psi(2h)$ are linear independent. Thus $\sin x = \cos(2x) = 0$, a contradiction. Therefore $L_g(f_2, x, h_3) < 0$ for some $h_3 \in (0, +\infty)$. Setting $h_2 = -h_3$ yields $L_g(f_2, x, h_2) > 0$ for some $h_2 \in (-\infty, 0)$.

b) If $L_g(f_2, x, \cdot) \leq 0$ on $(0, \infty)$, then $-L_g(f_2, x, \cdot) \geq 0$ on $(0, \infty)$. As in a), $L_g(f_2, x, h_4) > 0$ for some $h_4 \in (0, +\infty)$ and setting $h_1 = -h_4$ gives $L_g(f_2, x, h_1) < 0$ for some $h_1 \in (-\infty, 0)$.

Now it is easy to see that $f_2 \in V$.

□

Theorem 9. *Let f_1, f_2 be as in Theorem 8. Let $f_3(x) = \sin x + \cos(2x)$. Then at least one of the functions f_1, f_2, f_3 is in W.*

PROOF. On account of Theorem 7 at most one of the functions ϕ and ψ is constant. If $\phi(h) = 0$ for every h, then by Theorem 8 $f_2 \in V$. So $f_2 \in W$. Thus we may assume $\phi(h) \neq 0$ for some h.

1) Suppose $\psi(h) \neq 0$ for some h and let $x \in \mathbb{R}$. If $L_g(\cos, x, h) = 0$ for each $h > 0$, then by (5) and Theorem 4, $a_0 \cos x = 0$ and if $\cos x \neq 0$, the

function $\phi = 0$, otherwise $\cos x = 0$ in which case $\psi = 0$; i.e., one of the functions ϕ, ψ is identically zero contrary to the assumption $\phi \neq 0$ and $\psi \neq 0$. Now we can see easily that $f_1 = \cos \in W$.

2) Suppose $\psi(h) = 0$ for all $h \in \mathbb{R}$ and let $x \in \mathbb{R}$. Then, by (5) and (6), $L_g(f_3, x, h) = \phi(h)\sin x + \phi(2h)\cos(2x)$. As was shown in the proof of Theorem 8, the functions $\psi(h)$ and $\psi(2h)$ are linearly independent. Similarly, the functions $\phi(h)$ and $\phi(2h)$ are linearly independent. This easily implies that $f_3 \in W$.

□

Theorem 10. *If $V \neq \emptyset$, then $V^* \neq \emptyset$.*

Proof. Let $\Phi \in V$. Theorem 5 (where we take $J = [0, 2\pi]$) implies that there are positive numbers η, δ and H such that for each $x \in \mathbb{R}$ there are $h_1, h_2 \in [-H, -\delta]$ and $h_3, h_4 \in [\delta, H]$ such that

$$(-1)^i L_g(\Phi, x, h_i) \geq \eta \quad (i = 1, 2, 3, 4). \tag{8}$$

Set $\delta = \dfrac{\eta}{2(|\beta_0| + \cdots + |\beta_r|)}$. There is a trigonometric polynomial P such that $|\Phi - P| < \delta$ on $\mathbb{R}$. There are positive numbers λ, μ (see Theorem 2) such that $|L_g(P, x, h)| \leq \min(\mu |h|^r, \lambda)$ for all x, h. Set $A = 6\dfrac{\mu H^r}{\eta}$, $B = 6\dfrac{\lambda}{\eta}$. Choose a natural number $a > 1 + A$ such that

$$a^r > (1 + A)(1 + B) \tag{9}$$

and define

$$b = \frac{(1 + A)}{a^r}. \tag{10}$$

Obviously $a^{r-1}b = \dfrac{(1 + A)}{a} < 1$ and for $s = 0, \ldots, r - 1$,

$$b^k (a^k)^s = \Big(\frac{1 + A}{a^{r-s}}\Big)^k \leq \Big(\frac{1 + A}{a}\Big)^k.$$

Since $\displaystyle\sum_{k=0}^{\infty} \Big(\frac{1 + A}{a}\Big)^k < \infty$, we have $\displaystyle\sum_{k=0}^{\infty} b^k (a^k)^s < \infty$, for $s = 0, \ldots, r - 1$. Thus, taking $s = 0$ we may define

$$f(x) = \sum_{k=0}^{\infty} b^k P(a^k x) \quad (x \in \mathbb{R}).$$

Because the sum defining f converges uniformly on $\mathbb{R}$ and because each term is 2π-periodic, f is 2π-periodic on $\mathbb{R}$.

