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A CONSTRUCTIVE DEFINITION OF THE n-DIMENSIONAL $\nu(S)$-INTEGRAL IN TERMS OF RIEMANN SUMS

Abstract

In a former paper ([Ju-No 1]) we introduced an axiomatic approach to the theory of non-absolutely convergent integrals in $\mathbb{R}^n$. A specialization of this abstract concept leads to the well-behaved $\nu(S)$-integral over quite general sets A which yields the divergence theorem in its presently most general form. (See [Ju-No 3].) While the definition of the $\nu(S)$-integral is of descriptive type (i.e. in terms of an additive almost everywhere differentiable set function) we prove in this paper that it can equivalently be defined using Riemann sums. As an application we show that any function being variationally integrable over A in the sense of [Pf 3] is also $\nu(S)$-integrable over A and both integrals coincide.

1 Introduction

Suppose $F : \mathbb{R} \to \mathbb{R}$ to be differentiable everywhere and assume that we seek an integration process which always integrates $f = F'$ to the expected value. Since not every derivative is absolutely integrable, we need an extension of Lebesgue's integral. Denjoy (1912, 1916, 1917) was the first to solve this problem by a transfinite construction, and shortly later Lusin (1912, 1916, 1917) and Hincin (1916, 1917, 1918) gave much simpler descriptive definitions of this integral by characterizing the associated interval function $F(b) - F(a)$. A partially constructive definition was given by Perron (1914), but a (directly) constructive definition in terms of Riemann sums was given only much later

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by Kurzweil (1957) and Henstock (1961). A slight modification also yields a constructive definition of the Lebesgue integral (McShane (1969)).

Analogous results for dimension $n > 1$ are very desirable, and here we consider, e.g., a vector field $\vec{v}$ which is differentiable everywhere on $\mathbb{R}^n$ and reasonably general sets $A \subseteq \mathbb{R}^n$. Again we seek an integration process which always integrates $\operatorname{div} \vec{v}$ over A to the expected value.

Ideally we look for a constructive definition (in terms of Riemann sums) of such an integral, associating with a point function f a single real number, which also allows an equivalent descriptive definition associating with f an additive set function whose set derivative equals f almost everywhere. This occurred first in [Ju-Kn] for intervals A and again in [No 1] for quite general sets A (i.e. compact sets A with $|\partial A|_{n-1} < \infty$), cf. also [Pf 1-2].

In [Ju-No 1] an axiomatic descriptive theory of non-absolutely convergent integrals in $\mathbb{R}^n$ is introduced, and the resulting ν-integral shows all the usual properties as linearity, additivity, extension of the Lebesgue integral etc.. Furthermore, a version of the Saks-Henstock-Lemma is proved which characterizes integrable functions and can be seen in between a descriptive and constructive definition of the integral.

A first specialization of this abstract concept (See [Ju-No 2].) leads to the well-behaved ν_1-integral over compact intervals which also allows a constructive definition and which yields the divergence theorem in its presently most general form (with regards to the analytic assumptions), including singularities where $\vec{v}$ is only assumed to fulfill a Lipschitz condition with a negative exponent ($> 1 - n$), cf. also [Ju-No 4, No 2].

Our second specialization leads to the $\nu(S)$-integral (depending upon a set $S \subseteq \mathbb{R}^n$ of potential singularities) over quite general sets A (See [Ju-No 3].) and using this integration theory the divergence theorem can be proved in a form analogous to the one for the ν_1-integral but now considering general sets A not just intervals. It is the aim of this paper to give an equivalent constructive definition of the $\nu(S)$-integral in terms of Riemann sums. This will be achieved via the Saks-Henstock-Lemma, and the main ideas first occurred in [No 1]. The proof of our main theorem requires several steps and is presented in Section 4. In particular, we state an Approximation Lemma, which essentially says that the $\nu(S)$-integral of a point function over A can be obtained as the limit of the integrals over certain figures contained in the interior of A. (See Sec. 2.5.) Since we work with (complete) partitions of the set A fulfilling various conditions, we establish in Section 3 the existence of such partitions using the Decomposition Theorem in [Ju]. As an application of our constructive definition we prove in Section 5 that any function being variationally integrable over A in the sense of [Pf 3] is $\nu(S)$-integrable over A with the same value.

2 Preliminaries

As usual $\mathbb{R}$ (resp. $\mathbb{R}^+$) denotes the set of all real (resp. all positive real) numbers, n is assumed to be a fixed positive integer, and we work in $\mathbb{R}^n$ with the usual inner product $x \cdot y = \sum x_i y_i$ ($x = (x_i), y = (y_i) \in \mathbb{R}^n$) and the associated norm $\| \cdot \|$. For $x \in \mathbb{R}^n$ and $r > 0$ we let $B(x,r) = \{y \in \mathbb{R}^n : \|y - x\| \le r\}$.

If $x \in \mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$, we denote by E°, $\operatorname{cl} E$, ∂E, $d(E)$ and $\operatorname{dist}(x, E)$ the interior, closure, boundary, diameter of E and the distance from the point x to the set E.

By $|\cdot|_s$ ($0 \le s \le n$) we denote the s-dimensional normalized outer Hausdorff measure in $\mathbb{R}^n$ which coincides for integral s on $\mathbb{R}^s$ ($\subseteq \mathbb{R}^n$) with the s-dimensional outer Lebesgue measure ($|\cdot|_0$ being the counting measure). $E \subseteq \mathbb{R}^n$ is called an s-null set if $|E|_s = 0$.

An *interval* I in $\mathbb{R}^n$ is always assumed to be compact and non-degenerate, and a *cube* is an interval with all sides having equal length. Finitely many intervals are said to be *non-overlapping* if they have pairwise disjoint interiors, and the union of finitely many non-overlapping intervals is called a *figure* (in $\mathbb{R}^n$). Given an interval I we call a finite sequence of non-overlapping intervals whose union is I a *decomposition of* I, and we denote by $r(I)$ the ratio of the smallest and the largest of the edges of I.

Finally, if f is a function defined on a subset E of $\mathbb{R}^n$, we denote for $F \subseteq E$ the restriction of f to F by $f|_F$ or simply by f again.

3 A General Existence Theorem

Throughout the paper we assume S to be an arbitrary but fixed subset of $\mathbb{R}^n$. The associated $\nu(S)$-integration theory is discussed in detail in [Ju-No 3].

We denote by $\mathcal{A}$ the system of all compact subsets A of $\mathbb{R}^n$ with $|\partial A|_{n-1} < \infty$, and given $\rho > 0$ a set $M \subseteq \mathbb{R}^n$ is called *ρ-regulated* if $|B(x,r) \cap M|_{n-1} \le \rho r^{n-1}$ holds for all $x \in \mathbb{R}^n$, $r > 0$. We let $\mathcal{A}(S)$ consist of those $A \in \mathcal{A}$ for which there is a $\rho > 0$ such that for any $x \in S \cap \partial A$ there exists a neighborhood U of x with $U \cap \partial A$ being ρ-regulated. For $\rho > 0$ we set $\mathcal{A}'_\rho = \{A \in \mathcal{A} : \partial A \text{ is } \rho\text{-regulated}\}$ and $\mathcal{A}_\rho(S) = \{A \in \mathcal{A}(S) : d(A)^n \le \rho |A|_n, \quad |\partial A|_{n-1} \le \rho d(A)^{n-1}\}$.

Recall ([Ju-No 3, Remark 1.1]) that there is a positive constant ρ^* ($\ge 2n^n$), depending only on n, such that each cube belongs to $\mathcal{A}_{\rho^*}(S)$, and each interval belongs to $\mathcal{A}'_{\rho^*}$. Furthermore, for each $\rho > 0$ and $A \in \mathcal{A}'_\rho$ we have $A \in \mathcal{A}(S)$ and $|\partial A|_{n-1} \le (1+\rho) d(A)^{n-1}$, and if $A, B \in \mathcal{A}(S)$ with corresponding parameters ρ_A, ρ_B, then $A \cup B$, $A \cap B$, $A - B^\circ \in \mathcal{A}(S)$ with a corresponding parameter $\rho_A + \rho_B$.

Given $E \subseteq \mathbb{R}^n$ and $\delta : E \to \mathbb{R}^+$ a finite sequence of pairs $\{(x_k, A_k)\}$ with $x_k \in A_k \in \mathcal{A}(S)$, $A_i^\circ \cap A_j^\circ = \emptyset$ ($i \ne j$), $x_k \in E$ and $d(A_k) < \delta(x_k)$ is called

(E,δ)-*fine*. If additionally $E = \bigcup A_k$, we call $\{(x_k, A_k)\}$ a δ-*fine partition of* E.

A *control condition* C associates with any positive numbers K and Δ a class $C(K,\Delta)$ of finite sequences $\{A_k\}$ with $A_k \in \mathcal{A}(S)$. Furthermore, with C there is associated a system $\mathcal{E}(C)$ of subsets of $\mathbb{R}^n$ and the control conditions $C^\alpha_{1,2}$ $(0 \le \alpha < n)$, C^n we use here are defined as follows, cf. [Ju-No 3, Sec. 1a]:

For $0 \le \alpha < n-1$ the control condition C^α_1 (resp. C^α_2) associates with any positive numbers K and Δ the system of all finite sequences $\{A_k\}$ with $A_k \in A'_K$ such that each $x \in S$ is contained in at most K of the A_k and such that $\sum d(A_k)^\alpha \le K$ (resp. $\sum d(A_k)^\alpha \le \Delta$). By $\mathcal{E}(C^\alpha_1)$ (resp. $\mathcal{E}(C^\alpha_2)$) we denote the system of all $E \subseteq S$ with $|E|_\alpha < \infty$ (resp. $|E|_\alpha = 0$).

