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A GENERALIZATION OF THE FUNDAMENTAL THEOREM OF CALCULUS

Abstract

After introducing the concepts of φ-derivatives and φ-integrals inside the dual real number algebra, we prove a new generalization of the fundamental theorem of calculus.

Except for the complex number field and the direct product of two real number fields, the only remaining 2-dimensional real associative algebra is the dual real number algebra $\mathcal{R}^{(2)}$ which has zero divisors. It turns out that the well-known theory of Riemann integrals can be rewritten by replacing the real number field $\mathcal{R}$ with the dual real number algebra $\mathcal{R}^{(2)}$. The purpose of this paper is to present a new way of rewriting the fundamental theorem of calculus inside the dual real number algebra $\mathcal{R}^{(2)}$.

Using Fréchet derivatives is a well-known way of introducing differentiability of functions with values in real associative algebras which have zero-divisors. Fréchet's way of introducing differentiability avoids the problem produced from zero-divisors effectively, but it ignores the invertible elements of a real associative algebra even if the zero-divisors of the real associative algebra can be controlled easily. Being dissatisfied at this aspect of Fréchet derivatives, we give a new way of introducing differentiability inside real associative algebras which have zero-divisors. The key idea in this new way is to use the topology transferred from the topology on a field to introduce differentiability inside real associative algebras which have zero-divisors. Based on this idea, we get the concept of φ-derivatives inside the real associative algebra $\mathcal{R}^{(2)}$, which is defined by using both invertible elements of $\mathcal{R}^{(2)}$ and the

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topology transferred from the topology on the real number field $\mathcal{R}$. Except for φ-derivatives, another fundamental concept introduced in this paper is the concept of φ-integrals. Unlike the counterparts of Riemann integrals in other generalizations of single variable calculus such as multivariable calculus, complex analysis and Lebesgue integration, the concept of φ-integrals is defined by generalizing the order relation on the real number field and replacing the length function on intervals with a function whose values are not always in the set of non-negative real numbers. This paper consists of five sections. In Section 1, we generalize the order relation on the real number field. In Section 2 and Section 3 we introduce the concepts of the φ-derivatives and φ-integrals. In Section 4, we give the basic properties of the φ-integrals. In Section 5, we prove the new generalization of the Fundamental Theorem of Calculus.

1 Two generalized order relations on $\mathcal{R}^{(2)}$

The multiplication on the dual real number algebra $\mathcal{R}^{(2)} = \mathcal{R} \oplus \mathcal{R}$ (as real vector space) is defined by

$$(a_1,\, a_2)(b_1,\, b_2) := (a_1b_1,\, a_1b_2 + a_2b_1) \quad \text{for } (a_1,\, a_2),\, (b_1,\, b_2) \in \mathcal{R}^{(2)}$$

We denote the element $(1, 0)$ by 1, and the element $(0, 1)$ by ℓ. Then every element $a = (a_1, a_2)$ of $\mathcal{R}^{(2)}$ can be expressed in a unique way as a linear combination of 1 and ℓ:

$$a = (a_1,\, a_2) = a_1\, 1 + a_2\ell = a_1 + a_2\ell \quad \text{for } a_1,\, a_2 \in \mathcal{R},$$

where $Re\, a := a_1$ is called the *real part* of a, and $Ze\, a := a_2$ is called the *zero-divisor part* of a. If S is a non-empty subset of $\mathcal{R}^{(2)}$, we defined the *real part* $Re\, S$ and the *zero-divisor part* $Ze\, S$ of S by

$$Re\, S := \{\, Re\, x \,|\, x \in S\,\} \quad \text{and} \quad Ze\, S := \{\, Ze\, x \,|\, x \in S\,\}.$$

One nice algebraic property of $\mathcal{R}^{(2)}$ is that the zero-divisors of $\mathcal{R}^{(2)}$ can be characterized in a convenient way.

Proposition 1. *Let x be a non-zero element of $\mathcal{R}^{(2)}$. Then*

(i) x is a zero-divisor if and only if $Re\, x = 0$;

(ii) x is invertible if and only if $Re\, x \neq 0$, in which case, the inverse $x^{-1} = \dfrac{1}{x}$ is given by

$$x^{-1} = \frac{1}{Re\, x} - \frac{Ze\, x}{(Re\, x)^2}\,\ell\,.$$

PROOF. This proposition follows from the definition of the multiplication on $\mathcal{R}^{(2)}$.

□

Unlike the complex field, there are two generalized order relations on $\mathcal{R}^{(2)}$ which are compatible with the multiplication in $\mathcal{R}^{(2)}$.

Definition 1.1. *Let x and y be two elements of $\mathcal{R}^{(2)}$.*

(i) We say that x is type 1 greater than y (or y is type 1 less than x) and we write $x \overset{1}{>} y$ (or $y \overset{1}{<} x$) if

$$\text{either} \quad \begin{cases} Re\, x > Re\, y \\ Ze\, x \geq Ze\, y \end{cases} \quad \text{or} \quad \begin{cases} Re\, x = Re\, y \\ Ze\, x > Ze\, y \end{cases}$$

(ii) We say that x is type 2 greater than y (or y is type 2 less than x) and we write $x \overset{2}{>} y$ (or $y \overset{2}{<} x$) if

$$\text{either} \quad \begin{cases} Re\, x > Re\, y \\ Ze\, y \geq Ze\, x \end{cases} \quad \text{or} \quad \begin{cases} Re\, x = Re\, y \\ Ze\, y > Ze\, x \end{cases}$$

We use $x \overset{\theta}{\geq} y$ when $x \overset{\theta}{>} y$ or $x = y$ for $\theta = 1, 2$. By Definition 1.1, if $Re\, x = Re\, y$, then $x \overset{1}{>} y \Longleftrightarrow y \overset{2}{>} x$; if $Ze\, x = Ze\, y$, then $x \overset{1}{>} y \Longleftrightarrow x \overset{2}{>} y$. The following proposition gives the basic properties of the two generalized order relations.

Proposition 2. *Let x, y and z be elements of $\mathcal{R}^{(2)}$ and $\theta = 1, 2$. Then*

(i) one of the following holds:

$$x \overset{1}{>} y, \quad y \overset{1}{>} x, \quad x = y, \quad x \overset{2}{>} y, \quad y \overset{2}{>} x;$$

(ii) if $x \overset{\theta}{>} y$ and $y \overset{\theta}{>} z$, then $x \overset{\theta}{>} z$;

(iii) if $x \overset{\theta}{>} y$, then $x + z \overset{\theta}{>} y + z$;

(iv) if $x \overset{\theta}{>} 0$ and $y \overset{\theta}{>} 0$, then $xy \overset{\theta}{\geq} 0$;

(v) if $x \overset{\theta}{>} y$, then $-x \overset{\theta}{<} -y$.

PROOF. Clear.

□

2 φ-Derivatives

In the remaining part of this paper, let φ be a real-valued function $\varphi : \mathcal{R} \to \mathcal{R}$.

