RESEARCH

Real Analysis Exchange
Vol. 21(2), 1995–96, pp. 629–636

Héctor Cuenya [*], Departamento de Matemáticas, Universidad Nacional de Río Cuarto, 5.800 Río Cuarto, Argentina
e-mail: hcuenya@@unrccc.edu.ar, mmarano@@unrccc.edu.ar

Miguel Marano, Departamento de Matemáticas, Universidad de Jaén, 23071 Jaén, Spain e-mail: mmarano@@jabalcuz.ujaen.es

A CHARACTERIZATION OF ORLICZ FUNCTIONS PRODUCING AN ADDITIVE PROPERTY

Abstract

It is shown that the only Luxemburg functionals that satisfy a very simply formulated property are induced by pth-power functions, $0 < p < \infty$. The known result that Orlicz spaces cannot be normed analogously to L_p-spaces follows as a consequence.

1 Introduction

Let $\Psi : [0,\infty) \to [0,\infty]$ be a nondecreasing function, $\Psi(0) = 0$, $\Psi(x) \to \infty$ as $x \to \infty$, and such that if $0 < a < b$, $0 < \Psi(a)$, $\Psi(b) < \infty$, then Ψ is strictly increasing on $[a,b]$ and continuous on $[0,b]$. Such a function is called an *O-function*. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Identify real valued functions on Ω that differ only on a set of measure zero. Let $\mathcal{M}$ denote the corresponding set of congruence classes. It is known [2, 3] that the pair $\{\Psi, \mu\}$ induces the *Luxemburg functional* on the *Orlicz space*

$$\mathcal{L}_\Psi(\mu) := \left\{ f \in \mathcal{M} : \int_\Omega \Psi(\alpha|f|)\, d\mu < \infty \text{ for some } \alpha > 0 \right\}.$$

Key Words: Orlicz function, Luxemburg norm, L_p-spaces
Mathematical Reviews subject classification: Primary: 46E30
Received by the editors August 27, 1995
[*]Partially supported by CONICOR and Univ. Nac. de Río Cuarto, Argentina.

Its expression is

$$\rho_{\Psi,\mu}(f) := \inf\left\{\lambda > 0 : \int_{\Omega} \Psi(|f|/\lambda)\, d\mu \le 1\right\}.$$

A well known example is the L_p-norm (with abuse of language for $0 < p < 1$), induced by any μ and $\Psi(x) \equiv cx^p$, $c > 0$, for $0 < p < \infty$, which gives $\mathcal{L}_\Psi(\mu) = L_p(\mu) = \{f \in \mathcal{M} : \int_\Omega |f|^p\, d\mu < \infty\}$, $\rho_{\Psi,\mu}(f) = (c\int_\Omega |f|^p\, d\mu)^{1/p}$, and induced by any μ and a function Ψ that satisfies $\Psi(x) = 0$ if $0 \le x \le a$, $\Psi(x) = \infty$ if $x > a$, $a > 0$, for $p = \infty$, which gives $\mathcal{L}_\Psi(\mu) = L_\infty(\mu) = \{f \in \mathcal{M} : \text{ess sup}\,|f| < \infty\}$, $\rho_{\Psi,\mu}(f) = \|f\|_\infty = a\,\text{ess sup}\,|f|$. The L_p-norms satisfy the following property, applicable to any functional ρ defined on a quite arbitrary real function space.

For any set $B \in \mathcal{A}$ and f, g simple functions, if $\rho(f\chi_B) = \rho(g\chi_B)$ and $\rho(f\chi_{\Omega\setminus B}) = \rho(g\chi_{\Omega\setminus B})$, then $\rho(f) = \rho(g)$. $(*)$