The condition C^{n-1}_1 (resp. C^{n-1}_2) associates with K, $\Delta > 0$ the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{A}(S)$ and $\sum |\partial A_k|_{n-1} \le K$ (resp. $\sum |\partial A_k|_{n-1} \le \Delta$) and we let $\mathcal{E}(C^{n-1}_1)$ (resp. $\mathcal{E}(C^{n-1}_2)$) be the system of all $E \subseteq \mathbb{R}^n$ with $|E|_{n-1} < \infty$ (resp. $|E|_{n-1} = 0$).

If $n-1 < \alpha < n$, the control condition C^α_1 (resp. C^α_2) associates with K, $\Delta > 0$ the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{A}_K(S)$ and $\sum d(A_k)^\alpha \le K$ (resp. $\sum d(A_k)^\alpha \le \Delta$). $\mathcal{E}(C^\alpha_1)$ (resp. $\mathcal{E}(C^\alpha_2)$) consists of all $E \subseteq \mathbb{R}^n$ with $|E|_\alpha < \infty$ (resp. $|E|_\alpha = 0$).

Finally the condition C^n associates with any positive K the system of all finite sequences $\{A_k\}$ with $A_k \in \mathcal{A}_K(S)$, and we let

$$\mathcal{E}(C^n) = \{E \subseteq \mathbb{R}^n : |E|_n = 0\}.$$

We set $\Gamma = \{C^\alpha_i : 0 \le \alpha \le n-1,\ i = 1,2\}$, $\dot\Gamma = \{C^n\} \cup \{C^\alpha_i : n-1 < \alpha < n,\ i = 1,2\}$.

A *division of a set* $A \in \mathcal{A}(S)$ consists of a set $\dot E$ and a sequence $(E_i, C_i)_{i\in\mathbb{N}}$ such that $\dot E \subseteq A^\circ$, $|A - \dot E|_n = 0$, $C_i \in \Gamma \cup \dot\Gamma$, $E_i \in \mathcal{E}(C_i)$ and A is the disjoint union of all the sets E_i and $\dot E$.

In the proof of Theorem 4.1 we will need the existence of special δ-fine partitions of any set $A \in \mathcal{A}(S)$. It's the aim of this section to establish the existence of such partitions.

3a. A Modified Version of the Decomposition Theorem

The following theorem is proved in [Ju].

Decomposition Theorem. *Suppose an n-dimensional interval I to be the disjoint union of countably many sets E_i with $|E_i|_{\alpha_i} < \infty$ $(0 \le \alpha_i \le n)$ and that positive numbers ε_i and $\delta : I \to \mathbb{R}^+$ are given. Then there are finitely many intervals I_k, being similar to I and points x_k such that $\{(x_k, I_k)\}$ forms a δ-fine partition of I and $\sum\limits_{x_k \in E_i} d(I_k)^{\alpha_i} \le \frac{c(n)}{r(I)^n}|E_i|_{\alpha_i} + \varepsilon_i$, $i \in \mathbb{N}$, where $c(n)$ denotes an absolute constant.*

We deduce the following assertion.

Theorem 3.1 *Assume I to be an n-dimensional interval, let $\dot{E}$, $(E_i, C_i)_{i\in\mathbb{N}}$ be a division of I and suppose $M \subseteq I$ with $|M|_{n-1} < \infty$, $1 \geq \varepsilon' > 0$, $\Delta_i > 0$ and $\delta : I \to \mathbb{R}^+$ to be given. Then there exist finitely many intervals I_k, being similar to I, and points x_k such that $\{(x_k, I_k)\}$ is a δ-fine partition of I with*

$$\sum_{x_k\in M} d(I_k)^{n-1} \leq \frac{c(n)}{r(I)^n}|M|_{n-1} + \varepsilon',$$

$$\sum_{x_k\in E_i} d(I_k)^{\alpha} \leq \begin{cases} \frac{c(n)}{r(I)^n}(|E_i|_{n-1} + |M|_{n-1}) + \varepsilon' & \text{if} \quad C_i = C_1^\alpha,\ \alpha = n-1 \\ \frac{c(n)}{r(I)^n}|E_i|_\alpha + \varepsilon' & \text{if} \quad C_i = C_1^\alpha,\ \alpha \neq n-1 \\ \Delta_i & \text{if} \quad C_i = C_2^\alpha. \end{cases}$$

PROOF. Of course we want to apply the Decomposition Theorem and we therefore express I as the disjoint union of the following sets:

$$\dot{E} \setminus M,\ E_i \setminus M \text{ if } C_i \in \dot{\Gamma} \cup \{C_1^{n-1}\},\ E_i \text{ if } C_i \in \Gamma \setminus \{C_1^{n-1}\},$$
$$\tilde{M} = M \setminus \bigcup\{E_i : C_i \in \Gamma \setminus \{C_1^{n-1}\}\}.$$

Now, having chosen the ε_i appropriately and making δ smaller if necessary, the Decomposition Theorem yields a δ-fine partition $\{(x_k, I_k)\}$ of I having all the desired properties. For example let us show how the inequality corresponding to M can be obtained. Obviously

$$\begin{aligned} \sum_{x_k\in M} d(I_k)^{n-1} &\leq \sum_{x_k\in\tilde{M}} d(I_k)^{n-1} + \sum_{\substack{C_i=C_{1,2}^\alpha \\ \alpha<n-1}} \sum_{x_k\in M\cap E_i} \delta(x_k)^{n-1-\alpha} d(I_k)^\alpha \\ &\quad + \sum_{C_i=C_2^{n-1}} \sum_{x_k\in M\cap E_i} d(I_k)^{n-1} \\ &\leq \tfrac{c(n)}{r(I)^n}|\tilde{M}|_{n-1} + \tilde{\varepsilon} + \sum_{\substack{C_i=C_{1,2}^\alpha \\ \alpha<n-1}} \delta_i\Big(\tfrac{c(n)}{r(I)^n}|E_i|_\alpha + \varepsilon_i\Big) + \sum_{C_i=C_2^{n-1}} \varepsilon_i, \end{aligned}$$

where we assumed $\delta(\cdot)^{n-1-\alpha} \leq \delta_i$ on $M \cap E_i$ ($C_i = C_{1,2}^\alpha : \alpha < n-1$) and choosing $\tilde{\varepsilon}, \delta_i$ and ε_i suitable we get the required inequality. □

3b. The Existence Theorem

We now show the existence of special δ-fine partitions of $A \in \mathcal{A}(S)$. The following theorem (as well as its proof) should be compared with the Decomposition Lemma in our abstract theory [Ju-No 1].

Theorem 3.2 *Let I be a cube in $\mathbb{R}^n$, assume $\dot{E}$, $(E_i, C_i)_{i\in\mathbb{N}}$ to be a division of I and let $A \in \mathcal{A}(S)$ be a subset of I. Then there exist positive numbers $\tilde{K}_i(A) = \tilde{K}_i(A, E_i)$ such that for any parameter $\rho > 0$ corresponding to $A \in \mathcal{A}(S)$, any choice of numbers $\varepsilon > 0$, $1 \geq \varepsilon' > 0$, $\Delta_i > 0$ and any $\delta : A \to \mathbb{R}^+$ there is a δ-fine partition $\{(x_k, A_k)\}$ of A with the following properties:*

(i) If $x_k \in A^\circ$, then A_k is a cube contained in A°, and if $x_k \in \partial A$, then A_k is the intersection of a cube with A;

$$\Big|\bigcup_{x_k\in\partial A} A_k\Big|_n \leq \varepsilon,\ \Big|\partial \bigcup_{x_k\in A^\circ} A_k\Big|_{n-1} \leq \rho^* c(n)|\partial A|_{n-1} + \varepsilon'$$

(ii) if $x_k \in S \cap \partial A$, then $A_k \in \mathcal{A}'_{\rho+\rho^*}$ and any $x \in S$ is contained in at most 2^n of the A_k

(iii) $\sum_{x_k\in\partial A} |\partial A_k|_{n-1} \leq (1 + \rho^* c(n))|\partial A|_{n-1} + \varepsilon'$

(iv) $\sum_{x_k\in E_i} d(A_k)^\alpha \leq \begin{cases} \tilde{K}_i(A) & \text{if} \quad C_i = C_1^\alpha,\ \alpha \neq n-1 \\ \Delta_i & \text{if} \quad C_i = C_2^\alpha,\ \alpha \neq n-1 \end{cases}$

$$\sum_{x_k\in E_i} |\partial A_k|_{n-1} \leq \begin{cases} \tilde{K}_i(A) & \text{if} \quad C_i = C_1^{n-1} \\ \Delta_i & \text{if} \quad C_i = C_2^{n-1} \end{cases}.$$

PROOF. The $\tilde{K}_i(A)$ will be determined within the proof. We may assume $\delta(\cdot) \leq 1$ on A as well as $B(x, \delta(x)) \subseteq A^\circ$ for $x \in A^\circ$ and we extend δ to I by setting $\delta(x) = \operatorname{dist}(x, A)$ if $x \in I \setminus A$.