A set $S \subseteq \mathcal{R}^{(2)}$ is called a *φ-set* if $Ze\,x = \varphi(Re\,x)$ for all $x \in S$. Clearly,

$$\mathcal{R}_\varphi := \{\, x \,|\, x \in \mathcal{R}^{(2)} \quad \text{and} \quad Ze\,x = \varphi(Re\,x)\,\}$$

is the largest φ-set in $\mathcal{R}^{(2)}$. A φ-set S is called an open (or closed) *φ-interval* if $Re\,S$ is an open (or closed) interval in the real number field $\mathcal{R}$. For $a,\, b \in \mathcal{R}^{(2)}$ with $Re\,a < Re\,b$, we use $(a,\, b)_\varphi$ (or $[a,\, b]_\varphi$) to denote the open (or closed) φ-interval such that $Re\,(a,\, b)_\varphi = (Re\,a,\, Re\,b)$ (or $Re\,[a,\, b]_\varphi = [Re\,a,\, Re\,b]$). The topology of the real number field $\mathcal{R}$ can be transferred to the largest φ-set $\mathcal{R}_\varphi$ by employing open φ-intervals.

The usual matrix norm $||x|| = \sqrt{(Re\,x)^2 + (Ze\,x)^2}$ with $x \in \mathcal{R}^{(2)}$ does not make $\mathcal{R}^{(2)}$ into a normed algebra, but there are many other norms on $\mathcal{R}^{(2)}$ which make $\mathcal{R}^{(2)}$ into a normed algebra. In this paper, we use the *taxi norm* $||x||_1$ to make $\mathcal{R}^{(2)}$ into a normed algebra, where the taxi norm is defined by

$$||x||_1 = |Re\,x| + |Ze\,x| \quad \text{for } x \in \mathcal{R}^{(2)}.$$

If $f : S \to \mathcal{R}^{(2)}$ is a function with $S \subseteq \mathcal{R}^{(2)}$, then

$$f(x) = f_{Re}(x) + \ell f_{Ze}(x) \quad \text{for } x \in S,$$

where $f_{Re}(x) := Re\,f(x)$ and $f_{Ze}(x) := Ze\,f(x)$ are real-valued functions of x or, equivalently, of $Re\,x$ and $Ze\,x$. Sometimes, $f_\clubsuit(x)$ is also denoted by $f_\clubsuit(Re\,x,\, Ze\,x)$ to emphasize that $f_\clubsuit$ is regarded as a real-valued function of two real variables $Re\,x$ and $Ze\,x$ for $\clubsuit \in \{\, Re,\, Ze\,\}$. We say that a function $f : S \to \mathcal{R}^{(2)}$ with $S \subseteq \mathcal{R}^{(2)}$ is *bounded* if there exists a positive real number M such that $M \geq |f_\clubsuit(x)|$ for all $x \in S$ and $\clubsuit \in \{\, Re,\, Ze\,\}$.

Definition 2.1. *Let I be an open φ-interval containing $c \in \mathcal{R}^{(2)}$, and let f be a function defined everywhere on I except possibly at c. We say that an element $L \in \mathcal{R}^{(2)}$ is the φ-limit of f at c, and we write $\lim\limits_{Re\,x \overset{\varphi}{\to} Re\,c} f(x) = L$ if for every $\eta > 0$ there exists a $\delta > 0$ such that*

$$0 < |Re\,x - Re\,c| < \delta \implies ||f(x) - L||_1 < \eta. \tag{1}$$

Since $f_\clubsuit(x) = f_\clubsuit(Re\,x,\, \varphi(Re\,x))$ is a function of the single variable $Re\,x$ on a φ-interval, $\lim\limits_{Re\,x \overset{\varphi}{\to} Re\,c} f(x) = L$ if and only if $\lim\limits_{Re\,x \to Re\,c} f_{Re}(x) = Re(L)$ and $\lim\limits_{Re\,x \to Re\,c} f_{Ze}(x) = Ze(L)$.

Replacing $0 < |Re\,x - Re\,c| < \delta$ by $0 < Re\,x - Re\,c < \delta$ in (1), we get the concept of *right-hand φ-limit* $\lim_{Re\,x \overset{\varphi}{\to} Re\,c^+} f(x) = L$ of f at c. Replacing $0 < |Re\,x - Re\,c| < \delta$ by $-\delta < Re\,x - Re\,c < 0$ in (1), we get the concept of *left-hand φ-limit* $\lim_{Re\,x \overset{\varphi}{\to} Re\,c^-} f(x) = L$ of f at c. Clearly, $\lim_{Re\,x \overset{\varphi}{\to} Re\,c} f(x) = L$ if and only if both one-sided φ-limits exist and are equal to L.

Definition 2.2. *Let $f : S \to \mathcal{R}^{(2)}$ be a function defined on φ-set $S \subseteq \mathcal{R}^{(2)}$. We say that f is φ-continuous at $c \in S$ if for every $\eta > 0$ there exists a $\delta > 0$ such that*

$$|Re\,x - Re\,c| < \delta \quad \textit{and} \quad x \in S \implies ||f(x) - f(c)||_1 < \eta.$$

If $f : S \to \mathcal{R}^{(2)}$ is φ-continuous at every point of S, then f is said to be φ-continuous on S.

Using Proposition 1, we now introduce the concept of φ-derivatives in the following

Definition 2.3. *Let $f : I \to \mathcal{R}^{(2)}$ be a function defined on an open φ-interval I containing $c \in \mathcal{R}^{(2)}$. If the φ-limit*

$$f'_\varphi(c) := \lim_{Re\,x \overset{\varphi}{\to} Re\,c} \frac{f(x) - f(c)}{x - c}$$

exists as an element of $\mathcal{R}^{(2)}$, then we say that f has a φ-derivative $f'_\varphi(c)$ at c (or is φ-differentiable at c). If f is φ-differentiable at each point of the open φ-interval I, then f is said to be φ-differentiable on I.

The next proposition gives one of the basic properties of φ-derivatives.

Proposition 3. *Suppose that $f : I \to \mathcal{R}^{(2)}$ is a function defined on an open φ-interval I containing $c \in \mathcal{R}^{(2)}$, where $\varphi : Re\,I \to \mathcal{R}$ is differentiable at $Re\,c$.*

(i) If $f(x) = f_{Re}(x) + \ell f_{Ze}(x)$ is φ-differentiable at c, then both the function f_{Re} and the function f_{Ze} of the single variable $Re\,x$ are differentiable at $Re\,c$, and their derivatives at $Re\,c$ are given by

$$\left.\frac{df_{Re}}{d(Re\,x)}\right|_{x=c} = Re\,(f'_\varphi(c))$$

and

$$\left.\frac{df_{Ze}}{d(Re\,x)}\right|_{x=c} = \varphi'(Re\,c) Re\,(f'_\varphi(c)) + Ze\,(f'_\varphi(c)),$$

where $f_\clubsuit$ *is regarded as the function of the single variable* $Re\,x$ *defined by*

$$Re\,x \mapsto f_\clubsuit(Re\,x,\, \varphi(Re\,x)) \quad \textit{for } x \in [a,\, b]_\varphi \textit{ and } \clubsuit \in \{\, Re,\, Ze \,\}.$$

(ii) If the first-order partial derivatives $\dfrac{\partial f_{Re}}{\partial(Re\,x)}$, $\dfrac{\partial f_{Re}}{\partial(Ze\,x)}$, $\dfrac{\partial f_{Ze}}{\partial(Re\,x)}$ *and* $\dfrac{\partial f_{Ze}}{\partial(Ze\,x)}$ *exist in a neighborhood of* $(Re\,c,\, Ze\,c)$ *and are continuous at* $(Re\,c,\, Ze\,c)$, *then* f *is* φ*-differentiable at* c *and the* φ*-derivative* $f'_\varphi(c)$ *is given by*

$$f'_\varphi(c) = \left(\frac{\partial f_{Re}}{\partial(Re\,x)} + \varphi'(Re\,x) \frac{\partial f_{Re}}{\partial(Ze\,x)} \right)\Bigg|_{x=c} +$$

$$+\ell \left(\frac{\partial f_{Ze}}{\partial(Re\,x)} + \varphi'(Re\,x) \Big(\frac{\partial f_{Ze}}{\partial(Ze\,x)} - \frac{\partial f_{Re}}{\partial(Re\,x)} - \varphi'(Re\,x) \frac{\partial f_{Re}}{\partial(Ze\,x)} \Big) \right)\Bigg|_{x=c}.$$

PROOF. This proposition follows from Proposition 1 and the Chain Rule in multivariable calculus.