To prove that f is $r-1$ times continuously differentiable, first note that for each $s = 0, 1, \ldots, r-1$ there is a number $M_s > 0$ such that $\left|P^{(s)}(x)\right| \le M_s$ for all $x \in \mathbb{R}$. Thus for each $k \in \mathbb{N}$

$$\begin{aligned}\left|\frac{d^s}{dx^s} b^k P(a^k x)\right| &= \left|b^k a^{ks} P^{(s)}(a^k x)\right| \\ &\le b^k a^{ks} M_s \le (ba^{r-1})^k M_s \le \left(\frac{1+A}{a}\right)^k M_s.\end{aligned}$$

Because $\left(\dfrac{1+A}{a}\right) < 1$, $\displaystyle\sum_{k=0}^{\infty} \frac{d^s}{dx^s} b^k P(a^k x)$ converges uniformly to a continuous function and hence, $f^{(s)}$ exists and is continuous for $s = 0, 1, \ldots, r-1$.

Now let $x \in \mathbb{R}$. By (8) for $n \in \mathbb{N}$ there is a $T_n \in [\delta, H]$ such that

$$L_g(\Phi, a^n x, T_n) \ge \eta.$$

Set $t_n = T_n / a^n$. Then

$$0 < t_n \le \frac{H}{a^n}. \tag{11}$$

It follows from the choice of δ that $|L_g(\Phi, \xi, h) - L_g(P, \xi, h)| < \dfrac{\eta}{2}$ for all ξ and h; thus

$$L_g(P, a^n x, T_n) \ge \frac{\eta}{2}. \tag{12}$$

Now fix an n and set

$$\begin{aligned} S_1 &= \sum_{k=0}^{n-1} b^k L_g(P, a^k x, a^k t_n), \\ Z &= b^n L_g(P, a^n x, T_n), \text{ and} \\ S_2 &= \sum_{k=n+1}^{\infty} b^k L_g(P, a^k x, a^k t_n). \end{aligned}$$

It is easy to see that $L_g(f, x, t_n) = S_1 + Z + S_2$. Hence, we have (see (10), (11) and (12))

$$\begin{aligned} |S_1| &\le \sum_{k=0}^{n-1} b^k \mu a^{kr} t_n^r = \mu t_n^r \sum_{k=0}^{n-1} (1+A)^k \\ &= \frac{\mu}{A} t_n^r \big((1+A)^n - 1\big) < \frac{\mu}{A} H^r \left(\frac{1+A}{a^r}\right)^n = \frac{\eta}{6} b^n, \end{aligned}$$

$$Z \geq \frac{\eta}{2} b^n, \text{ and } |S_2| \leq \lambda \sum_{k=n+1}^{\infty} b^k = \frac{\lambda b^{n+1}}{1-b}.$$

By (9) and (10) we have $1 > b(1+B)$; whence $\dfrac{b}{1-b} < \dfrac{1}{B}$, $\quad |S_2| < \dfrac{\lambda}{B} b^n = \dfrac{\eta}{6} b^n$. It follows that $L_g(f,x,t_n) \geq Z - |S_1 + S_2| \geq \dfrac{\eta}{6} b^n$. As (see (11) and (10)) $\dfrac{b^n}{t_n^r} \geq b^n \dfrac{a^{nr}}{H^r} = \dfrac{(1+A)^n}{H^r}$, we have $t_n^{-r} L_g(f,x,t_n) \to \infty$. This shows that

$$\limsup_{h \searrow 0} h^{-r} L_g(f,x,h) = \infty.$$

The remaining equalities in (4) can be proved similarly. $\square$

Corollary 1. *If $\psi(h) \neq 0$ for some h and if either $\phi = 0$ on $\mathbb{R}$ or $|a_k| > |a_0|$ for some k, then $V^* \neq \emptyset$.*