We recall that a simple function is one of form $\sum_{i=1}^n c_i\chi_{A_i}$, $\mu(A_i) < \infty$, where χ_A denotes the characteristic function of the set A. If A has, exactly, none, one or two disjoint sets of finite and positive measure, then the class of all simple functions can be identified with $\{0\}$, $\mathbb{R}$ or $\mathbb{R}^2$, respectively, and in these cases any homogeneous functional defined on $\mathcal{L}_\Psi(\mu)$, depending on $|f|$, (e.g. a Luxemburg functional) satisfies $(*)$. We show in this paper that a rather different result follows when $\mathcal{A}$ has at least three disjoint sets of finite and positive measure μ. The function Ψ is said to satisfy property P_μ if $\{\Psi, \mu\}$ induces a Luxemburg functional on $\mathcal{L}_\Psi(\mu)$ satisfying $(*)$. We shall give a description of such functions. In all cases they yield a L_p-norm, $0 < p \le \infty$. In case that μ is σ-finite and Ψ is convex, this latter result can also be obtained from a classical theorem of H. F. Bohnenblust [1, 4]. As a consequence of that theorem, for $\dim \mathcal{L}_\Psi(\mu) \ge 3$ it is obtained that homogeneous functionals on $\mathcal{L}_\Psi(\mu)$ that satisfy $(*)$ are p-additive, $0 < p \le \infty$. For $p \ge 1$ this fact implies in turn that $\rho_{\Psi,\mu}(f)$ is a L_p-norm. However we do not follow the ideas of that theorem neither use the p-additive condition. Depending on a general measure μ, in each case our proofs directly lead to the characterization of Ψ.

We consider in Section 2 a Luxemburg functional induced by a continuous O-function Ψ. In Section 3 we assume that Ψ is not continuous and that μ is in addition a σ-finite measure. As a consequence of Theorem 1 we get in Section 4 the known fact that the space $\mathcal{L}'_\Psi(\mu) := \{f \in \mathcal{M} : \int_\Omega \Psi(\alpha|f|)\, d\mu < \infty$ for all $\alpha > 0\}$ cannot be normed analogously to L_p-spaces, $p > 0$, whenever $\dim \mathcal{L}'_\Psi(\mu) \ge 2$.

We say that $(\Omega, \mathcal{A}, \mu)$ is *infinitely divisible* if there are measurable subsets of Ω with positive and arbitrarily small measure. If $(\Omega, \mathcal{A}, \mu)$ is not infinitely divisible and $\mathcal{A}$ has at least one set of finite and positive measure, then we let

$$r_0 = \inf\{\mu(A),\, A \in \mathcal{A},\, \mu(A) > 0\} > 0.$$

2 Characterization of a Continuous O-function Ψ

In this section we assume that Ψ satisfies $\Psi(\mathbb{R}_+) \supseteq \mathbb{R}_+$. Hence its right inverse function $\Psi^{-1} : (0,\infty) \to \mathbb{R}_+$ exists, is continuous and satisfies $\Psi(\Psi^{-1}(x)) = x$ for all $x > 0$. We say that a Luxemburg functional induced by any measure and such a function Ψ is a *L-functional*.

Theorem 1 *Assume that $\mathcal{A}$ has at least three disjoint sets of finite and positive measure.*

(a) If $(\Omega, \mathcal{A}, \mu)$ is infinitely divisible, then Ψ satisfies property P_μ if and only if $\Psi(x) \equiv cx^p$ on $[0,\infty)$, $c > 0$, $p > 0$.

(b) If $(\Omega, \mathcal{A}, \mu)$ is not infinitely divisible, then Ψ satisfies property P_μ if and only if Ψ verifies $\Psi(x) = cx^p$ for any $x \in [0, \Psi^{-1}(1/r_0)]$, $p > 0$, $c > 0$.

In both cases the functional induced is a L_p-norm and therefore these are the only L-functionals that verify property $(*)$.

Proof. Assume that Ψ satisfies property P_μ. Take $F \in \mathcal{A}$, $0 < \mu(F) < \infty$, such that there are two disjoint sets G and E of finite and positive measure, $G \cup E \subseteq \Omega \backslash F$, $\mu(E) \geq \mu(F)$. Such a set F always exists due to the assumptions on $(\Omega, \mathcal{A}, \mu)$. Assume first $\mu(F) = \Psi(1) = 1$. Take $h \in \mathbb{R}$, $h > 0$, such that $\rho_{\Psi,\mu}(h\chi_{F\cup G}) = \rho_{\Psi,\mu}(\chi_F) = 1$. So property $(*)$ implies $\rho_{\Psi,\mu}(h\chi_{F\cup G} + s\chi_E) = \rho_{\Psi,\mu}(\chi_F + s\chi_E) =: \delta_s$ for any $s \in [0,\infty)$. Therefore, by definition of $\rho_{\Psi,\mu}$ and the continuity of Ψ (on $[0,1)$), we get $\mu(F \cup G)\Psi(h) = 1$ on the one hand and on the other hand