□

3 Upper and Lower φ-Sums and φ-Integrals

A closed φ-interval $[a,\, b]_\varphi$ is called *monotone* if $\varphi : [Re\,a,\, Re\,b] \to \mathcal{R}$ is either nondecreasing or nonincreasing. For convenience, we also use $\varphi \nearrow$ and $\varphi \searrow$ to indicate that the function $\varphi : [Re\,a,\, Re\,b] \to \mathcal{R}$ is nondecreasing and nonincreasing, respectively.

Let $[a,\, b]_\varphi$ be a monotone closed φ-interval in $\mathcal{R}^{(2)}$. A *partition* P of $[a,\, b]_\varphi$ is a finite set of points $\{\, x_0,\, x_1,\, \dots,\, x_n \,\}$ in $[a,\, b]_\varphi$ such that

$$Re\,a = Re\,x_0 < Re\,x_1 < \cdots < Re\,x_{n-1} < Re\,x_n = Re\,b.$$

If P and P^* are two partitions of a monotone closed φ-interval $[a,\, b]_\varphi$ with $P \subseteq P^*$, then P^* is called a *refinement* of P.

In the following, we assume that $f : [a,\, b]_\varphi \to \mathcal{R}^{(2)}$ is a bounded function, $P = \{\, x_0,\, x_1,\, \dots,\, x_n \,\}$ is a partition of $[a,\, b]_\varphi$, and $[a,\, b]_\varphi$ is a monotone closed φ-interval in $\mathcal{R}^{(2)}$. Let $\Delta x_h := x_h - x_{h-1}$ for $h = 1,\, 2,\, \dots,\, n$. Then

$$x_h \overset{\theta_\varphi}{>} 0 \quad \text{for } h = 1,\, 2,\, \dots,\, n,$$

where the notation θ_φ is defined by

$$\theta_\varphi := \begin{cases} 1 & \text{for } \varphi \nearrow \\ 2 & \text{for } \varphi \searrow \end{cases}$$

Δx_h is a generalization of the length function of an interval. Clearly, Δx_h is a positive real number if and only if $\varphi(Re\, x_h) = \varphi(Re\, x_{h-1})$.

Since $f : [x_{h-1},\, x_h]_\varphi \to \mathcal{R}^{(2)}$ is bounded, both

$$\sup_h f_\clubsuit := \sup\{\, f_\clubsuit(x) \,|\, x \in [x_{h-1},\, x_h]_\varphi \,\}$$

and

$$\inf_h f_\clubsuit := \inf\{\, f_\clubsuit(x) \,|\, x \in [x_{h-1},\, x_h]_\varphi \,\}$$

exist for $\clubsuit \in \{\, Re,\, Ze \,\}$. We define the *upper φ-sum* $U_\varphi(P,\, f)$ of f with respect to the partition P to be

$$U_\varphi(P,\, f) := \begin{cases} \displaystyle\sum_{h=1}^{n} \Big(\sup_h f_{Re} + \ell \sup_h f_{Ze} \Big) \Delta x_h & \text{for } \varphi \nearrow \\[2ex] \displaystyle\sum_{h=1}^{n} \Big(\sup_h f_{Re} + \ell \inf_h f_{Ze} \Big) \Delta x_h & \text{for } \varphi \searrow \end{cases}$$

and the *lower φ-sum* $L_\varphi(P,\, f)$ of f with respect to the partition P to be

$$L_\varphi(P,\, f) := \begin{cases} \displaystyle\sum_{h=1}^{n} \Big(\inf_h f_{Re} + \ell \inf_h f_{Ze} \Big) \Delta x_h & \text{for } \varphi \nearrow \\[2ex] \displaystyle\sum_{h=1}^{n} \Big(\inf_h f_{Re} + \ell \sup_h f_{Ze} \Big) \Delta x_h & \text{for } \varphi \searrow. \end{cases}$$

By the assumption that $f : [a,\, b]_\varphi \to \mathcal{R}^{(2)}$ is bounded, there exist real numbers m and M such that

$$M \geq f_\clubsuit(x) \geq m \quad \text{for all } x \in [a,\, b]_\varphi \text{ and } \clubsuit \in \{\, Re,\, Ze \,\}.$$

Let $\mathcal{P}$ be the set of all partitions of $[a,\, b]_\varphi$, i.e.,

$$\mathcal{P} := \{\, P \,|\, P \text{ is a partition of } [a,\, b]_\varphi \,\}.$$

It follows from Proposition 2 that if $P \in \mathcal{P}$, then

$$(M + \ell M)(b - a) \overset{1}{\geq} U_\varphi(P,\, f) \overset{1}{\geq} L_\varphi(P,\, f) \overset{1}{\geq} (m + \ell m)(b - a) \quad \text{for } \varphi \nearrow$$

and

$$(M + \ell m)(b - a) \overset{2}{\geq} U_\varphi(P,\, f) \overset{2}{\geq} L_\varphi(P,\, f) \overset{2}{\geq} (m + \ell M)(b - a) \quad \text{for } \varphi \searrow$$

which imply that the four sets $\{\, Re\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\}$, $\{\, Re\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\}$, $\{\, Ze\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\}$ and $\{\, Ze\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\}$ are bounded subsets of the real number field $\mathcal{R}$. We now define the *lower φ-integral* $\underline{\int_a^b} f(x) d_\varphi x$ and *upper φ-integral* $\overline{\int_a^b} f(x) d_\varphi x$ of $f(x)$ on $[a,\, b]_\varphi$ by

$$\begin{aligned} & \underline{\int_a^b} f(x) d_\varphi x \\ = \; & \begin{cases} \sup\{\, Re\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} + \ell \sup\{\, Ze\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} & \text{for } \varphi \nearrow \\ \sup\{\, Re\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} + \ell \inf\{\, Ze\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} & \text{for } \varphi \searrow \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \overline{\int_a^b} f(x) d_\varphi x \\ = \; & \begin{cases} \inf\{\, Re\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} + \ell \inf\{\, Ze\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} & \text{for } \varphi \nearrow \\ \inf\{\, Re\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} + \ell \sup\{\, Ze\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} & \text{for } \varphi \searrow \end{cases} \end{aligned}$$

If the lower φ-integral and the upper φ-integral of $f(x)$ on $[a,\, b]_\varphi$ are equal, i.e., if $\underline{\int_a^b} f(x) d_\varphi x = \overline{\int_a^b} f(x) d_\varphi x$, then we say that f is *φ-integrable* on $[a,\, b]_\varphi$, and we denote their common value by $\int_a^b f(x) d_\varphi x$ which is called the *φ-integral* of f on $[a,\, b]_\varphi$.