Since $|\partial A|_{n-1} < \infty$, we find an open set $G \supseteq \partial A$ with $|G|_n \leq \varepsilon$ and we may assume $B(x, \delta(x)) \subseteq G$ for $x \in \partial A$ (making δ smaller if necessary). Furthermore, if $C_i = C_2^{n-1}$, we choose an open set $G_i \supseteq E_i$ with $|G_i \cap \partial A|_{n-1} < \Delta_i/2$ (cf. [Ju-No 3, Remark 1.3]), and we assume $B(x, \delta(x)) \subseteq G_i$ for $x \in E_i$. Finally, if $x \in S \cap \partial A$, we assume $B(x, \delta(x)) \cap \partial A$ to be ρ-regulated.

By Theorem 3.1 there is a δ-fine partition $\{(x_k, I_k)\}$ of I, the I_k being cubes, fulfilling the following inequalities:

$$\sum_{x_k\in\partial A} d(I_k)^{n-1} \leq c(n)|\partial A|_{n-1} + \varepsilon'/\rho^*,$$

$$\sum_{x_k\in E_i} d(I_k)^\alpha \leq \begin{cases} c(n)(|E_i|_{n-1} + |\partial A|_{n-1}) + \varepsilon'/\rho^* & \text{if } C_i = C_1^\alpha, \alpha = n-1 \\ c(n)|E_i|_\alpha + \varepsilon'/\rho^* & \text{if } C_i = C_1^\alpha, \alpha \neq n-1 \\ \Delta_i/2\rho^* & \text{if } C_i = C_2^\alpha. \end{cases}$$

For $x_k \in A$ we set $A_k = A \cap I_k$ and obviously $\{(x_k, A_k)\}$ is a δ-fine partition of A. Furthermore, $A_k \subseteq B(x, \delta(x)) \subseteq G$ for $x_k \in \partial A$ what yields $|\bigcup_{x_k \in \partial A} A_k|_n \leq |G|_n \leq \varepsilon$ and since $\partial \bigcup_{x_k \in A^\circ} A_k \subseteq \bigcup_{x_k \in \partial A} \partial I_k$, we conclude

$$\begin{aligned}\Big|\partial \bigcup_{x_k \in A^\circ} A_k\Big|_{n-1} \leq \sum_{x_k \in \partial A} |\partial I_k|_{n-1} \leq \rho^* \sum_{x_k \in \partial A} d(I_k)^{n-1} \\ \leq& \rho^* c(n) |\partial A|_{n-1} + \varepsilon'.\end{aligned}$$

Now take an $x_k \in S \cap \partial A$ and observe that $\partial A_k \subseteq (I_k^\circ \cap \partial A) \cup \partial I_k \subseteq (B(x_k, \delta(x_k)) \cap \partial A) \cup \partial I_k$; consequently $A_k \in \mathcal{A}'_{\rho+\rho^*}$. Since each $x \in \mathbb{R}^n$ can at most be contained in 2^n of the I_k, any $x \in S$ lies in at most 2^n of the A_k, and it remains to prove the inequalities (iii) and (iv).

For (iii) observe

$$\begin{aligned}\sum_{x_k \in \partial A} |\partial A_k|_{n-1} \leq \sum_{x_k \in \partial A} |I_k^\circ \cap \partial A|_{n-1} + \sum_{x_k \in \partial A} |\partial I_k|_{n-1} \\ \leq& |\partial A|_{n-1} + \rho^* c(n) |\partial A|_{n-1} + \varepsilon'.\end{aligned}$$

To establish (iv) let us first treat the case $C_i = C_1^{n-1}$:

$$\begin{aligned}\sum_{x_k \in E_i} |\partial A_k|_{n-1} \leq& |\partial A|_{n-1} + \rho^* \sum_{x_k \in E_i} d(I_k)^{n-1} \\ \leq& |\partial A|_{n-1} + \rho^* c(n)(|E_i|_{n-1} + |\partial A|_{n-1}) + 1 = \tilde{K}_i(A, E_i).\end{aligned}$$

In case $C_i = C_2^{n-1}$ we have

$$\begin{aligned}\sum_{x_k \in E_i} |\partial A_k|_{n-1} \leq& \Big| \bigcup_{x_k \in E_i} I_k^\circ \cap \partial A \Big|_{n-1} + \sum_{x_k \in E_i} |\partial I_k|_{n-1} \leq |G_i \cap \partial A|_{n-1} \\ &+ \rho^* \Delta_i / 2\rho^* \leq \Delta_i.\end{aligned}$$

The remaining inequalities are obvious when setting $\tilde{K}_i(A, E_i) = c(n)|E_i|_\alpha + 1$ if $C_i = C_1^\alpha$, $\alpha \neq n-1$ and $\tilde{K}_i(A, E_i) = 1$ otherwise. $\square$

4 A Definition of the $\nu(S)$-integral in Terms of Riemann Sums

In this section we prove that the $\nu(S)$-integral (cf. [Ju-No 3]) can equivalently be defined by using Riemann sums.

Theorem 4.1 *Let $A \in \mathcal{A}(S)$ and $f : A \to \mathbb{R}$ be given. Then f is $\nu(S)$-integrable on A iff there exists a real number J and a division $\dot{E}$, $(E_i, C_i)_{i\in\mathbb{N}}$ of A with the following property:*

$$\forall\, \varepsilon > 0,\ K > 0,\ K_i > 0\ \exists\ \Delta_i > 0,\ \delta : A \to \mathbb{R}^+ \text{ such that}$$

$$\Big| J - \Big(\sum f(x_k)|A_k|_n + \sum f(x_k')|A_k'|_n \Big) \Big| \le \varepsilon \qquad (\star)$$

holds for any δ-fine partition $\{(x_k, A_k)\} \cup \{(x_k', A_k')\}$ of A with

(i) if $x_k \in \dot{E}$, then $A_k \in \mathcal{A}_K(S)$; $\{A_k : x_k \in E_i\} \in C_i(K_i, \Delta_i)$ $(i \in \mathbb{N})$

(ii) $\{A_k'\} \in C_1^{n-1}(K)$ and $x_k' \in \dot{E} \cup \bigcup\limits_{C_i \in \dot{\Gamma}} E_i$ for all k,

and in that case J is uniquely determined and $J = {}^{\nu(S)}\!\!\displaystyle\int_A f$.

Remark 4.1 *The necessity part of the theorem, i.e. starting with the $\nu(S)$-integrability of f, is a direct consequence of the abstract Saks-Henstock Lemma* ([Ju-No 1, Cor. 6.1] *and apply it to the concrete setting in* [Ju-No 3]), *and this also yields the uniqueness of J and $J = {}^{\nu(S)}\!\int_A f$.*

To prove the sufficiency part we will deduce the Saks-Henstock Lemma and since this gives a characterization of integrability, (cf. [Ju-No 1, Sec. 6]) the $\nu(S)$-integrability of f will follow. Here we proceed along the lines of [Ju-No 2, Theorem 3.1] where the corresponding constructive definition was given for the ν_1-integral. But there we only had to deal with intervals and now, using general sets, new difficulties must be overcome.

PROOF OF THEOREM 4.1. We assume $J \in \mathbb{R}$ and a division $\dot{E}, (E_i, C_i)_{i\in\mathbb{N}}$ of A to be given such that property $(\star)$ is fulfilled. The remainder of the proof is accomplished in several steps. In a first step we show that w.l.o.g. the set A can be assumed to be a cube.

1. Choose a cube I containing A in its interior and extend f to $I$to be zero. Then a division of I is given by $\dot{E} \cup (I^\circ \setminus A)$, $(\partial I, C_1^{n-1})$, $(E_i, C_i)_{C_i\in\Gamma}$, $(E_i \cap A^\circ, C_i)_{C_i\in\dot{\Gamma}}$, $(\partial A \cap \bigcup_{C_i\in\dot{\Gamma}} E_i, C_1^{n-1})$. We prove that property $(\star)$ is fulfilled for I and f corresponding to this division and J.

Given $\varepsilon > 0$, $K > 0$, $K_{\partial I} > 0$, $K_i > 0$ $(i \in \mathbb{N})$, $K_{\partial A} > 0$ we set $\tilde{K} = |\partial A|_{n-1} + K_{\partial A} + K$ and $\tilde{K}_i = K_i + \rho + |\partial A|_{n-1}$ and we determine $\tilde{\Delta}_i > 0$ $(i \in \mathbb{N})$, $\delta : A \to \mathbb{R}^+$ for $\varepsilon/2$, $\tilde{K}$, $\tilde{K}_i$ because of $(\star)$ for A. Here ρ denotes a parameter corresponding to $A \in \mathcal{A}(S)$. We let $\Delta_i = \tilde{\Delta}_i/2$ $(i \in \mathbb{N})$, $\Delta_{\partial I} = 1 = \Delta_{\partial A}$ and we may assume $B(x, \delta(x)) \subseteq A^\circ$ for $x \in A^\circ$ and $B(x, \delta(x)) \cap \partial A$ to be ρ-regulated if $x \in S \cap \partial A$. Furthermore, if $C_i = C_2^{n-1}$, we choose an open set $G_i \supseteq E_i$ with $|G_i \cap \partial A|_{n-1} \le \Delta_i$ and we assume $B(x, \delta(x)) \subseteq G_i$ for $x \in E_i$. Finally we set $\delta(x) = \operatorname{dist}(x, A)$ for $x \in I \setminus A$ and since $|\partial A|_n = 0$, we may assume $\sum |f(z_k)||D_k|_n \le \varepsilon/2$ for any $(\partial A, \delta)$-fine sequence $\{(z_k, D_k)\}$ (cf. Remark 5.2(iii)).