$$\mu(F \cup G)\Psi(h/\delta_s) + \mu(E)\Psi(s/\delta_s) = \Psi(1/\delta_s) + \mu(E)\Psi(s/\delta_s) = 1,$$

whence $\Psi(h/\delta_s) = \Psi(h)\Psi(1/\delta_s)$ for any $s \geq 0$. As $s \mapsto \delta_s$ maps continuously $[0,\infty)$ onto $[1,\infty)$, we have obtained that h is a *multiplier* for Ψ, i.e., h is a point $m \in [0,1]$ that satisfies

$$\Psi(m\gamma) = \Psi(m)\Psi(\gamma) \text{ for any } \gamma \in [0,1].$$

Observe that the former equation implies that $\Psi(h^n) = [\Psi(h)]^n$ for $n \in \mathbb{N}$. Moreover, h^n is a multiplier for Ψ. Since $0 < h < 1$, $0 < \Psi(h) < 1$, it follows that $h^n \downarrow 0$, $\Psi(h^n) \downarrow 0$ as $n \to \infty$ (and therefore $\Psi(x) > 0$ if $x > 0$). Take

$k_1 \in \mathbb{R}$, $k_1 > 0$, such that $\rho_{\Psi,\mu}(k_1\chi_F + mk_1\chi_G) = \rho_{\Psi,\mu}(\chi_F) = 1$, where m is a multiplier for Ψ. Hence $\Psi(k_1) + \mu(G)\Psi(mk_1) = 1$. As $k_1 < 1$, it follows that

$$\Psi(k_1)[1 + \mu(G)\Psi(m)] = 1.$$

On the other hand, property $(*)$ implies $\rho_{\Psi,\mu}(k_1\chi_F + mk_1\chi_G + s\chi_E) = \delta_s$ for any $s \geq 0$, whence $\Psi(k_1\gamma) = \Psi(k_1)\Psi(\gamma)$ for $\gamma \in [0,1]$. We have just proved that if m is a multiplier for Ψ, then $k_1 = \Psi^{-1}(1/[1 + \mu(G)\Psi(m)])$ is also a multiplier for Ψ. It follows that

$$m_n = m_n(\Psi, \mu) := \Psi^{-1}(1/[1 + \mu(G)\Psi(h^n)])$$

is a multipler for Ψ, $n \in \mathbb{N}$. As Ψ^{-1} is continuous and $\Psi^{-1}(1) = 1$, we get that $m_n \uparrow 1$ as $n \to \infty$.

The existence of the sequence $\{m_n\}$ implies that any $m \in [0,1]$ is in the collection $\mathcal{P}$ of multipliers for Ψ. Indeed, observe first that $\mathcal{P}$ is closed because Ψ is continuous. Hence

$$\beta_0(m) := \inf\{\beta \in \mathcal{P} : \beta > m\} \in \mathcal{P} \text{ for } m \in [0,1),$$

and $\beta_0(m) = m$ because $m_n\beta_0(m) \in \mathcal{P}$ for all $n \in \mathbb{N}$.

For $x \in (0,1]$ we have

$$[\Psi(xm_n) - \Psi(x)]/[x(m_n - 1)] = \Psi(x)[\Psi(m_n) - \Psi(1)]/[x(m_n - 1)]. \tag{1}$$

On the other hand, the obvious estimates below show that Ψ is absolutely continuous on $[\eta, 1]$ for all $\eta \in (0,1)$.

Let $x_i \in [\eta, 1]$, $1 \leq i \leq n+1$, $n \in \mathbb{N}$, and $x_1 < x_2 < \cdots < x_{n+1}$. Let $\gamma_i := x_i/x_{i+1}$, $1 \leq i \leq n$. Then

$$\eta \sum_{i=1}^{n} [1 - \gamma_i] < \sum_{i=1}^{n} x_{i+1}[1 - \gamma_i] = \sum_{i=1}^{n} [x_{i+1} - x_i],$$

$$\begin{aligned}
\sum_{i=1}^{n} [\Psi(x_{i+1}) - \Psi(x_i)] &= \sum_{i=1}^{n} [\Psi(x_{i+1}) - \Psi(\gamma_i x_{i+1})] \\
&= \sum_{i=1}^{n} \Psi(x_{i+1})[1 - \Psi(\gamma_i)] \leq \sum_{i=1}^{n} [1 - \Psi(\gamma_i)],
\end{aligned}$$

$$1 - \Psi(\gamma) \leq K[1 - \gamma] \text{ for some } K > 0 \text{ and any } \gamma < 1.$$