Proposition 4. *Let $f : [a,\, b]_\varphi \to \mathcal{R}^{(2)}$ be a bounded function on the monotone closed φ-interval $[a,\, b]_\varphi$.*

(i) If P and P^* are partitions of $[a,\, b]_\varphi$ and P^* is a refinement of P, then

$$U_\varphi(P,\, f) \overset{\theta_\varphi}{\geq} U_\varphi(P^*,\, f) \overset{\theta_\varphi}{\geq} L_\varphi(P^*,\, f) \overset{\theta_\varphi}{\geq} L_\varphi(P,\, f). \tag{2}$$

(ii) $\overline{\int_a^b} f(x) d_\varphi x \overset{\theta_\varphi}{\geq} \underline{\int_a^b} f(x) d_\varphi x.$

PROOF. The proof of this proposition is similar to the proof of the corresponding results in calculus.

□

The next proposition will play an important role in determining when a function is φ-integrable.

Proposition 5. *Let $f : [a, b]_\varphi \to \mathcal{R}^{(2)}$ be a bounded function on the monotone closed φ-interval $[a, b]_\varphi$.*

(i) Let $r \in \mathcal{R}$ be a fixed real number. If for each positive real number $\eta > 0$ there exists a partitions P of $[a, b]_\varphi$ such that

$$U_\varphi(P, f) - L_\varphi(P, f) \overset{\theta_\varphi}{<} \eta + r\eta\ell, \tag{3}$$

then f is φ-integrable on $[a, b]_\varphi$.

(ii) If f is φ-integrable on $[a, b]_\varphi$, then for each $\varepsilon \overset{\theta_\varphi}{>} 0$ with $(Re\,\varepsilon)(Ze\,\varepsilon) \neq 0$, there exists a partitions P of $[a, b]_\varphi$ such that $U_\varphi(P, f) - L_\varphi(P, f) \overset{\theta_\varphi}{<} \varepsilon$.

PROOF. (i) By Proposition 4 (ii), we have

$$U_\varphi(P, f) \overset{\theta_\varphi}{\geq} \overline{\int_a^b} f(x) d_\varphi x \overset{\theta_\varphi}{\geq} \underline{\int_a^b} f(x) d_\varphi x \overset{\theta_\varphi}{\geq} L_\varphi(P, f), \tag{4}$$

where P is any partition of $[a, b]_\varphi$. If P is a partition of $[a, b]_\varphi$ such that (3) holds, then (3) and (4) imply that

$$\eta + r\eta\ell \overset{\theta_\varphi}{>} U_\varphi(P, f) - L_\varphi(P, f) \overset{\theta_\varphi}{\geq} \overline{\int_a^b} f(x) d_\varphi x - \underline{\int_a^b} f(x) d_\varphi x\,. \tag{5}$$

If φ is non-increasing, then $0 \geq r$ and $\overline{\int_a^b} f(x) d_\varphi x \overset{2}{\geq} \underline{\int_a^b} f(x) d_\varphi x$ in this case. It follows from this fact and (5) that

$$\eta \geq Re\left(\overline{\int_a^b} f d_\varphi x - \underline{\int_a^b} f d_\varphi x\right) = Re\left(\overline{\int_a^b} f d_\varphi x\right) - Re\left(\underline{\int_a^b} f d_\varphi x\right) \geq 0$$

and

$$-r\eta \geq Ze\left(\underline{\int_a^b} f d_\varphi x - \overline{\int_a^b} f d_\varphi x\right) = Ze\left(\underline{\int_a^b} f d_\varphi x\right) - Ze\left(\overline{\int_a^b} f d_\varphi x\right) \geq 0$$

for any $\eta > 0$. Hence, we have

$$Re\left(\underline{\int_a^b} f d_\varphi x\right) \geq Re\left(\overline{\int_a^b} f d_\varphi x\right) \quad \text{and} \quad Ze\left(\overline{\int_a^b} f d_\varphi x\right) \geq Ze\left(\underline{\int_a^b} f d_\varphi x\right)$$

or

$$\underline{\int_a^b} f(x) d_\varphi x \overset{2}{\geq} \overline{\int_a^b} f(x) d_\varphi x \tag{6}$$

Similarly, if φ is non-decreasing, then (5) implies that

$$\underline{\int_a^b} f(x) d_\varphi x \overset{1}{\geq} \overline{\int_a^b} f(x) d_\varphi x \tag{7}$$

By (6) and (7), we get $\overline{\int_a^b} f(x) d_\varphi x = \underline{\int_a^b} f(x) d_\varphi x$, i.e., f is φ-integrable .

(ii) The proofs of (ii) are similar for $\varphi \nearrow$ and $\varphi \searrow$ are similar. Here, we prove (ii) for $\varphi \nearrow$. Since f is φ-integrable and φ is non-decreasing, we have

$$\sup\{\clubsuit L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\} = \inf\{\clubsuit U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\} \quad \text{for } \clubsuit \in \{Re,\, Ze\}. \tag{8}$$

Note that $\varepsilon \overset{1}{>} 0$ in this case. Hence, it follows from $(Re\,\varepsilon)(Ze\,\varepsilon) \neq 0$ that $Re\,\varepsilon > 0$ and $Ze\,\varepsilon > 0$. By these facts and (8), there exist four partitions P_{Re}, P_{Ze}, Q_{Re} and Q_{Ze} of $[a,\, b]_\varphi$ such that

$$\inf\{\clubsuit U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\} + \frac{\clubsuit(\varepsilon)}{2} > \clubsuit U_\varphi(P_\clubsuit,\, f) \tag{9}$$

and

$$\sup\{\clubsuit L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\} - \frac{\clubsuit(\varepsilon)}{2} < \clubsuit L_\varphi(Q_\clubsuit,\, f) \tag{10}$$

for each $\clubsuit \in \{Re,\, Ze\}$. Let $T = P_{Re} \cup P_{Ze} \cup Q_{Re} \cup Q_{Ze}$. Using (8), (9), (10) and Proposition 4 (i), we get

$$\begin{aligned}
\clubsuit L_\varphi(T,\, f) + \clubsuit(\varepsilon) \geq \clubsuit L_\varphi(Q_\clubsuit,\, f) + \clubsuit(\varepsilon) &= \clubsuit L_\varphi(Q_\clubsuit,\, f) + \frac{\clubsuit(\varepsilon)}{2} + \frac{\clubsuit(\varepsilon)}{2} \\
&> \sup\{\clubsuit L_\varphi(P,\, f) \,|\, P \in \mathcal{P}\} + \frac{\clubsuit(\varepsilon)}{2} \\
&= \inf\{\clubsuit U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\} + \frac{\clubsuit(\varepsilon)}{2} > \clubsuit U_\varphi(P_\clubsuit,\, f) \\
&\geq \clubsuit U_\varphi(T,\, f).
\end{aligned}$$

That is,

$$\clubsuit\,(U_\varphi(T,\,f)-L_\varphi(T,\,f))<\clubsuit(\varepsilon)\quad\text{for }\clubsuit\in\{Re,\,Ze\},$$

which proves that $U_\varphi(T,\,f)-L_\varphi(T,\,f)\overset{1}{<}\varepsilon$ for the partition T of $[a,\,b]_\varphi$.