Now let $\{(x_k, B_k) : 1 \le k \le p\} \cup \{(x'_k, B'_k) : p+1 \le k \le p+m\}$ $(p, m \in \mathbb{N}_0)$ be a δ-fine partition of I with the following properties:

- If $x_k \in \dot{E} \cup (I^\circ \setminus A)$, then $B_k \in \mathcal{A}_K(S)$; $\{B_k : x_k \in \partial I\} \in C_1^{n-1}(K_{\partial I})$,
  $\{B_k : x_k \in E_i\} \in C_i(K_i, \Delta_i)$ if $C_i \in \Gamma$,
  $\{B_k : x_k \in E_i \cap A^\circ\} \in C_i(K_i, \Delta_i)$ if $C_i \in \dot{\Gamma}$,
  $\{B_k : x_k \in \partial A \cap \bigcup_{C_i \in \dot{\Gamma}} E_i\} \in C_1^{n-1}(K_{\partial A})$.
- $\{B'_k\} \in C_1^{n-1}(K)$ and $x'_k \in \dot{E} \cup (I^\circ \setminus A) \cup \bigcup_{C_i \in \dot{\Gamma}} (E_i \cap A^\circ)$, $p+1 \le k \le p+m$.

Then we define a δ-fine partition $\{(y_i, A_i)\} \cup \{(y'_i, A'_i)\}$ of A as follows:
If $x_k \in A$ and $x_k \notin \partial A \cap \bigcup_{C_i \in \dot{\Gamma}} E_i$, then we let $y_k = x_k$ and $A_k = A \cap B_k$, while in case $x_k \in \partial A \cap \bigcup_{C_i \in \dot{\Gamma}} E_i$, we set $y'_k = x_k$ and $A'_k = A \cap B_k$; if $x'_k \in A$, we let $y'_k = x'_k$ and $A'_k = A \cap B'_k$. Now it is easy to see, using the choice of δ, and recalling that $\partial(A \cap B) \subseteq (B^\circ \cap \partial A) \cup \partial B$ for any $B \in \mathcal{A}(S)$, that the partition $\{(y_i, A_i)\} \cup \{(y'_i, A'_i)\}$ fulfills the requirements (i) and (ii) of Theorem 4.1 with parameters $\tilde{K}, \tilde{K}_i, \tilde{\Delta}_i$. Furthermore, since $f = 0$ on $I \setminus A$ and because of the δ-function,

$$\begin{aligned}
&\Big| J - \Big(\sum f(x_k)|B_k|_n + \sum f(x'_k)|B'_k|_n \Big) \Big| \\
\le &\Big| J - \Big(\sum f(y_i)|A_i|_n + \sum f(y'_i)|A'_i|_n \Big) \Big| + \sum_{x_k \in \partial A} |f(x_k)||D_k|_n \\
&+ \sum_{x'_k \in \partial A} |f(x'_k)||D'_k|_n \le \varepsilon,
\end{aligned}$$

where $D_k = B_k \cap (I \setminus A^\circ)$, $D'_k = B'_k \cap (I \setminus A^\circ)$.

Suppose now that the theorem has been established for any cube. Then it follows that f is $\nu(S)$-integrable on I and by [Ju-No 3, Prop. 1.1] f is $\nu(S)$-integrable on A with ${}^{\nu(S)}\!\int_A f = {}^{\nu(S)}\!\int_I f = J$.

Therefore, from now on, we assume A to be a cube, $J \in \mathbb{R}$ and $\dot{E}$, $(E_i, C_i)_{i \in \mathbb{N}}$ to be a division of A such that $(\star)$ is fulfilled. In the sequel we will write I instead of A and we will use the letter A to denote subsets of I.

2. Notation. Let $A \in \mathcal{A}(S)$ be a subset of I. For any positive numbers K, K_i, Δ_i and any $\delta : A \to \mathbb{R}^+$ we denote by $\mathcal{P}(A, K, K_i, \Delta_i, \delta)$ the system of all δ-fine partitions $\Pi = \{(x_k, A_k)\} \cup \{(x'_k, A'_k)\}$ of A fulfilling conditions (i) and

(ii) of Theorem 4.1. Obviously $\mathcal{P}(A, K, K_i, \Delta_i, \delta)$ is monotone increasing in K, K_i, Δ_i and δ (since all control conditions $C_i = C_i(K, \Delta)$ are non-decreasing in K and Δ) and for $\Pi \in \mathcal{P}(A, K, K_i, \Delta_i, \delta)$ as above we set $\mathcal{S}(f, \Pi, A) = \sum f(x_k)|A_k|_n + \sum f(x'_k)|A'_k|_n$.

Denoting by ρ a parameter corresponding to $A \in \mathcal{A}(S)$, we see by Theorem 3.2 that for any $K \geq K^*(A) = (1 + \rho^* c(n))|\partial A|_{n-1} + \rho^*$, $K_i \geq K_i^*(A, \rho) = \tilde{K}_i(A) + \rho + \rho^*$, $\Delta_i > 0$ and $\delta : A \to \mathbb{R}^+$ there is a $\Pi \in \mathcal{P}(A, K, K_i, \Delta_i, \delta)$, where the marked quantities are $\{(x_k, A_k) : x_k \in \partial A \cap (\dot{E} \cup \bigcup_{C_i \in \dot{\Gamma}} E_i)\}$. In the sequel we simply write $K_i^*(A)$ instead of $K_i^*(A, \rho)$, since the considered parameter ρ will always be clear from the context.

A set function F on I associates with every subset $A \in \mathcal{A}(S)$ of I a real number $F(A)$ and we call F *additive* if $F(A) = \sum F(A_k)$ holds for any subset $A \in \mathcal{A}(S)$ of I and every finite sequence $\{A_k\}$ with $A_k \in \mathcal{A}(S)$ having disjoint interiors and $A = \bigcup A_k$.

Next we will associate with each subset $A \in \mathcal{A}(S)$ of I a real number $\gamma(A)$ and thereby define an additive set function on I.

Fix a set $A \in \mathcal{A}(S)$, $A \subseteq I$:

3. $\forall\, \varepsilon > 0,\ K > 0,\ K_i > 0\ \exists\ \Delta_i > 0,\ \delta : A \to \mathbb{R}^+$:

$$|\mathcal{S}(f, \Pi_1, A) - \mathcal{S}(f, \Pi_2, A)| \leq \varepsilon \ \forall\, \Pi_{1,2} \in \mathcal{P}(A, K, K_i, \Delta_i, \delta).$$

Proof. Choose a common corresponding parameter $\rho > 0$ for A and $B = I \setminus A^\circ \quad (\in \mathcal{A}(S))$, let ε, K, K_i be given and set $\bar{K} = K + K^*(B)$ and $\bar{K}_i = K_i + K_i^*(B)$. Then, by our assumption, there are $\bar{\Delta}_i > 0$ and $\bar{\delta} : I \to \mathbb{R}^+$ such that $|\mathcal{S}(f, \Pi^1, I) - \mathcal{S}(f, \Pi^2, I)| \leq \varepsilon \quad \forall\, \Pi^{1,2} \in \mathcal{P}(I, \bar{K}, \bar{K}_i, \bar{\Delta}_i, \bar{\delta})$ and we let $\Delta_i = \bar{\Delta}_i/2$ and $\delta = \bar{\delta}|_A$. Fix an $\Pi \in \mathcal{P}(B, K^*(B), K_i^*(B), \Delta_i, \bar{\delta})$ ($\neq \emptyset$ by **2.**) and let $\Pi_{1,2} \in \mathcal{P}(A, K, K_i, \Delta_i, \delta)$. Then an easy check shows that $\Pi^{1,2} = \Pi \cup \Pi_{1,2} \in \mathcal{P}(I, \bar{K}, \bar{K}_i, \bar{\Delta}_i, \bar{\delta})$, where the marked parts of Π^i is just the union of the marked parts of Π and Π_i. Consequently

$$|\mathcal{S}(f, \Pi_1, A) - \mathcal{S}(f, \Pi_2, A)| = |\mathcal{S}(f, \Pi^1, I) - \mathcal{S}(f, \Pi^2, I)| \leq \varepsilon. \qquad \square$$

4. We define the following (extended) real numbers which may be seen as an upper and a lower integral.

$$\gamma(A)^+ = \sup_{(K, K_i)} \ \inf_{(\Delta_i, \delta)} \ \sup_{\Pi \in \mathcal{P}(A, K, K_i, \Delta_i, \delta)} \mathcal{S}(f, \Pi, A).$$

$$\gamma(A)^- = \inf_{(K, K_i)} \ \sup_{(\Delta_i, \delta)} \ \inf_{\Pi \in \mathcal{P}(A, K, K_i, \Delta_i, \delta)} \mathcal{S}(f, \Pi, A).$$

Obviously $\gamma(A)^- \leq \gamma(A)^+$ and by the Cauchy-property **3.** one sees that indeed $\gamma(A)^- = \gamma(A)^+ \in \mathbb{R}$. This real number will be denoted by $\gamma(A)$. As an immediate consequence we get

$$\forall\, \varepsilon > 0,\ K > 0,\ K_i > 0\ \exists\, \Delta_i > 0,\ \delta : A \to \mathbb{R}^+ :$$
$$|\gamma(A) - \mathcal{S}(f, \Pi, A)| \le \varepsilon\ \forall\, \Pi \in \mathcal{P}(A, K, K_i, \Delta_i, \delta).$$

Of course $\gamma(A)$ is the only real number with this property since $\mathcal{P}(A, K, K_i, \Delta_i, \delta)$ is not empty for sufficiently large K, K_i.