Therefore we get that the derivative $\Psi'(x)$ exists and is finite-valued for almost every x on $[0,1]$. Hence the left side in eq. (1) converges to $\Psi'(x)$ as $n \to \infty$ for almost every $x \in [0,1]$, and it follows that the right side in eq. (1) converges to $p\Psi(x)/x$ as $n \to \infty$ for any $x \in (0,1]$, where $p \geq 0$. Therefore $\Psi'(x)/\Psi(x) = p/x$ a.e. on $(0,1]$. As Ψ is not constant, we have $p > 0$. Since $\ln \Psi$ is absolutely continuous on $[\eta, 1]$, the integration of both sides of the former equation from x to 1 gives $\Psi(x) \equiv x^p$ on $[0,1]$. Observe that, so far, only the restriction of Ψ on [0,1] has been considered (cf. the end of Section 3).

Suppose now $\mu(F) = r > 0$, $\Psi(1) \geq 0$. The L-functional induced by $r\Psi$ and μ/r on $\mathcal{L}_\Psi(\mu)$ coincides with the L-functional induced by Ψ and μ. On the other hand, the L-functional induced by $\tilde{\Psi}(x) := r\Psi(\Psi^{-1}(1/r)x)$ and μ/r on $\mathcal{L}_\Psi(\mu)$ is $\Psi^{-1}(1/r)$ times the L-functional induced by $r\Psi$ and μ/r, and therefore it also satisfies property $(*)$, with $\tilde{\Psi}(1) = (\mu/r)(F) = 1$. So we get $\tilde{\Psi}(x) \equiv x^p$ on $[0,1]$, with $p > 0$, whence $\Psi(x) \equiv cx^p$ on $[0, \Psi^{-1}(1/r)]$, where $c = 1/[r(\Psi^{-1}(1/r))^p]$.

Under the hypothesis of (a) we can take $r \downarrow 0$. Then the case $\Psi^{-1}(1/r) \uparrow b$, $b < \infty$, leads to a contradiction, whence $\psi^{-1}(1/r) \uparrow \infty$ and the necessary part of (a) follows. The sufficiency of (a) is obvious. The necessity of (b) follows by taking $r \to r_0$. (Observe that this taking of limits in r is compatible with the assumption on F at the beginning of the proof). Conversely, if $\Psi(x) \equiv cx^p$ on $[0, \Psi^{-1}(1/r_0)]$, then $\tilde{\Psi}(x) := r_0\Psi(\Psi^{-1}(1/r_0)x) \equiv x^p$ on [0,1] and, as mentioned above, $\tilde{\Psi}$ and μ/r_0 induce, up to a multiplicative constant, the same L-functional as Ψ and μ. So, to conclude the proof, it suffices to observe that if $\Psi(x) \equiv x^p$ on $[0,1]$ and in addition $\mu(C) \geq 1$ for any measurable set C with $\mu(C) > 0$, then $\{\Psi, \mu\}$ induces the standard L_p-norm on $\mathcal{L}_\Psi(\mu)$ ($= L_p(\mu)$). Indeed, if $g \in \mathcal{M}$ and $\int_\Omega |g|\, d\mu \leq 1$, then $|g| \leq 1$ almost everywhere (μ) on Ω, whence $\int_\Omega \Psi(|f|/\lambda)\, d\mu \leq 1$ is equivalent to $\int_\Omega (|f|/\lambda)^p\, d\mu \leq 1$. Therefore $\rho_{\Psi,\mu}(f) = \inf\{\lambda : \int_\Omega (|f|/\lambda)^p\, d\mu = 1\} = \left(\int_\Omega |f|^p\, d\mu\right)^{1/p}$. □