□

4 Basic Properties of φ-Integrals

We beginning this section by proving the linearity property of φ-integrals.

Proposition 6. *If f and g are φ-integrable on a monotone closed φ-interval $[a,\,b]_\varphi$ and $k\in\mathcal{R}^{(2)}$, then both kf and $f+g$ are φ-integrable on $[a,\,b]_\varphi$, and following two equations hold*

$$\int_a^b kf(x)d_\varphi x=k\int_a^b f(x)d_\varphi x \tag{11}$$

$$\int_a^b (f(x)+g(x))d_\varphi x=\int_a^b f(x)d_\varphi x+\int_a^b g(x)d_\varphi x. \tag{12}$$

Proof. Clearly, Proposition 6 holds if we can prove that (11) holds for $k\in\mathcal{R}^{(2)}$ with either $Ze\,k=0$ or $Re\,k=0$ and (12) holds. Since (11) clearly holds for $k=0$, we assume that $k\neq 0$.

First, we prove (11) for $k\in\mathcal{R}^{(2)}$ with $Ze\,k=0$. In this case, $k=r\in\mathcal{R}$. Then there are four subcases:

Subcase 1: $r<0$ and φ is non-decreasing;

Subcase 2: $r>0$ and φ is non-decreasing;

Subcase 3: $r<0$ and φ is non-increasing;

Subcase 4: $r>0$ and φ is non-increasing.

The proofs of (11) are similar for these four subcases. As an example, we give the proof for Subcase 1. Note that $(rf)_\clubsuit=r(f_\clubsuit)$ for $r\in\mathcal{R}$ and each

$\clubsuit \in \{Re,\, Ze\}$. Let $P = \{\, x_0,\, x_1,\, \dots\, x_n \,\}$ be a partition of $[a,\, b]_\varphi$. We have

$$\begin{aligned} U_\varphi(P,\, rf) &= \sum_{h=1}^{n} \Big(\sup_h\, (rf)_{Re} + \ell \sup_h\, (rf)_{Ze} \Big) \Delta x_h \\ &= \sum_{h=1}^{n} \Big(\sup_h\, \big(r(f_{Re})\big) + \ell \sup_h\, \big(r(f_{Ze})\big) \Big) \Delta x_h \\ &= \sum_{h=1}^{n} \Big(r \inf_h\, f_{Re} + \ell r \inf_h\, f_{Ze} \Big) \Delta x_h = r \sum_{h=1}^{n} \Big(\inf_h\, f_{Re} + \ell \inf_h\, f_{Ze} \Big) \Delta x_h \\ &= r L_\varphi(P,\, f) \end{aligned}$$

or

$$\clubsuit\, U_\varphi(P,\, rf) = \clubsuit\big(r\, L_\varphi(P,\, f)\big) = r \clubsuit\, L_\varphi(P,\, f) \tag{13}$$

for $P \in \mathcal{P}$ and $\clubsuit \in \{Re,\, Ze\}$. It follows from (13) that

$$\begin{aligned} &\overline{\int_a^b} (rf)(x) d_\varphi x \\ &= \inf\{\, Re\, U_\varphi(P,\, rf) \,|\, P \in \mathcal{P} \,\} + \ell\, \inf\{\, Ze\, U_\varphi(P,\, rf) \,|\, P \in \mathcal{P} \,\} \\ &= \inf\{\, rRe\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P} \,\} + \ell\, \inf\{\, rZe\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P} \,\} \\ &= r \sup\{\, Re\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P} \,\} + \ell\, r \sup\{\, Ze\, L_\varphi(P,\, f) \,|\, P \in \mathcal{P} \,\} \\ &= r \underline{\int_a^b} f(x) d_\varphi x\,. \end{aligned} \tag{14}$$

Using the same method, we also have

$$\underline{\int_a^b} (rf)(x) d_\varphi x = r \overline{\int_a^b} f(x) d_\varphi x\,. \tag{15}$$

By the assumption that f is φ-integrable on $[a,\, b]_\varphi$, we get from (4) and (15) that

$$\overline{\int_a^b} (rf)(x) d_\varphi x = r \underline{\int_a^b} f(x) d_\varphi x = r \overline{\int_a^b} f(x) d_\varphi x = \underline{\int_a^b} (rf)(x) d_\varphi x\,,$$

which proves that (11) holds in Subcase 1.

Next, we prove (11) for $k \in \mathcal{R}^{(2)}$ with $Re\, k = 0$. In this case, $k = r\ell$ with $r \in \mathcal{R}$, and

$$(kf)_{Re} = 0 \quad \text{and} \quad (kf)_{Ze} = r f_{Re}. \tag{16}$$

We also have the four subcases. Let's prove (11) in *Subcase 2:* $r > 0$ and φ is non-decreasing. By (16), we have

$$U_\varphi(P,\, kf) = \sum_{h=1}^{n} \Big(\sup_h\, (kf)_{Re} + \ell \sup_h\, (kf)_{Ze} \Big) \Delta x_h$$
$$= \sum_{h=1}^{n} \ell\big(\sup_h\ (r f_{Re})\big) \Delta x_h = \sum_{h=1}^{n} \ell\, r\big(\sup_h\ f_{Re}\big) \Delta x_h$$
$$= \sum_{h=1}^{n} \ell\, r\Big(\sup_h\ f_{Re} + \ell \sup_h\ f_{Ze} \Big) \Delta x_h = \ell\, r \sum_{h=1}^{n} \Big(\sup_h\ f_{Re} + \ell \sup_h\ f_{Ze} \Big) \Delta x_h$$
$$= \ell\, r U_\varphi(P,\, f) = k\, U_\varphi(P,\, f),$$

which implies

$$Re\, U_\varphi(P,\, kf) = Re\big(k\, U_\varphi(P,\, f)\big) = 0 \tag{17}$$

and

$$Ze\, U_\varphi(P,\, kf) = Ze\big(k\, U_\varphi(P,\, f)\big) = r\, Re\, U_\varphi(P,\, f). \tag{18}$$

It follows from (17) and (18) that

$$\overline{\int_a^b} (kf) d_\varphi x$$
$$= \inf\{\, Re\, U_\varphi(P,\, kf) \,|\, P \in \mathcal{P}\,\} + \ell\, \inf\{\, Ze\, U_\varphi(P,\, kf) \,|\, P \in \mathcal{P}\,\}$$
$$= \ell\, \inf\{\, r\, Re\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} = \ell r\, \inf\{\, Re\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\}$$
$$= \ell\, r\, \Big(\inf\{\, Re\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} + \ell\, \inf\{\, Ze\, U_\varphi(P,\, f) \,|\, P \in \mathcal{P}\,\} \Big)$$
$$= \ell\, r \overline{\int_a^b} f d_\varphi x = k \overline{\int_a^b} f d_\varphi x\,. \tag{19}$$

Similarly, we have

$$\underline{\int_a^b} (kf) d_\varphi x = k \underline{\int_a^b} f d_\varphi x\,. \tag{20}$$

By (4) and (20), we get that (11) with $Re\, k = 0$ holds in Subcase 2.