By the considerations in **3.** and **4.** we are in the position to define a set function F on I by $F(A) = \gamma(A)$ for each $A \in \mathcal{A}(S)$, $A \subseteq I$. Note that in particular $F(I) = J$. To prove the additivity of F it suffices to show that $F(A) = F(A_1) + F(A_2)$ whenever $A, A_1, A_2 \in \mathcal{A}(S)$ are subsets of I with $A_1^\circ \cap A_2^\circ = \emptyset$ and $A = A_1 \cup A_2$. For this let $\varepsilon > 0$ be arbitrary and choose suitable parameters for A_1, A_2 and correspondingly for A such that $\mathcal{S}(f, \Pi_i, A_i)$ and $\mathcal{S}(f, \Pi, A)$ approximate $F(A_i)$ (resp. $F(A)$) up to ε. Taking $\Pi = \Pi_1 \cup \Pi_2$ yields $|F(A) - (F(A_1) + F(A_2))| \le 3\varepsilon$ which implies $F(A) = F(A_1) + F(A_2)$.

Before we can show the Saks-Henstock (SH) property, we prove a general approximation lemma, which seems to be of some interest for itself, as well as two restricted versions of the (SH)-Lemma.

5. Approximation Lemma. *Assume $A \in \mathcal{A}(S)$ to be a subset of I with a non-empty interior. Then there exists a sequence of figures R_m ($m \in \mathbb{N}$) fulfilling the following conditions:*

(i) $R_m \subseteq A^\circ$ *for all* m *and* $|A \setminus R_m|_n \to 0 \quad (m \to \infty)$

(ii) $|\partial R_m|_{n-1} \le (1 + \rho^* c(n))|\partial A|_{n-1}$ *for all* m

(iii) $F(R_m) \to F(A) \quad (m \to \infty)$.

Furthermore, if $A \in \mathcal{A}_K(S)$ for some $K > 0$, we have in addition $\{x\} \cup R_m \in \mathcal{A}_{\underline{K}}(S)$ for all $x \in A$ and all $m \in \mathbb{N}$, where $\underline{K} = 2(1 + \rho^* c(n))(K + \rho^*)^2$.

Proof. Denote by ρ $(\ge \rho^*)$ a corresponding parameter for A and set $\hat{K} = \rho^* + 2(2 + \rho^* c(n))^2 |\partial A|_{n-1}$, $\hat{K}_i = \tilde{K}_i(A) + 1 + 2\rho^2 + 2^n + \rho^* c(n)|E_i|_\alpha + (1 + \rho^* c(n))^2 |\partial A|_{n-1}$ if $C_i = C_j^\alpha$ $(0 \le \alpha \le n,\ j = 1, 2)$ with the understanding that $C_{1,2}^\alpha = C^n$ if $\alpha = n$. (Here $\tilde{K}_i(A)$ are the numbers from Theorem 3.2.) Then by **4.** we can determine for $\varepsilon(m) = 1/m$ $(m \in \mathbb{N})$, $\hat{K}$, $\hat{K}_i$ numbers $\Delta_i(m) > 0$ and a $\delta_m : A \to \mathbb{R}^+$ such that

$$|F(A) - \mathcal{S}(f, \Pi, A)| \le \varepsilon(m) \quad \forall\, \Pi \in \mathcal{P}(A, \hat{K}, \hat{K}_i, \Delta_i(m), \delta_m).$$

Writing $\partial A = \bigcup_{j \in \mathbb{N}} T_j$ with $T_j = \{x \in \partial A : j - 1 \le |f(x)| < j\}$, we choose open sets $G_j \supseteq T_j$ with $|G_j|_n \le \varepsilon(m)/j2^{j+1}$ and we may assume $B(x, \delta_m(x)) \subseteq G_j$ for $x \in T_j$. Since $|A|_n \le d(A)|\partial A|_{n-1}$ (which holds for each set from $\mathcal{A}$; see Remark 5.1), we see that $|\partial A|_{n-1} > 0$ ($A^\circ \neq \emptyset$) and applying Theorem 3.2 with $\varepsilon = \varepsilon(m)$, $\varepsilon' = \min(1, |\partial A|_{n-1})$, $\Delta_i(m)/2$ and δ_m there is a δ_m-fine partition $\Pi_m = \{(x_k^m, A_k^m)\}$ of A with the properties (i)–(iv) stated there. Setting $R_m = \bigcup_{x_k^m \in A^\circ} A_k^m$ we see that the R_m are figures lying in the interior of A, $|A - R_m|_n \leq |\bigcup_{x_k^m \in \partial A} A_k^m|_n \leq \varepsilon(m)$ and that $|\partial R_m|_{n-1} \leq (1 + \rho^* c(n))|\partial A|_{n-1}$. We proceed by proving $F(R_m) \to F(A)$.

First note that for any figure R we have $R \in \mathcal{A}(S)$ with a (possible) corresponding parameter ${\rho^*}^2$. For, if R is the finite union of the non-overlapping intervals I_i and if $x \in S \cap \partial R$ there are at most 2^n of the intervals I_i containing x. Denote those by I_i^* and choose a neighborhood U of x not intersecting any of the other intervals. Then for any $z \in \mathbb{R}^n$ and any $r > 0$

$$|U \cap \partial R \cap B(z,r)|_{n-1} \leq \sum |U \cap \partial I_i^* \cap B(z,r)|_{n-1} \leq 2^n \rho^* r^{n-1} \leq {\rho^*}^2 r^{n-1}$$

since any interval belongs to $\mathcal{A}'_{\rho^*}$.

Now let $\tilde{\varepsilon} > 0$ be arbitrary, choose an $m_0 \in \mathbb{N}$ with $1/m_0 \leq \tilde{\varepsilon}/2$ and let $m \geq m_0$. Again by **4.** we determine $\Delta_i > 0$ and a $\delta : R_m \to \mathbb{R}^+$ such that

$$|F(R_m) - \mathcal{S}(f, \Pi, R_m)| \leq \tilde{\varepsilon}/4 \; \forall \, \Pi \in \mathcal{P}(R_m, K^*(R_m), K_i^*(R_m), \Delta_i, \delta)$$

(cf. **2.**). Fix a $\Pi^* \in \mathcal{P}(R_m, K^*(R_m), K_i^*(R_m), \Delta_i^*, \delta^*)$ ($\neq \emptyset$) where $\Delta_i^* = \min(\Delta_i(m), \Delta_i)/2$ and $\delta^* = \min(\delta_m, \delta)$ and let $\Pi = \Pi^* \cup \{(x_k^m, A_k^m) \in \Pi_m : x_k^m \in \partial A\}$. Then $\Pi \in \mathcal{P}(A, \hat{K}, \hat{K}_i, \Delta_i(m), \delta_m)$ where the marked quantities are those of Π^* unified with those pairs (x_k^m, A_k^m) where $x_k^m \in \partial A \cap (\dot{E} \cup \bigcup_{C_i \in \dot{\Gamma}} E_i)$. For example we get for the marked quantities

$$\begin{aligned} \sum |\partial(\cdot)|_{n-1} \leq & \sum_{x_k^m \in \partial A} |\partial A_k^m|_{n-1} + K^*(R_m) \\ \leq & (2 + \rho^* c(n))|\partial A|_{n-1} + (1 + \rho^* c(n))|\partial R_m|_{n-1} + \rho^* \leq \hat{K} \end{aligned}$$

by the properties of Π_m, by **2.** and the upper bound for $|\partial R_m|_{n-1}$.

Analogously one checks the other conditions and thus we obtain

$$\begin{aligned} |F(R_m) - F(A)| \leq & |F(R_m) - \mathcal{S}(f, \Pi^*, R_m)| + |F(A) - \mathcal{S}(f, \Pi, A)| \\ & + \sum_{x_k^m \in \partial A} |f(x_k^m)||A_k^m|_n \\ \leq & \frac{\tilde{\varepsilon}}{4} + \varepsilon(m) + \sum_{j \in \mathbb{N}} \sum_{x_k^m \in T_j} j|A_k^m|_n \leq \frac{3}{4}\tilde{\varepsilon} + \sum_j j|G_j|_n \leq \tilde{\varepsilon}. \end{aligned}$$

Now choose an $n_0 \in \mathbb{N}$ such that $|R_m|_n \geq |A|_n/2$ for all $m \geq n_0$ and assume in addition that $A \in \mathcal{A}_K(S)$ for some $K > 0$ (which always holds for sufficiently large K). Take any $x \in A$, let $\tilde{R}_m = \{x\} \cup R_m$ ($\in \mathcal{A}(S)$), $m \geq n_0$, and observe that $d(\tilde{R}_m)^n \leq d(A)^n \leq K|A|_n \leq 2K|\tilde{R}_m|_n \leq \underline{K}|\tilde{R}_m|_n$. To derive the second desired inequality assume $n \geq 2$ (the case $n = 1$ being trivial); consequently

$$\begin{aligned}|\partial\tilde{R}_m|_{n-1} \leq& |\partial R_m|_{n-1} \leq (1+\rho^* c(n))|\partial A|_{n-1}\\ \leq& K(1+\rho^* c(n))d(A)^{n-1} \leq K^2(1+\rho^* c(n))|A|_n/d(\tilde{R}_m)\\ \leq& 2K^2(1+\rho^* c(n))d(\tilde{R}_m)^{n-1} \leq \underline{K}d(\tilde{R}_m)^{n-1}\end{aligned}$$