3 Characterization of a Discontinuous O-function Ψ

Now we suppose, and only in this section, that Ψ is not $(\Omega, \mathcal{A}, \mu)$ is in addition a σ-finite measure space. So we consider an O-function Ψ jumping to infinity at a, $a > 0$. We say that the Luxemburg functional induced by such a pair $\{\Psi, \mu\}$ on $\mathcal{L}_\Psi(\mu)$ is a L^*-functional. Observe that when $(\Omega, \mathcal{A}, \mu)$ is not infinitely divisible, a L^*-functional may coincide with a L-functional. Since μ is σ-finite, the condition required to $\mathcal{A}$ in Theorem 1 is now equivalent to $\dim \mathcal{L}_\Psi(\mu) \geq 3$. We can suppose without loss of generality that Ψ is left continuous. An example of such a function is: $\Psi(x) = 0$ if $0 \leq x \leq a$, $\Psi(x) = \infty$ if $x > a$. It is easy to see that $\{\Psi, \mu\}$, with this function Ψ, induces the *a essential sup* norm on $\mathcal{L}_\Psi(\mu)$, which we call, as usual, a L_∞-norm. Moreover, it is easy to show that if $\mu(\Omega)\Psi(a) \leq 1$, then $\{\Psi, \mu\}$ induces a L_∞-norm. Observe also that if $\mu(C) \geq 1$ for any measurable set C of positive measure and Ψ satisfies $\Psi(x) \equiv x^p$ on $[0,1]$ (e.g., $\Psi(x) \equiv x^p$ on $[0,1]$, $\Psi(x) = \infty$ for $x > 1$), then $\{\Psi, \mu\}$ induces the standard L_p-norm on $\mathcal{L}_\Psi(\mu)$.

Assume now that $\{\Psi, \mu\}$ induces a L^*-functional, where $\mu(\Omega)\Psi(a) > 1$. Under this condition we consider two exhaustive cases. Suppose first that there exists $F \in \mathcal{A}$, $0 < \mu(F)\Psi(a) < 1$. For instance, this is the case if $(\Omega, \mathcal{A}, \mu)$ is infinitely divisible. Assume also that there exists $B \in \mathcal{A}$, $B \supseteq F$, $\mu(B) < \mu(\Omega)$, $\mu(B)\Psi(a) > 1$. (This is the case if $\mu(\Omega) = \infty$). Then, since Ψ is a continuous function on $[0, a)$, it follows that there exists b, $0 < b < a$, such that $\mu(F)\Psi(a) + \mu(B \setminus F)\Psi(b) = 1$. Then we have $\rho_{\Psi,\mu}(a\chi_F) = \rho_{\Psi,\mu}(a\chi_F + b\chi_{B\setminus F}) = 1$. Take $E \in \mathcal{A}$, $E \subseteq \Omega \setminus B$, $0 < \mu(E) < \infty$. Therefore there exists c, $0 < c \leq a$, $\Psi(c) > 0$, such that $\mu(F)\Psi(a) + \mu(E)\Psi(c) \leq 1$, whence $\rho_{\Psi,\mu}(a\chi_F + c\chi_E) = 1$ but $\rho_{\Psi,\mu}(a\chi_F + b\chi_{B\setminus F} + c\chi_E) > 1$. So property $(*)$ does not hold.

If for any set D in $\mathcal{D} = \{D \in \mathcal{A},\, D \supseteq F,\, \mu(D) < \mu(\Omega)\}$ is $\mu(D)\Psi(a) \leq 1$, then consider a set $B \in \mathcal{D}$ such that $\mu(B) = \sup\{\mu(D),\, D \in \mathcal{D}\}$. Hence $\mu(B) < \mu(\Omega) < \infty$ and $\Omega \setminus B$ is an atom, whence B is not an atom. At this point we can assume that F satisfies at least one of the following conditions.

(1) There exists $G \in \mathcal{A}$ such that $F \subseteq G$, $\mu(F)\Psi(a) < \mu(G)\Psi(a) < 1$.

(2) F is an atom.