Finally, (12) can be proved by using the following properties of real-valued functions:

$$\sup\ f_1(D) + \sup\ f_2(D) \ge \sup\ \big((f_1 + f_2)(D)\big)$$

and

$$\inf\ \big((f_1 + f_2)(D)\big) \ge \inf\ f_1(D) + \inf\ f_2(D),$$

where D is a subset of the real number field $\mathcal{R}$ and $f_i : D \to \mathcal{R}$ is a real-valued function with the range $f_i(D)$ for $i = 1, 2$.

□

Next, we give other algebraic properties of φ-integrals.

Proposition 7. *Let $[a, b]_\varphi$ be a monotone closed φ-interval.*

(i) If f is φ-integrable on both $[a, c]_\varphi$ and $[c, b]_\varphi$ with $c \in [a, b]_\varphi$, then f is φ-integrable on $[a, b]_\varphi$ and $\displaystyle\int_a^b f d_\varphi x = \int_a^c f d_\varphi x + \int_c^b f d_\varphi x$.

(ii) If f and g are φ-integrable on both $[a, b]_\varphi$ and $f(x) \overset{\theta_\varphi}{\geq} g(x)$ for all $x \in [a, b]_\varphi$, then $\displaystyle\int_a^b f d_\varphi x \overset{\theta_\varphi}{\geq} \int_a^b g d_\varphi x$.

PROOF. The proof of this proposition is similar to the proof of the corresponding properties of Riemann integrals.

□

Finally, we prove that φ-continuous functions are φ-integrable.

Proposition 8. *If $f : [a, b]_\varphi \to \mathcal{R}^{(2)}$ is φ-continuous on a monotone closed φ-interval $[a, b]_\varphi$, then f is φ-integrable on $[a, b]_\varphi$.*

PROOF. We prove this proposition by cases:

Case 1. φ is non-decreasing and $Ze(b - a) > 0$;

Case 2. φ is non-decreasing and $Ze(b - a) = 0$;

Case 3. φ is non-increasing and $Ze(b - a) < 0$;

Case 4. φ is non-increasing and $Ze(b - a) = 0$.

The proofs are similar in the four cases. We prove Proposition 8 in Case 1 to explain the way of doing the proof.

In Case 1, $Ze(b - a)$ is a positive real number. Since f is φ-continuous on $[a, b]_\varphi$, the real-valued function $f_\clubsuit$ of the single variable $Re\, x$ is uniformly continuous on the closed interval $[Re\, a, Re\, b]$, where $\clubsuit \in \{Re, Ze\}$. Hence, for any positive real number $\eta > 0$, there exists a positive real number $\delta > 0$ such that

$$x, y \in [a,\, b]_\varphi \text{ and } |Re\, x - Re\, y| < \delta$$

$$\Longrightarrow \begin{cases} |f_{Re}(x) - f_{Re}(y)| < \dfrac{\eta}{Re\,(b-a)} \\[2ex] |f_{Ze}(x) - f_{Ze}(y)| < \dfrac{\eta\, Ze\,(b-a)}{\big(Re\,(b-a)\big)^2} \end{cases} \tag{21}$$

Let $P = \{\, x_0,\, x_1,\, \dots\, x_n \,\}$ be a partition of $[a,\, b]_\varphi$ with $||P|| < \delta$, where

$$||P|| := max\,\{\, Re\, x_h - Re\, x_{h-1} \,|\, n \geq h \geq 1 \,\}.$$

By the properties of continuous functions, both $f_{Re}(x)$ and $f_{Ze}(x)$ assume their maximum and minimum on each subinterval $[Re\, x_{h-1}\,, Re\, x_h]$. Thus, there exist $s_h^{\clubsuit},\, t_h^{\clubsuit} \in [x_{h-1},\, x_h]_\varphi$ such that

$$\sup_h\, f_{\clubsuit} = f_{\clubsuit}(s_h^{\clubsuit}) \quad \text{and} \quad \inf_h\, f_{\clubsuit} = f_{\clubsuit}(t_h^{\clubsuit}) \quad \text{for } \clubsuit \in \{Re,\, Ze\}.$$

Since $\delta > ||P|| \geq |Re\, s_h^{\clubsuit} - Re\, t_h^{\clubsuit}|$, we get from (21) that

$$\begin{aligned}
&U_\varphi(P,\, f) - L_\varphi(P,\, f) \\
&= \sum_{h=1}^{n} \Big(\sup_h\, f_{Re} + \ell \sup_h\, f_{Ze} \Big) \Delta x_h - \sum_{h=1}^{n} \Big(\inf_h\, f_{Re} + \ell \inf_h\, f_{Ze} \Big) \Delta x_h \\
&= \sum_{h=1}^{n} \Big(f_{Re}(s_h^{Re}) + \ell f_{Ze}(s_h^{Ze}) \Big) \Delta x_h - \sum_{h=1}^{n} \Big(f_{Re}(t_h^{Re}) + \ell f_{Ze}(t_h^{Ze}) \Big) \Delta x_h \\
&= \sum_{h=1}^{n} \Big(f_{Re}(s_h^{Re}) - f_{Re}(t_h^{Re}) \Big) \Delta x_h + \ell \sum_{h=1}^{n} \Big(f_{Ze}(s_h^{Ze}) - f_{Ze}(t_h^{Ze}) \Big) \Delta x_h \\
&= \sum_{h=1}^{n} \Big| f_{Re}(s_h^{Re}) - f_{Re}(t_h^{Re}) \Big| \Delta x_h + \ell \sum_{h=1}^{n} \Big| f_{Ze}(s_h^{Ze}) - f_{Ze}(t_h^{Ze}) \Big| \Delta x_h \\
&= \sum_{h=1}^{n} \Big(\Big| f_{Re}(s_h^{Re}) - f_{Re}(t_h^{Re}) \Big| + \ell \Big| f_{Ze}(s_h^{Ze}) - f_{Ze}(t_h^{Ze}) \Big| \Big) \Delta x_h \\
&\overset{1}{<} \sum_{h=1}^{n} \left(\frac{\eta}{Re\,(b-a)} + \ell \frac{\eta\, Ze\,(b-a)}{\big(Re\,(b-a)\big)^2} \right) \Delta x_h \\
&= \left(\frac{\eta}{Re\,(b-a)} + \ell \frac{\eta\, Ze\,(b-a)}{\big(Re\,(b-a)\big)^2} \right) (b-a) = \eta + \left(\frac{2\, Ze\,(b-a)}{Re\,(b-a)} \right) \eta \ell\,,
\end{aligned}$$

which proves that f is φ-integrable by Proposition 5 (i).

□

5 Fundamental Theorem of Calculus$^\varphi$

Our way of rewriting the First Fundamental Theorem of Calculus is given in the following proposition.

Proposition 9. *Let $f : [a,\, b]_\varphi \to \mathcal{R}^{(2)}$ be φ-integrable on a monotone closed φ-interval $[a,\, b]_\varphi$, and let $F : [a,\, b]_\varphi \to \mathcal{R}^{(2)}$ be defined by*

$$F(x) := \int_a^x f(t) d_\varphi t \quad \text{for } x \in [a,\, b]_\varphi .$$

If $\varphi : [Re\, a,\, Re\, b] \to \mathcal{R}$ is Lipschitz continuous on the interval $[Re\, a,\, Re\, b]$, and if f is φ-continuous at $c \in [a,\, b]_\varphi$, then F is φ-differentiable at c and $F'_\varphi(c) = f(c)$.