(Note that $|B|_n \leq d(B)^n$ holds for each $B \in \mathcal{A}$.) and this implies $\tilde{R}_m \in \mathcal{A}_{\underline{K}}(S)$ for $m \geq n_0$. Thus the sequence R_m, $m \geq n_0$ fulfills all the desired conditions. $\square$

6. Fix a $j \in \mathbb{N}$ such that $C_j = C_1^\alpha$ resp. C_2^α with $0 \leq \alpha < n-1$. Then the following holds.

$$\forall\, \varepsilon > 0,\ K_j > 0\ \exists\ \Delta_j > 0,\ \delta : E_j \to \mathbb{R}^+ :\ \sum |F(A_k) - f(x_k)|A_k|_n| \leq \varepsilon$$

for each (E_j, δ)-fine sequence $\{(x_k, A_k)\}$ with $A_k \subseteq I$ and $\{A_k\} \in C_j(K_j, \Delta_j)$.
PROOF. Given ε and K_j set $\hat{K} = (1+\rho^* c(n))(1+|\partial I|_{n-1}) + \rho^*$, $\hat{K}_i = 1+\hat{K}+\rho^*+K_j+K_j^2+\rho^* c(n)|E_i|_\alpha$ for $C_i = C_{1,2}^\alpha$ (again with the understanding that $C_{1,2}^n = C^n$) and choose $\hat{\Delta}_i > 0$, $\delta : I \to \mathbb{R}^+$ such that

$$|F(I) - \mathcal{S}(f,\Pi,I)| \leq \varepsilon/4\ \forall\, \Pi \in \mathcal{P}(I,\hat{K},\hat{K}_i,\hat{\Delta}_i,\delta).$$

Let $\Delta_j = \min(1, \hat{\Delta}_j/2)$. Assume $\delta(\cdot)^{n-1-\alpha} \leq 1/(1+K_j)^2$ on E_j and let $\{(x_k, A_k) : 1 \leq k \leq m\}$ be a (E_j,δ)-fine sequence with $A_k \subseteq I$ and $\{A_k\} \in C_j(K_j, \Delta_j)$. After rearranging (if necessary) we find a $0 \leq \mu \leq m$ such that $f(x_k)|A_k|_n \geq F(A_k)$ if $1 \leq k \leq \mu$ and $f(x_k)|A_k|_n < F(A_k)$ otherwise and we let $B_1 = I \setminus (\bigcup_{k=1}^{\mu} A_k)^\circ$ and $B_2 = I \setminus (\bigcup_{k=\mu+1}^{m} A_k)^\circ$. Since all $A_k \in \mathcal{A}'_{K_j}$, we have $|\partial A_k|_{n-1} \leq (1+K_j)d(A_k)^{n-1}$ and thus

$$\sum |\partial A_k|_{n-1} \leq (1+K_j)\sum \delta(x_k)^{n-1-\alpha}d(A_k)^\alpha \leq \sum d(A_k)^\alpha/(1+K_j) \leq 1$$

by the choice of Δ_j and δ. Consequently $|\partial B_j|_{n-1} \leq 1 + |\partial I|_{n-1}$.

Furthermore, $\bigcup_{k=1}^{\mu} A_k \in \mathcal{A}(S)$ with a corresponding parameter K_j^2. For, if $x \in S \cap \partial(\bigcup_{k=1}^{\mu} A_k)$, there are at most K_j of the A_k containing x and we choose a neighborhood U of x intersecting at most these A_k; consequently for $z \in \mathbb{R}^n$ and $r > 0$ we have

$$\Big|U \cap \partial\Big(\bigcup_{k=1}^{\mu} A_k\Big) \cap B(z,r)\Big|_{n-1} \leq \sum_{U \cap A_k \neq \emptyset} |B(z,r) \cap \partial A_k|_{n-1} \leq K_j^2 r^{n-1}.$$

Since the same is true for $\bigcup_{k=\mu+1}^{m} A_k$, we see that $B_j \in \mathcal{A}(S)$ with a corresponding parameter $K_j^2 + \rho^*$.

Next we choose $\Delta_i^* > 0$ and $\delta^* : B_1 \to \mathbb{R}^+$ such that

$$|F(B_1) - \mathcal{S}(f, \Pi_1, B_1)| \le \varepsilon/4 \;\forall\, \Pi_1 \in \mathcal{P}(B_1, K^*(B_1), K_i^*(B_1), \Delta_i^*, \delta^*),$$

we fix a $\Pi_1 \in \mathcal{P}(B_1, K^*(B_1), K_i^*(B_1), \min(\Delta_i^*, \hat{\Delta}_i)/2, \min(\delta^*, \delta))$ ($\neq \emptyset$ by **2.**) and we let $\Pi = \Pi_1 \cup \{(x_k, A_k) : 1 \le k \le \mu\}$. Then, recalling the definition of the numbers $K^*(B_1)$ and $K_i^*(B_1)$, one immediately sees that $\Pi \in \mathcal{P}(I, \hat{K}, \hat{K}_i, \hat{\Delta}_i, \delta)$ and consequently

$$\frac{\varepsilon}{4} \ge \mathcal{S}(f, \Pi, I) - F(I) = \sum_{k=1}^{\mu} |F(A_k) - f(x_k)|A_k|_n| + \mathcal{S}(f, \Pi_1, B_1) - F(B_1)$$

and thus $\sum_{k=1}^{\mu} |F(A_k) - f(x_k)|A_k|_n| \le \varepsilon/2$. Analogously it follows that

$$\sum_{k=\mu+1}^{m} |F(A_k) - f(x_k)|A_k|_n| \le \varepsilon/2. \qquad \square$$

7. Set $\tilde{\Gamma} = \dot{\Gamma} \cup \{C_1^{n-1}, C_2^{n-1}\}$. Then the following holds:

$$\forall\, \varepsilon > 0,\;\; K > 0,\;\; K_i > 0\;\; (C_i \in \tilde{\Gamma}) \quad \exists\;\; \Delta_i > 0\;\; (C_i \in \tilde{\Gamma}),\;\; \delta : I \to \mathbb{R}^+ \;:$$

$$\sum |F(R_k) - f(x_k)|R_k|_n| \;+\; \sum |F(R'_k) - f(x'_k)|R'_k|_n| \le \varepsilon$$

for each finite sequence of pairs $\{(x_k, R_k)\} \cup \{(x'_k, R'_k)\}$, where the R_k and R'_k are figures contained in I with pairwise disjoint interiors, $x_k, x'_k \in I$ and $d(\tilde{R}_k) < \delta(x_k)$, $d(\tilde{R}'_k) < \delta(x'_k)$ with $\tilde{R}_k = \{x_k\} \cup R_k$, $\tilde{R}'_k = \{x'_k\} \cup R'_k$, fulfilling the following conditions:

(i) $x_k \in \dot{E} \cup \bigcup_{C_i \in \tilde{\Gamma}} E_i$ for all k, and if $x_k \in \dot{E}$, then $\tilde{R}_k \in \mathcal{A}_K(S)$;
$\{\tilde{R}_k : x_k \in E_i\} \in C_i(K_i, \Delta_i) \quad (C_i \in \tilde{\Gamma})$

(ii) $\{\tilde{R}'_k\} \in C_1^{n-1}(K)$ and $x'_k \in \dot{E} \cup \bigcup_{C_i \in \dot{\Gamma}} E_i$ for all k.

Proof. The proof is closely related to what has been shown in 3. within the proof of Theorem 3.1 in [Ju-No 2].

Recall the following well-known facts:

Given $0 \le \alpha \le n$ and $M \subseteq \mathbb{R}^n$ the inequality $\sum_i |M \cap I_i|_\alpha \le 2^n |M|_\alpha$ holds for any finite sequence of non-overlapping intervals, $\{I_i\}$. Furthermore, if $\{I_i\}$ denotes any finite sequence of non-overlapping intervals each $x \in \mathbb{R}^n$ can at most be contained in 2^n of the I_i. For any interval J and any $0 < r < 1$ there exists a decomposition $\{I_k\}$ of J with $r(I_k) \geq r$ for all k. Now let ε, K and K_i ($C_i \in \tilde{\Gamma}$) be given, let $K_i = 1$ for $C_i \notin \tilde{\Gamma}$, set $\bar{K} = \max(K, \rho^*)$,

$$\bar{K}_i = \begin{cases} 1 + \bar{K} + K_i + \frac{2nc(n)2^n}{r^n}|E_i|_\alpha & \text{if } C_i = C_1^\alpha \\ \bar{K} + K_i & \text{otherwise} \end{cases}, \text{ where } r = \frac{\sqrt{n}}{\sqrt[n]{\bar{K}}} \ (< 1),$$

and choose $\Delta_i > 0$ and $\delta : I \to \mathbb{R}^+$ such that

$$|F(I) - \mathcal{S}(f, \Pi, I)| \leq \varepsilon/2 \ \forall\, \Pi \in \mathcal{P}(I, \bar{K}, \bar{K}_i, 2\Delta_i, \delta) .$$

Denote by $\{(x_k, R_k)\} \cup \{(x'_k, R'_k)\}$ a finite non-empty sequence according to **7.** and assume w.l.o.g., cf. [Ju-No 2], $f(x_k)|R_k|_n \geq F(R_k)$ and $f(x'_k)|R'_k|_n \geq F(R'_k)$ for all k.