PROOF. This follows from Theorems 8 and 10. $\square$

Corollary 2. *Let $r, \alpha_j, \beta_j, j = 0, 1, \ldots, r$ be as in Theorem 1. If $\alpha_j \neq 0$ for all $j = 0, 1, \ldots, r$, then the corresponding $V^* \neq \emptyset$.*

PROOF. If $\psi(h) = 0$ for every h, then from Theorem 7, r is even, $r = 2q$, and $\alpha_q = 0$, a contradiction. Therefore

$$\psi(h) \neq 0 \text{ for some } h.$$

Because $\alpha_j \neq 0$ for all $j = 0, 1, \ldots, r$, it follows that $a_0 = 0$. If there is a k with $|a_k| > 0$, then by Theorem 10, $V^* \neq \emptyset$. If $a_k = 0$ for all k, then $\phi(h) = 0$ for all h and hence by Theorem 10, $V^* \neq \emptyset$. $\square$

Corollary 3. *Let r be odd and let $\alpha_j = 2j - r$, $j = 0, 1, \ldots, r$. Then the corresponding $V^* \neq \emptyset$.*

PROOF. This follows from Corollary 2. $\square$

Remark 1. *If $r = 1, 3$, then Corollary 3 is a generalization of [1] and [2] respectively.*

Corollary 4. *Let r be natural number $r \geq 3$, let $\alpha_j = j$, $j = 0, 1, \ldots, r$. Then the corresponding $V^* \neq \emptyset$.*

PROOF. From Theorem 1

$$\beta_j = \frac{r!}{\prod_{\substack{i=0\\ i\neq j}}^{r}(\alpha_j - \alpha_i)} = \frac{r!}{\prod_{\substack{i=0\\ i\neq j}}^{r}(j-i)} = (-1)^{r+j}\binom{r}{j}.$$

Then

$$\phi(h) = \sum_{j=0}^{r} \beta_j \cos(\alpha_j h) = \frac{2\beta_0}{2} + \sum_{j=1}^{r} \beta_j \cos(\alpha_j h) = \frac{a_0}{2} + \sum_{j=1}^{r} a_j \cos(\gamma_j h),$$

$$\psi(h) = \sum_{j=0}^{r} \beta_j \sin(\alpha_j h) = \sum_{j=1}^{r} b_j \sin(\gamma_j h),$$

with

$$a_0 = 2\beta_0 = 2(-1)^r \binom{r}{0} = 2(-1)^r, \qquad a_j = \beta_j = (-1)^{r+j}\binom{r}{j},\ j = 1, 2, \ldots, r$$

and

$$\gamma_j = \alpha_j = j,\ j = 1, 2, \ldots, r, \qquad b_j = \beta_j = (-1)^{r+j}\binom{r}{j},\ j = 1, 2, \ldots, r.$$

Since $r \geq 3$, we have $|a_1| > |a_0|$ and

$$\psi(h) = \sum_{j=1}^{r} \beta_j \sin(\alpha_j h) = \sum_{j=1}^{r} (-1)^{r+j}\binom{r}{j} \sin(jh). \qquad \square$$

Theorem 11. *We always have $W^* \neq \emptyset$.*

PROOF. By Theorem 9 we have $W \neq \emptyset$. Now we proceed as in the proof of Theorem 10. $\square$

Acknowledgement. The author wishes to thank the referees for their valuable comments and suggestions.

References

[1] L. Filipczak, *Exemple d'une Fonction Continue Prívée de Dérivée Symetrique Partout*, Colloquium Math., vol. **xx**, (1969), Fasc **2**, 143–153.

[2] John C. Georgiou, *On an Example of a Function with a Derivative which does not have Third Order Symmetric Riemann Derivative Anywhere*, Real Anal. Exchange, **37(1)** (2011-2012).

[3] Jan Mařik, *Generalized Derivatives and one-dimensional Integrals*, Unpublished Notes, M.S.U., (1973).