Proof. We prove this proposition for the case where φ is nondecreasing. The proof of this proposition for the case where φ is nonincreasing can be obtained in a similar way.

Since $\varphi : [Re\, a,\, Re\, b] \to \mathcal{R}$ is Lipschitz continuous on the interval $[Re\, a,\, Re\, b]$, there exists a positive real number M such that

$$M|Re\, x - Re\, y| \geq |\varphi(Re\, x) - \varphi(Re\, y)| = |Ze\, x - Ze\, y| \quad \text{for } x,\, y \in [a,\, b]_\varphi . \tag{22}$$

If $Re\, x > Re\, c$, by Proposition 6 and Proposition 7 (i), we have

$$\begin{aligned} & \frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x - c}\left(\int_a^x f(t) d_\varphi t - \int_a^c f(t) d_\varphi t\right) - f(c) \\ = \quad & \frac{1}{x - c}\int_c^x f(t) d_\varphi t - \frac{1}{x - c}\int_c^x f(c) d_\varphi t = \frac{1}{x - c}\int_c^x \big(f(t) - f(c)\big) d_\varphi t\,, \end{aligned}$$

which implies

$$\left|\frac{1}{x - c}\right|_1 \left|\int_c^x \big(f(t) - f(c)\big) d_\varphi t\right|_1 \geq \left|\frac{F(x) - F(c)}{x - c} - f(c)\right|_1 . \tag{23}$$

By the assumption that f is φ-continuous at c, for any $\eta > 0$ there exists $\delta > 0$ such that

$$t \in [a,\, b]_\varphi \quad \text{and} \quad |Re\, t - Re\, c| < \delta \Longrightarrow |f(t) - f(c)|_1 < \eta\, . \tag{24}$$

It follows from (24) and Proposition 6 that

$$\begin{aligned}
& t \in [a,\, b]_\varphi \quad \text{and} \quad |Re\, t - Re\, c| < \delta \\
\Longrightarrow\quad & -\eta - \eta\ell \overset{1}{<} f(t) - f(c) \overset{1}{<} \eta + \eta\ell \\
\Longrightarrow\quad & \int_c^x (\eta + \eta\ell) d_\varphi t \overset{1}{\geq} \int_c^x \big(f(t) - f(c)\big) d_\varphi t \overset{1}{\geq} \int_c^x (-\eta - \eta\ell) d_\varphi t \\
\Longrightarrow\quad & (\eta + \eta\ell)(x - c) \overset{1}{\geq} \int_c^x \big(f(t) - f(c)\big) d_\varphi t \overset{1}{\geq} (-\eta - \eta\ell)(x - c)\,,
\end{aligned}$$

which implies

$$\eta Re\,(x - c) \geq \left| Re\,\left(\int_c^x \big(f(t) - f(c)\big) d_\varphi t \right) \right| \tag{25}$$

and

$$\eta Re\,(x - c) + \eta Ze\,(x - c) \geq \left| Ze\,\left(\int_c^x \big(f(t) - f(c)\big) d_\varphi t \right) \right| \tag{26}$$

It follows from (22), (23), (25) and (26) that if $x \in [a,\, b]_\varphi$ and $0 < Re\, x - Re\, c < \delta$, then

$$\begin{aligned}
& x \in [a,\, b]_\varphi \text{ and } 0 < Re\, x - Re\, c < \delta \\
\Longrightarrow\ & |Re\, t - Re\, c| < \delta \text{ for } t \in [c,\, x]_\varphi \\
\Longrightarrow\ & \eta(2 + 3M + M^2) = 2\eta + 3\eta M + \eta M^2 \\
& \geq 2\eta + 3\eta \left| \frac{Ze\,(x - c)}{Re\,(x - c)} \right| + \eta \left| \frac{Ze\,(x - c)}{Re\,(x - c)} \right|^2 \\
& = 2\eta + 3\eta \frac{Ze\,(x - c)}{Re\,(x - c)} + \eta \left(\frac{Ze\,(x - c)}{Re\,(x - c)} \right)^2 \\
& = \left(\frac{1}{Re\,(x - c)} + \frac{Ze\,(x - c)}{\big(Re\,(x - c)\big)^2} \right) \Big(2\eta Re\,(x - c) + \eta Ze\,(x - c) \Big) \\
& = \left(\left| \frac{1}{Re\,(x - c)} \right| + \left| \frac{Ze\,(x - c)}{\big(Re\,(x - c)\big)^2} \right| \right) \Big(2\eta Re\,(x - c) + \eta Ze\,(x - c) \Big) \\
& \geq \left| \frac{1}{x - c} \right|_1 \left| \int_c^x \big(f(t) - f(c)\big) d_\varphi t \right|_1 \geq \left| \frac{F(x) - F(c)}{x - c} - f(c) \right|_1 ,
\end{aligned}$$

which implies

$$\lim_{Re\, x \overset{\varphi}{\to} Re\, c^+} \frac{F(x) - F(c)}{x - c} = f(c)\,. \tag{27}$$

Similarly, we have

$$\lim_{Re\,x \xrightarrow{\varphi} Re\,c^-} \frac{F(x) - F(c)}{x - c} = f(c)\,. \tag{28}$$

By (27) and (28), we get $F'_\varphi(c) = \lim_{Re\,x \xrightarrow{\varphi} Re\,c} \dfrac{F(x) - F(c)}{x - c} = f(c)$.

□

Our way of rewriting the Second Fundamental Theorem of Calculus is given in the following proposition.

Proposition 10. *Suppose that $f : [a,\, b]_\varphi \to \mathcal{R}^{(2)}$ is φ-differentiable on a monotone closed φ-interval $[a,\, b]_\varphi$, and f'_φ is φ-integrable on $[a,\, b]_\varphi$. Then*

$$\int_a^b f'_\varphi(x) d_\varphi x = f(b) - f(a)$$

if one of the following is true:

(i) $\varphi : [Re\,a,\, Re\,b] \to \mathcal{R}$ *is constant;*

(ii) Both φ and f_{Re} are continuously differentiable on $[Re\,a,\, Re\,b]$.

PROOF. The proof of this proposition consists of three parts:

Part 1: Prove that Proposition 10 holds if $\varphi : [Re\,a,\, Re\,b] \to \mathcal{R}$ is constant;

Part 2: Prove that Proposition 10 holds if both φ and f_{Re} are continuously differentiable on $[Re\,a,\, Re\,b]$ and φ is non-increasing;

Part 3: Prove that Proposition 10 holds if both φ and f_{Re} are continuously differentiable on $[Re\,a,\, Re\,b]$ and φ is non-decreasing.

The proofs in these three parts are similar. Here, we use Part 3 to give the way of doing the proofs. In the following, we assume that both φ and f_{Re} are continuously differentiable on $[Re\,a,\, Re\,b]$ and φ is non-decreasing.