We can express each R_k, R'_k as a finite union of non-overlapping intervals, and if necessary we add intervals $J_1, \ldots, J_p$ ($p \in \mathbb{N}_0$) with $r(J_i) \geq r$ such that all occurring intervals form a decomposition of I.

Fix a subinterval J of I with $r(J) \geq r$ and let

$$K_i(J) = \begin{cases} \frac{2nc(n)}{r^n}|E_i \cap J|_\alpha + \bar{K} & \text{if } C_i = C_1^\alpha \\ \bar{K} & \text{otherwise.} \end{cases}$$

By **4.** we may determine $\Delta_i(J) > 0$ and $\delta(J) : J \to \mathbb{R}^+$ such that

$$|F(J) - \mathcal{S}(f, \Pi, J)| \leq \varepsilon/2(p+1) \quad \forall\, \Pi \in \mathcal{P}(J, \bar{K}, K_i(J), \Delta_i(J), \delta(J)).$$

Setting $\Delta_i^*(J) = \min(\Delta_i(J), \Delta_i)/2n(p+1), \delta^*(J) = \min(\delta(J), \delta)$ there is, by the Decomposition Theorem, a $\delta^*(J)$-fine partition $\Pi(J) = \{(y_k, L_k)\}$ of J, the L_k being intervals with $r(L_k) = r(J)$ ($\geq r$) and with

$$\sum_{y_k \in E_i \cap J} d(L_k)^\alpha \leq \begin{cases} \frac{c(n)}{r^n}|E_i \cap J|_\alpha + \frac{1}{2n(p+1)} & \text{if} \quad C_i = C_1^\alpha \\ \Delta_i^*(J) & \text{if} \quad C_i = C_2^\alpha. \end{cases}$$

Now it is easy to see that $\Pi(J) \in \mathcal{P}(J, \bar{K}, K_i(J), \Delta_i(J), \delta(J))$ without any marked quantities (Note, e.g., that all $L_k \in \mathcal{A}_{\bar{K}}(S)$ by the choice of r.) and therefore

$$|F(J) - \mathcal{S}(f, \Pi(J), J)| \leq \varepsilon/2(p+1). \tag{$\star$}$$

Setting $\Pi = \{(x'_k, \tilde{R}'_k)\} \cup \{(x_k, \tilde{R}_k)\} \cup \bigcup_{i=1}^p \Pi(J_i)$, where the only marked quantities are given by $\{(x'_k, \tilde{R}'_k)\}$, one easily verifies that $\Pi \in \mathcal{P}(I, \bar{K}, \bar{K}_i, 2\Delta_i, \delta)$. Thus

$$\begin{aligned} \frac{\varepsilon}{2} \geq & \mathcal{S}(f, \Pi, I) - F(I) = \sum |F(R_k) - f(x_k)|R_k|_n| \\ & + \sum |F(R'_k) - f(x'_k)|R'_k|_n| + \sum_{i=1}^p [\mathcal{S}(f, \Pi(J_i), J_i) - F(J_i)] \end{aligned}$$

and using $(\star)$ completes the proof. $\square$

The final step gives the (SH) property for our set function F and the division $\dot{E}, (E_i, C_i)_{i\in\mathbb{N}}$ of I.

8. $\forall\, \varepsilon > 0,\ \ K > 0,\ \ K_i > 0 \quad \exists\ \Delta_i > 0,\ \ \delta : I \to \mathbb{R}^+$ such that

$$\sum \Big| F(A_k) - f(x_k)|A_k|_n \Big| + \sum \Big| F(A'_k) - f(x'_k)|A'_k|_n \Big| \le \varepsilon$$

holds for any (I,δ)-fine sequence $\{(x_k, A_k)\} \cup \{(x'_k, A'_k)\}$ with A_k, A'_k being subsets of I and

(i) if $x_k \in \dot{E}$, then $A_k \in \mathcal{A}_K(S)$; $\{A_k : x_k \in E_i\} \in C_i(K_i, \Delta_i)$ $(i \in \mathbb{N})$

(ii) $\{A'_k\} \in C_1^{n-1}(K)$ and $x'_k \in \dot{E} \cup \bigcup_{C_i \in \dot{\Gamma}} E_i$ for all k.

PROOF. Let ε, K, K_i be given positive numbers and if $C_i = C_1^\alpha$ (resp. C_2^α) with $0 \le \alpha < n-1$ determine $\Delta_i > 0$ and $\delta_i : E_i \to \mathbb{R}^+$ for $\varepsilon/2^{i+2}$ and K_i according to **6.**. Furthermore, we can find positive numbers $\bar{\Delta}_i > 0$ $(C_i \in \tilde{\Gamma})$ and $\delta : I \to \mathbb{R}^+$ by **7.** for $\varepsilon/4$, $\bar{K} = 2(1+\rho^* c(n))(K+\rho^*)^2$ and $\bar{K}_i = 2(1+\rho^* c(n))(K_i+\rho^*)^2$ $(C_i \in \tilde{\Gamma})$. We may assume $\delta \le \delta_i$, and we set $\Delta_i = \bar{\Delta}_i/(2+\rho^* c(n))$ for $C_i \in \tilde{\Gamma}$.

Let $\{(x_k, A_k)\} \cup \{(x'_k, A'_k)\}$ be a (I,δ)-fine sequence having the properties stated in **8.**. We obviously may assume $A_k^\circ, A_k^{'\circ} \ne \emptyset$ for all k since for an n-null set $A \subseteq I$ $(A \in \mathcal{A}(S))$, we have $F(A) = 0$ by the additivity of F. By **6.** we have

$$\sum_{x_k \in E_i} \Big| F(A_k) - f(x_k)|A_k|_n \Big| \le \varepsilon/2^{i+2} \quad \text{for } C_i \notin \tilde{\Gamma}. \tag{1}$$

For the remaining A_k (resp. all the A'_k) we can determine corresponding figures R_k (resp. R'_k) according to **5.** such that

$$|A_k - R_k|_n \le \frac{\varepsilon|A_k|_n}{4(1+|f(x_k)|)|I|_n},\ |F(A_k) - F(R_k)| \le \frac{\varepsilon|A_k|_n}{4|I|_n}$$

$$\Big(\text{resp. } |A'_k - R'_k|_n \le \frac{\varepsilon|A'_k|_n}{4(1+|f(x'_k)|)|I|_n},\ |F(A'_k) - F(R'_k)| \le \frac{\varepsilon|A'_k|_n}{4|I|_n}.\Big)$$

Obviously

$$\begin{aligned}
&\Big| F(A_k) - f(x_k)|A_k|_n \Big| \\
\le &\Big| F(A_k) - F(R_k) \Big| + \Big| F(R_k) - f(x_k)|R_k|_n \Big| + |f(x_k)||A_k - R_k|_n \\
\le &\Big| F(R_k) - f(x_k)|R_k|_n \Big| + \varepsilon|A_k|_n/2|I|_n
\end{aligned}$$

and since the same inequality holds for all marked sets A'_k, we have

$$\begin{aligned}\sum\Big|F(A_k)-f(x_k)|A_k|_n\Big|+\sum\Big|F(A'_k)-f(x'_k)|A'_k|_n\Big| \\ \le \varepsilon/2+\sum\Big|F(R_k)-f(x_k)|R_k|_n\Big|+\sum\Big|F(R'_k)-f(x'_k)|R'_k|_n\Big|,\end{aligned} \tag{2}$$

where we only sum over those (x_k, A_k) not occurring in (1).

Now, using further properties of the R_k and R'_k as stated in **5.**, it is easy to see that the sequence $\{(x_k, R_k)\} \cup \{(x'_k, R'_k)\}$ fulfills the requirements of **7.** with parameters $\bar{K}, \bar{K}_i, \bar{\Delta}_i$ (Note, e.g., that $|\partial\tilde{R}'_k|_{n-1} \le (2+\rho^* c(n))|\partial A'_k|_{n-1}$, since in case $n=1$, we have $|\partial A'_k|_{n-1} \ge 1$.) and thus

$$\sum\Big|F(R_k)-f(x_k)|R_k|_n\Big|+\sum\Big|F(R'_k)-f(x'_k)|R'_k|_n\Big| \le \varepsilon/4. \tag{3}$$

Consequently, considering (1)–(3), we get

$$\sum\Big|F(A_k)-f(x_k)|A_k|_n\Big|+\sum\Big|F(A'_k)-f(x'_k)|A'_k|_n\Big| \le \varepsilon$$

as desired. □

5 Every Variationally Integrable Function is $\nu(S)$-integrable

In this section we assume $A \in \mathcal{A}(S)$ and $f : A \to \mathbb{R}$ to be fixed and using Theorem 4.1 we will show that if f is v-integrable in the sense of [Pf 3, Def. 5.1], then f is $\nu(S)$-integrable and both integrals coincide.