In either case it follows that $\mu(F) < \mu(B)$. We have $\rho_{\Psi,\mu}(a\chi_F) = \rho_{\Psi,\mu}(a\chi_B) = 1$. As $\mu(F) + \mu(\Omega \setminus B) < \mu(\Omega)$, we get that $F \cup (\Omega \setminus B) \in \mathcal{D}$, i.e., $\mu(F)\Psi(a) + \mu(\Omega \setminus B)\Psi(a) \leq 1$. So $\rho_{\Psi,\mu}(a\chi_F + a\chi_{\Omega\setminus B}) = 1$ but $\rho_{\Psi,\mu}(a\chi_B + a\chi_{\Omega\setminus B}) > 1$, whence property $(*)$ does not hold.

It remains to consider the case where $(\Omega, \mathcal{A}, \mu)$ is not infinitely divisible and $r_0\Psi(a) \geq 1$. In this case Ψ^{-1} is well defined and continuous on the interval $(0, 1/r_0]$, whence the proof in Theorem 1(b) applies without changes. Thus, in this case Ψ satisfies property P_μ if and only if $\Psi(x) \equiv cx^p$ on $[0, \Psi^{-1}(1/r_0)]$. So we have the following.

Theorem 2 *If* $\dim \mathcal{L}_\Psi(\mu) \geq 3$*, then the* L^**-functional induced by* Ψ *and* μ *satisfies property* $(*)$ *if it is necessarily a* L_p*-norm,* $0 < p \leq \infty$*.*

4 $\mathcal{L}'_\Psi(\mu)$ cannot be normed analogously to L_p-spaces

Let $\mathcal{L}'_\Phi(\mu) = \{f \in \mathcal{M} : \int_\Omega \Phi(a|f|)\, d\mu < \infty \text{ for all } \alpha > 0\}$, where Φ is a finite-valued convex O-function. $\mathcal{L}'_\Phi(\mu)$ is a linear subspace of $\mathcal{L}_\Phi(\mu)$. A natural way for trying to provide $\mathcal{L}'_\Phi(\mu)$ with a norm, analogously to the L_p-norm, is to consider the expression $\Phi^{-1}\left(\int_\Omega \Phi(|f|)\, d\mu\right)$. We refer to [6] for a historical survey of this and related questions. In this article it is proved that this attempt is possible only if $\Phi(x) \equiv cx^p$, in the case where μ is the Lebesgue measure on the real line. This result was later extended [5] to the linear space $\mathcal{L}'_\Psi(\mu)$, where Ψ is a finite-valued strictly increasing O-function and $(\Omega, \mathcal{A}, \mu)$ is such that $\dim \mathcal{L}'_\Psi(\mu) \geq 2$. More precisely, it is proved in that paper that if $\varphi_{\Gamma,\Psi,\mu}(f) := \Gamma\left(\int_\Omega \Psi(|f|)\, d\mu\right)$ is a homogeneous functional on $\mathcal{L}'_\Psi(\mu)$, being Γ and Ψ finite-valued strictly increasing O-functions, then $\Psi(x) \equiv \Psi(1)x^p$, $\Gamma(x) \equiv \Gamma(1)x^{1/p}$, $p > 0$. Next we prove that this result is a consequence of Theorem 1. Assume first that $\dim \mathcal{L}'_\Psi(\mu) = 2$. Therefore $\mathcal{L}'_\Psi(\mu)$ can be identified with $\mathbb{R}^2 = \{(x_1, x_2),\, x_1, x_2 \in \mathbb{R}\}$, and where

$$\varphi_{\Gamma,\Psi,\mu}(x_1, x_2) = \Gamma(a_1\Psi(|x_1|) + a_2\Psi(|x_2|)),\ a_1, a_2 > 0.$$

Assume that $\varphi_{\Gamma,\Psi,\mu}(x_1, x_2)$ is a homogeneous functional. For any $x > 0$ we have

$$\varphi_{\Gamma,\Psi,\mu}(x, 0) = \Gamma(a_1\Psi(x)) = x\varphi_{\Gamma,\Psi,\mu}(1, 0) = x\Gamma(a_1\Psi(1)).$$