Since φ' is continuous on the closed interval $[Re\,a,\, Re\,b]$, there exists a positive real number M such that

$$M \geq |\varphi'(Re\,x)| \quad \text{for all } x \in [a,\, b]_\varphi.$$

Let $\eta > 0$ be any fixed positive real number. By Proposition 3 (i), both f_{Re} and f_{Ze} are differentiable on $[Re\,a,\, Re\,b]$. Since $\dfrac{df_{Re}}{d(Re\,x)}$ is continuous on $[Re\, a,\, Re\, b]$, $\dfrac{df_{Re}}{d(Re\, x)}$ is uniformly continuous on $[Re\, a,\, Re\, b]$. Hence, there exists $\delta > 0$ such that

$$\begin{aligned} & s,\, t \in [a,\, b]_\varphi \text{ and } |Re\, s - Re\, t| < \delta \\ \Longrightarrow\ & \left| \left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=s} - \left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=t} \right| < \frac{\eta}{M(Re\, b - Re\, a)}\,. \end{aligned} \tag{29}$$

Let Q be any partition of $[a,\, b]_\varphi$, and let $P = \{\, x_0,\, x_1,\, \dots\, x_n \,\}$ be a refinement of the partition Q such that $||P|| < \delta$. After using Proposition 3 and applying the mean value theorem to each subinterval $[Re\, x_{h-1},\, Re\, x_h]$ twice, we obtain points $t_h,\, s_h \in [x_{h-1},\, x_h]_\varphi$ such that

$$f_{Re}(x_h) - f_{Re}(x_{h-1}) = Re\,\big((f'_\varphi(t_h)\big)Re\,(x_h - x_{h-1}) \tag{30}$$

and

$$\begin{aligned} & f_{Ze}(x_h) - f_{Ze}(x_{h-1}) \\ = \ & Re\,\big((f'_\varphi(t_h)\big)Ze\,(x_h - x_{h-1}) + Ze\,\big(f'_\varphi(s_h)\big)Re\,(x_h - x_{h-1}) + \\ & + \left(\left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=s_h} - \left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=t_h} \right) \varphi'(Re\, s_h)Re\,(x_h - x_{h-1}), \end{aligned} \tag{31}$$

where $n \geq h \geq 1$. It follows from (30) and (31) that

$$\begin{aligned} & f(x_h) - f(x_{h-1}) = \big[f_{Re}(x_h) - f_{Re}(x_{h-1})\big] + \ell\big[f_{Ze}(x_h) - f_{Ze}(x_{h-1})\big] \\ = \ & \Big\{ Re\,\big((f'_\varphi(t_h)\big)Re\,(x_h - x_{h-1}) + \\ & +\ell\big[Re\,\big((f'_\varphi(t_h)\big)Ze\,(x_h - x_{h-1}) + Ze\,\big(f'_\varphi(s_h)\big)Re\,(x_h - x_{h-1})\big]\Big\} + \\ & +\ell \left(\left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=s_h} - \left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=t_h} \right) \varphi'(Re\, s_h)Re\,(x_h - x_{h-1}) \\ = \ & \big[Re\,\big((f'_\varphi(t_h)\big) + \ell Ze\,\big(f'_\varphi(s_h)\big)\big]\big[Re\,(x_h - x_{h-1}) + \ell Ze\,(x_h - x_{h-1})\big] + \\ & +\ell \left(\left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=s_h} - \left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=t_h} \right) \varphi'(Re\, s_h)Re\,(x_h - x_{h-1}) \\ = \ & \big[Re\,\big((f'_\varphi(t_h)\big) + \ell Ze\,\big(f'_\varphi(s_h)\big)\big](x_h - x_{h-1}) + \\ & +\ell \left(\left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=s_h} - \left.\frac{df_{Re}}{d(Re\, x)}\right|_{x=t_h} \right) \varphi'(Re\, s_h)Re\,(x_h - x_{h-1})\,. \end{aligned} \tag{32}$$

By (30), we have

$$\eta = \sum_{h=1}^{n} \frac{\eta}{M(Re\, b - Re\, a)} M Re\,(x_h - x_{h-1})$$
$$> \sum_{h=1}^{n} \left| \frac{df_{Re}}{d(Re\, x)}\bigg|_{x=s_h} - \frac{df_{Re}}{d(Re\, x)}\bigg|_{x=t_h} \right| |\varphi'(Re\, s_h)|\, |Re\,(x_h - x_{h-1})|$$
$$\geq \left| \sum_{h=1}^{n} \left(\frac{df_{Re}}{d(Re\, x)}\bigg|_{x=s_h} - \frac{df_{Re}}{d(Re\, x)}\bigg|_{x=t_h} \right) \varphi'(Re\, s_h) Re\,(x_h - x_{h-1}) \right|. \quad (33)$$

Note that

$$U_\varphi(P,\, f'_\varphi) \overset{1}{\geq} \sum_{h=1}^{n} \left[Re\left((f'_\varphi(t_h)\right) + \ell Ze\left(f'_\varphi(s_h)\right)\right](x_h - x_{h-1}) \overset{1}{\geq} L_\varphi(P,\, f'_\varphi) \quad (34)$$

and

$$f(b) - f(a) = \sum_{h=1}^{n} f(x_h) - f(x_{h-1})$$
$$= \sum_{h=1}^{n} \left[Re\left((f'_\varphi(t_h)\right) + \ell Ze\left(f'_\varphi(s_h)\right)\right](x_h - x_{h-1}) +$$
$$+ \ell \sum_{h=1}^{n} \left(\frac{df_{Re}}{d(Re\, x)}\bigg|_{x=s_h} - \frac{df_{Re}}{d(Re\, x)}\bigg|_{x=t_h} \right) \varphi'(Re\, s_h) Re\,(x_h - x_{h-1}). \quad (35)$$

It follows from (5), (34) and (5) that

$$U_\varphi(P,\, f'_\varphi) + \eta\ell \overset{1}{\geq} f(b) - f(a) \overset{1}{\geq} L_\varphi(P,\, f'_\varphi) - \eta\ell \quad \text{for any } \eta > 0,$$

which implies that

$$U_\varphi(P,\, f'_\varphi) \overset{1}{\geq} f(b) - f(a) \overset{1}{\geq} L_\varphi(P,\, f'_\varphi). \quad (36)$$

Since P is a a refinement of the partition Q, we get from (5) and Proposition 4 that

$$U_\varphi(Q,\, f'_\varphi) \overset{1}{\geq} f(b) - f(a) \overset{1}{\geq} L_\varphi(Q,\, f'_\varphi) \quad \text{for any partition } Q \text{ of } [a,\, b]_\varphi. \quad (37)$$

Using (37), for each $\clubsuit \in \{Re,\, Ze\}$, we get

$$\inf\{\clubsuit U_\varphi(T,\, f'_\varphi) \,|\, T \in \mathcal{P}\} \geq \clubsuit\big(f(b) - f(a)\big) \geq \sup\{\clubsuit L_\varphi(T,\, f'_\varphi) \,|\, T \in \mathcal{P}\}$$

or

$$\overline{\int_a^b} f'_\varphi d_\varphi x \overset{1}{\geq} f(b) - f(a) \overset{1}{\geq} \underline{\int_a^b} f'_\varphi d_\varphi x \,. \tag{38}$$

By the assumption that f'_φ is φ-integrable on $[a,\, b]_\varphi$, we get from (38) that

$$f(b) - f(a) = \overline{\int_a^b} f'_\varphi d_\varphi x = \underline{\int_a^b} f'_\varphi d_\varphi x = \int_a^b f'_\varphi d_\varphi x \,.$$

□

Since a closed φ-interval is the union of monotone closed φ-intervals, the concept of φ-integrals can be introduced on any closed φ-interval, and the results about φ-integrals in this paper are also true for the φ-integrals on any closed φ-interval.

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