For $x = (x_i) \in \mathbb{R}^n$ and $r > 0$ set $C(x, r) = \{y = (y_i) \in \mathbb{R}^n : |x_i - y_i| < r, 1 \le i \le n\}$. Let $E \subseteq \mathbb{R}^n$ be $|\cdot|_n$-measurable, $x \in \mathbb{R}^n$. Then we call x a density (resp. a dispersion) point of E if

$$\liminf_{r\to 0}\frac{|E\cap C(x,r)|_n}{(2r)^n} = 1 \text{ (resp. } \limsup_{r\to 0}\frac{|E\cap C(x,r)|_n}{(2r)^n} = 0).$$

We denote the set of all density points of E by $\mathrm{int}_e E$ and $\mathrm{cl}_e E$ denotes the complement of the set of all dispersion points of E. By [Saks] the sets E, $\mathrm{int}_e E$, $\mathrm{cl}_e E$ differ at most by n-null sets, we obviously have the inclusions $E^\circ \subseteq \mathrm{int}_e E \subseteq \mathrm{cl}_e E \subseteq \mathrm{cl} E$ and we set $\partial_e E = \mathrm{cl}_e E \setminus \mathrm{int}_e E \subseteq \partial E$.

A bounded $|\cdot|_n$-measurable set $B \subseteq \mathbb{R}^n$ is called a BV *set* if $|\partial_e B|_{n-1}$ is finite (cf. [Fed]) and for any BV set B we define its regularity by $r(B) = |B|_n / d(B)|\partial_e B|_{n-1}$ if $d(B)|\partial_e B|_{n-1} > 0$ and by $r(B) = 0$ otherwise. We denote by BV_A the system of all BV sets contained in A. (Note that A itself is a BV set.) A function $F : BV_A \to \mathbb{R}$ is called *continuous* if for every $\varepsilon > 0$

there is a $\delta > 0$ such that $|F(B)| < \varepsilon$ for any $B \in BV_A$ with $|B|_n < \delta$ and $|\partial_e B|_{n-1} < 1/\varepsilon$. Furthermore, a function $F : BV_A \to \mathbb{R}$ is called *additive* if $F(B) = \sum F(B_k)$ for any $B \in BV_A$ and any finite sequence of disjoint BV sets $\{B_k\}$ whose union is B (compare with Section 4, Subsection 2).

Remark 5.1 *According to* [Fed] *we can associate with any BV set B a Borel vector function $\vec{n}_B : \mathbb{R}^n \to \mathbb{R}^n$ ($\|\vec{n}_B\| \le 1$), the so called exterior normal of B, such that $\int_{\partial_e B} \vec{v} \cdot \vec{n}_B \, d|\cdot|_{n-1} = \int_B \operatorname{div} \vec{v}$ holds for any vector function $\vec{v}$ which is continuously differentiable in a neighborhood of $\operatorname{cl} B$. From this we deduce at once for any $B \in \mathcal{A}$ the inequality $|B|_n \le d(B)|\partial_e B|_{n-1}$.*

Remark 5.2 *Let $F : BV_A \to \mathbb{R}$ be a continuous additive function, $E \subseteq A$.*

(i) Observe that, because of the continuity, $F(B) = 0$ for any $B \in BV_A$ with $|B|_n = 0$. Consequently F is additive in the sense of Section 4, Subsection 2.

(ii) $\forall\ \varepsilon > 0,\ K > 0\ \ \exists\ \delta : E \to \mathbb{R}^+$ such that $\sum |F(B_k)| \le \varepsilon$ for any (E, δ)-fine sequence $\{(x_k, B_k)\}$ with $B_k \subseteq A$ and $\{B_k\} \in C_1^{n-1}(K)$, i.e. $\sum |\partial B_k|_{n-1} \le K$. (In the language of [Ju-No 3, Sec. 1c] *we would say that F satisfies $\mathcal{N}(C_1^{n-1}, E)$.) For, if $\varepsilon > 0$, $K > 0$ are given, we set $\varepsilon' = \frac{1}{2}\min(\varepsilon, 1/K)$, and we determine for ε' a $\delta' > 0$ according to the continuity of F. Now we let $\delta(\cdot) = \delta'/2K$ on E and we assume $\{(x_k, B_k)\}$ to be a (E, δ)-fine sequence with $B_k \subseteq A$ and $\sum |\partial B_k|_{n-1} \le K$. Then $|\bigcup B_k|_n \le \sum d(B_k)|\partial B_k|_{n-1} \le \sum \delta(x_k)|\partial B_k|_{n-1} < \delta'$, $|\bigcup \partial B_k|_{n-1} \le \sum |\partial B_k|_{n-1} \le K < 1/\varepsilon'$. Hence*

$$\sum |F(B_k)| = F(\bigcup_{F(B_k) \ge 0} B_k) - F(\bigcup_{F(B_k) < 0} B_k) \le 2\varepsilon' \le \varepsilon.$$

(iii) Assume in addition $|E|_n = 0$ and let $\varepsilon > 0$ be given. Write $E = \bigcup_{j \in \mathbb{N}} E_j$ with $E_j = \{x \in E : j - 1 \le |f(x)| < j\}$, choose open sets $G_j \supseteq E_j$ with $|G_j|_n \le \varepsilon/j2^j$ and determine for $x \in E_j$ a $\delta(x) > 0$ with $B(x, \delta(x)) \subseteq G_j$ what defines a function $\delta : E \to \mathbb{R}^+$. Then we have for any (E, δ)-fine sequence $\{(x_k, B_k)\}$ with $B_k \subseteq A$ the inequality

$$\sum |f(x)||B_k|_n \le \sum_{j \in \mathbb{N}} \sum_{x_k \in E_j} j|B_k|_n \le \sum_{j \in \mathbb{N}} j|G_j|_n \le \varepsilon.$$

Proposition 5.1 *Suppose the function f to be v-integrable on A. Then f is $\nu(S)$-integrable on A and both integrals coincide.*

PROOF. Since f is v-integrable on A, there is according to [Pf 3, Def. 5.1, Cor. 5.5] a uniquely determined continuous additive function $F : BV_A \to \mathbb{R}$ and $F(A)$ is called the v-integral of f on A. Furthermore, by [Pf 3, Prop. 5.15] there is a certain set $T \subseteq \mathbb{R}^n$ which can be expressed as a countable union of sets with finite outer $|\cdot|_{n-1}$-measure and consequently, we may write $A \cap (T \cup \partial A) = \bigcup_{i\in\mathbb{N}} E_i$ with disjoint sets E_i, $|E_i|_{n-1} < \infty$. Hence a division of A is obviously given by $\dot{E} = A \setminus (T \cup \partial A)$, $(E_i, C_1^{n-1})_{i\in\mathbb{N}}$ and we assume $\varepsilon > 0$, $K > 0$, $K_i > 0$ to be given according to Theorem 4.1. We set $\Delta_i = 1$, $\varepsilon' = \frac{1}{5}\min(\varepsilon, 1/K^2)$ and for ε' we choose a $\delta : \dot{E} \to \mathbb{R}^+$ according to [Pf 3, Prop. 5.15]. (Note that $\dot{E} = A^\circ \setminus T \subseteq \mathrm{cl}_e A \setminus T$.) By Remark 5.2(iii) we can find for $\varepsilon/5$ a $\delta : A \setminus \dot{E} \to \mathbb{R}^+$ and we obviously may assume $\delta(\cdot) \leq \varepsilon/5K(1+|f(\cdot)|)$ on A. Finally we determine for $\varepsilon/5\,2^i$, K_i (resp. for $\varepsilon/5$), K a $\delta_i : E_i \to \mathbb{R}^+$ (resp. $\dot{\delta} : \dot{E} \to \mathbb{R}^+$) according to Remark 5.2(ii), and again we may assume $\delta(\cdot) \leq \delta_i(\cdot)$ on E_i (resp. $\delta(\cdot) \leq \dot{\delta}(\cdot)$) on $\dot{E}$. Thus a function $\delta : A \to \mathbb{R}^+$ is determined and we assume $\{(x_k, A_k)\} \cup \{(x'_k, A'_k)\}$ to be a δ-fine partition of A fulfilling the following two conditions:

(i) if $x_k \in \dot{E}$, then $A_k \in \mathcal{A}_K(S)$; $\{A_k : x_k \in E_i\} \in C_1^{n-1}(K_i)$ $(i \in \mathbb{N})$

(ii) $\{A'_k\} \in C_1^{n-1}(K)$ and all $x'_k \in \dot{E}$.

Then, using Remark 5.2(i) and the choice of δ, we get

$$\begin{aligned}
&\Big|F(A) - \Big(\sum f(x_k)|A_k|_n + \sum f(x'_k)|A'_k|_n\Big)\Big| \\
\leq &\sum_{x_k\in\dot{E}} |F(A_k^\circ) - f(x_k)|A_k^\circ|_n| + \sum_{i\in\mathbb{N}}\sum_{x_k\in E_i} |F(A_k)| + \sum |F(A'_k)| \\
&+ \sum_{x_k\in A-\dot{E}} |f(x_k)||A_k|_n + \sum |f(x'_k)||A'_k|_n \\
\leq &\varepsilon' + \sum_{i\in\mathbb{N}} \frac{\varepsilon}{5\,2^i} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \sum |f(x'_k)|\delta(x'_k)|\partial A'_k|_{n-1} \leq \varepsilon.
\end{aligned}$$

Here we applied [Pf 3, Prop. 5.15] to the pairs $\{(x_k, A_k^\circ) : x_k \in \dot{E},\ A_k^\circ \neq \emptyset\}$ and it suffices to see that $\partial_e(A_k^\circ \cup \{x_k\}) = \partial_e A_k$. Hence using Remark 5.1 and the fact that $A_k \in \mathcal{A}_K(S)$, we have $r(A_k^\circ \cup \{x_k\}) \geq 1/K^2 > \varepsilon'$. $\square$

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