As $\varphi_{a\Gamma,\Psi,\mu}(= a\varphi_{\Gamma,\Psi,\mu})$ is also a homogeneous functional for all $a > 0$, we can suppose without loss of generality that $\Gamma(a_1\Psi(1)) = 1$. Under this assumption, $a_1\Psi \equiv \Gamma^{-1}$. Define on $\mathbb{R}^3$ the homogeneous functional

$$\begin{aligned}
\varphi'(x_1, x_2, x_3) =& \varphi_{\Gamma,\Psi,\mu}(\varphi_{\Gamma,\Psi,\mu}(x_1, x_2), x_3) \\
=& \Gamma(a_1\Psi(\Gamma(a_1\Psi(|x_1|) + a_2\Psi(|x_2|))) + a_2\Psi(|x_3|)) \\
=& \Gamma(a_1\Psi(|x_1|) + a_2\Psi(|x_2|) + a_2\Psi(|x_3|)).
\end{aligned}$$

This functional is of the form $\varphi_{\Gamma,\Psi,\mu'}$, where $\dim \mathcal{L}'_\Psi(\mu') = 3$. This fact shows that it suffices to consider a homogeneous functional $\varphi_{\Gamma,\Psi,\mu}$ defined on $\mathcal{L}'_\Psi(\mu)$, $\dim \mathcal{L}'_\Psi(\mu) \geq 3$. After dividing Γ by $\Gamma(1)$ we can suppose $\Gamma(1) = 1$. For any $f \in \mathcal{L}'_\Psi(\mu)$ we have that $\varphi_{\Gamma,\Psi,\mu}(f) = 1$ if and only if $\rho_{\Psi,\mu}(f) = \inf\{\lambda : \int_\Omega \Psi(|f|/\lambda)\, d\mu \leq 1\} = 1$, and since these two functionals are homogeneous, it follows that $\varphi_{\Gamma,\Psi,\mu}(f) = \rho_{\Psi,\mu}(f)$ for all $f \in \mathcal{L}'_\Psi(\mu)$. Note that $\dim \mathcal{L}_\Psi(\mu) \geq 3$ implies that there exist three measurable sets of finite and positive measure, since Ψ is strictly increasing. As the simple functions belong to $\mathcal{L}'_\Psi(\mu)$ and $\varphi_{\Gamma,\Psi,\mu}$ satisfies property $(*)$, we get that Theorem 1 implies that $\varphi_{\Gamma,\Psi,\mu}(f) = \left(c\int_\Omega |f|^p\, d\mu\right)^{1/p}$ for all $f \in \mathcal{L}'_\Psi(\mu)$, and also $\Psi(x) \equiv cx^p$ in the case where $(\Omega, \mathcal{A}, \mu)$ is infinitely divisible, and $\Psi(x) \equiv cx^p$ on $[0, \Psi^{-1}(1/r_0)]$ in the case where $(\Omega, \mathcal{A}, \mu)$ is not infinitely divisible, $c > 0$, $p > 0$. For $\epsilon > 0$ define $\Gamma^*(x) = \Gamma(x/\epsilon)/\Gamma(1/\epsilon)$. Then applying in the latter case the same conclusion to the homogeneous functional $\varphi_{\Gamma^*,\psi,\epsilon\mu} = [1/\Gamma(1/\epsilon)]\varphi_{\Gamma,\Psi,\mu}$, and taking $\epsilon \downarrow 0$, it follows that also in this case $\Psi(x) = cx^p$ for any x in $[0, \infty)$. Therefore $\Gamma\left(c\int_\Omega |f|^p\, d\mu\right) = \left(c\int_\Omega |f|^p\, d\mu\right)^{1/p}$ for all $f \in \mathcal{L}'_\Psi(\mu)$, whence it is easy to show that $\Gamma(x) \equiv x^{1/p}$.

References

[1] H. F. Bohnenblust, *An axiomatic characterization of L_p-spaces*, Duke Math. J., **6** (1940), 627–640.

[2] M. A. Krasnoselskii and Ya. B. Rutikii, *Convex functions and Orlicz spaces*, P. Noordhoff, Groningen, 1961.

[3] W. Luxemburg, *Banach function spaces*, Thesis, Technische Hogeschool te Delft, 1955.

[4] P. Meyer-Nieberg, *Banach Lattices*, Springer Verlag, Berlin and New York, 1991.

[5] W. Wnuk, *Orlicz spaces cannot be normed analogously to L_p-spaces*, Indag. Math., **87**(3) (1984) 357–359.

[6] A. C. Zaanen, *Some remarks about the definition of an Orlicz space*, Lecture Notes in Mathematics, **945**, Springer Verlag, Berlin and New York, 1982, pp. 263–268.